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SOLVED EXERCISES FROM EXAMS IN MATHEMATICAL MODELLING

Study material

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Introduction

The exercises on the following pages appeared on theoretical exams in the study years 2018/19–2021/22 for the course Mathematical Modelling, which is an elective course for students enrolled in the undergraduate programme of Computer and Information Science at the University of Ljubljana.

The exercises are divided in chapters, which are covered in the course, so that you can try to solve them immediately after studying the appropriate chapter.

This study material is not peer-reviewed and so there can be some mistakes in the solutions. If you notice a mistake, please send me an e-mail to aljaz.zalar@fri.uni-lj.si.

For easier navigation within the document there are some shortcuts. If you click on the symbol ⇩ after the exercise, you will move to the corresponding solution. If you click on the symbol ⇧ at the solution, you will return to the text of the exercise. For the TeX template of the document I am grateful to dr. Aleksandra Franc.

Notation

$\mathbb{N}$	...	natural numbers
$\mathbb{R}$	...	real numbers
$\mathbb{C}$	...	complex numbers
i	...	imaginary unit $i = \sqrt{-1}$
$f'(x)$	...	a derivative of a function of one variable
$f(x, y, z)$	...	a function of three variables
$f_x(x, y, z)$	...	a partial derivative of f with respect to x
$\operatorname{grad} f(x, y, z)$	...	a gradient of a function of three variables
$\mathbb{R}^{m\times n}$	...	the set of $m \times n$ real matrices
$\operatorname{diag}(d_1, d_2, \ldots, d_r)$	...	a diagonal $r \times r$ matrix with $d_1, d_2, \ldots, d_r$ on the diagonal
$0_{m,n}$	...	a $m \times n$ matrix with only zero entries
I_n	...	a $n \times n$ identity matrix
$A_{i,j}$	...	the entry in the i-th row and j-th column of the matrix A
$A^\dagger$	...	the Moore-Penrose inverse of the matrix A
$\ker A$	...	the kernel of the matrix A
$(Jf)(x)$	...	the Jacobian matrix of the vector function f evaluated in the point x
$\langle u, v\rangle$	...	the inner product between the vectors u and v
$\|u\|$	...	the Euclidean norm of the vector u

Contents

Part 1

Exercises

CHAPTER 1

Linear systems

TASK. 1.
Compute the singular value decomposition (SVD) of the matrix

$$B = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

TASK. 2.
We are given the matrix A and the vector b:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}.$$

a. What is the rank of A?
b. Compute one generalized inverse of A.
c. Determine all solutions of the system $Ax = b$.

TASK. 3.
Let A, B be $m \times n$ matrices, $m, n \in \mathbb{N}$, such that $A^T B = 0$ and $BA^T = 0$. Verify the following statements:

a. Every column of A is perpendicular to every column of B.
Hint: What is the meaning of the entry in the i-th row and j-th column of $A^T B$?

b. $A^\dagger B = B^\dagger A = 0$.
Hint: Remember the geometric meaning of $A^\dagger b$ (resp. $B^\dagger a$), where b (resp. a) is a column in $\mathbb{R}^m$, and use this for every column of the matrix B (resp. A).

c. Every column of A^T is perpendicular to every column of B^T.
Hint: What is the meaning of the entry in the i-th row and j-th column of $(B^T)^T A^T = BA^T$?

d. $BA^\dagger = AB^\dagger = 0$.
Hint: Assuming (b) is true, this statement can be proved by plugging A^T and B^T into the appropriate variables in (b).

e. $(A + B)^\dagger = A^\dagger + B^\dagger$.

TASK. 4.

a. Prove that if $AA^\intercal$ is an invertible matrix, then $A^\intercal(AA^\intercal)^{-1}$ is the Moore–Penrose inverse $A^\dagger$ of the matrix A.

b. Find the point on the intersection of the planes $x+y+z=0$ and $x-y=1$ that is closest to the origin following the next two steps:
   (a) Write down the matrix of the system for the intersection and find its Moore-Penrose inverse.
   (b) Among all solutions of the system find the one closest to the origin.

TASK. 5.
Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

a. Find the Moore–Penrose inverse $A^\dagger$ of the matrix A.
b. Does the system $Ax = b$ have a solution?
c. If the system is solvable, find the solution closest to the origin. If the system does not have a solution, find the vector x^+ such that the error $\|Ax^+ - b\|_2$ is the smallest possible.
d. Find the Moore–Penrose inverse $(A^\dagger)^\dagger$ of $A^\dagger$.

TASK. 6.
We are given the following four points:

$$(0,1),\ (-1,0),\ (1,2),\ (2,3).$$

We would like to fit a function of the form $ax^2 + bx$ to these points.

a. Write down the matrix A of the corresponding system of linear equations.
b. Find the Moore-Penrose inverse $A^\dagger$.
c. Find the function of the above form that fits the points best according to the least squares criterion.
d. Find one more generalized inverse of A.

TASK. 7.
The system of equations

$$\begin{aligned} 2x - y + z &= 3, \\ -x + 2y - z &= 1 \end{aligned}$$

can be expressed in the form $Ax = b$.

a. Find the Moore-Penrose inverse of A.
b. Describe the property uniquely characterizing the point $A^\dagger b$ with respect to the system.
c. Construct a matrix, which has the following matrices as their generalized inverses:

$$\begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

TASK. 8.

a. Construct any non-diagonal 3×2 matrix A whose singular values are 2 and 1.
b. Find the Moore-Penrose inverse $A^\dagger$ of A.
c. Let $b \in \mathbb{R}^3$. Describe the property uniquely characterizing point $A^\dagger \cdot b$ with respect to the system $Ax = b$.

TASK. 9.
Let

$$A = \begin{bmatrix} -2 & 2 \\ -3 & -2 \\ -2 & -3 \end{bmatrix}.$$

a. Find the matrix $B \in \mathbb{R}^{3\times 2}$ of rank 1, which is the closest to A in the Frobenius norm.
b. Calculate $\|A - B\|_F$.

TASK. 10.
Let

$$A = \begin{bmatrix} 0 & 2 \\ -2 & -1 \\ -2 & 0 \end{bmatrix}$$

be a matrix.

a. Compute the truncated singular value decomposition of A.
b. Does there exist a matrix $B \in \mathbb{R}^{3\times 2}$ of rank 1 such that $\|A - B\|_F = 1$? If yes, compute it, otherwise justify, why it does not exist.

TASK. 11.
Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times r}$ and $C \in \mathbb{R}^{m\times r}$ be matrices. Consider the solutions of the matrix equations:

$$AXB = C. \tag{1}$$

Let $G_1 \in \mathbb{R}^{n\times m}$ and $G_2 \in \mathbb{R}^{r\times m}$ be generalized inverses of A and B, respectively.

a. Assume that $C = AG_1CG_2B$. Check that G_1CG_2 solves (1).
b. Prove that if (1) is solvable, then $C = AG_1CG_2B$ holds.
   Hint: Multiply (1) from left and from right by appropriate matrices and use the definitions of G_1, G_2.
c. Assume that (1) is solvable. Check that

$$X = G_1CG_2 + Z - G_1AZBG_2$$

solves (1) for any $Z \in \mathbb{R}^{n\times p}$.

CHAPTER 2

Nonlinear systems

Task. 12.

Perform one step of Gauss-Newton method to approximate the least squares solution of the system
$$f(x,y) = (2,3,1),$$
where
$$f : \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(x,y) = (x^2 + y^3 + 2, x + e^{y-1}, \sin x + \frac{1}{2}y^2 - 3).$$
For the initial approximation take $(x_0, y_0) = (0,1)$.

Task. 13.

Using one step of Newton's method approximate the solution of the system
$$\sin x + \cos y + e^{xy} = \arctan(x+y) - xy = 0,$$
with the initial approximation $(x_0, y_0) = (0,0)$.

Task. 14.

Let
$$F(x,y) := \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} x^2 + y^2 - 10x + y \\ x^2 - y^2 - x + 10y \end{bmatrix}$$
be a vector function and $a = (2,4) \in \mathbb{R}^2$ a point.

a. Calculate the Jacobian matrix of the function F in the point a.
b. Calculate the linear approximation of F in the point a.
c. Perform one step of Newton's method to find the approximate solution of the system
$$F(x,y) = \begin{bmatrix} 1 \\ 25 \end{bmatrix}$$
with the initial approximation a.

Task. 15.

Let
$$\begin{aligned} x^2 + y &= 37, \\ x - y^2 &= 5, \\ x + y + z &= 3 \end{aligned}$$
be a nonlinear system and $v^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ a vector.

a. Compute the approximation $v^{(1)}$ of the solution of the system using one step of Newton's method.

b. Compute the tangent plane to the surface given by the equation $z = f(x, y)$, where
$$f(x, y) = 8xy + 4,$$
in the point $(1, 1)$.

TASK. 16.
Let $f(x, y, z) = x^2 + 3xy + yz^3$ be a function of three variables.

a. Compute the gradient ∇f.
b. Perform one step of Newton's method to approximate the stationary point of f using the initial approximation $(x_0, y_0, z_0) = \big(1, 0, \frac{1}{\sqrt{3}}\big)$.

TASK. 17.
Let
$$\sqrt{\pi}\ln(x_1^2 + x_2^2) - \frac{1}{\sqrt{\pi}}\sin(x_1 x_2) = \ln(2\pi),$$
$$e^{x_1 - x_2} + \frac{1}{\sqrt{\pi}}\cos(x_1 x_2) = 0,$$
be a nonlinear system and $v^{(0)} = \begin{bmatrix} \sqrt{\pi} & \sqrt{\pi} \end{bmatrix}^T$ a vector. Compute the approximation $v^{(1)}$ of the solution of the system using one step of Newton's method.

CHAPTER 3

Curves and surfaces

TASK. 18.
Let S be a surface given by $z = g(x, y)$, where

$$g(x, y) = x^3 - x^2 y + y^2 - 2x + 3y - 2$$

is a differentiable function. Determine the tangent plane to S in the point $(-1, 3)$ in the parametric and implicit form.

Hint: Note that the parametric equation $f : \mathbb{R}^2 \to \mathbb{R}^3$ of the surface S is

$$f(x, y) = (x, y, g(x, y)).$$

TASK. 19.
Sketch the closed curves given in polar coordinates by

$$r_1(\varphi) = 1 + \cos\varphi \qquad \text{and} \qquad r_2(\varphi) = 1 + \sin\varphi.$$

Compute the area of the intersection of the bounded regions determined by the curves.

Hint: You will need the formulas $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ to compute the area.

TASK. 20.
Let

$$r(t) = (2\sin(2t), 2\cos(2t), 3t)$$

be the curve in parametric coordinates with $t \in [0, 2\pi]$.

a. Sketch the curve in $\mathbb{R}^3$.
b. Sketch all three projections of the curve in the xy-, xz- and yz-coordinate planes.
c. Compute the arc length of the curve.

TASK. 21.
For the parametric curve

$$f(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2t - t^2 \\ 3t - t^3 \end{bmatrix},$$

where $t \in \mathbb{R}$, solve the following:

a. Find intersections with coordinate axes.
b. Find points at which the tangent is horizontal or vertical.
c. Find points where $x'(t) = y'(t) = 0$.
d. Determine the asymptotic behaviour (limits as $t \to \pm\infty$).

e. Show that there are no self-intersections.

Hint: To notice that the curve does not have any self-intersections verify that the equality $1 - x(t) = 1 - x(s)$ implies that $s = 2 - t$ and plug this into the equation $y(t) = y(s)$.

f. Plot the curve.

TASK. 22.
Let
$$r_1(\varphi) = 2\sin\varphi \qquad \text{and} \qquad r_2(\varphi) = 2\cos\varphi$$
be curves in polar coordinates.

a. Prove that both curves are circles and provide a sketch.
Hint. Try multiplying each equation by r_i and expressing r_i^2, $r_i \sin\varphi$ and $r_i \cos\varphi$ by x and y.

b. Compute the area of the region that lies inside both circles.

TASK. 23.
Let
$$\vec{r}(t) = (x(t), y(t)) = (t^3 - 4t, t^2 - 4)$$
be the parametric curve.

a. Find all points where it intersects the coordinate axes.
b. Find the tangent to the curve at $t = 1$.
c. Find the points on the curve where the tangent is horizontal or vertical.
d. If there is a self-intersection, find it and compute the area inside the loop formed by the curve.
e. Sketch the curve.

TASK. 24.
Let
$$\gamma(t) = (t^3 - t + 1, t^2)$$
be the parametric curve.

a. Find self-intersections of γ.
b. Find the angle at which γ intersects itself in the self-intersections.
c. Find the point at which γ reaches its global minimum in the direction of y-axis.

TASK. 25.
Let
$$\gamma(t) = (2\cos(t), 2\sin(t), -t)^T$$
be the parametric curve.

a. Sketch γ.
b. Parametrize γ with a natural parameter.
c. Find the length of γ between points $(2, 0, 0)$ and $(2, 0, 2\pi)$.

Task. 26.

Two surfaces in the upper halfspace $z > 0$ are given by the following equations:

$$\Pi : x^2 + y^2 = \frac{z^2}{2} \qquad \Sigma : x^2 + y^2 = z.$$

Curve γ is the intersection of surfaces Π and Σ. Let $P = (1, 1, 2) \in \gamma$.

a. Find the angle at which the surfaces intersect at P.
b. Find the line tangent to γ at P.
c. Find the plane that is tangent to Σ at $(1, 2, 5)$.

Task. 27.

Let

$$f(t) = (\sin t, \cos(3t)), \quad t \in [0, 2\pi]$$

be the parametric curve.

a. Find all points where the curve intersects the coordinate axes.
b. Find all points on the curve where the tangent is horizontal or vertical.
c. Sketch the curve.

Task. 28.

Let

$$f(t) = (t^3 - 5t^2 + 3t + 11, t^2 - 2t + 3), \quad t \in \mathbb{R}$$

be the parametric curve.

a. Find all points on the curve, where the tangent is horizontal or vertical.
b. Find all self-intersections.
c. Sketch the curve.

Task. 29.

Sketch the curve given in polar coordinates by

$$r(\varphi) = 2 + 4\sin(\varphi)$$

and compute the area of the smaller bounded region determined by the curve.

CHAPTER 4

Differential equations

TASK. 30.
Let

$$2xy - 9x^2 + (2y + x^2 + 1) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \tag{2}$$

be a differential equation.

a. Rewrite (2) in the form $M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 0$ and prove that this DE is exact by checking the necessary and sufficient conditions involving partial derivatives of M and N.
b. Solve the DE (2) with an initial condition $y(0) = -3$.

TASK. 31.
Convert the differential equation

$$y'' + 11y' + 24y = 0 \tag{3}$$

into the system of first order DEs, solve this system and recover the solution of the initial DE (3).

TASK. 32.
Solve the differential equation

$$3y' \cos x + y \sin x - \frac{1}{y^2} = 0,$$

given the initial condition $y(0) = 1$.

Hint: Note that this DE can be transformed into a first order linear nonhomogeneous DE by multiplying it with an appropriate factor and introducing a new variable. To compute $\int \tan x \, dx$ use the substitution $u = \cos x$. Also remember that $\displaystyle\int \frac{1}{(\cos x)^2} \, dx = \tan x + C$.

TASK. 33.
Find the general solution of the system

$$\dot{x} = 2x - 3y,$$
$$\dot{y} = x - 2y,$$

and sketch the phase potrait.

Task. 34.
Solve the differential equation

$$y'' - 4y' + 5y = 8\cos x.$$

Find a solution to this DE which has a local extremum in the point $(0, 2)$.

Task. 35.
Find the solution $x(t), y(t)$ of the nonautonomous system of first order linear differential equations

$$\begin{aligned} \dot{x} &= 2x - y, \\ \dot{y} &= -2x + y + 18t, \end{aligned}$$

which satisfies $x(0) = 1$, $y(0) = 0$.

Hint: One of the particular solutions of the system above is of the form

$$x_p(t) = At^2 + Bt + C, \qquad y_p(t) = Dt^2 + Et + F,$$

where A, B, C, D, E, F are constants.

Task. 36.
Let

$$y' = 2xy^2 \tag{4}$$

be the differential equation with an initial condition $y(0) = 1$.

a. Find the exact solution of the DE (4).
b. Use Euler's method with step size 0.2 to estimate $y(0.4)$ and compare the result to the exact value $y(0.4)$.

Task. 37.
Find the general solution of the nonhomogeneous second order linear equation

$$\ddot{x} + \dot{x} - 2x = t^2.$$

Task. 38.
Find the general solution of the differential equation

$$y' = 2x(1 + y^2)$$

and the particular solution that satisfies $y(1) = 0$.

Task. 39.
For the system of nonlinear differential equations

$$\dot{x} = xy + 1, \quad \dot{y} = x + xy,$$

solve the following:

a. Find stationary points.
b. Classify each stationary point as a saddle, source, sink or center.
c. Sketch the phase portrait of the system around each stationary point.

Task. 40.
Solve the differential equation

$$xy' = y + 2x^3$$

with the initial condition $y(2) = 3$.

Task. 41.
Solve the differential equation

$$y'' + y' - 6y = 36x$$

with the initial conditions $y(0) = y'(0) = 1$.

Task. 42.
Find the solution y of the differential equation

$$x^2y' + xy + 3 = 0$$

with the initial condition $y(1) = 1$.

Task. 43.
Solve the following system of differential equations:

$$\begin{aligned} x'(t) &= -2x(t) + 5y(t), \\ y'(t) &= x(t) + 2y(t), \end{aligned}$$

with the initial conditions $x(0) = y(0) = 1$.

Task. 44.
Solve the following exact differential equation

$$2xy + (x^2 + 3y^2)y' = 0.$$

Task. 45.
Solve the differential equation

$$y'' + 9y = 2x^2 - 1$$

with the initial condition $y(0) = y'(0) = 1$.

Task. 46.
Consider the system of nonlinear differential equations

$$\begin{aligned} \dot{x} &= x(3 - x - 2y), \\ \dot{y} &= y(4 - 3x - y). \end{aligned}$$

a. Find the stationary points of the system.
b. Compute the linearization of the system around the nontrivial stationary point, i.e., the one with both coordinates being nonzero.
c. Solve the linear system from the previous question and sketch its phase portrait.

Task. 47.

Let

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix},$$

where $A \in \mathbb{R}^{3\times 3}$, be a system of differential equations with the following three solutions:

$$e^{-t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad e^{t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad e^{2t}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

a. Write down a general solution of the system.
b. Determine the matrix A.
c. Write down a third order differential equation with constants coefficients, which is transformed into the above system.

Task. 48.

Solve the differential equation

$$\ddot{x} - \dot{x} - 4x = 2t + e^{t}. \tag{5}$$

Part 2

Solutions

Linear systems

Solution of the task 1.

We have to compute the orthogonal matrices

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \in \mathbb{R}^{2\times 2}, \qquad V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

and a diagonal rectangular matrix $\Sigma \in \mathbb{R}^{2\times 3}$ such that

$$B = U\Sigma V^T.$$

We have that

$$BB^T = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix},$$

which implies

$$\det(BB^T - \lambda I_2) = (11-\lambda)^2 - 1 = (11-\lambda-1)(11-\lambda+1) = (12-\lambda)(10-\lambda).$$

So the eigenvalues of BB^T are $\lambda_1 = 12$, $\lambda_2 = 10$ and hence

$$\Sigma = \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}.$$

The kernel of

$$BB^T - \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

contains the vector $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and hence

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The kernel of

$$BB^T - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

contains the vector $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$ and hence

$$u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Now the first two columns of V are

$$v_1 = \frac{1}{\sqrt{12}} \cdot B^T u_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad v_2 = \frac{1}{\sqrt{10}} \cdot B^T u_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

Finally,

$$v_3 = \frac{v_1 \times v_2}{\|v_1 \times v_2\|} = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}.$$

SOLUTION OF THE TASK 2.

a. The left upper 2×2 submatrix has determinant -1 and hence the first two columns are linearly independent. The third column is the sum of the first two and so $\operatorname{rank} A = 2$.

b. We choose an invertible 2×2 submatrix of A, replace it with its transposed inverse, replace the other entries with 0s and transpose the matrix obtained:

$$G = \begin{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1}\right)^T & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

c. The candidate for the solution is

$$Gb = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

We can check that $A(Gb) = b$ and hence Gb is really a solution of the system. All solutions are of the form

$$\begin{aligned} Gb + (GA - I)z &= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 + z_3 \\ -1 + z_3 \\ -z_3 \end{bmatrix}, \end{aligned}$$

where $z_1, z_2, z_3 \in \mathbb{R}$.

SOLUTION OF THE TASK 3.

a.

$$\begin{aligned} 0 = (A^T B)_{i,j} &= \text{dot product of the } i\text{-th row of } A^T \text{ and } j\text{-th column of } B \\ &= \text{dot product of the } i\text{-th column of } A \text{ and } j\text{-th column of } B. \end{aligned}$$

b. Two possible solutions to this part:

Geometrical solution: $A^\dagger b$ is the vector with the smallest norm among all vectors from the set

$$\mathcal{S}(A, b) := \left\{ x \in \mathbb{R}^n \colon \|b - Ax\| = \min_{x' \in \mathbb{R}^n} \|b - Ax'\| \right\}.$$

Since every column b_j of B is perpendicular to the span of the columns of A,

$$\min_{x'\in\mathbb{R}^n} \|b_j - Ax'\| = \|b_j\|$$

and hence $0 \in \mathcal{S}(A, b_j)$. Thus $0 = A^\dagger b_j$.

Computational solution: Let $A = U\Sigma V^T$ be the singular value decomposition of A, where $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ are orthogonal matrices,

$$\Sigma = \begin{bmatrix} D & 0_{r,(n-r)} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

and $D = \operatorname{diag}(\sigma_1, \dots, \sigma_r) \in \mathbb{R}^{r\times r}$ is a diagonal matrix with singular values σ_i of A on the diagonal. We have that

$$B = U \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} V^T,$$

where $B_{11} \in \mathbb{R}^{r\times r}$, $B_{12} \in \mathbb{R}^{r\times(n-r)}$, $B_{21} \in \mathbb{R}^{(m-r)\times r}$ and $B_{22} \in \mathbb{R}^{(m-r)\times(n-r)}$. Now we calculate $A^\dagger B$:

$$\begin{aligned} A^\dagger B &= \left(V\Sigma^\dagger U^T\right)\left(U \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} V^T\right) \\ &= V \begin{bmatrix} D^{-1} & 0_{r,(m-r)} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} V^T \qquad (6) \\ &= V \begin{bmatrix} D^{-1}B_{11} & D^{-1}B_{12} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix} V^T \end{aligned}$$

We know that $A^T B = 0$:

$$\begin{aligned} 0 = A^T B &= \left(V\Sigma U^T\right)\left(U \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} V^T\right) \\ &= V \begin{bmatrix} D & 0_{r,(m-r)} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} V^T \qquad (7) \\ &= V \begin{bmatrix} DB_{11} & DB_{12} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix} V^T \end{aligned}$$

Multiplying (7) with V^{-1} from the left side and $\left(V^T\right)^{-1}$ from the right side we get

$$0 = \begin{bmatrix} DB_{11} & DB_{12} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix}.$$

In particular, $DB_{11} = 0$ and $DB_{12} = 0$. Since D is invertible this implies that $B_{11} = 0$ and $B_{12} = 0$. Using this in (6) we conclude that $A^\dagger B = 0$.

c.

$$\begin{aligned} 0 = (BA^T)_{i,j} &= \text{dot product of the } i\text{-th row of } B \text{ and } j\text{-th column of } A^T \\ &= \text{dot product of the } i\text{-th column of } B^T \text{ and } j\text{-th column of } A^T. \end{aligned}$$

d. Plugging A^T, B^T for A, B in (b) we obtain

$$\begin{aligned} 0 &= (A^T)^\dagger B^T = (A^\dagger)^T B^T = (BA^\dagger)^T && \Rightarrow && 0 = BA^\dagger, \\ 0 &= (B^T)^\dagger A^T = (B^\dagger)^T A^T = (AB^\dagger)^T && \Rightarrow && 0 = AB^\dagger. \end{aligned}$$

e. We have to check four properties of the Moore-Penrose inverse:

$$\begin{aligned} (A+B)(A^\dagger + B^\dagger)(A+B) &= (AA^\dagger + AB^\dagger + BA^\dagger + BB^\dagger)(A+B) \\ &= (AA^\dagger + BB^\dagger)(A+B) \\ &= AA^\dagger A + AA^\dagger B + BB^\dagger A + BB^\dagger B \\ &= A^\dagger + 0 + 0 + B^\dagger \\ &= A^\dagger + B^\dagger, \\ (A^\dagger + B^\dagger)(A+B)(A^\dagger + B^\dagger) &= (A^\dagger A + A^\dagger B + B^\dagger A + B^\dagger B)(A+B) \\ &= (A^\dagger A + B^\dagger B)(A^\dagger + B^\dagger) \\ &= A^\dagger A A^\dagger + A^\dagger A B^\dagger + B^\dagger B A^\dagger + B^\dagger B B^\dagger \\ &= A + 0 + 0 + B \\ &= A + B, \\ ((A+B)(A^\dagger + B^\dagger))^T &= (A^\dagger + B^\dagger)^T (A+B)^T \\ &= (A^\dagger)^T A^T + (A^\dagger)^T B^T + (B^\dagger)^T A^T + (B^\dagger)^T B^T \\ &= (AA^\dagger)^T + 0 + 0 + (BB^\dagger)^T \\ &= AA^\dagger + BB^\dagger, \\ ((A^\dagger + B^\dagger)(A+B))^T &= (A+B)^T (A^\dagger + B^\dagger)^T \\ &= A^T (A^\dagger)^T + A^T (B^\dagger)^T + B^T (A^\dagger)^T + B^T (B^\dagger)^T \\ &= (A^\dagger A)^T + 0 + 0 + (B^\dagger B)^T \\ &= A^\dagger A + B^\dagger B. \end{aligned}$$

Solution of the task 4.

a. We denote $G := A^T(AA^T)^{-1}$. There are four requirements to check for G to be equal to $A^\dagger$:

$$\begin{aligned} AGA &= A\left(A^T\left(AA^T\right)^{-1}\right)A = \left(AA^T\right)\left(AA^T\right)^{-1}A = IA = A, \\ GAG &= GA\left(A^T\left(AA^T\right)^{-1}\right) = G\left(A^TA\right)\left(A^TA\right)^{-1} = G, \\ (AG)^T &= \left(A\left(A^T\left(AA^T\right)^{-1}\right)\right)^T = \left(\left(AA^T\right)\left(AA^T\right)^{-1}\right)^T = I = AG, \\ (GA)^T &= \left(\left(A^T\left(AA^T\right)^{-1}\right)A\right)^T = \left(A^T\left(\left(AA^T\right)^{-1}\right)^T\left(A^T\right)^T\right) = GA, \end{aligned}$$

where we used that $\left(A^TA\right)^{-1}$ is symmetric in the last equality.

b. (a) The matricial form of the system is the following:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b}.$$

The Moore-Penrose inverse of A is the following:

$$A^\dagger = A^T(AA^T)^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}.$$

(b) The solution of the system of the smallest norm is

$$x^+ = A^\dagger b = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}.$$

Solution of the task 5.

a. Since $\operatorname{rank} A = 2$, the matrix $A^TA \in \mathbb{R}^{2\times s}$ is invertible and hence $A^\dagger = \left(A^TA\right)^{-1} A^T$. So,

$$A^\dagger = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}^{-1} \cdot A^T = \begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} \end{bmatrix}^{-1} \cdot A^T = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}.$$

b. If the systema $Ax = b$ has a solution, then one of the solutions is $x^+ = A^\dagger b$. So, we have to compute Ax^+ and see, if the result is b.

$$x^+ = A^\dagger b = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \qquad \Rightarrow \qquad Ax^+ = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \neq b.$$

So the system $Ax = b$ does not have a solution.

c. The vector x^+ such that the error $\|Ax^+ - b\|_2$ is the smallest possible, is

$$A^\dagger b = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

d. The Moore–Penrose inverse of $A^\dagger$ is always A by the symmetry in A and $A^\dagger$ in the conditions the MP inverse must satisfy.

Solution of the task 6.

a. The matricial form of the system is the following:

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 1 & 1 \\ 4 & 2 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}}_{c}.$$

b. Since rank $A = 2$, also rank$(A^T A) = 2$ and hence $A^\dagger$ is equal to

$$\begin{aligned} A^\dagger = (A^T A)^{-1} A^T &= \begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 1 & 4 \\ 0 & -1 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{22} & -\frac{2}{11} \\ -\frac{2}{11} & \frac{9}{22} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 4 \\ 0 & -1 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{7}{22} & -\frac{1}{22} & \frac{2}{11} \\ 0 & -\frac{13}{22} & \frac{5}{22} & \frac{1}{11} \end{bmatrix}. \end{aligned}$$

c. The solution a, b such that $ax^2 + bx$ fits the data best w.r.t. the least squares error method is

$$A^\dagger c = \begin{bmatrix} 0 & \frac{7}{22} & -\frac{1}{22} & \frac{2}{11} \\ 0 & -\frac{13}{22} & \frac{5}{22} & \frac{1}{11} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{5}{11} \\ \frac{8}{11} \end{bmatrix}.$$

d. Another generalized inverse of A is

$$G = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \left(\begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}^{-1} \right)^T \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 2 & -\frac{1}{2} \end{bmatrix}.$$

Solution of the task 7.

a. The matricial form of the system is the following:

$$\underbrace{\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{b}.$$

Since $\operatorname{rank} A = 2$, also $\operatorname{rank}(AA^T) = 2$ and hence $A^\dagger$ is equal to

$$A^\dagger = A^T(AA^T)^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 6 & -5 \\ -5 & 6 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{6}{11} & \frac{5}{11} \\ \frac{5}{11} & \frac{6}{11} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{11} & \frac{4}{11} \\ \frac{4}{11} & \frac{7}{11} \\ \frac{1}{11} & -\frac{1}{11} \end{bmatrix}.$$

b. Since $A \in \mathbb{R}^{2\times 3}$ and $\operatorname{rank} A = 2$, it follows that the system $Ax = b$ is solvable and the kernel of A is one-dimensional. Hence, there is a one-dimensional family of solutions of the system $Ax = b$. The vector $A^\dagger b$ is the solution of the system of the smallest norm among all solutions.

c. The matrix A is of size 4×2. By construction of some generalized inverses, the matrix

$$\begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

is a generalized inverse of any matrix of the form

$$\begin{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}^{-1} \\ X \end{bmatrix},$$

where $X \in \mathbb{R}^{2\times 2}$ is any matrix, and the matrix

$$\begin{bmatrix} 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

is a generalized inverse of any matrix of the form

$$\begin{bmatrix} Y \\ \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}^{-1} \end{bmatrix},$$

where $Y \in \mathbb{R}^{2\times 2}$ is any matrix. Hence,

$$A = \begin{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}^{-1} \\ \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}^{-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{3}{5} \end{bmatrix}.$$

Solution of the task 8.

a. One possible solution is

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \\ 0 & 0 \end{bmatrix}.$$

b. The Moore-Penrose inverse of A is

$$A^{\dagger} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^{T} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{T} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}.$$

c. If the system $Ax = b$ is solvable, then the vector $A^{\dagger}b$ is the solution of the system of the smallest norm among all solutions. Otherwise the vector $A^{\dagger}b$ is the unique solution of the system of the smallest norm w.r.t. the least squares method, i.e.,

$$\left\|A(A^{\dagger}b) - b\right\| = \min\left\{\|Ax - b\| : x \in \mathbb{R}^2\right\}.$$

Solution of the task 9.
Let

$$A = U\Sigma V^T$$

be the truncated SVD of A, where

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \in \mathbb{R}^{3\times 2} \quad \text{with } U^T U = I_2,$$
$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \in \mathbb{R}^{2\times 2} \quad \text{with } V^T V = I_2$$

and

$$\Sigma = \operatorname{diag}(\sigma_1, \sigma_2), \quad \text{where } \sigma_1 \geq \sigma_2 \geq 0 \text{ are the singular values of } A.$$

By the Eckart-Young theorem it follows that

$$B = \sigma_1 u_1 v_1^T,$$

and hence

$$\|A - B\|_F = \|\sigma_2 u_2 v_2^T\|_F = \sigma_2 \|u_2 v_2^T\|_F.$$

We write $v_2 = \begin{bmatrix} v_{2,1} & v_{2,2} \end{bmatrix}^T$. Then

$$\|u_2 v_2^T\|_F = \|[u_2 v_{2,1}, u_2 v_{2,2}]\|_F = \sqrt{\|u_2 v_{2,1}\|_F^2 + \|u_2 v_{2,2}\|_F^2}$$
$$= \sqrt{(v_{2,1})^2 + (v_{2,2})^2} = \|v_2\|_F = 1,$$

where in the third equality we used that $\|u_2\|_F = 1$. So we need to compute only $\sigma_1, \sigma_2, v_1, u_1$ to determine B and $\|A - B\|_F$. We have that

$$A^T A = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix},$$

which implies

$$\det(A^T A - \lambda I_2) = (17 - \lambda)^2 - 8^2 = (17 - \lambda - 8)(17 - \lambda + 8) = (9 - \lambda)(25 - \lambda).$$

So the eigenvalues of $A^T A$ are $\lambda_1 = 25$, $\lambda_2 = 9$ and hence

$$\Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}.$$

The kernel of

$$A^TA - \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = \begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix}$$

contains the vector $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and hence

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Now the first column of U is

$$u_1 = \frac{1}{5} A v_1 = \frac{1}{5\sqrt{2}} \begin{bmatrix} 0 \\ -5 \\ -5 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

Finally,

$$B = 5u_1v_1^T = 5 \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{5}{2} \begin{bmatrix} 0 & 0 \\ -1 & -1 \\ -1 & -1 \end{bmatrix}$$

and

$$\|A - B\|_F = 3.$$

Solution of the task 10.

a. We have that

$$A^TA = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix},$$

which implies

$$\det(A^TA - \lambda I_2) = (8-\lambda)(5-\lambda) - 2^2 = \lambda^2 - 13\lambda + 36 = (\lambda - 9)(\lambda - 4).$$

So the eigenvalues of A^TA are $\lambda_1 = 9$, $\lambda_2 = 4$. Hence, Σ in the SVD of $A = U\Sigma V^T$ is equal to

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

The kernel of

$$A^TA - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$$

contains the vector $\begin{bmatrix} 2 & 1 \end{bmatrix}^T$ and hence

$$v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Further on,

$$u_1 = \frac{1}{3} A v_1 = \frac{1}{3\sqrt{5}} \begin{bmatrix} 2 \\ -5 \\ -4 \end{bmatrix}.$$

The kernel of

$$A^T A - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

contains the vector $\begin{bmatrix} 1 & -2 \end{bmatrix}^T$ and hence

$$v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Further on,

$$u_2 = \frac{1}{2} A v_2 = \frac{1}{2\sqrt{5}} \begin{bmatrix} -4 \\ 0 \\ -2 \end{bmatrix}.$$

So, the truncated SVD of A is equal to

$$A = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T.$$

b. By the Eckart-Young theorem the matrix B of rank 1, which minimizes the norm $\|A - B\|_F$ is equal to $\sigma_1 u_1 v_1^2$. The distance $\|A - B\|_F$ is $\|\sigma_2 u_2 v_2^T\|_F = \sigma_2 = 2$. Hence, there does not exist a matrix B of rank 1 satisfying $\|A - B\|_F = 1$.

Solution of the task 11. ⇧

a.

$$A(G_1 C G_2) B = A G_1 C G_2 B = C.$$

b. We multiply (1) from left by AG_1 and from right by $G_2 B$ to obtain

$$AG_1 AXBG_2 B = AG_1 CG_2 B. \tag{8}$$

Since G_1 (resp. G_2) is a generalized inverse of A (resp. B), we have that $AG_1 A = A$ (resp. $BG_2 B = B$). Hence, (8) implies that

$$C = AXB = AG_1 CG_2 B, \tag{9}$$

where in the first equality we used (1). This proves (b).

c.

$$\begin{aligned} AXB &= A(G_1 CG_2 + Z - G_1 AZBG_2)B \\ &= AG_1 CG_2 B + A(Z - G_1 AZBG_2)B \\ &= C + AZB - AG_1 AZBG_2 B \\ &= C + AZB - AZB = C, \end{aligned}$$

where we used (b) in the second equality and the definitions of G_1, G_2 in the third equality.

Note: If (1) is solvable, then all solutions are of the form given in (c). Indeed, if X solves (1), then $X = G_1 C G_2 + X - G_1 AXBG_2$, which means that $Z = X$ is one appropriate choice.

Nonlinear systems

Solution of the task 12.

We define a vector function $F : \mathbb{R}^2 \to \mathbb{R}^3$ by the rule

$$F(x, y) := f(x, y) - (2, 3, 1) = (x^2 + y^3, x + e^{y-1} - 3, \sin x + \frac{1}{2}y^2 - 4).$$

After one step of Gauss-Newton method the approximate of the least squares solution of the system $F(x, y) = (0, 0, 0)$ will be

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - ((JF)(x_0, y_0))^\dagger F(x_0, y_0).$$

We have

$$(JF)(x, y) = \begin{bmatrix} 2x & 3y^2 \\ 1 & e^{y-1} \\ \cos x & y \end{bmatrix} \qquad \Rightarrow \qquad (JF)(0, 1) = \begin{bmatrix} 0 & 3 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since

$$((JF)(0, 1))^T \cdot (JF)(0, 1) = \begin{bmatrix} 0 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 11 \end{bmatrix}$$

is invertible, it follows that

$$\begin{aligned} ((JF)(0, 1))^\dagger &= \Big(((JF)(0, 1))^T \cdot (JF)(0, 1) \Big)^{-1} ((JF)(0, 1))^T \\ &= \begin{bmatrix} 2 & 2 \\ 2 & 11 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{18} \begin{bmatrix} 11 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{18} \begin{bmatrix} -6 & 9 & 9 \\ 6 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -\frac{1}{3} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -\frac{7}{2} \end{bmatrix} = \begin{bmatrix} \frac{37}{12} \\ \frac{2}{3} \end{bmatrix}.$$

SOLUTION OF THE TASK 13.
We define a vector function $f : \mathbb{R}^2 \to \mathbb{R}^2$ by the rule

$$f(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} \sin x + \cos y + e^{xy} \\ \arctan(x+y) - xy \end{bmatrix}.$$

We are approximating the zero $f(x,y) = 0$ of f using one step of Newton's method with the initial approximation $(x_0, y_0) = (0,0)$. The Jacobian $Jf(x,y)$ of f is

$$Jf(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos x + ye^{xy} & -\sin y + xe^{xy} \\ \frac{1}{(x+y)^2+1} - y & \frac{1}{(x+y)^2+1} - x \end{bmatrix}.$$

So

$$\begin{aligned} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \left(Jf(x_0,y_0)\right)^{-1} f(x_0,y_0) \\ &= -\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= -\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}. \end{aligned}$$

SOLUTION OF THE TASK 14.

a.

$$JF(a) = \begin{bmatrix} 2x-10 & 2y+1 \\ 2x-1 & -2y+10 \end{bmatrix}(a) = \begin{bmatrix} -6 & 9 \\ 3 & 2 \end{bmatrix}.$$

b.

$$\begin{aligned} L_{F,a}(x,y) &= F(a) + JF(a) \begin{bmatrix} x-2 \\ y-4 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 26 \end{bmatrix} + \begin{bmatrix} -6 & 9 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x-2 \\ y-4 \end{bmatrix} \\ &= \begin{bmatrix} -6x+9y-20 \\ 3x+2y+12 \end{bmatrix}. \end{aligned}$$

c. We are searcing for zeroes of the vector function

$$G(x,y) = F(x,y) - \begin{bmatrix} 1 \\ 25 \end{bmatrix}$$

with the initial approximation $(x_0, y_0) = a$. One step of Newton's method:

$$\begin{aligned}
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - JG(a)^{-1}G(x_0, y_0) \\
&= \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} -6 & 9 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \frac{1}{39} \begin{bmatrix} 2 & -9 \\ -3 & -6 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\
&= \frac{1}{13} \begin{bmatrix} 25 \\ 47 \end{bmatrix}.
\end{aligned}$$

SOLUTION OF THE TASK 15.

a. We define a vector function of a vector variable $F : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(x, y, z) = \begin{bmatrix} F_1(x, y, z) \\ F_2(x, y, z) \\ F_3(x, y, z) \end{bmatrix} = \begin{bmatrix} x^2 + y - 37 \\ x - y^2 - 5 \\ x + y + z - 3 \end{bmatrix}.$$

We are searching for the solution of $F(x, y, z) = 0$ using Newton's method. We have that

$$v^{(1)} = v^{(0)} - \left(JF(v^{(0)})\right)^{-1} F(v^{(0)}),$$

where

$$JF(x, y, z) = \begin{bmatrix} \frac{\partial F_1(x,y,z)}{\partial x} & \frac{\partial F_1(x,y,z)}{\partial y} & \frac{\partial F_1(x,y,z)}{\partial z} \\ \frac{\partial F_2(x,y,z)}{\partial x} & \frac{\partial F_2(x,y,z)}{\partial y} & \frac{\partial F_2(x,y,z)}{\partial z} \\ \frac{\partial F_3(x,y,z)}{\partial x} & \frac{\partial F_3(x,y,z)}{\partial y} & \frac{\partial F_3(x,y,z)}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & 1 & 0 \\ 1 & -2y & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

is the Jacobian matrix of F. So

$$JF(v^{(0)}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

We compute $\left(JF(v^{(0)})\right)^{-1}$ using Gaussian elimination:

$$\underbrace{\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]}_{\left[\, JF(v^{(0)}) \,\middle|\, I_3 \,\right]} \underbrace{\sim}_{\ell_1 \leftrightarrow \ell_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underbrace{\sim}_{\ell_3 = \ell_3 - \ell_1 - \ell_2} \underbrace{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]}_{\left[\, I_3 \,\middle|\, \left(JF(v^{(0)})\right)^{-1} \,\right]}.$$

So

$$v^{(1)} = -\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -37 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 37 \\ -39 \end{bmatrix}.$$

b. The surface is parametrized by

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix} = \begin{bmatrix} x \\ y \\ 8xy + 4 \end{bmatrix}.$$

The tangent plane to the surface in the point $(1,1)$ is

$$\begin{aligned} L(x,y) &= \begin{bmatrix} 1 \\ 1 \\ f(1,1) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \frac{\partial f(1,1)}{\partial x} \end{bmatrix} (x-1) + \begin{bmatrix} 0 \\ 1 \\ \frac{\partial f(1,1)}{\partial y} \end{bmatrix} (y-1) \\ &= \begin{bmatrix} 1 \\ 1 \\ (8xy+4)(1,1) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ (8y)(1,1) \end{bmatrix} (x-1) + \begin{bmatrix} 0 \\ 1 \\ (8x)(1,1) \end{bmatrix} (y-1) \\ &= \begin{bmatrix} 1 \\ 1 \\ 12 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix} (x-1) + \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} (y-1) \\ &= \begin{bmatrix} x \\ y \\ -4 + 8x + 8y \end{bmatrix}, \end{aligned}$$

or $z = -4 + 8x + 8y$.

SOLUTION OF THE TASK 16.

a.

$$\nabla f(x,y,z) = \begin{bmatrix} \frac{\partial f}{\partial x}(x,y,z) \\ \frac{\partial f}{\partial y}(x,y,z) \\ \frac{\partial f}{\partial z}(x,y,z) \end{bmatrix} = \begin{bmatrix} 2x+3y \\ 3x+z^3 \\ 3yz^2 \end{bmatrix}.$$

b. We are searching for the solution of $\nabla f(x,y,z) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ using Newton's method. We have that

$$v^{(1)} = v^{(0)} - \big(J(\nabla f)(v^{(0)})\big)^{-1}(\nabla f)(v^{(0)}),$$

where

$$J(\nabla f)(x,y,z) = \begin{bmatrix} \frac{\partial^2 f(x,y,z)}{\partial x^2} & \frac{\partial^2 f(x,y,z)}{\partial x \partial y} & \frac{\partial f(x,y,z)}{\partial x \partial z} \\ \frac{\partial^2 f(x,y,z)}{\partial x \partial y} & \frac{\partial^2 f(x,y,z)}{\partial y^2} & \frac{\partial f(x,y,z)}{\partial y \partial z} \\ \frac{\partial^2 f(x,y,z)}{\partial x \partial x} & \frac{\partial^2 f(x,y,z)}{\partial y \partial z} & \frac{\partial f(x,y,z)}{\partial z^2} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 0 & 3z^2 \\ 0 & 3z^2 & 6yz \end{bmatrix}$$

is the Jacobian matrix of ∇f. So

$$J(\nabla f)(v^{(0)}) = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We compute $\big(J(\nabla f)(v^{(0)})\big)^{-1}$ using Gaussian elimination:

$$\underbrace{\left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]}_{\left[\, J(\nabla f)(v^{(0)}) \,\middle|\, I_3 \,\right]} \underbrace{\sim}_{\ell_2 \leftrightarrow \ell_3} \left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\underbrace{\sim}_{\ell_1 = \frac{1}{2}(\ell_1 - 3\ell_2)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\underbrace{\sim}_{\ell_3 = \ell_3 - 3\ell_1} \underbrace{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{3}{2} & 1 & \frac{9}{2} \end{array}\right]}_{\left[\, I_3 \,\middle|\, \left(J(\nabla f)(v^{(0)})\right)^{-1} \,\right]}.$$

So

$$v^{(1)} = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & 0 & 1 \\ -\frac{3}{2} & 1 & \frac{9}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 + \frac{1}{3\sqrt{3}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{2}{3\sqrt{3}} \end{bmatrix}.$$

SOLUTION OF THE TASK 17.

We define a vector function of a vector variable $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x_1, x_2) = \begin{bmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} \sqrt{\pi} \ln(x_1^2 + x_2^2) - \frac{1}{\sqrt{\pi}} \sin(x_1 x_2) - \ln(2\pi) \\ e^{x_1 - x_2} + \frac{1}{\sqrt{\pi}} \cos(x_1 x_2) \end{bmatrix}.$$

We are searching for the solution of $F(x_1, x_2) = 0$ using Newton's method. We have that

$$v^{(1)} = v^{(0)} - \left(JF(v^{(0)})\right)^{-1} F(v^{(0)}),$$

where

$$\begin{aligned} JF(x_1, x_2) &= \begin{bmatrix} \frac{\partial F_1(x_1,x_2)}{\partial x_1} & \frac{\partial F_1(x_1,x_2)}{\partial x_2} \\ \frac{\partial F_2(x_1,x_2)}{\partial x_1} & \frac{\partial F_2(x_1,x_2)}{\partial x_2} \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{\pi} \frac{2x_1}{x_1^2+x_2^2} - \frac{x_2}{\sqrt{\pi}} \cos(x_1 x_2) & \sqrt{\pi} \frac{2x_2}{x_1^2+x_2^2} - \frac{x_1}{\sqrt{\pi}} \cos(x_1 x_2) \\ e^{x_1 - x_2} - \frac{x_2}{\sqrt{\pi}} \sin(x_1 x_2) & -e^{x_1 - x_2} - \frac{x_1}{\sqrt{\pi}} \sin(x_1 x_2) \end{bmatrix} \end{aligned}$$

is the Jacobian matrix of F. So

$$JF(v^{(0)}) = \begin{bmatrix} 1 - (-1) & 1 - (-1) \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}.$$

We compute $\left(JF(v^{(0)})\right)^{-1}$ using Gaussian elimination:

$$\begin{aligned} \underbrace{\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right]}_{\left[\, JF(v^{(0)}) \,\middle|\, I_2 \,\right]} &\underbrace{\sim}_{\substack{\ell_1 = \frac{\ell_1}{2}, \\ \ell_2 = \ell_2 - \frac{1}{2}\ell_1}} \left[\begin{array}{cc|cc} 1 & 1 & \frac{1}{2} & 0 \\ 0 & -2 & -\frac{1}{2} & 1 \end{array}\right] \\ &\underbrace{\sim}_{\substack{\ell_2 = -\frac{1}{2}\ell_2, \\ \ell_1 = \ell_1 + \frac{1}{2}\ell_2}} \underbrace{\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} & -\frac{1}{2} \end{array}\right]}_{\left[\, I_2 \,\middle|\, \left(JF(v^{(0)})\right)^{-1} \,\right]}. \end{aligned}$$

So

$$v^{(1)} = \begin{bmatrix} \sqrt{\pi} \\ \sqrt{\pi} \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \sqrt{\pi}\ln(2\pi) - \ln(2\pi) \\ 1 - \frac{1}{\sqrt{\pi}} \end{bmatrix}$$
$$= \begin{bmatrix} \sqrt{\pi} - \frac{1}{4}\sqrt{\pi}\ln(2\pi) + \frac{1}{4}\ln(2\pi) - \frac{1}{2} + \frac{1}{2\sqrt{\pi}} \\ \sqrt{\pi} - \frac{1}{4}\sqrt{\pi}\ln(2\pi) + \frac{1}{4}\ln(2\pi) + \frac{1}{2} - \frac{1}{2\sqrt{\pi}} \end{bmatrix}$$
$$\approx \begin{bmatrix} 1.20 \\ 1.64 \end{bmatrix}.$$

Curves and surfaces

Solution of the task 18.

The parametric form $r : \mathbb{R}^2 \to \mathbb{R}^3$ of the tangent plane to S in the point $(x, y) = (-1, 3)$ is

$$\begin{aligned} r(u, v) &= f(-1, 3) + uf_x(-1, 3) + vf_y(-1, 3) \\ &= (-1, 3, g(-1, 3)) + u \cdot (1, 0, g_x(-1, 3)) + v \cdot (0, 1, g_y(-1, 3)). \end{aligned}$$

We have

$$\begin{aligned} g(-1, 3) &= (-1)^3 - (-1)^2 \cdot 3 + 3^2 - 2(-1) + 3 \cdot 3 - 2 = 14, \\ g_x(x, y) &= 3x^2 - 2xy - 2 \quad \Rightarrow \quad g_x(-1, 3) = 3 - 2(-1)3 - 2 = 7, \\ g_y(x, y) &= -x^2 + 2y + 3 \quad \Rightarrow \quad g_y(-1, 3) = -1 + 2 \cdot 3 + 3 = 8. \end{aligned}$$

Hence,

$$r(u, v) = (-1, 3, 14) + u(1, 0, 7) + v(0, 1, 8) = (-1 + u, 3 + v, 14 + 7u + 8v).$$

For the implicit form of the tangent plane we need its normal

$$\vec{n} := f_x(-1, 3) \times f_y(-1, 3) = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} = \begin{bmatrix} -7 \\ -8 \\ 1 \end{bmatrix}.$$

Hence, the implicit form

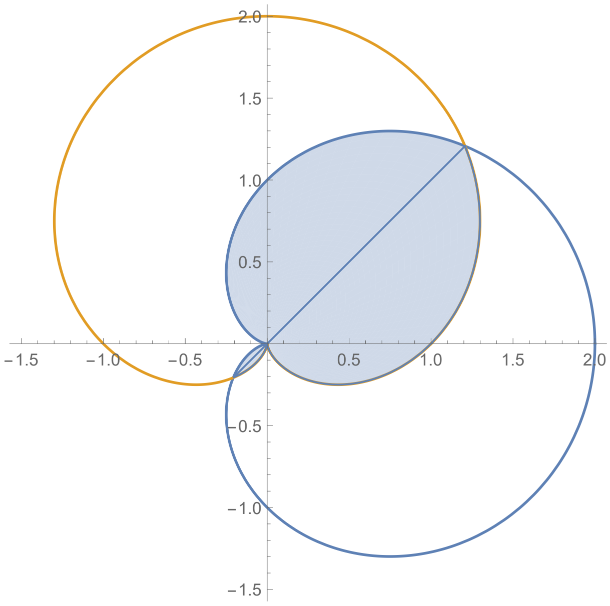
$$(x, y, z) \cdot \vec{n} = f(-1, 3) \cdot \vec{n}$$

is

$$-7x - 8y + z = 7 - 24 + 14 = -3.$$

Solution of the task 19.

The sketch of the curves and the bounded region in the intersection is the following, where $r_1(\varphi)$ is the blue curve and $r_2(\varphi)$ is the orange one:



Since $\sin\varphi$ and $\cos\varphi$ are periodic function with a period 2π, it is enough to restrict ourselves to the interval $[0, 2\varphi]$. First we have to determine the points, where the curves intersect. As seen from the sketch, one of the points is the origin $(0,0)$, where both polar radii are 0. This is true for $\varphi = \pi$ for r_1 and $\varphi = \frac{3\pi}{2}$ for r_2. The other intersections can occur for nonzero radii, in which case they have to be the same at the same angle. We have that

$$r_1(\varphi) = r_2(\varphi) \quad \Leftrightarrow \quad \sin\varphi = \cos\varphi \quad \Leftrightarrow \quad \varphi \in \left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}.$$

So the other two intersections are points

$$A = \left(r_1\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right), r_1\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)\right)$$

and

$$B = \left(r_1\left(\frac{5\pi}{4}\right)\cos\left(\frac{5\pi}{4}\right), r_1\left(\frac{5\pi}{4}\right)\sin\left(\frac{5\pi}{4}\right)\right).$$

We see from the sketch that the intersection consists of the area enclosed by $r_1(\varphi)$ on the interval $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ and the area enclosed by $r_2(\varphi)$ on the union of intervals $\left[0, \frac{\pi}{4}\right] \cup \left[\frac{5\pi}{4}, 2\pi\right]$. Hence,

$$\text{area} = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (r_1(\varphi))^2\, d\varphi + \frac{1}{2}\int_{\frac{5\pi}{4}}^{2\pi} (r_2(\varphi))^2\, d\varphi + \frac{1}{2}\int_0^{\frac{\pi}{4}} (r_2(\varphi))^2\, d\varphi.$$

Further on,

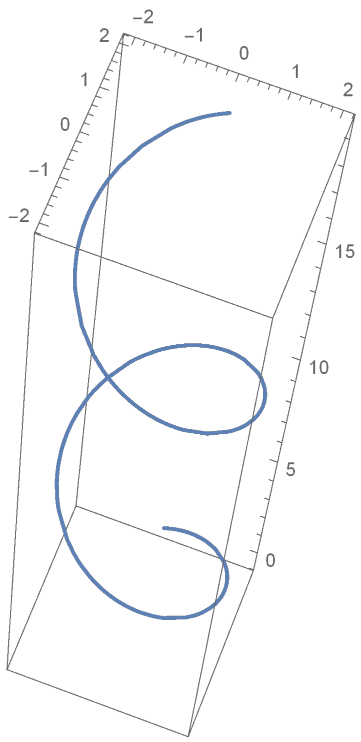
$$\begin{aligned}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1+\cos\varphi)^2\, d\varphi &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1 + 2\cos\varphi + \cos^2\varphi)\, d\varphi \\ &= \left[\frac{3}{2}\varphi + 2\sin\varphi + \frac{1}{4}\sin 2\varphi\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = \frac{3}{2}\pi - 2\sqrt{2},\end{aligned}$$

where we used $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ in the second equality. Similarly

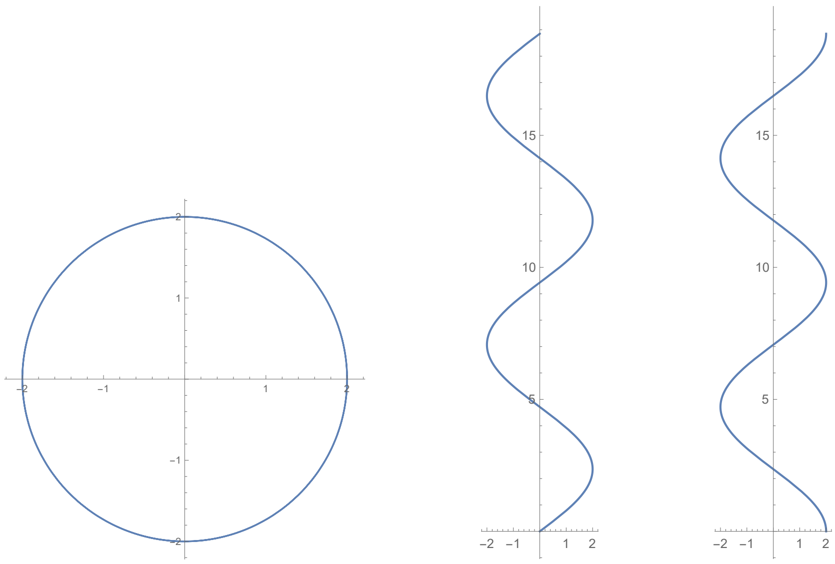
$$\begin{aligned}
&\int_{\frac{5\pi}{4}}^{2\pi} (1 + \sin\varphi)^2 \, d\varphi + \int_0^{\frac{\pi}{4}} (1 + \sin\varphi)^2 \, d\varphi = \\
&= \int_{-3\frac{\pi}{4}}^{0} (1 + \sin\varphi)^2 \, d\varphi + \int_0^{\frac{\pi}{4}} (1 + \sin\varphi)^2 \, d\varphi \\
&= \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} (1 + \sin\varphi)^2 \, d\varphi = \left[\frac{3}{2}\varphi - 2\cos\varphi - \frac{1}{4}\sin 2\varphi\right]_{\frac{-3\pi}{4}}^{\frac{\pi}{4}} \\
&= \frac{3}{2}\pi - 2\sqrt{2},
\end{aligned}$$

where we used $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ in the third equality. So, area $= \frac{3}{2}\pi - 2\sqrt{2}$.

Solution of the task 20.
The sketch of the curve is the following:



The sketch of the projections to xy-, xz- and yz-planes are:



The arc length is the following:

$$\begin{aligned}\int_0^{2\pi} \|r'(t)\|\,dt &= \int_0^{2\pi} \|(4\cos(2t), -4\sin(2t), 3)\|\,dt\\ &= \int_0^{2\pi} \sqrt{16\cos^2(2t) + 16\sin^2(2t) + 9}\,dt\\ &= \int_0^{2\pi} \sqrt{25}\,dt = 10\pi.\end{aligned}$$

Solution of the task 21.

a.

$$\begin{aligned} x(t) = 0 \quad &\Leftrightarrow \quad 2t - t^2 = 0 \quad \Leftrightarrow \quad t(t-2) = 0 \quad \Leftrightarrow \quad t \in \{0, 2\},\\ y(t) = 0 \quad &\Leftrightarrow \quad 3t - t^3 = 0 \quad \Leftrightarrow \quad t(3-t^2) = 0 \quad \Leftrightarrow \quad t \in \{-\sqrt{3}, 0, \sqrt{3}\}.\end{aligned}$$

Intersections with y-axis: $(0,0)$, $(0,-2)$.

Intersections with x-axis: $(-2\sqrt{3}-3, 0)$, $(0,0)$, $(2\sqrt{3}-3, 0)$.

b.

$$\begin{aligned} \dot{x}(t) = 0 \quad &\Leftrightarrow \quad 2 - 2t = 0 \quad \Leftrightarrow \quad t = 1,\\ \dot{y}(t) = 0 \quad &\Leftrightarrow \quad 3 - 3t^2 = 0 \quad \Leftrightarrow \quad 1 - t^2 = 0 \quad \Leftrightarrow \quad t \in \{-1, 1\}.\end{aligned}$$

Horizontal tangent: $(-3,-2)$, vertical tangents: none.

c. From the part above $(1,2)$.

d. $\lim\limits_{t\to-\infty} f(t) = \begin{bmatrix} -\infty \\ \infty \end{bmatrix}$, $\lim\limits_{t\to\infty} f(t) = \begin{bmatrix} -\infty \\ -\infty \end{bmatrix}$.

e. Assume that $t \neq s$.

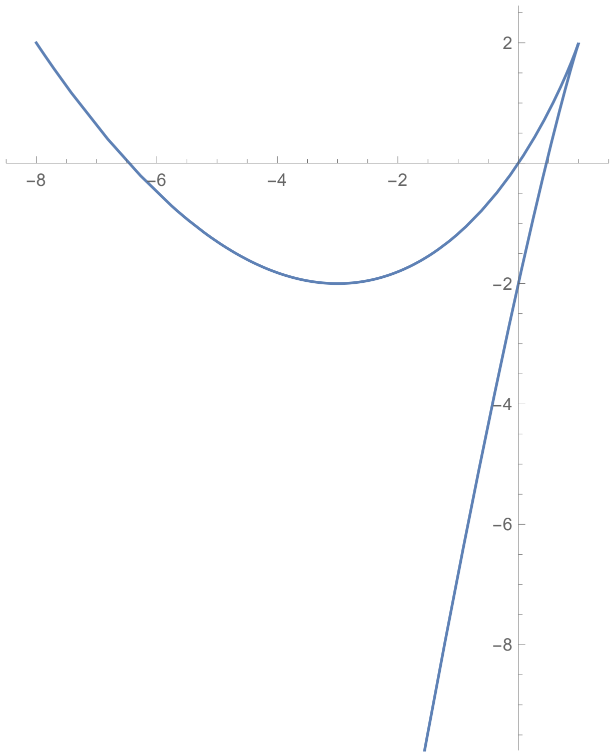
$$\begin{aligned} 1 - x(t) = 1 - x(s) \quad &\Leftrightarrow \quad 1 - 2t + t^2 = 1 - 2s + s^2 \\ &\Leftrightarrow \quad (1-t)^2 = (1-s)^2 \\ &\Leftrightarrow \quad 1 - t \in \{1 - s, s - 1\}. \end{aligned}$$

Since $t \neq s$, $1 - t = s - 1$ and hence $s = 2 - t$. Thus

$$\begin{aligned} y(t) = y(2-t) \quad &\Leftrightarrow \quad 3t - t^3 = 3(2-t) + (2-t)^3 \\ &\Leftrightarrow \quad t^3 - 3t^2 + 3t - 1 = 0 \\ &\Leftrightarrow \quad (t-1)^3 = 0 \quad \Leftrightarrow \quad t = 1. \end{aligned}$$

But then $s = 2 - 1 = 1$ and $s = t$.

f. Using the information above, the sketch of the curve is the following:

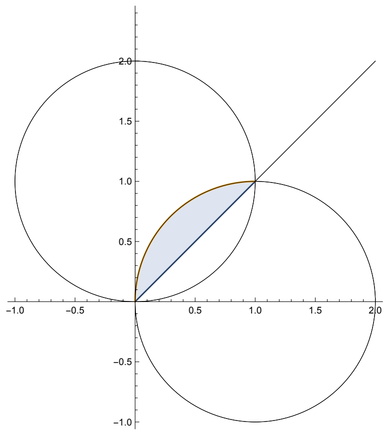


SOLUTION OF THE TASK 22.

a. We have

$$\begin{aligned} r_1^2 = r_1 \cdot 2\sin\varphi \quad &\Rightarrow \quad x^2 + y^2 = 2y \quad \Leftrightarrow \quad x^2 + (y-1)^2 = 1, \\ r_2^2 = r_2 \cdot 2\cos\varphi \quad &\Rightarrow \quad x^2 + y^2 = 2x \quad \Leftrightarrow \quad (x-1)^2 + y^2 = 1, \end{aligned}$$

where we used $r_i^2 = x^2 + y^2$, $r_i \cos\varphi = x$ and $r_i \sin\varphi = y$, $i = 1, 2$. The first equation $r_1(\varphi)$ is the equation of the circle with center at $(0, 1)$ and radius 1, and the second $r_2(\varphi)$ is the circle with center at $(1, 0)$ and radius 1. The sketch is the following



b. The area can be computed using elementary geometry or using the formula for integrals of regions in polar coordinates. If A is the shaded region in the plot, then

$$A = \frac{1}{2}\int_{\pi/4}^{\pi/2} (2\cos\varphi)^2\, d\varphi = \int_{\pi/4}^{\pi/2} (1+\cos(2\varphi))\, d\varphi$$
$$= \left[\varphi + \frac{\sin(2\varphi)}{2}\right]_{\pi/4}^{\pi/2} = \frac{\pi}{4} - \frac{1}{2},$$

and the total area is $2A = \dfrac{\pi}{2} - 1$.

SOLUTION OF THE TASK 23.

a. The intersections with the x-axis correspond to $y(t) = 0$:

$$y(t) = 0 \quad \Leftrightarrow \quad t^2 - 4 = 0 \quad \Leftrightarrow \quad t = t_1 = -2,\ t = t_2 = 2.$$

Hence, there are two intersections with the x-axis:

$$P = \vec{r}(-2) = (0,0), \qquad \vec{r}(2) = (0,0).$$

The intersections with the y-axis correspond to $x(t) = 0$:

$$x(t) = 0 \quad \Leftrightarrow \quad t^3 - 4t = 0 \quad \Leftrightarrow \quad t(t^2-4) = 0 \quad \Leftrightarrow \quad t = t_1,\ t = t_2,\ t = t_3 = 0.$$

Hence, there are three intersections with the y-axis:

$$\vec{r}(-2) = (0,0), \qquad \vec{r}(2) = (0,0), \qquad Q = \vec{r}(0) = (0,-4).$$

b. The tangent to the curve at the point $t = t_0$ is

$$\vec{s}(\lambda) = \vec{r}(t_0) + \lambda\, (\vec{r})'\,(t_0).$$

Hence,

$$\vec{s}(\lambda) = (-3,-3) + \lambda(3t^2-4, 2t)(1) = (-3,-3) + \lambda(-1,2) = (-3-\lambda, -3+2\lambda).$$

c. The tangent to the curve is horizontal at the points where $y'(t) = 0$. Hence,

$$y'(t) = 0 \quad \Leftrightarrow \quad 2t = 0 \quad \Leftrightarrow \quad t = 0,$$

and the point is $Q = (0,-4)$.

The tangent to the curve is vertical at the points where $x'(t) = 0$. Hence,

$$x'(t) = 0 \quad \Leftrightarrow \quad 3t^2 - 4 = 0 \quad \Leftrightarrow \quad t_4 = \frac{2}{\sqrt{3}},\ t_5 = -\frac{2}{\sqrt{3}},$$

and the points are

$$R=\left(-\frac{16}{3\sqrt{3}},-\frac{8}{3}\right)\approx(3.079,-2.67),\qquad S=\left(\frac{16}{3\sqrt{3}},-\frac{8}{3}\right)\approx(-3.079,-2.67).$$

d. Self-intersections correspond to different values of t_1, t_2, where $\vec{r}(t_1)=\vec{r}(t_2)$:

$$\begin{aligned}\vec{r}(t_1)=\vec{r}(t_2)\quad&\Leftrightarrow\quad t_1^3-4t_1=t_2^3-4t_2\quad\text{and}\quad t_1^2-4=t_2^2-4\\&\Leftrightarrow\quad t_1^3-4t_1=t_2^3-4t_2\quad\text{and}\quad t_1^2=t_2^2\\&\Leftrightarrow\quad t_1^3-4t_1=(-t_1)^3+4t_1\quad\text{and}\quad t_2=-t_1\\&\Leftrightarrow\quad 2t_1(t_1^2-4)=0\quad\text{and}\quad t_2=-t_1\\&\Leftrightarrow\quad t_1=2,t_2=-2\quad\text{or}\quad t_1=-2,t_2=2.\end{aligned}$$

Hence, the only self-intersection is the point P.

The area A inside the loop formed by the curve between $t_1=-2$ and $t_2=2$ is

$$\begin{aligned}A&=\frac{1}{2}\int_{-2}^{2}\left|(t^3-4t)2t-(t^2-4)(3t^2-4)\right|\,dt\\&=\frac{1}{2}\int_{-2}^{2}\left|-t^4+8t^2-16\right|\,dt.\end{aligned}$$

Since

$$t^4-8t^2+16=(t^2-4)^2=(t-2)^2(t+2)^2,$$

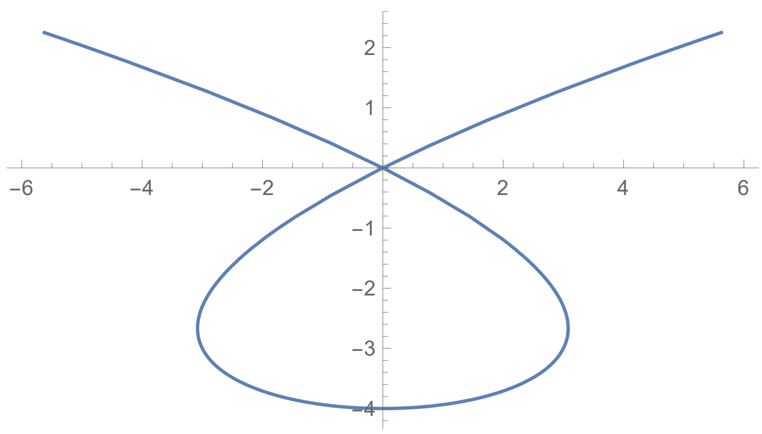
we have that

$$\left|-t^4+8t^2-16\right|=t^4-8t^2+16$$

for every $t\in\mathbb{R}$. Hence,

$$A=\frac{1}{2}\int_{-2}^{2}\left(t^4-8t^2+16\right)\,dt=\frac{1}{2}\left[\frac{t^5}{5}-8\frac{t^3}{3}+16t\right]_{-2}^{2}=\frac{512}{30}.$$

e. The sketch of the curve is the following:



Solution of the task 24.

a. Self-intersections are points such that $\gamma(t_1) = \gamma(t_2)$ for some $t_1 \neq t_2$:

$$\begin{aligned}
\gamma(t_1) = \gamma(t_2) \quad &\Leftrightarrow \quad t_1^3 - t_1 + 1 = t_2^3 - t_2 + 1 \quad \text{and} \quad t_1^2 = t_2^2,\\
&\Leftrightarrow \quad t_1^3 - t_1 + 1 = -t_1^3 + t_1 + 1 \quad \text{and} \quad t_2 = -t_1,\\
&\Leftrightarrow \quad 2t_1(t_1^2 - 1) \quad \text{and} \quad t_1^2 = t_2^2,\\
&\Leftrightarrow \quad t_1 = 0, t_2 = 0 \quad \text{or} \quad t_1 = -t_2 = 1 \quad \text{or} \quad t_1 = -t_2 = -1.
\end{aligned}$$

Hence, the self-intersection is $\gamma(1) = \gamma(-1) = (1, 1)$.

b. The angle at which γ intersects itself in the self-intersection is the angle between the tangents γ_1, γ_2 in $(1, 1)$:

$$\begin{aligned}
\gamma_1(\lambda) &= (1,1) + \lambda \cdot \dot\gamma(1) = (1,1) + \lambda\left((3t^2 - 1, 2t)(1)\right) = (1,1) + \lambda(2,2),\\
\gamma_2(\lambda) &= (1,1) + \lambda \cdot \dot\gamma(-1) = (1,1) + \lambda\left((3t^2 - 1, 2t)(-1)\right) = (1,1) + \lambda(-4,-2).
\end{aligned}$$

Hence,

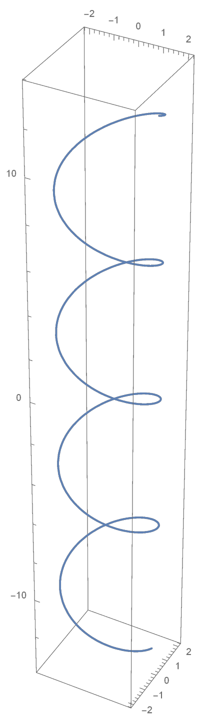
$$\arccos\left(\frac{\langle (2,2), (-4,-2)\rangle}{\|(2,2)\|\,\|(-4,-2)\|}\right) = \arccos\left(\frac{-8-4}{\sqrt{8}\sqrt{20}}\right) = \arccos\left(\frac{-3}{\sqrt{10}}\right) \approx 2.82.$$

So, the angle between the tangents is $\pi - 2.82 \approx 0.32$.

c. The point at which γ reaches a a global minimum in the y-directions satisfies $\frac{d}{dt}(t^2) = 0$. Hence, $2t = 0$ and $t = 0$. The point is $(1, 0)$.

Solution of the task 25.

a. The sketch of γ is the following:



b. The natural parameter $s(t)$, which measures the arc length between the points $\gamma(0)$ and $\gamma(t)$ is

$$\begin{aligned} s(t) = \int_0^t \|\gamma'(u)\| \ du &= \int_0^t \|(-2\sin u, 2\cos u, -1\| \ du \\ &= \int_0^t \sqrt{4\sin^2 u + 4\cos^2 u + 1}\, du \\ &= \int_0^t \sqrt{5}\, du = t\sqrt{5}. \end{aligned}$$

Hence, $t(s) = \frac{s}{\sqrt{5}}$ and the parametrization of the curve with the natural parameter s is

$$\gamma(s) = \left(2\cos\left(\frac{s}{\sqrt{5}}\right), 2\sin\left(\frac{s}{\sqrt{5}}\right), -\frac{s}{\sqrt{5}}\right).$$

c. The point $(2,0,0)$ corresponds to $t=0$, while $(2,0,2\pi)$ to $t=-2\pi$. Hence, the arc length between this points equals by symmetry to $s(2\pi) = 2\pi\sqrt{5}$.

Solution of the task 26.

a. The angle at which the surfaces intersect at P is the angle between the normals to their tangent planes at the point P, i.e., between their gradients at the point P:

$$\begin{aligned} &\arccos\left(\frac{\langle(\operatorname{grad}\Pi)(P), (\operatorname{grad}\Sigma)(P)\rangle}{\|\operatorname{grad}\Pi)(P)\|\ \|\operatorname{grad}\Sigma)(P)\|}\right) \\ &= \arccos\left(\frac{\langle(2x,2y,-z)(P), (2x,2y,-1)(P)\rangle}{\|(2x,2y,-z)(P)\|\ \|(2x,2y,-1)(P)\|}\right) \\ &= \arccos\left(\frac{\langle(2,2,-2),(2,2,-1)\rangle}{\|(2,2,-2)\|\ \|(2,2,-1)\|}\right) = \arccos\left(\frac{10}{\sqrt{12}\sqrt{9}}\right) \\ &= \arccos\left(\frac{5}{3\sqrt{3}}\right) \approx 0.28. \end{aligned}$$

b. The intersection of the surfaces satisfies

$$x^2+y^2 = \frac{(x^2+y^2)^2}{2} \quad \Rightarrow \quad 2 = x^2+y^2 \quad \Rightarrow \quad z = 2.$$

So this is a circle with the parametrization

$$\gamma(t) = \left(\sqrt{2}\cdot\cos t, \sqrt{2}\cdot\sin t, 2\right).$$

The tangent to this circle in the point $(1,1,2)$, which corresponds to $t=\frac{\pi}{4}$, is

$$\begin{aligned} \ell(\lambda) &= (1,1,2) + \lambda\cdot\left(\gamma'\left(\frac{\pi}{4}\right)\right) \\ &= (1,1,2) + \lambda\cdot\left(\left(-\sqrt{2}\cdot\sin t, \sqrt{2}\cdot\cos t, 0\right)\left(\frac{\pi}{4}\right)\right) \\ &= (1,1,2) + \lambda\cdot(-1,1,0). \end{aligned}$$

c. The tangent plane to Σ at $Q := (a,b,c) = (1,2,5)$, which is given implicitly by the equation

$$F(x,y,z) = 0,$$

where

$$F(x,y,z) := x^2+y^2-z,$$

is determined by

$$\begin{aligned}0 &= \left(\frac{\partial F}{\partial x}(Q)\right)\cdot(x-a)+\left(\frac{\partial F}{\partial y}(Q)\right)\cdot(y-b)+\left(\frac{\partial F}{\partial z}(Q)\right)\cdot(z-c)\\ &= ((2x)(Q))\cdot(x-1)+((2y)(Q))\cdot(y-2)+((-1)(Q))\cdot(z-5)\\ &= 2(x-1)+4(y-2)-(z-5).\end{aligned}$$

SOLUTION OF THE TASK 27.

a. The intersections with the x-axis correspond to $y(t)=0$:

$$\cos(3t)=0 \quad\Leftrightarrow\quad t\in\left\{\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6},\frac{7\pi}{6},\frac{3\pi}{2},\frac{11\pi}{6}\right\}.$$

The corresponding points are

$$\begin{aligned} f\Big(\frac{\pi}{6}\Big) &= f\Big(\frac{5\pi}{6}\Big)=\Big(\sin\frac{\pi}{6},0\Big)=\Big(\frac{1}{2},0\Big),\\ f\Big(\frac{\pi}{2}\Big) &= \Big(\sin\frac{\pi}{2},0\Big)=(1,0),\\ f\Big(\frac{7\pi}{6}\Big) &= f\Big(\frac{11\pi}{6}\Big)=\Big(\sin\frac{7\pi}{6},0\Big)=\Big(-\frac{1}{2},0\Big),\\ f\Big(\frac{3\pi}{2}\Big) &= \Big(\sin\frac{3\pi}{2},0\Big)=(-1,0).\end{aligned}$$

The intersections with the y-axis correspond to $x(t)=0$:

$$\sin t=0 \quad\Leftrightarrow\quad t\in\{0,\pi,2\pi\}.$$

The corresponding points are

$$\begin{aligned} f(0)&=f(2\pi)=(0,1),\\ f(\pi)&=(0,-1).\end{aligned}$$

b. The tangent to the curve is horizontal at the points where $y'(t)=0$. Hence,

$$(\cos(3t))'=0 \quad\Leftrightarrow\quad -3\sin(3t)=0 \quad\Leftrightarrow\quad t\in\left\{0,\frac{\pi}{3},\frac{2\pi}{3},\pi,\frac{4\pi}{3},\frac{5\pi}{3},2\pi\right\}.$$

The corresponding points are

$$\begin{aligned} f(0)=f(2\pi)&=(0,1),\\ f\Big(\frac{\pi}{3}\Big)&=\Big(\frac{\sqrt{3}}{2},-1\Big),\\ f\Big(\frac{2\pi}{3}\Big)&=\Big(\frac{\sqrt{3}}{2},1\Big),\\ f(\pi)&=\Big(0,-1\Big),\\ f\Big(\frac{4\pi}{3}\Big)&=\Big(-\frac{\sqrt{3}}{2},1\Big),\\ f\Big(\frac{5\pi}{3}\Big)&=\Big(-\frac{\sqrt{3}}{2},-1\Big).\end{aligned}$$

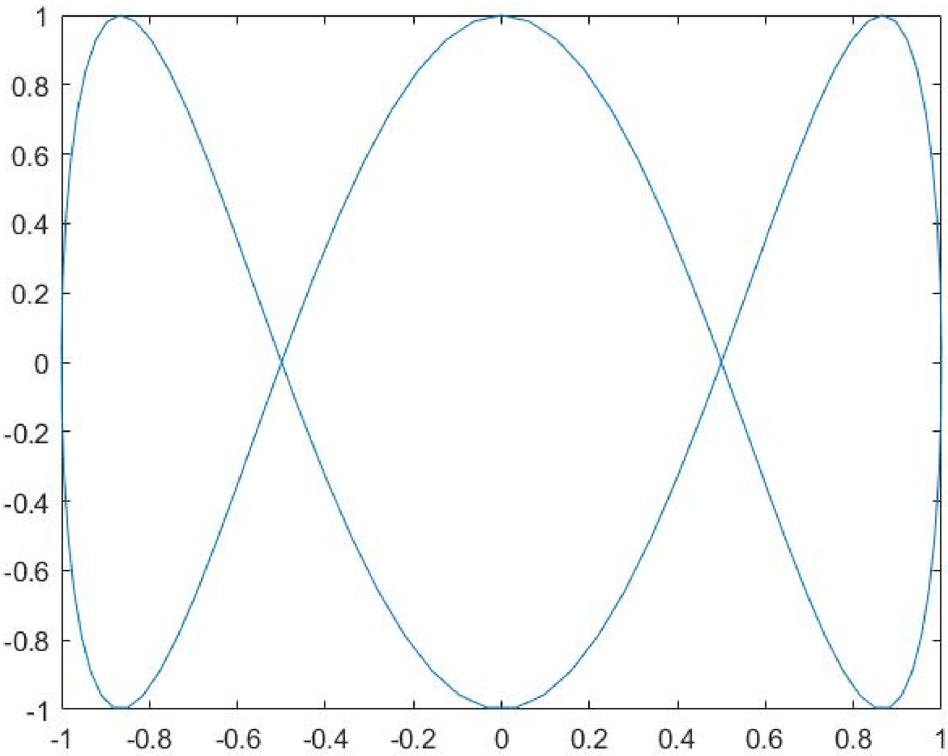
The tangent to the curve is vertical at the points where $x'(t)=0$. Hence,

$$(\sin t)'=0 \quad\Leftrightarrow\quad \cos t=0 \quad\Leftrightarrow\quad t\in\left\{\frac{\pi}{2},\frac{3\pi}{2}\right\}.$$

The corresponding points are

$$f\left(\frac{\pi}{2}\right) = (1, 0),$$
$$f\left(\frac{3\pi}{2}\right) = (-1, 0).$$

c. The sketch of the curve is the following:



Solution of the task 28.

a. The tangent is horizontal in the points where $y'(t) = 0$:

$$2t - 2 = 0 \quad \Leftrightarrow \quad t = 1.$$

The corresponding point is

$$f(1) = (1 - 5 + 3 + 11, 1 - 2 + 3) = (10, 2).$$

The tangent is horizontal in the points where $x'(t) = 0$:

$$3t^2 - 10t + 3 = 0 \quad \Leftrightarrow \quad t_{1,2} = \frac{10 \pm \sqrt{100 - 36}}{6} \in \left\{3, \frac{1}{3}\right\}.$$

The corresponding points are

$$f(3) = (27 - 45 + 9 + 11, 9 - 6 + 3) = (2, 6),$$
$$f(3^{-1}) = (3^{-3} - 5 \cdot 3^{-2} + 1 + 11, 3^{-2} - 2 \cdot 3^{-1} + 3) = \left(\frac{310}{27}, \frac{22}{9}\right) \approx (11.5, 2.4).$$

b. The curve has self-intersections, where $f(t) = f(s)$ for $t \neq s$. We have that

$$\begin{aligned} & t^3 - 5t^2 + 3t + 11 = s^3 - 5s^2 + 3s + 11 \\ \Leftrightarrow \quad & t^3 - s^3 = 5(t^2 - s^2) - 3(t - s) \\ \Leftrightarrow \quad & t^2 + ts + s^2 = 5(t + s) - 3, \end{aligned} \tag{10}$$

where we divided by $t - s$ in the last line. Further on,

$$\begin{aligned} & t^2 - 2t + 3 = s^2 - 2s + 3 \\ \Leftrightarrow \quad & t^2 - s^2 = 2(t - s) \\ \Leftrightarrow \quad & t + s = 2, \end{aligned} \tag{11}$$

where we divided by $t - s$ in the last line. We use (11) in (10):

$$t^2 + t(2 - t) + (2 - t)^2 = 10 - 3 = 7$$

and hence

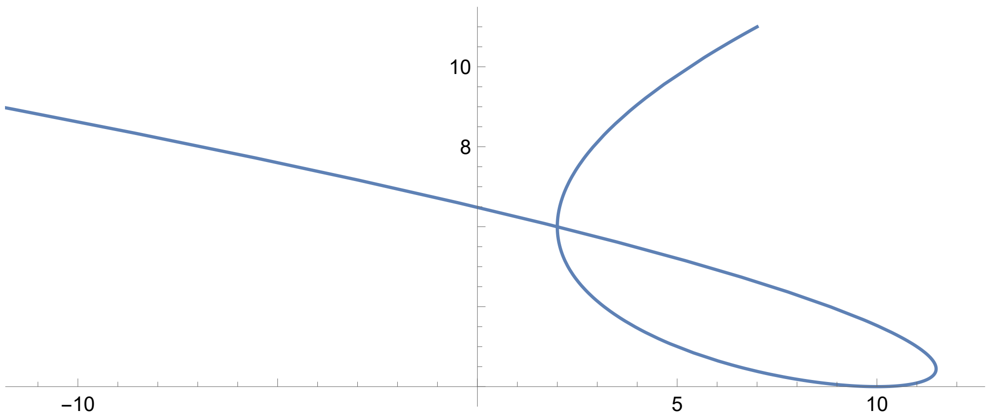
$$0 = t^2 - 2t - 3 = (t - 3)(t + 1).$$

The solutions are $t_1 = 3$, $t_2 = -1$ with the correspoding $s_1 = -1$ and $s_2 = 3$. So the only point of self-intersection is $f(-1) = f(3) = (2, 6)$.

c. We compute

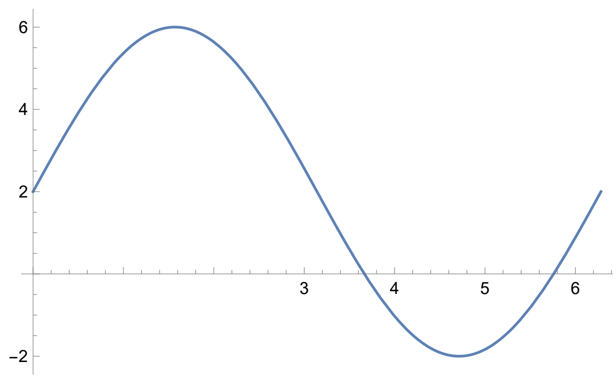
$$\lim_{t\to-\infty} f(t) = (-\infty, \infty) \qquad \text{and} \qquad \lim_{t\to\infty} f(t) = (\infty, \infty).$$

The sketch of the curve is the following:



Solution of the task 29.

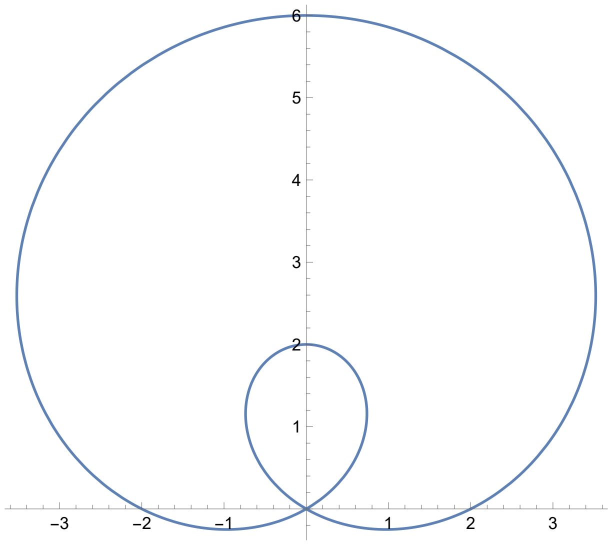
Since $r(\varphi)$ is periodic with a period 2π, we can restrict $r(\varphi)$ to the interval $[0, 2\pi]$. The sketch of $r(\varphi)$ is the following:



Let us write down $r(\varphi)$ for various φ:

φ	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}$	$\frac{7\pi}{6}$	$\frac{8\pi}{6}$	$\frac{9\pi}{6}$	$\frac{10\pi}{6}$	$\frac{11\pi}{6}$
$r(\varphi)$	2	4	$\underbrace{2(1+\sqrt{3})}_{\approx 5.46}$	6	$\underbrace{2(1+\sqrt{3})}_{\approx 5.46}$	4	2	0	$\underbrace{2(1-\sqrt{3})}_{\approx -1.46}$	-2	$\underbrace{2(1-\sqrt{3})}_{\approx -1.46}$	0

Using this calculations we can sketch the curve:



We see from the sketch that the smaller bounded region enclosed by $r(\varphi)$ is obtained by restricting φ to the interval $[\frac{7\pi}{6}, \frac{11\pi}{6}]$. Its area A is

$$
\begin{aligned}
A &= \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} r(\varphi)^2 \, d\varphi = \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (2+4\sin\varphi)^2 \, d\varphi \\
&= \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (4+16\sin\varphi+16\sin^2\varphi) \, d\varphi \\
&= 2\,[\varphi]_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} + 8\,[-\cos\varphi]_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} + \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (8-8\cos(2\varphi)) \, d\varphi \\
&= \frac{4\pi}{3} - 8\sqrt{3} + 4\,[\varphi]_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} - 2\,[\sin(2\varphi)]_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} = \frac{12\pi}{3} - 8\sqrt{3},
\end{aligned}
$$

where we used that $\sin^2\varphi = \frac{1}{2}(1-\cos(2\varphi))$ in the fourth equality.

Differential equations

SOLUTION OF THE TASK 30.

a. An equivalent form of the DE (2) is

$$\underbrace{(2xy-9x^2)}_{M(x,y)}\,\mathrm{d}x + \underbrace{(2y+x^2+1)}_{N(x,y)}\,\mathrm{d}y = 0.$$

Since $M(x,y)$ and $N(x,y)$ are differentiable functions for every $(x,y) \in \mathbb{R}^2$, the DE (2) is exact if and only if

$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y).$$

We have
$$\frac{\partial M}{\partial y}(x,y) = 2x = \frac{\partial N}{\partial x}(x,y)$$
and hence the DE (2) is exact.

b. Solutions of the exact DE are level curves $u(x,y) = C$, $C \in \mathbb{R}$, of the potential function $u(x,y)$, i.e., a function which satisfies
$$\frac{\partial u}{\partial x}(x,y) = M(x,y) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(x,y) = N(x,y).$$
Hence,
$$u(x,y) = \int M(x,y)\mathrm{d}x = x^2y - 3x^3 + C(y),$$
$$u(x,y) = \int N(x,y)\mathrm{d}y = y^2 + x^2y + y + D(x),$$
where $C(y)$ and $D(x)$ are functions of y and x, respectively. So,
$$u(x,y) = x^2y - 3x^3 + y^2 + y + E,$$
where $E \in \mathbb{R}$ is a constant. The level curves of $u(x,y)$ are given by the equations
$$x^2y - 3x^3 + y^2 + y = C, \quad C \in \mathbb{R}. \tag{12}$$
The solution satisfying $y(0) = -3$ is the level curve (12)
$$0^2 \cdot (-3) - 3 \cdot 0^3 + (-3)^2 + (-3) = 6 = C$$
with $C = 6$.

Solution of the task 31.

We introduce the functions $x_1(t) := y(t)$ and $x_2(t) := x_1'(t)$. The DE (3) converts into the system
$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= -24x_1(t) - 11x_2(t). \end{aligned} \tag{13}$$
The matricial form of (13) is
$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -24 & -11 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$
The eigenvalues of the matrix A are the roots of the following determinant
$$\det\left(A - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = (-\lambda)(-11-\lambda) + 24 = \lambda^2 + 11\lambda + 24 = (\lambda+8)(\lambda+3).$$
So, $\lambda_1 = -8$ and $\lambda_2 = -3$.

The kernel of
$$A - \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ -24 & -3 \end{bmatrix}$$
contains the vector
$$u_1 = \begin{bmatrix} 1 \\ -8 \end{bmatrix}.$$

The kernel of

$$A - \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -24 & -8 \end{bmatrix}$$

contains the vector

$$u_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

So, the general solution of the system (13) is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 \cdot e^{-8t} \cdot \begin{bmatrix} 1 \\ -8 \end{bmatrix} + C_2 \cdot e^{-3t} \cdot \begin{bmatrix} 1 \\ -3 \end{bmatrix},$$

where C_1 and C_2 are constants. Hence, the solution of the initial DE is

$$y(t) = Ce^{-8t} + De^{-3t},$$

where C, D are constants.

Solution of the task 32.

Multiplying the DE with y^2 we get

$$3y^2y' \cos x + y^3 \sin x - 1 = 0. \tag{14}$$

Introducing a new variable $z = y^3$, (14) becomes

$$z' \cos x + z \sin x - 1 = 0. \tag{15}$$

The homogeneous part

$$z' \cos x + z \sin x = 0$$

can be solved by separation of variables. We get

$$\frac{dz}{z} = -\tan x \, dx$$

and hence

$$\log|z| = -\int \tan x \, dx = \int \frac{du}{u} = \log|u| + \log K = \log(K \cos x),$$

where we used the substitution $u = \cos x$ in the third equality and K is a constant. So the solution of the homogeneous part of (15) is

$$z_h(x) = K \cos x.$$

To find one particular solution we use variation of constants, i.e.,

$$z_p(x) = K(x) \cos x.$$

Plugging $y_p(x)$ into (15) we get

$$K'(x)(\cos x)^2 = 1. \tag{16}$$

(16) is a separable DE:

$$1\, dK = \frac{1}{(\cos x)^2}\, dx \qquad \Rightarrow \qquad K = \tan x.$$

Hence,

$$z_p(x) = \tan x \cdot \cos x = \sin x,$$

and the general solution of (14) is

$$y(x) = K(\cos x)^{\frac{1}{3}} + (\sin x)^{\frac{1}{3}}.$$

Using that $y(0) = 1$ we get $K = 1$.

Solution of the task 33.
The matrix form of the system is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The eigenvalues of the matrix A are the roots of the following determinant

$$\det\left(A - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = (2-\lambda)(-2-\lambda) + 3 = \lambda^2 - 4 = (\lambda - 1)(\lambda + 1).$$

So, $\lambda_1 = 1$ and $\lambda_2 = -1$.

The kernel of

$$A - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix}$$

contains the vector

$$u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

The kernel of

$$A - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}$$

contains the vector

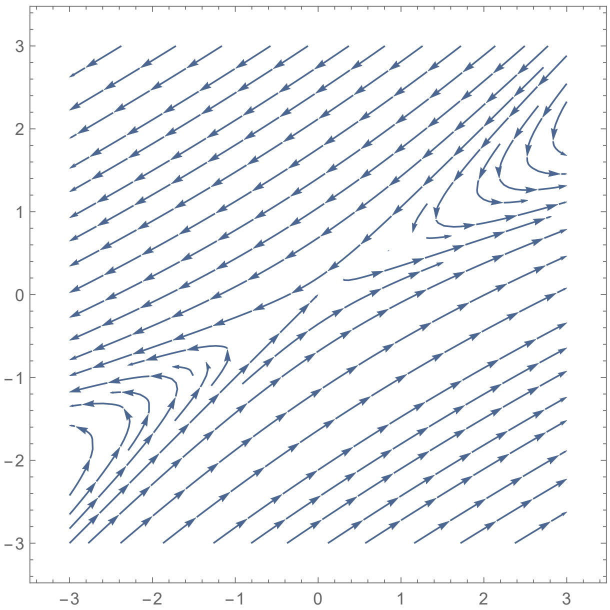
$$u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So, the general solution of the system is

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 \cdot e^t \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_2 \cdot e^{-t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where C_1 and C_2 are constants.

The phase portait is the following:



Solution of the task 34.

First we solve the homogeneous part of the DE. The characteristic polynomial is $\lambda^2-4\lambda+5$ with zeroes

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i.$$

So, two linearly independent solutions to the homogeneous part are

$$y_1(t) = e^{(2+i)t} = e^{2t}e^{it} \qquad \text{and} \qquad y_2(t) = e^{(2-i)t} = e^{2t}e^{-it}.$$

Two *real* linearly independent solutions are

$$y_{\mathbb{R},1}(t) = e^{2t}\cos t \qquad \text{and} \qquad y_{\mathbb{R},2}(t) = e^{2t}\sin t.$$

A general solution of the homogeneous part is

$$y_h(t) = Ay_{\mathbb{R},1}(t) + By_{\mathbb{R},2}(t), \qquad \text{where} \quad A, B \in \mathbb{R}.$$

To find one particular solution of the DE we can use the form

$$y_p(t) = C\cos t + D\sin t,$$

where C, D are some constants. Plugging this form into the initial DE we obtain

$$(-C\cos t - D\sin t) - 4\,(-C\sin t + D\cos t) + 5(C\cos t + D\sin t) = 8\cos x.$$

Comparing the coefficients at $\cos t$ and $\sin t$ on both sides we obtain the system

$$4C - 4D = 8 \qquad \text{and} \qquad 4D + 4C = 0,$$

with the solution $C = -D = 1$. So a general solution to the DE is

$$y(t) = y_h(t) + y_p(t) = e^{2t}(A\cos t + B\sin t) + \cos t - \sin t.$$

The solution with a local extremum in the point $(0, 2)$ is determined by the conditions

$$y(0) = 2 \qquad \text{and} \qquad y'(0) = 0.$$

We get

$$\begin{aligned} 2 &= y(0) = A + 1, \\ 0 &= y'(0) = \big(2e^{2t}(A\cos t + B\sin t) + e^{2t}(-A\sin t + B\cos t) - (\sin t + \cos t)\big)\,(0) \\ &= 2A + B - 1. \end{aligned}$$

So $A = 1$ and $B = -1$.

Solution of the task 35.

The system in the matricial form is the following:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 18t \end{bmatrix} =: \mathcal{A} \begin{bmatrix} x \\ y \end{bmatrix} + f(t).$$

First we will solve the homogeneous part of the system.

$$\det(\mathcal{A} - \lambda I) = \det \begin{bmatrix} 2-\lambda & -1 \\ -2 & 1-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda) - 2 = \lambda(\lambda - 3).$$

Hence, $\det(\mathcal{A} - \lambda I) = 0$ for $\lambda_1 = 0$ and $\lambda_2 = 3$. Further on,

$$\ker \mathcal{A} = \ker \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = \left\{ \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} : \alpha \in \mathbb{R} \right\},$$

$$\ker(\mathcal{A} - 3I) = \ker \begin{bmatrix} -1 & -1 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} = \left\{ \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} : \alpha \in \mathbb{R} \right\}.$$

Thus the solution $(x_h(t), y_h(t))$ of the homogeneous part is the following:

$$\begin{bmatrix} x_h(t) \\ y_h(t)) \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

where $\alpha, \beta \in \mathbb{R}$ are constants.

Now we will find one particular solution $(x_p(t), y_p(t))$ of the system using the hint. Plugging the form of the particular solution from the hint into our system we obtain:

$$\begin{aligned} 2At + B &= 2(At^2 + Bt + C) - (Dt^2 + Et + F) \\ &= (2A - D)t^2 + (2B - E)t + (2C - F), \\ 2Dt + E &= -2(At^2 + Bt + C) + (Dt^2 + Et + F) + 18t \\ &= (-2A + D)t^2 + (-2B + E + 18)t + (-2C + F). \end{aligned} \tag{17}$$

By comparing the coefficients at $1, t, t^2$ in (17) we get the system:

$$2A - D = 0, \quad 2A = 2B - E, \quad B = 2C - F, \quad 2D = -2B + E + 18, \quad E = 2C - F,$$

with a one parametric family of solutions

$$(A, B, C, D, E, F) = (3, 2, C, 6, -2, -2 + 2C),$$

where $C \in \mathbb{R}$ is a constant. Choosing $C = 0$ we get

$$(x_p(t), y_p(t)) = (3t^2 + 2t, 6t^2 - 2t - 2).$$

Finally, a general solution is

$$\begin{bmatrix} x(t) \\ y(t)) \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 3t^2 + 2t \\ 6t^2 - 2t - 2 \end{bmatrix}. \tag{18}$$

We plug $x(0) = 1$ and $y(0) = 0$ in (18) and obtain $\alpha = 1$, $\beta = 0$.

Solution of the task 36.

a. The exact solution can be obtained using separation of variables:

$$\frac{dy}{dx} = 2xy^2 \quad \Rightarrow \quad \frac{dy}{y^2} = 2x\,dx \quad \Rightarrow \quad -\frac{1}{y} = x^2 + C \quad \Rightarrow \quad y = -\frac{1}{x^2 + C}.$$

The solution passing through the point $(0, 1)$ is obtained by

$$y(0) = 1 = -\frac{1}{C} \quad \Rightarrow \quad C = -1 \quad \Rightarrow \quad y = \frac{1}{1 - x^2}.$$

b. Using Euler's method to estimate $y(0.4)$ we get:

$$y_1 = y(0.2) = 1 + 0.2 \cdot (0 \cdot 1^2) = 1,$$
$$y_2 = y(0.4) = 1 + 0.2 \cdot (2 \cdot 0.2 \cdot 1^2) = 1.08.$$

The exact solution is

$$y(0.4) = \frac{1}{1 - 0.16} \approx 1.2.$$

Solution of the task 37.

First we solve the homogenenous part

$$\ddot{x} + \dot{x} - 2x = 0.$$

The characterstic polynomial is $p(\lambda) := \lambda^2 + \lambda - 2$ and hence

$$p(\lambda) = 0 \quad \Leftrightarrow \quad (\lambda - 2)(\lambda + 1) = 0.$$

So, the solution of the homogeneous part is

$$x_h(t) = Ce^{2t} + De^{-t},$$

where $C, D \in \mathbb{R}$ are constants. To find one particular solution we use the form

$$x_p = At^2 + Bt + C.$$

Hence, $\dot{x}_p = 2At + B$ and $\ddot{x}_p = 2A$. Plugging this into the DE we obtain

$$2A + (2At + B) - 2\left(At^2 + Bt + C\right) = t^2. \tag{19}$$

Comparing the coefficients at $1, t, t^2$ on both sides of (19) we get the system

$$2A + B - 2C = 0, \qquad 2A - 2B = 0, \qquad -2A = 1,$$

with the solution

$$A = -\frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = -\frac{3}{4}.$$

Hence, the general solution of the DE is

$$x(t) = Ce^{2t} + De^{-t} - \frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4}.$$

Solution of the task 38.

The DE can be solved by separation of variables:

$$\frac{dy}{1 + y^2} = 2x\ dx \quad \Rightarrow \quad \arctan y = x^2 + C.$$

The particular solution, which goes through the point $(1, 0)$, is

$$\arctan y(1) = \arctan(0) = 0 = 1^2 + C \quad \Rightarrow \quad C = -1 \quad \Rightarrow \quad \arctan y = x^2 - 1.$$

Solution of the task 39.

a. Stationary points satisfy $\dot{x} = \dot{y} = 0$. Hence,

$$\begin{aligned} \dot{x} = xy + 1 = 0 \quad \text{and} \quad \dot{y} = x + xy = 0 \quad &\Leftrightarrow \quad xy = -1 \quad \text{and} \quad x = 1 \\ &\Leftrightarrow \quad y = -1 \quad \text{and} \quad x = 1. \end{aligned}$$

b. To classify the stationary point $(1, -1)$ we have to linearize the system. We denote the right side of the system by

$$f(x, y) = (xy + 1, x + xy).$$

Hence,

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} &\approx ((Jf)(1, -1)) \cdot \begin{bmatrix} x - 1 \\ y + 1 \end{bmatrix} \\ &= \left(\begin{bmatrix} y & x \\ 1 + y & x \end{bmatrix} (1, -1) \right) \cdot \begin{bmatrix} x - 1 \\ y + 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x - 1 \\ y + 1 \end{bmatrix} \\ &= \begin{bmatrix} -x + y + 2 \\ y + 1 \end{bmatrix}. \end{aligned}$$

The behaviour of the system depends on the eigenvalues of $(Jf)(1, -1)$:

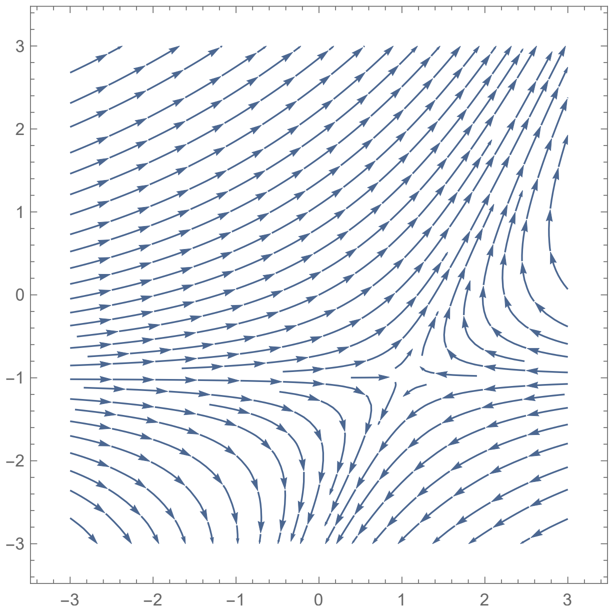
$$\det\left((Jf)(1, -1) - \lambda I_2\right) = \begin{bmatrix} -1 - \lambda & 1 \\ 0 & 1 - \lambda \end{bmatrix} = (-1 - \lambda)(1 - \lambda).$$

Hence, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$. The solution of the system is of the form

$$C \cdot e^{t} \cdot v_1 + D \cdot e^{-t} \cdot v_2 + \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where v_1, v_2 are the eigenvectors of $(Jf)(1, -1)$ for the eigenvalues $1, -1$ and C, D are constants. The point $(1, -1)$ is a saddle.

c. The sketch of the phase portrait is the following:



Solution of the task 40.

First we solve the homogeneous part of the DE:

$$xy' = y \quad \Rightarrow \quad \frac{dy}{y} = \frac{dx}{x} \quad \Rightarrow \quad \ln|y| = \ln|x| + k \quad \Rightarrow \quad y_h(x) = Kx,$$

where $k, K \in \mathbb{R}$ are constants. Now we have to determine one particular solution. By the form of the DE we can try with the form

$$y_p(x) = Ax^3 + Bx^2 + Cx + D,$$

where $A, B, C, D \in \mathbb{R}$ are constants. Hence,

$$y_p'(x) = 3Ax^2 + 2Bx + C$$

and plugging into the DE we get

$$x(3Ax^2 + 2Bx + C) = Ax^3 + Bx^2 + Cx + D + 2x^3. \tag{20}$$

Comparing the coefficients at $x^3, x^2, x, 1$ on both sides of (20) we obtain the system

$$3A = A + 2, \quad 2B = B, \quad C = C, \quad 0 = D.$$

Hence,

$$A = 1, \quad B = D = 0 \quad \text{and} \quad C \in \mathbb{R} \text{ is arbitrary.}$$

We choose $C = 0$ and get

$$y_p(x) = x^3.$$

The general solution of the DE is

$$y(x) = y_h(x) + y_p(x) = Kx + x^3.$$

The solution which passes through the point $(2, 3)$ is

$$y(2) = 3 = 2K + 27 \quad \Rightarrow \quad K = -12 \quad \Rightarrow \quad y(x) = -12x + x^3.$$

Solution of the task 41.

First we solve the homogeneous part of the DE:

$$y'' + y' - 6y = 0.$$

The characteristic polynomial is

$$p(\lambda) = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$$

with zeroes $\lambda_1 = -3$, $\lambda_2 = 2$. Hence, the solution of the homogeneous part is

$$y_h(x) = Ce^{-3x} + De^{2x},$$

where $C, D \in \mathbb{R}$ are constants.

To obtain a particular solution we can try with the form

$$y_p(x) = ax^2 + bx + c \quad \Rightarrow \quad y_p'(x) = 2ax + b \quad \Rightarrow \quad y_p''(x) = 2a. \tag{21}$$

Plugging (21) into the DE we obtain

$$2a + (2ax + b) - 6(ax^2 + bx + c) = 36x. \tag{22}$$

Comparing the coefficients at $x^2, x, 1$ on both sides of (22) we obtain a system

$$-6a = 0, \quad -6b + 2a = 36, \quad 2a + b - 6c = 0,$$

with the solution

$$a = 0, \quad b = -6, \quad c = -1.$$

Hence, the general solution of the DE is

$$y(x) = y_h(x) + y_p(x) = Ce^{-3x} + De^{2x} - 6x - 1.$$

The one satisfying the initial conditions

$$\begin{aligned} y(0) &= C + D - 1 = 1 \\ y'(0) &= -3C + 2D - 6 = 1, \end{aligned}$$

is the one with

$$C = -\frac{3}{5}, \quad D = \frac{13}{5}.$$

So the final solution is

$$y(x) = -\frac{3}{5}e^{-3x} + \frac{13}{5}e^{2x} - 6x - 1.$$

Solution of the task 42.

First we solve the homogeneous part of the DE:

$$x^2y' + xy = 0 \quad \Rightarrow \quad -\frac{dy}{y} = \frac{dx}{x} \quad \Rightarrow \quad -\ln|y| = \ln|x| + k \quad \Rightarrow \quad y_h(x) = \frac{K}{x},$$

where $k, K \in \mathbb{R}$ are constants. Now we have to determine one particular solution. By variation of constants the form of the particular solution is

$$y_p(x) = \frac{K(x)}{x},$$

where $K(x)$ is a function of x. Thus,

$$y_p'(x) = \frac{K'(x)x - K(x)}{x^2} \tag{23}$$

and plugging (23) into the initial DE we get

$$x^2 \cdot \frac{K'(x)x - K(x)}{x^2} + x\frac{K(x)}{x} + 3 = 0.$$

Equivalently,

$$K'(x)x + 3 = 0. \tag{24}$$

We solve the DE (24) by separation of variables:

$$-\frac{dK}{3} = \frac{dx}{x} \quad \Rightarrow \quad -\frac{1}{3}K = \ln|x| \quad \Rightarrow \quad K = \ln\frac{1}{|x|^3}.$$

Since in the initial conditin $x > 0$, we have $K = \ln \frac{1}{x^3}$ and $y_p(x) = \ln \frac{1}{x^3} \cdot \frac{1}{x}$. So, the general solution of the DE is

$$y(x) = y_h(x) + y_p(x) = \left(K + \ln \frac{1}{x^3}\right) \frac{1}{x}.$$

The solution which passes through the point $(1, 1)$ is

$$y(1) = 1 = K + \ln 1 \quad \Rightarrow \quad K = 1 \quad \Rightarrow \quad y(x) = \left(1 + \ln \frac{1}{x^3}\right) \frac{1}{x}.$$

SOLUTION OF THE TASK 43.
The matricial form of the system is the following:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & 5 \\ 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

We compute the eigenvalues of A:

$$\det\left(A - \lambda I_2\right) = \det \begin{bmatrix} -2-\lambda & 5 \\ 1 & 2-\lambda \end{bmatrix} = (-2-\lambda)(2-\lambda) - 5 = \lambda^2 - 9 = (\lambda - 3)(\lambda + 3).$$

Thus the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -3$. The kernel of

$$A - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 5 \\ 1 & -1 \end{bmatrix}$$

contains the vector

$$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The kernel of

$$A - \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 1 & 5 \end{bmatrix}$$

contains the vector

$$u_2 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$

So, the general solution of the system is

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 \cdot e^{3t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cdot e^{-3t} \cdot \begin{bmatrix} -5 \\ 1 \end{bmatrix},$$

where C_1 and C_2 are constants. The solution, which satisfies $x(0) = y(0) = 1$, is:

$$C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad C_1 - 5C_2 = 1, \quad C_1 + C_2 = 1$$

$$\Rightarrow \quad C_1 = 1,\ C_2 = 0.$$

Solution of the task 44.
The DE is of the form
$$2xy\ dx + (x^2 + 3y^2)\ dy = 0.$$
It is indeed exact, since
$$\frac{d(2xy)}{dy} = \frac{d(x^2 + 3y^2)}{dx} = 2x.$$
We have that
$$\int 2xy\ dx = x^2 y + C(y),$$
$$\int (x^2 + 3y^2)\ dx = x^2 y + y^3 + D(x),$$
where $C(y)$ and $D(x)$ are functions of y and x. Hence, the solution of the DE is a family of functions
$$u(x, y, K) = x^2 y + y^3 + K,$$
where $K \in \mathbb{R}$ is a constant.

Solution of the task 45.
First we solve the homogeneous part of the DE:
$$y'' + 9y = 0.$$
The characteristic polynomial is
$$p(\lambda) = \lambda^2 + 9 = (\lambda - 3i)(\lambda + 3i)$$
with zeroes $\lambda_1 = 3i$, $\lambda_2 = -3i$. Hence, the solution of the homogeneous part is
$$y_h(x) = Ce^{3ix} + De^{-3ix},$$
where $C, D \in \mathbb{C}$ are constants. Another way of expressing all solutions of the DE is
$$y_h(x) = C \cos(3x) + D \sin(3x),$$
where $C, D \in \mathbb{C}$ are constants.

To obtain a particular solution we can try with the form
$$y_p(x) = ax^2 + bx + c \quad \Rightarrow \quad y_p'(x) = 2ax + b \quad \Rightarrow \quad y_p''(x) = 2a. \tag{25}$$
Plugging (25) into the DE we obtain
$$2a + 9(ax^2 + bx + c) = 2x^2 - 1. \tag{26}$$
Comparing the coefficients at $x^2, x, 1$ on both sides of (26) we obtain a system
$$9a = 2, \quad 9b = 0, \quad 2a + 9c = -1,$$
with the solution
$$a = \frac{2}{9}, \quad b = 0, \quad c = -\frac{13}{81}.$$
Hence, the general solution of the DE is
$$y(x) = y_h(x) + y_p(x) = C \cos(3x) + D \sin(3x) + \frac{2}{9}x^2 - \frac{13}{81}.$$
The one satisfying the initial conditions
$$y(0) = C - \frac{13}{81} = 1,$$
$$y'(0) = 3D = 1,$$

is the one with

$$C = \frac{94}{81}, \quad D = \frac{1}{3}.$$

So the final solution is

$$y(x) = \frac{94}{81}\cos(3x) + \frac{1}{3}\sin(3x) + \frac{2}{9}x^2 - \frac{13}{81}.$$

Solution of the task 46.

a. The stationary points satisfy $\dot{x} = \dot{y} = 0$. Hence:

$$\begin{aligned} x(3 - x - 2y) = 0 \quad &\Rightarrow \quad x = 0 \quad \text{or} \quad x = 3 - 2y, \\ y(4 - 3x - y) = 0 \quad &\Rightarrow \quad y = 0 \quad \text{or} \quad y = 4 - 3x. \end{aligned}$$

If $x \neq 0$ and $y \neq 0$, then $y = 4 - 3(3 - 2y)$ and thus, $y = 1$ and $x = 1$. So the stationary points are:

$$(0,0),\ (0,4),\ (3,0),\ (1,1).$$

b. The right side of the system is

$$\begin{aligned} f_1(x,y) &= 3x - x^2 - 2xy, \\ f_2(x,y) &= 4y - 3xy - y^2. \end{aligned}$$

The linearization of the system around $(1,1)$ is

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} &\approx \begin{bmatrix} f_1(1,1) \\ f_2(1,1) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1(1,1)}{\partial x} & \frac{\partial f_1(1,1)}{\partial y} \\ \frac{\partial f_2(1,1)}{\partial x} & \frac{\partial f_2(1,1)}{\partial y} \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 1 \end{bmatrix} \\ &= \begin{bmatrix} (3 - 2x - 2y)(1,1) & (-2x)(1,1) \\ (-3y)(1,1) & (4 - 3x - 2y)(1,1) \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -2 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 1 \end{bmatrix}. \end{aligned}$$

c. Introducing the new variables $X = x - 1$ and $Y = y - 1$, one obtains the following autonomous linear system:

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & -2 \\ -3 & -1 \end{bmatrix}}_{A} \begin{bmatrix} X \\ Y \end{bmatrix}$$

We have to compute the eigenvalues of A:

$$\begin{aligned} \det(A - \lambda I_2) = \det \begin{bmatrix} -1 - \lambda & -2 \\ -3 & -1 - \lambda \end{bmatrix} &= (-1 - \lambda)^2 - 6 \\ &= (\lambda + 1 - \sqrt{6})(\lambda + 1 + \sqrt{6}). \end{aligned}$$

Hence, the eigenvalues are $\lambda_1 = \sqrt{6} - 1$ and $\lambda_2 = -\sqrt{6} - 1$. The corresponding eigenspaces are:

$$\ker(A - \lambda_1 I_2) = \ker \begin{bmatrix} -\sqrt{6} & -2 \\ -3 & -\sqrt{6} \end{bmatrix} = \mathrm{Lin}\left\{ \underbrace{\begin{bmatrix} 1 \\ -\frac{\sqrt{6}}{2} \end{bmatrix}}_{v_1} \right\}$$

and

$$\ker(A - \lambda_2 I_2) = \ker \begin{bmatrix} \sqrt{6} & -2 \\ -3 & \sqrt{6} \end{bmatrix} = \operatorname{Lin}\Bigg\{ \underbrace{\begin{bmatrix} 1 \\ \frac{\sqrt{6}}{2} \end{bmatrix}}_{v_2} \Bigg\}.$$

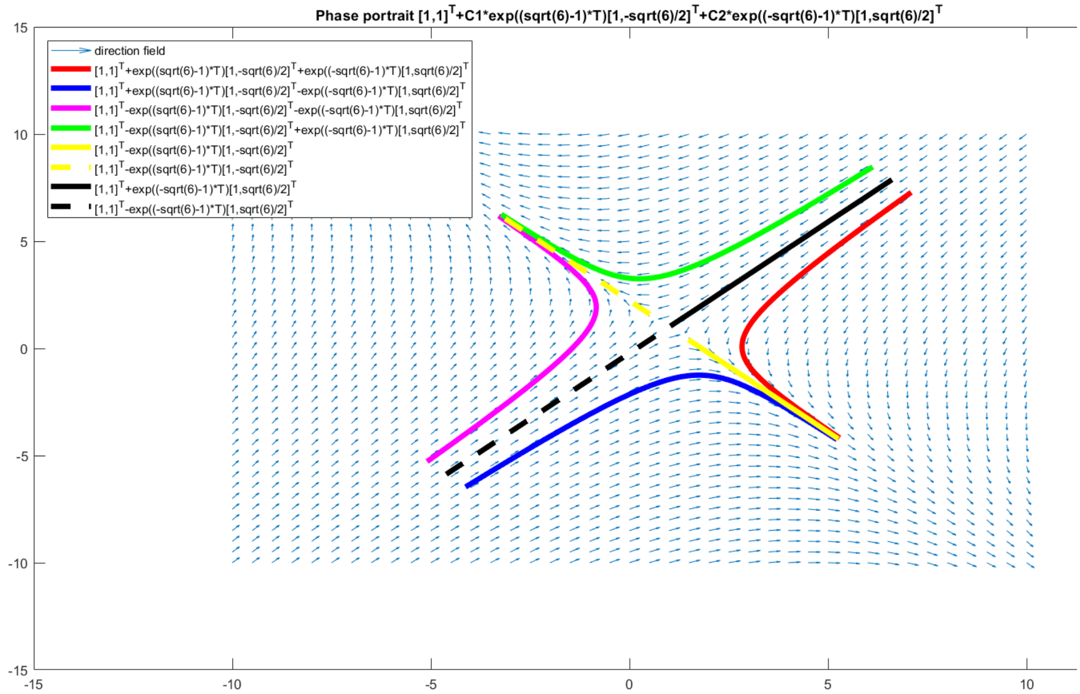
The solutions of the system are

$$C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2,$$

where $C_1, C_2 \in \mathbb{R}$. Hence, the solutions of the original system are

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2,$$

where $C_1, C_2 \in \mathbb{R}$. The sketch of the phase portrait of the original system is the following:



Solution of the task 47.

a. A general solution of the system is

$$C_1 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_3 e^{2t} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix},$$

where C_1, C_2, C_3 are constants.

b. The matrix A has eigenpairs $(-1, v_1)$, $(1, v_2)$, $(2, v_3)$, where

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

Hence,

$$A = \underbrace{\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}}_{P} \cdot \operatorname{diag}(-1,1,-2) \cdot P^{-1}.$$

Let us compute P^{-1} using Guassian elimination:

$$\underbrace{\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]}_{\left[\, P \,\middle|\, I_3 \,\right]} \underbrace{\sim}_{\substack{\ell_2=\ell_2+\ell_1 \\ \ell_3=\ell_3-\ell_1}} \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right]$$

$$\underbrace{\sim}_{\substack{\ell_1=\ell_1-\frac{1}{3}\ell_3 \\ \ell_2=\ell_2-\ell_3}} \left[\begin{array}{rrr|rrr} 1 & 1 & 0 & \frac{4}{3} & 0 & -\frac{1}{3} \\ 0 & 2 & 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right]$$

$$\underbrace{\sim}_{\substack{\ell_1=\ell_1-\frac{1}{2}\ell_2 \\ \ell_2=\frac{1}{2}\ell_2 \\ \ell_3=\frac{1}{3}\ell_3}} \underbrace{\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} \end{array}\right]}_{\left[\, I_3 \,\middle|\, P^{-1} \,\right]}.$$

So

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 4 \\ -1 & 1 & 8 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 0 & 2 \end{bmatrix}. \end{aligned}$$

c. Since the eigenvalues of the matrix A are $-1, 1, 2$, the corresponding third order polynomial is

$$(\lambda+1)(\lambda-1)(\lambda-2) = (\lambda^2-1)(\lambda-2) = \lambda^3 - 2\lambda^2 - \lambda + 2$$

and hence the differential equation with constants coefficients is

$$x^{(3)} = 2x^{(2)} + x^{(2)} - 2x.$$

Solution of the task 48.

First we solve the homogenenous part

$$\ddot{x} - \dot{x} - 4x = 0.$$

The characterstic polynomial is $p(\lambda) := \lambda^2 - \lambda - 4$ and hence

$$p(\lambda) = 0 \quad \Leftrightarrow \quad \lambda_{1,2} = \frac{1 \pm \sqrt{1+16}}{2} = \frac{1 \pm \sqrt{17}}{2}.$$

So, the solution of the homogeneous part is

$$x_h(t) = Ae^{\frac{1+\sqrt{17}}{2}t} + Be^{\frac{1-\sqrt{17}}{2}t},$$

where $A, B \in \mathbb{R}$ are constants. To find one particular solution we use the form

$$x_p = x_{p_1}(t) + x_{p_2}(t),$$

where

$$x_{p_1}(t) = Ct + D, \qquad x_{p_2}(t) = Ee^t.$$

Hence, $\dot{x}_{p_1} = C$, $\ddot{x}_p = 0$ and $\dot{x}_{p_2} = \ddot{x}_{p_2} = Ee^t$. Plugging this into the DE we obtain

$$Ee^t - (C + Ee^t) - 4(Ct + D + Ee^t) = 2t + e^t. \tag{27}$$

Comparing the coefficients at $1, t, e^t$ on both sides of (27) we get the system

$$-C - 4D = 0, \qquad -4C = 2, \qquad -4E = 1,$$

with the solution

$$D = \frac{1}{8}, \quad C = -\frac{1}{2}, \quad E = -\frac{1}{4}.$$

Hence, the general solution of the DE is

$$x(t) = Ae^{\frac{1+\sqrt{17}}{2}t} + Be^{\frac{1-\sqrt{17}}{2}t} - \frac{1}{2}t + \frac{1}{8} - \frac{1}{4}e^t.$$