A gap between positive maps (resp. copositive matrices) and completely positive ones

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joint work with

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Outline

1. Preliminaries and our results
   - quantitative estimates on volumes of cones
   - algorithms to produce examples

2. Converting to polynomials
   - biquadratic biforms
   - even quartic forms

   } real algebraic geometry

3. Proofs
   - asymptotic convex analysis
   - harmonic analysis

1. Preliminaries

Positive and completely positive maps

Definitions

$\mathcal{S} \subseteq M_n(\mathbb{R})$, $\mathcal{T} \subseteq M_m(\mathbb{R})$ linear subspaces containing identity matrix and invariant under transpose.

A linear map

$$\Phi : \mathcal{S} \to \mathcal{T}$$

such that $\Phi(A^T) = \Phi(A)^T$ for all $A \in \mathcal{S}$, is:

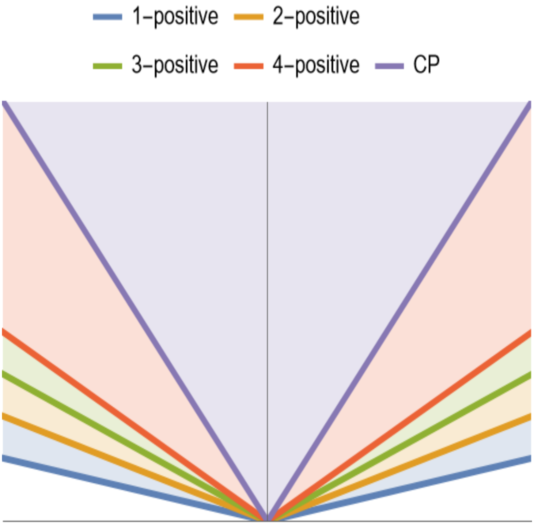
- positive if $A \succeq 0 \Rightarrow \phi(A) \succeq 0$.
- k–positive if

$$\phi_k\left(\begin{pmatrix} A_{11} & \dots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \dots & A_{kk} \end{pmatrix} \right) = \begin{pmatrix} \phi(A_{11}) & \dots & \phi(A_{1k}) \\ \vdots & \ddots & \vdots \\ \phi(A_{k1}) & \dots & \phi(A_{kk}) \end{pmatrix}$$

  is positive.
- completely positive (CP) if it is k–positive for every $k \in \mathbb{N}$.

Positive and completely positive maps

Mental picture



Positive and completely positive maps

A breadth of applications

- matrix theory
- operator theory and operator algebra
- real algebraic geometry
- quantum physics
- quantum information theory
- free probability

Positive and completely positive maps

Our results
with I. Klep, S. McCullough, K. Šivic: There are many more positive maps than completely positive maps, Int. Math. Res. Not. 11 (2019)

1. Quantitave bounds on the fraction of positive maps that are CP.
   (exact asymptotics)
   real algebraic geometry
   convex analysis
   harmonic analysis
2. An algorithm to produce positive maps that are not CP.
   (from random input data)
   algebraic geometry

Positive and completely positive maps

A small sample of existing literature

Theorem (Arveson, 2009)

Let $n, m \geq 2$. Then the probability p that a positive map $\varphi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is cp satisfies $0 < p < 1$.

- Szarek, Werner, Życzkowski (2008) and Auburn, Szarek, Ye (2014): for the case $m = n$ provide quantitative bounds on p and establish its asymptotic behaviour.
- Collins, Hayden, Nechita (2017): random techniques for constructing k-positive maps that are not $(k+1)$-positive in large dimensions.

Copositive and completely positive matrices

Definitions

$\mathbb{S}_n\dots$ real symmetric $n \times n$ matrices

A matrix

$$A = (a_{ij})_{i,j} \in \mathbb{S}_n$$

is:

- positive semidefinite (PSD) if $v^T A v \geq 0$ for every $v \in \mathbb{R}^n$.

Copositive and completely positive matrices

Definitions

$\mathbb{S}_n \ldots$ real symmetric $n \times n$ matrices

A matrix

$$A = (a_{ij})_{i,j} \in \mathbb{S}_n$$

is:

- copositive (COP) if $v^T A v \geq 0$ for every $v \in \mathbb{R}^n_{\geq 0}$.
- positive semidefinite (PSD) if $v^T A v \geq 0$ for every $v \in \mathbb{R}^n$.

Copositive and completely positive matrices

Definitions

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- positive semidefinite (PSD) if $v^T A v \geq 0$ for every $v \in \mathbb{R}^n$.
- completely positive (CP) if $A = BB^T$ for some $B \in \mathbb{R}^{n \times k}_{\geq 0}$.

Copositive and completely positive matrices

Definitions

$\mathbb{S}_n$... real symmetric $n \times n$ matrices

A matrix

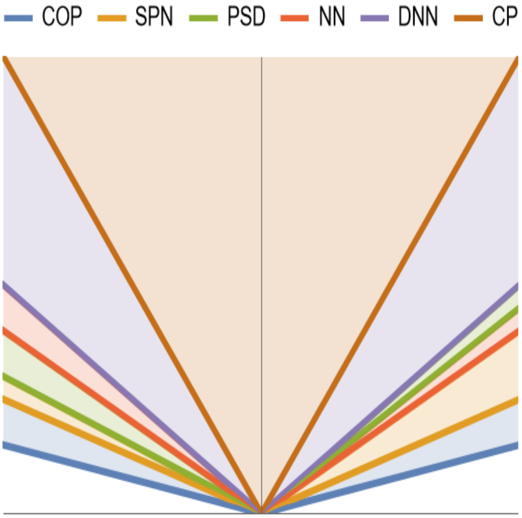
$$A = (a_{ij})_{i,j} \in \mathbb{S}_n$$

is:

- copositive (COP) if $v^T A v \geq 0$ for every $v \in \mathbb{R}^n_{\geq 0}$.
- positive semidefinite (PSD) if $v^T A v \geq 0$ for every $v \in \mathbb{R}^n$.
- nonnegative (NN) if $a_{ij} \geq 0$ for every i, j.
- SPN if $A = P + N$ for some P PSD and N NN.
- doubly nonnegative (DNN) if $A = P \cap N$ for some P PSD and N NN.
- completely positive (CP) if $A = BB^T$ for some $B \in \mathbb{R}^{n \times k}_{\geq 0}$.

Copositive and completely positive matrices

Mental picture



Copositive and completely positive matrices

A breadth of applications

- matrix theory
- optimization
- graph theory
- combinatorics
- quantum information theory

Copositive vs completely positive matrices

Our results
with I. Klep, T. Štrekelj: A random copositive matrix is completely positive

with positive probability, in preparation

1. Quantitave bounds on the fraction of COP matrices that are CP.
   (exact asymptotics)
   real algebraic geometry
   convex analysis
   harmonic analysis

2. An algorithm to produce COP matrices that are not CP.
   free probability inspired construction

Copositive and completely positive matrices

A small sample of existing literature

- Maxfield, Minc (1962) and Hall, Newman (1963): $\mathrm{COP}_n = \mathrm{SPN}_n$ holds only for $n \leq 4$.

- Murty, Kadaby (1987) and Dickinson, Gijben (2014): Deciding containment in COP (resp. CP) is co-NP-complete (resp. NP-hard).

- Parrilo (2000): $\mathrm{int}(\mathrm{COP}_n) \subseteq \bigcup_r K_n^{(r)}$, where $(\mathrm{x}^2 = (x_1^2, \ldots, x_n^2))$

$$K_n^{(r)} := \{A \in \mathbb{S}_n : (\sum_{i=1}^{n} x_i^2)^r \cdot (\mathrm{x}^2)^T A \mathrm{x}^2 \text{ is a sum of squares of forms}\}.$$

- Dickinson, Dür, Gijben, Hildebrand (2013): $\mathrm{COP}_5 \neq K_5^{(r)}$ for any $r \in \mathbb{N}$.

- Laurent, Schweighofer, Vargas (2022, 2023+): $\mathrm{COP}_5 = \bigcup_r K_5^{(r)}$ and $\mathrm{COP}_6 \neq \bigcup_r K_6^{(r)}$.

- Berman, Shaked-Monderer (2021): Copositive and completely positive matrices, World Scientific Publishing Co.

Quantitative bounds

Theorem (Klep, McCullough, Šivic, Z, 2019)

For integers $n, m \geq 3$ the probability $p_{n,m}$ that a positive map $\Phi : \mathbb{S}_n \to \mathbb{S}_m$ is CP, is

$$p_{n,m} \in \Theta(\min(n, m)^{-d/2}),$$

where $d = \binom{n+1}{2}\binom{m+1}{2} - 1$.

Theorem (Klep, Štrekelj, Z, 2023+)

For every integer $n > 4$ the probability p_n that a copositive matrix $A \in \mathbb{S}_n$ is CP, is

$$2^{-13} \leq p_n \leq 1.$$

2. Converting to polynomials

Positive maps meet real algebraic geometry (RAG)

$\mathcal{L}(\mathbb{S}_n, \mathbb{S}_m)$	...	the vector space of all linear maps from $\mathbb{S}_n$ to $\mathbb{S}_m$,
$\mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2}$	...	biforms in $\mathrm{x} = (x_1, \ldots, x_n)$ and $\mathrm{y} = (y_1, \ldots, y_m)$ of bidegree $(2, 2)$

There is a natural bijection

$$\Gamma : \mathcal{L}(\mathbb{S}_n, \mathbb{S}_m) \to \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2}, \quad \Phi \mapsto p_\Phi(\mathrm{x}, \mathrm{y}) := \mathrm{y}^T \Phi(\mathrm{x}\mathrm{x}^T)\mathrm{y}.$$

Proposition

Let $\Phi : \mathbb{S}_n \to \mathbb{S}_m$ be a linear map. Then:

1. Φ is positive iff p_Φ is nonnegative.
2. Φ is completely positive iff p_Φ is a sum of squares (SOS). (Choi-Kraus theorem)

Corollary

The following probabilities (w.r.t. the corresponding distributions) are equal:

1. *The probability that a positive map $\Phi \in \mathcal{L}(\mathbb{S}_n, \mathbb{S}_m)$ is CP.*
2. *The probability that a nonnegative biform $p \in \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2}$ is SOS.*

Copositive matrices meet RAG

$\mathbb{R}[\mathrm{x}^2]_{4,e}$ ... forms in $\mathrm{x}^2 = (x_1^2, \ldots, x_n^2)$ of degree 4, i.e., *quartic even forms*.

There is a natural bijection

$$\Gamma : \mathbb{S}_n \to \mathbb{R}[\mathrm{x}]_{4,e}, \quad A \mapsto q_A(\mathrm{x}) := (\mathrm{x}^2)^T A \mathrm{x}^2 = \sum_{i,j=1}^{n} a_{ij} x_i^2 x_j^2.$$

Copositive matrices meet RAG

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Proposition

Let $A \in \mathbb{S}_n$ be a matrix. Then:

1. A is COP iff q_A is nonnegative. ($q_A \ldots POS$)
2. A is PSD iff q_A is of the form $\sum_i \left(\sum_j f_{ij} x_j^2\right)^2$. ($q_A \ldots \ell\text{-}SOS$)

6. A is CP iff q_A is of the form $\sum_i \left(\sum_j f_{ij} x_j^2\right)^2$ with $f_{ij} \geq 0$. ($q_A \ldots CP$)

Copositive matrices meet RAG

$\mathbb{R}[\mathrm{x}^2]_{4,e}$ ... forms in $\mathrm{x}^2 = (x_1^2, \ldots, x_n^2)$ of degree 4, i.e., *quartic even forms*.

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Proposition

Let $A \in \mathbb{S}_n$ be a matrix. Then:

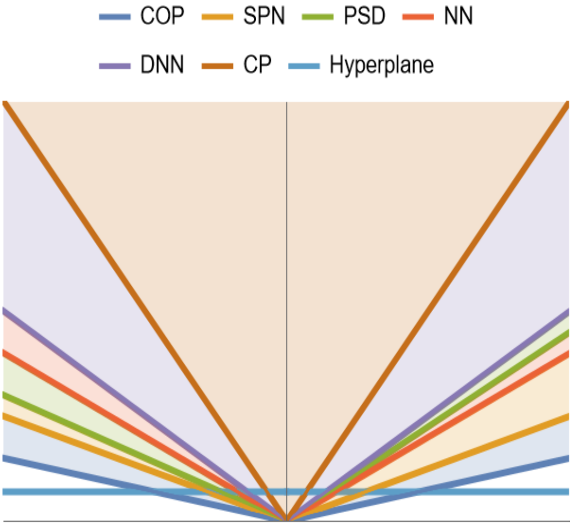
1. A is COP iff q_A is nonnegative. *(q_A ... POS)*
2. A is PSD iff q_A is of the form $\sum_i \left(\sum_j f_{ij} x_j^2\right)^2$. *($q_A$ ... ℓ-SOS)*
3. A is NN iff q_A has nonnegative coefficients. *(q_A ... NN)*
4. A is SPN iff q_A is of the form $\sum_i \left(\sum_{j,k} f_{ijk} x_j x_k\right)^2$ (Parrilo, 00') *(q_A ... SOS)*
5. A is DNN iff q_A is ℓ-SOS and NN. *(q_A ... DNN)*
6. A is CP iff q_A is of the form $\sum_i \left(\sum_j f_{ij} x_j^2\right)^2$ with $f_{ij} \geq 0$. *(q_A ... CP)*

Corollary. The gaps between COP/PSD/NN/SPN/DNN/CP matrices correspond to the gaps between POS/ℓ-SOS/NN/SOS/DNN/CP even quartics.

3. Proofs

Cones in question

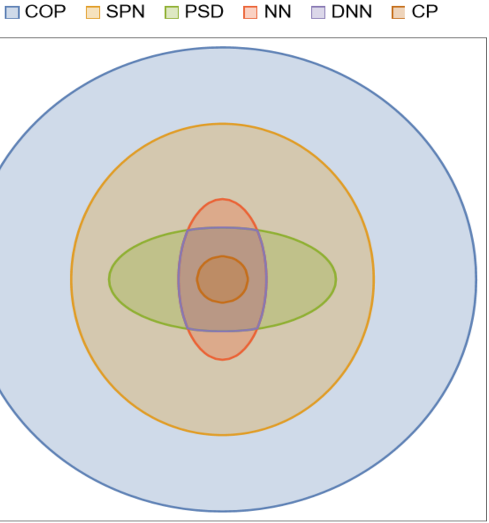
Intersect with some hyperplane



Constraint: A hyperplane should be chosen such that the intersections with cones are compact and hence finite.

Cones in question

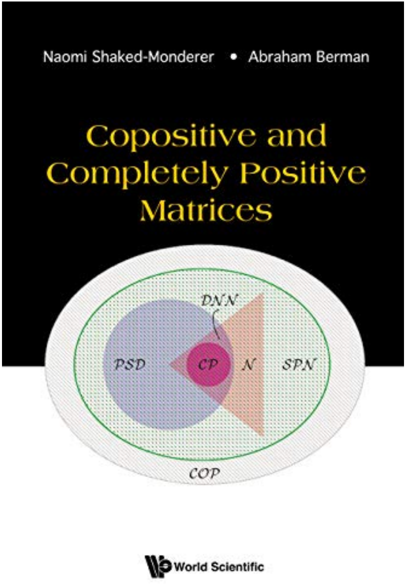
Compact bases of the cones



Perspective: Use results of real algebraic geometry, convex analysis and harmonic analysis to estimate the volumes from both sides.

Cones in question

Or maybe a proper mental picture for Problem 2 is the following...



Volume radius

Proper measure of the sizes of convex cones

The volume radius $\mathtt{vrad}(C)$ of a compact set $C \subseteq \mathbb{R}^n$, equipped with an inner product $\langle \cdot, \cdot \rangle$ and a measure μ, is

$$\mathrm{vrad}(C) = \left(\frac{\mathrm{Vol}(C)}{\mathrm{Vol}(B)} \right)^{1/n},$$

where B is the unit ball in $\langle \cdot, \cdot \rangle$.

- Indeed, since we are concerned with the asymptotic behavior as n goes to infinity, we need to eliminate the dimension effect when dilating K by some factor c.
- A dilation multiplies the volume of C by c^n, but a more appropriate effect would be multiplication by c.

A general procedure to obtain the volume estimates

Input: a convex cone K in $\mathbb{R}^n$.

Output: Bounds on the size of K.

Procedure:

1. Choose an inner product $\langle \cdot, \cdot \rangle$: ...to equip $\mathbb{R}^n$.
2. Choose an affine hyperplane $\mathcal{H}$: ...such that $K' = K \cap \mathcal{H}$ is bounded.
3. Translate $\mathcal{H}$ for $-z$ to $\mathcal{M}$:...such that $\mathcal{M}$ is a hyperplane ($0 \in \mathcal{M}$). Write $\widetilde{K} := K' - z$.
4. Equip $\mathcal{M}$ with a pushforward measure of the Lebesgue measure.
5. Estimate $\mathrm{vrad}(\widetilde{K})$ from both sides.

Blaschke-Santaló inequality and its reverse

Statement

$\langle \cdot, \cdot \rangle$	...	the inner product on $\mathbb{R}^n$
B	...	the unit ball w.r.t. $\langle \cdot, \cdot \rangle$
K	...	a bounded convex set with a non-empty interior in $\mathbb{R}^n$
K°	...	the polar dual of a set $K \subseteq \mathbb{R}^n$:

$$K^\circ = \{y \in \mathbb{R}^n \colon \langle x, y \rangle \leq 1 \quad \forall x \in K\}$$

Theorem (Bourgain, Milman, '87, Kuperberg, 2008; Blaschke, 1917, Santaló, 49')

If K is 'central enough', then

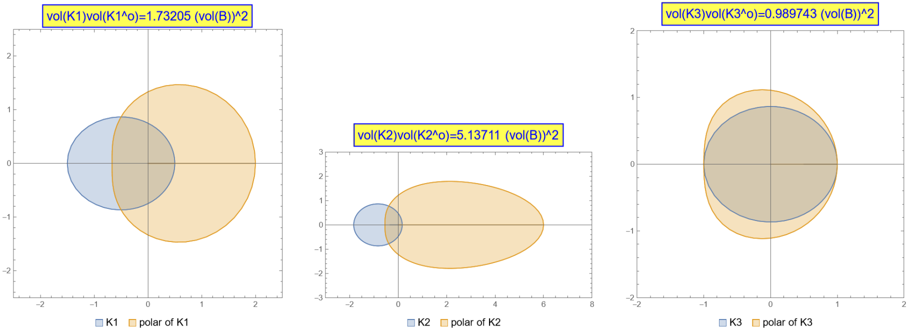
$$4^{-n}(\mathrm{Vol}(B))^2 \leq \mathrm{Vol}(K)\,\mathrm{Vol}(K^\circ) \leq (\mathrm{Vol}(B))^2,$$

Remark: The left inequality holds also without the centrality assumption, but with the origin in the interior.

Blaschke-Santaló inequality and its reverse

Geometric picture

K_1 ... the convex hull of the ellipse with a polar equation $r(\varphi) = \frac{3}{4}(1 + \frac{1}{2}\cos\varphi)^{-1}$,
$K_2 = K_1 - (\frac{1}{3}, 0)$, $\quad K_3 = K_1 + (\frac{1}{2}, 0)$,



- The set K_1 is centered in different points on each of the pictures. The first two centers are not close enough to the origin for the BS to hold, while in the third one it is.
- The translation of the body (i.e., Santaló point) so that the BS holds is difficult to determine, unless the body has enough symmetries, fixing only one point which then must be the Santaló one.

Procedure (from 3 slides above) applied to our Problem 2

1. $\mathbb{R}[\mathrm{x}]_{4,e}$ is equipped with the natural L^2 inner product

$$\langle f, g\rangle = \int_{S^{n-1}} fg \,\mathrm{d}\sigma,$$

   where and σ is the rotation invariant probability measures on the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

2. $\mathcal{H}$ is the affine hyperplane of forms from $\mathbb{R}[\mathrm{x}]_{4,e}$ of average 1 on S^{n-1}:

$$\mathcal{H} = \left\{ f \in \mathbb{R}[\mathrm{x}]_{4,e} \colon \int_{S^{n1}} f \,\mathrm{d}\sigma = 1 \right\}.$$

3. $z := \left(\sum_{i=1}^{n} x_i^2\right)^2$ and thus

$$\mathcal{M} = \mathcal{H} - z = \left\{ f \in \mathbb{R}[\mathrm{x}]_{4,e} \colon \int_{S^{n-1}} f \,\mathrm{d}\sigma = 0 \right\}.$$

4. Let μ the pushforward of the Lebesgue measure on $\mathbb{R}^{\dim \mathcal{M}}$ to $\mathcal{M}$.

Procedure applied to our problems

5. It is crucial to make the following two observations:

Observation 1: $\widetilde{(\mathrm{NN})^*_d} = \widetilde{\mathrm{NN}}$ and $\widetilde{(\mathrm{LF})^*_d} = \widetilde{\mathrm{POS}}$.

Here d stands for the differential inner product and $*$ for the dual,

$$\mathrm{LF} := \left\{ \mathrm{pr}(f) \in \mathbb{R}[\mathrm{x}]_{4,e} \colon f = \sum_i f_i^4 \quad \text{for some } f_i \in \mathbb{R}[\mathrm{x}]_1 \right\}$$

and $\mathrm{pr} : \mathbb{R}[\mathrm{x}]_4 \to \mathbb{R}[\mathrm{x}]_{4,e}$ is projection defined by:

$$\mathrm{pr}\left(\sum_{1 \le i \le j \le k \le \ell \le n} a_{ijk\ell} x_i x_j x_k x_\ell \right) = \sum_{1 \le i \le j \le n} a_{iijj} x_i^2 x_j^2. \qquad (1)$$

Observation 2: $\widetilde{\mathrm{LF}}$ is central enough.

Observation 3: $\widetilde{\mathrm{CP}} \subseteq \widetilde{\mathrm{LF}} \subseteq \widetilde{\mathrm{NN}} \subseteq 4(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}})$.

The differential (also apolar) inner product

From Observation 1

For

$$f(\mathrm{x}) = \sum_{1 \leq i,j,k,\ell \leq n} a_{ijk\ell} x_i x_j x_k x_\ell \in \mathbb{R}[\mathrm{x}]_4$$

the differential operator $D_f : \mathbb{R}[\mathrm{x}]_4 \to \mathbb{R}$ is defined by

$$D_f(g) = \sum_{1 \leq i,j,k,\ell \leq n} a_{ijk\ell} \frac{\partial^4 g}{\partial x_i \partial x_j \partial x_k \partial x_\ell}.$$

The differential inner product on $\mathbb{R}[\mathrm{x}]_4$ is given by

$$\langle f, g \rangle_d = D_f(g).$$

Blaschke-Santaló inequality and its reverse in $\langle \cdot, \cdot \rangle_d$

For a cone $K \subseteq \mathbb{R}[\mathrm{x}]_{4,e}$ let K_d^* be its dual in $\langle \cdot, \cdot \rangle_d$:

$$K_d^* = \{f \in \mathbb{R}[\mathrm{x}]_{4,e} \colon \langle f, g \rangle_d \geq 0 \quad \forall g \in K\}$$

Theorem (BS$_d$ inequality and its reverse; Blekherman, 06')

Let K be any of the cones from our Problem 2. Then

$$\frac{1}{2n^2} \underbrace{\leq}_{n \geq 5} \frac{2}{(n+4)(n+6)} \leq \operatorname{vrad}(\widetilde{K}) \operatorname{vrad}(\widetilde{K_d^*}).$$

Moreover, if $\widetilde{K}$ is 'central enough', then

$$\operatorname{vrad}(\widetilde{K}) \operatorname{vrad}(\widetilde{K_d^*}) \leq \left(\frac{8}{(n+4)(n+6)} \right)^{1 - \frac{2n-1}{n^2+n-1}} \underbrace{\leq}_{n \geq 5} \frac{32}{n^2}.$$

The proof uses representation theory, i.e., SO(n) acting on $\mathbb{R}[\mathrm{x}]_{4,e}$ by rotation of coordinates.

Observation 3: $\widetilde{\mathrm{NN}} \subseteq 4(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}})$

Follows from $2ab = (a+b)^2 - a^2 - b^2$

Let $r = \left(\sum_{k=1}^n x_k^2\right)^2$. The extreme points of $\widetilde{\mathrm{NN}}_{\mathcal{Q}}$ are of two types:

$$\frac{n(n+2)}{3}x_i^4 - r \qquad \text{and} \qquad n(n+2)x_i^2x_j^2 - r,\ i \neq j.$$

The first type clearly belong to $\widetilde{\mathrm{CP}}$, while the second type to $4(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}})$:

$$\begin{aligned}
& n(n+2)x_i^2x_j^2 - r = \\
&= \frac{n(n+2)}{2}\left((x_i^2 + x_j^2)^2 - x_i^4 - x_j^4\right) - r \\
&= 4\underbrace{\left(\frac{n(n+2)}{8}(x_i^2 + x_j^2)^2 - r\right)}_{p_1} - \frac{3}{2}\underbrace{\left(\frac{n(n+2)}{3}x_i^4 - r\right)}_{p_2} - \frac{3}{2}\underbrace{\left(\frac{n(n+2)}{3}x_j^4 - r\right)}_{p_3} \\
&= p_1 + \frac{3}{2}(p_1 - p_2) + \frac{3}{2}(p_1 - p_3) \\
&\in \widetilde{\mathrm{CP}}_{\mathcal{Q}} + \frac{3}{2}(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}}) + \frac{3}{2}(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}}) \subseteq 4(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}}).
\end{aligned}$$

Roger's-Shepard inequality

Crucial for Observation 3 to be applicable

K ... a bounded convex set with a non-empty interior in $\mathbb{R}^n$

The difference body $\mathrm{Diff}(K)$ of K is defined by

$$\mathrm{Diff}(K) := K - K.$$

Theorem (Roger's-Shepard inequality, 1957)

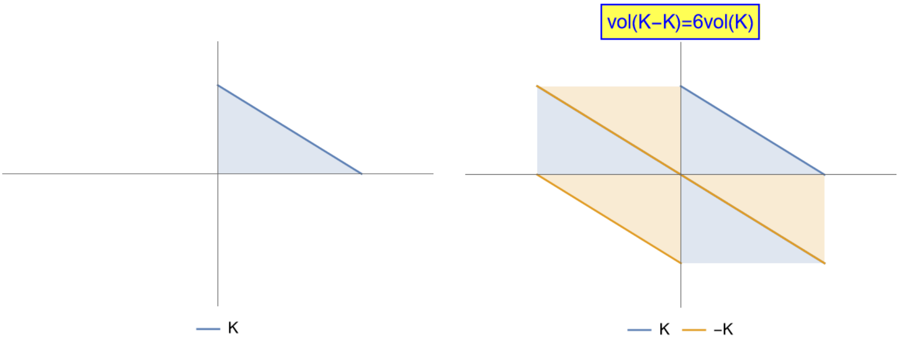
$$\mathrm{Vol}(\mathrm{Diff}(K)) \leq \binom{2n}{n} \mathrm{Vol}(K)$$

Hence,

$$\mathrm{vrad}(\mathrm{Diff}(K)) \leq 4\,\mathrm{vrad}(K).$$

Roger's-Shepard inequality

Geometric picture



Remark: Working with Diff K instead of K is one of the crucial steps to obtain our volume estimates for the problem of copositive matrices.

Proof of the gap for Problem 2

Theorem (Klep, Štrekelj, Z, 2023+)

Let $n \geq 5$. *For all* $K \in \mathcal{C} := \{\mathrm{POS}, \mathrm{SOS}, \mathrm{NN}, \mathrm{PSD}, \mathrm{DNN}, \mathrm{LF}, \mathrm{CP}\}$ *we have that*

$$\mathrm{vrad}(\widetilde{K}) = \Theta(n^{-1}). \tag{2}$$

Proof:

1. By $\widetilde{(\mathrm{NN})^*_d} = \widetilde{\mathrm{NN}}$ and the reverse BS_d inequality:

$$\frac{1}{2n^2} \leq \left(\mathrm{vrad}(\widetilde{\mathrm{NN}})\right)^2.$$

2. By $\widetilde{\mathrm{CP}} \subseteq \widetilde{\mathrm{NN}} \subseteq 4(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}})$ and the RS inequality:

$$\frac{1}{16\sqrt{2}n} \leq \frac{1}{16}\mathrm{vrad}(\widetilde{\mathrm{NN}}) \leq \mathrm{vrad}(\widetilde{\mathrm{CP}}), \tag{3}$$

3. By $\widetilde{(\mathrm{LF})^*_d} = \widetilde{\mathrm{POS}}$ and the BS_d inequality:

$$\mathrm{vrad}(\widetilde{\mathrm{POS}}) \leq \frac{32}{n^2}(\mathrm{vrad}(\widetilde{\mathrm{LF}}))^{-1} \leq \frac{32}{n^2}(\mathrm{vrad}(\widetilde{\mathrm{CP}}))^{-1} \leq 2^9\sqrt{2}\frac{1}{n}. \tag{4}$$

4. Now by observing that

$$\mathrm{CP} \subseteq K \subseteq \mathrm{POS},$$

the inequalities (3) and (4) imply that for all cones $K \in \mathcal{C}$ the statement (2) holds.

4*. Algorithms and Examples

4.1. Positive but not CP maps

Positive polynomials that are not SOS

Algorithm by Blekherman, Smith, Velasco, 2013

1. The setting:

   $X \subseteq \mathbb{P}^n \ldots$ a nondegenerate (not contained in a hyperplane),
   $\ldots$ totally-real (real points $X(\mathbb{R})$ are Zariski dense),
   $\ldots$ irreducible variety,
   $\ldots$ $\deg(X) > \operatorname{codim}(X) + 1$,

   $R = \mathbb{R}[x_0, \ldots, x_n]/I(X) \ldots$ the coordinate ring of X.

2. Step 1:
   - Choose linear forms $h_1, \ldots, h_{\dim(X)}$ intersecting in $\deg(X)$ distinct points with at least $\operatorname{codim}(X) + 1$ real and smooth ones, $p_1, \ldots, p_{\operatorname{codim}(X)+1}$.
   - Choose a linear form h_0 vanishing in $p_1, \ldots, p_{\operatorname{codim}(X)}$, but not in $p_{\operatorname{codim}(X)+1}$.
   - Let $I = \langle h_0, \ldots, h_m \rangle$.
3. Step 2: Choose a quadratic form $f \in R \setminus I^2$ vanishing of order > 1 in $p_1, \ldots, p_{\operatorname{codim}(X)}$.
4. Step 3: For $\delta > 0$ small enough, $\delta f + h_0^2 + \ldots + h_m^2$ is nonnegative on X but not SOS.

Positive but not sos biquadratic biforms

Algorithm

1. The setting:

$$X = \sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) \subseteq \mathbb{P}^{nm-1}, \quad \sigma_{n,m} \text{ Segre embedding}$$

$$\sigma_{n,m} : ([x_1 : \ldots : x_n], [y_1 : \ldots : y_m]) \mapsto [x_1 y_1 : x_1 y_2 : \ldots : x_n y_m],$$

$$\mathrm{z} = (z_{11}, z_{12}, \ldots, z_{1m}, \ldots, z_{nm}),$$

$I_{n,m} \ldots$ the ideal generated by 2×2 minors of $(z_{ij})_{i,j}$,

$$\sigma^{\#}_{n,m} : \mathbb{C}[\mathrm{z}]/I_{n,m} \to \mathbb{C}[\mathrm{x}, \mathrm{y}], \quad \sigma^{\#}_{n,m}(z_{ij} + I_{n,m}) = x_i y_j \quad \text{ring homomorphism},$$

$\dim(X) = n + m - 2$, $\mathrm{codim}(X) = (n-1)(m-1)$.

2. Step 1:
   - Choose $\mathrm{codim}(X) + 1$ random points $x^{(i)} \in \mathbb{R}^n$, $y^{(i)} \in \mathbb{R}^m$ and compute $z^{(i)} = x^{(i)} \otimes y^{(i)} \in \mathbb{R}^{nm}$.
   - Choose $\dim(X) = n + m - 2$ random vectors $v_1, \ldots v_{\dim(X)} \in \mathbb{R}^{nm}$ from the kernel of the matrix

     $$\begin{pmatrix} z^{(1)} & \ldots & z^{(\mathrm{codim}(X)+1)} \end{pmatrix}^*$$

     and define

     $$h_j(\mathrm{z}) = v_j^* \cdot \mathrm{z} \in \mathbb{R}[\mathrm{z}] \quad \text{for } j = 1, \ldots, \dim(X).$$
   - Let $I = \langle h_0, \ldots, h_{\dim(X)} \rangle$.

Positive but not sos biquadratic biforms

Algorithm

3. Step 2:
   3.1 Let $g_1(\mathrm{z}), \ldots, g_{\binom{n}{2}\binom{m}{2}}(\mathrm{z})$ be the generators of the ideal $I_{n,m}$. For each $i = 1, \ldots, \operatorname{codim}(X)$ compute a basis $\{w_1^{(i)}, \ldots, w_{\dim(X)+1}^{(i)}\} \subseteq \mathbb{R}^{nm}$ of the kernel of the matrix
   $$\begin{pmatrix} \nabla g_1(z^{(i)}) & \cdots & \nabla g_{\binom{n}{2}\binom{m}{2}}(z^{(i)}) \end{pmatrix}^* .$$
   3.2 Choose a random vector $v \in \mathbb{R}^{n^2m^2}$ from the intersection of the kernels of the matrices
   $$\begin{pmatrix} z^{(i)} \otimes w_1^{(i)} & \cdots & z^{(i)} \otimes w_{\dim(X)+1}^{(i)} \end{pmatrix}^* \quad \text{for } i = 1, \ldots, \operatorname{codim}(X)$$
   with the kernels of the matrices
   $$\left(\mathrm{e}_i \otimes \mathrm{e}_j - \mathrm{e}_j \otimes \mathrm{e}_i\right)^* \quad \text{for } 1 \le i < j \le nm$$
   and define
   $$f(\mathrm{z}) = v^* \cdot (\mathrm{z} \otimes \mathrm{z}) \in \mathbb{R}[\mathrm{z}]/I_{n,m}.$$
4. Step 3: Calculate the greatest $\delta_0 > 0$ such that $\delta_0 f + \sum_{i=0}^{\operatorname{codim}(X)} h_i^2$ is nonnegative on $V_{\mathbb{R}}(I_{n,m})$. Then
   $$(\delta f + \sum_i h_i^2)(\mathrm{z}) \in \mathrm{POS} \setminus \mathrm{SOS} \quad \text{for every } 0 < \delta < \delta_0.$$

Positive but not sos biquadratic biforms

Example

$$\begin{aligned} p_\Phi(x,y) = {} & 104x_1^2y_1^2 + 283x_1^2y_2^2 + 18x_1^2y_3^2 - 310x_1^2y_1y_2 + 18x_1^2y_1y_3 + 4x_1^2y_2y_3 + \\ & 310x_1x_2y_1^2 - 18x_1x_3y_1^2 - 16x_1x_2y_2^2 + 52x_1x_3y_2^2 + 4x_1x_2y_3^2 - 26x_1x_3y_3^2 \\ & - 610x_1x_2y_1y_2 - 44x_1x_3y_1y_2 + 36x_1x_2y_1y_3 - 200x_1x_3y_1y_3 - 44x_1x_2y_2y_3 \\ & + 322x_1x_3y_2y_3 + 285x_2^2y_1^2 + 16x_3^2y_1^2 + 4x_2x_3y_1^2 + 63x_2^2y_2^2 + 9x_3^2y_2^2 + 20x_2x_3y_2^2 \\ & + 7x_2^2y_3^2 + 125x_3^2y_3^2 - 20x_2x_3y_3^2 + 16x_2^2y_1y_2 + 4x_3^2y_1y_2 - 60x_2x_3y_1y_2 \\ & + 52x_2^2y_1y_3 + 26x_3^2y_1y_3 - 330x_2x_3y_1y_3 - 20x_2^2y_2y_3 + 20x_3^2y_2y_3 - 100x_2x_3y_2y_3. \end{aligned}$$

Positive but not CP map

Example $\Phi : \mathbb{S}_3 \to \mathbb{S}_3$

$$\Phi(E_{11}) = \begin{bmatrix} 104 & -155 & 9 \\ -155 & 283 & 2 \\ 9 & 2 & 18 \end{bmatrix}, \quad \Phi(E_{22}) = \begin{bmatrix} 285 & 8 & 26 \\ 8 & 63 & -10 \\ 26 & -10 & 7 \end{bmatrix},$$

$$\Phi(E_{33}) = \begin{bmatrix} 16 & 2 & 13 \\ 2 & 9 & 10 \\ 13 & 10 & 125 \end{bmatrix}, \quad \Phi(E_{12} + E_{21}) = \begin{bmatrix} 310 & -305 & 18 \\ -305 & -16 & -22 \\ 18 & -22 & 4 \end{bmatrix},$$

$$\Phi(E_{13}+E_{31}) = \begin{bmatrix} -18 & -22 & -100 \\ -22 & 52 & 161 \\ -100 & 161 & -26 \end{bmatrix}, \quad \Phi(E_{23}+E_{32}) = \begin{bmatrix} 4 & -30 & -165 \\ -30 & 20 & -50 \\ -165 & -50 & -20 \end{bmatrix}.$$

4.2. Exceptional DNN and exceptional COP matrices

DNN matrices that are not CP of size $n \geq 5$

Algorithm

1. The setting:

$$L^2[0,1] \ldots \quad \text{an ambient space,}$$

$$\mathcal{B} := \{1\} \cup \{\sqrt{2}\cos(2k\pi) \colon k \in \mathbb{N}\} \cup \{\sqrt{2}\sin(2k\pi) \colon k \in \mathbb{N}\} \ldots \text{ a basis,}$$

$$M_f : L^2[0,1] \to L^2[0,1],\ M_f(g) = fg \ldots \quad \text{the multiplication operator.}$$

2. The idea: Find a closed infinite dimensional subspace $\mathcal{H}$ and $f \in \mathcal{H}$ such that

$$M_f^{\mathcal{H}} := P_{\mathcal{H}} M_f P_{\mathcal{H}}$$

   has all finite principal submatrices DNN but not CP, where $P_{\mathcal{H}} : L^2[0,1] \to \mathcal{H}$ the orthogonal projection onto $\mathcal{H}$.

3. Choice of $\mathcal{H}$ and $f \in \mathcal{H}$:

$$\mathcal{H} \subseteq L^2[0,1] \ldots \quad \text{a closed subspace spanned by } \cos(2k\pi), k \in \mathbb{N}_0,$$

$$f \text{ is of the form } 1 + 2\sum_{k=1}^{m} a_k \cos(2k\pi), \quad m \in \mathbb{N},$$

DNN matrices that are not CP of size $n \geq 5$

Algorithm

4. Certificates:
   4.1 NN: $a_1 \geq 0, \dots, a_m \geq 0$.
   4.2 PSD: $f = \sum_i h_i^2$.
   4.3 Not CP:

$\mathcal{H}_n \dots$ a subspace spanned by $1, \cos(2\pi), \dots, \cos(2(n-1)\pi)$,

$P_n : \mathcal{H} \to \mathcal{H}_n \dots$ the orthogonal projection onto $\mathcal{H}_n$,

$A^{(n)} := P_n M_f^{\mathcal{H}} P_n$,

$$H = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix} \in \mathrm{COP} \setminus \mathrm{SPN},$$

(Horn matrix; Hall, Newman, 1963)

We demand

$$\langle A^{(5)}, H \rangle < 0,$$

with $\langle \cdot, \cdot \rangle$ the usual Frobenius inner product on symmetric matrices.

DNN matrices that are not CP of size $n \geq 5$

Justification of the certificates

1. NN is certified by the following equation:

$$\int_0^1 \cos(2j\pi x)\cos(2k\pi x)\cos(2\ell\pi x)dx = \begin{cases} \frac{1}{2}, & \text{if } j=\ell, k=0, \\ \frac{1}{4}, & \text{if } k \neq 0 \text{ and } j \in \{\ell+k, \ell-k\}, \\ 0, & \text{otherwise.} \end{cases}$$

   In particular,

$$A^{(5)} = \begin{pmatrix} 1 & \sqrt{2}a_1 & \sqrt{2}a_2 & \sqrt{2}a_3 & \sqrt{2}a_4 \\ \sqrt{2}a_1 & a_2+1 & a_1+a_3 & a_2+a_4 & a_3+a_5 \\ \sqrt{2}a_2 & a_1+a_3 & a_4+1 & a_1+a_5 & a_2+a_6 \\ \sqrt{2}a_3 & a_2+a_4 & a_1+a_5 & 1+a_6 & a_1 \\ \sqrt{2}a_4 & a_3+a_5 & a_2+a_6 & a_1 & 1 \end{pmatrix}.$$

2. PSD is certified by

$$M_f^{\mathcal{H}} = \sum_i (M_{h_i}^{\mathcal{H}})^2 = \sum_i M_{h_i}^{\mathcal{H}} (M_{h_i}^{\mathcal{H}})^*.$$

3. Not CP is certified by

$$\mathrm{COP}^* = \mathrm{CP} \quad \text{(in the Frobenius inner product).}$$

DNN matrices that are not CP of size $n \geq 5$

Implementation and an example

Let $m = 6$. The feasibility semidefinite program (SDP) implements the algorithm above:

$$\begin{aligned}
&\mathrm{tr}(A^{(5)}H) = -\epsilon, \\
&f = v^{\mathrm{T}}Bv \quad \text{with} \quad B \succeq 0 \text{ of size } m' \times m', \\
&a_i \geq 0, \quad i = 1, \ldots, 6,
\end{aligned}$$

where $\epsilon > 0$ is predetermined (small enough) and

$$v^T = \begin{pmatrix} 1 & \cos(2\pi x) & \cdots & \cos(2m'\pi x) \end{pmatrix}.$$

Solving this SDP for different values of ϵ and $m' \leq 6$, we get (for $\epsilon = 1/20$)

$$A^{(5)} = \begin{pmatrix}
1 & \frac{16\sqrt{2}}{27} & \frac{\sqrt{2}}{123} & \frac{1}{147\sqrt{2}} & \frac{5\sqrt{2}}{21} \\
\frac{16\sqrt{2}}{27} & \frac{124}{123} & \frac{1577}{2646} & \frac{212}{861} & \frac{1205}{8526} \\
\frac{\sqrt{2}}{123} & \frac{1577}{2646} & \frac{26}{21} & \frac{572}{783} & \frac{1777340\sqrt{2}-2413803}{3254580} \\
\frac{1}{147\sqrt{2}} & \frac{212}{861} & \frac{572}{783} & \frac{1777340\sqrt{2}+814317}{3254580} & \frac{16}{27} \\
\frac{5\sqrt{2}}{21} & \frac{1205}{8526} & \frac{1777340\sqrt{2}-2413803}{3254580} & \frac{16}{27} & 1
\end{pmatrix}.$$

COP matrices that are not SPN of size $n \geq 5$

Algorithm and an example

Let $A^{(n)}$ be a DNN not CP matrix. To obtain a matrix $C \in \mathrm{COP} \setminus \mathrm{SPN}$ of size $n \times n$ we demand

$$\langle A^{(n)}, C \rangle < 0, \tag{5}$$

$$\Big(\sum_{i=1}^{n} x_i^2\Big)^k \big((\mathrm{x}^2)^T C \mathrm{x}^2\big) \quad \text{is SOS for some } k \in \mathbb{N}. \tag{6}$$

(5) certifies C is not SPN due to

$$\mathrm{SPN}^* = \mathrm{DNN} \quad \text{(in the Frobenius inner product)},$$

while (6) certifies C is COP.

This is again a feasibility SDP. Using $A^{(5)}$ as above we obtain

$$C = \begin{pmatrix} 17 & -\frac{91}{5} & \frac{33}{2} & \frac{38}{3} & -\frac{36}{5} \\ -\frac{91}{5} & \frac{59}{3} & -\frac{53}{4} & 8 & \frac{33}{4} \\ \frac{33}{2} & -\frac{53}{4} & \frac{39}{4} & -\frac{13}{2} & 8 \\ \frac{38}{3} & 8 & -\frac{13}{2} & \frac{16}{3} & -\frac{13}{3} \\ -\frac{36}{5} & \frac{33}{4} & 8 & -\frac{13}{3} & \frac{1373628701}{353935575} \end{pmatrix}.$$

Open questions

Maps:

- Estimate the gap between k-positive vs $(k+1)$-positive vs cp maps for fixed k.
- Construct an algorithm for producing random k-positive not $(k+1)$-positive maps.
- Can the algorithm produce extreme rays of the cone of positive maps?

Matrices:

- Estimate precisely the constants for volume radius of a Parrilo cone $K_n^{(r)}$ for fixed r.
- Construct an algorithm for producing matrices from $K_n^{(r)} \setminus K_n^{(r-1)}$ for fixed r.
- Construct an algorithm for producing matrices from $\mathrm{COP}_n \setminus \bigcup_r K_n^{(r)}$ for $n \geq 6$.

Thank you for your attention!