Matrix Polynomials and Matrix Positiv/Nichtnegativstellensätze

Winter School 2025
Moments, Non-Negative Polynomials, and Algebraic Statistics

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Konstanz

Outline of the talk

1. Notation

2. Noncommutative Fejér-Riesz theorems
   - matrix version for $\mathbb{T}$ and $\mathbb{R}$
   - operator version for $\mathbb{T}$
   - operator version for $\mathbb{T}^d$

3. Matrix versions of classical theorems of real algebra
   - matrix Artin
   - matrix Schmüdgen
   - matrix Putinar
   - matrix Krivine-Stengle

4. Matrix versions of Nichtnegativstellensätze in $\mathbb{R}$

1. Notation

Scalar polynomials

$\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$

$\mathbb{F}[\underline{\mathrm{x}}]$ ... usual multivariate polynomials

$$\underline{\mathrm{x}} := (\mathrm{x}_1, \dots, \mathrm{x}_d); \quad (\mathrm{x}_{i_1} \cdots \mathrm{x}_{i_k})^* = \mathrm{x}_{i_1} \cdots \mathrm{x}_{i_k}$$

$\mathbb{F}[\underline{\mathrm{z}}, \frac{1}{\underline{\mathrm{z}}}]$ ... Laurent polynomials

$$\underline{\mathrm{z}} := (\mathrm{z}_1, \dots, \mathrm{z}_d), \quad \frac{1}{\underline{\mathrm{z}}} := (\frac{1}{\mathrm{z}_1}, \dots, \frac{1}{\mathrm{z}_d}); \quad (\mathrm{z}_{i_1}^{j_1} \cdots \mathrm{z}_{i_k}^{j_k})^* = \mathrm{z}_{i_1}^{-j_1} \cdots \mathrm{z}_{i_k}^{-j_k}$$

Let $\underline{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}_0^d / \mathbb{Z}^d$ we write

$$\underline{\mathrm{x}}^{\underline{\alpha}} := \mathrm{x}_1^{\alpha_1} \mathrm{x}_2^{\alpha_2} \cdots \mathrm{x}_d^{\alpha_d}, \qquad \underline{\mathrm{z}}^{\underline{\alpha}} := \mathrm{z}_1^{\alpha_1} \mathrm{z}_2^{\alpha_2} \cdots \mathrm{z}_d^{\alpha_d}.$$

Matrix polynomials

$M_n(\mathbb{F})$: $n \times n$ matrices over $\mathbb{F}$.

Involution $*$ on $M_n(\mathbb{F})$: $F^* = \begin{cases} \overline{F}^T, & \mathbb{F} = \mathbb{C}, \\ F^T, & \mathbb{F} = \mathbb{R}. \end{cases}$

$F \succ 0 \ldots F$ is positive definite. $F \succeq 0 \ldots F$ is positive semidefinite.

$M_n(\mathbb{F}[\underline{x}]) \ldots$ usual matrix polynomials,

$$\Big(\sum_{\text{finite}} F_{\underline{\alpha}} \underline{x}^{\underline{\alpha}}\Big)^* = \sum_{\text{finite}} F_{\underline{\alpha}}^* \underline{x}^{\underline{\alpha}}.$$

$M_n(\mathbb{F}[\underline{z}, \frac{1}{\underline{z}}]) \ldots$ matrix Laurent polynomials

$$\Big(\sum_{\text{finite}} A_{\underline{\alpha}} \underline{z}^{\underline{\alpha}}\Big)^* = \sum_{\text{finite}} A_{\underline{\alpha}}^* \underline{z}^{-\underline{\alpha}}.$$

Operator polynomials

$\mathcal{H}$ ... separable Hilbert space over $\mathbb{F}$.

$B(\mathcal{H})$ ... bounded linear operators on $\mathcal{H}$.

Involution $*$ ... the usual hermitian adjoint.

Replacing matrices $F_{\underline{\alpha}}$ and A_{α} with operators from $B(\mathcal{H})$:

$B(\mathcal{H})[\underline{x}]$ ... operator polynomials

$B(\mathcal{H})[\underline{z}, \frac{1}{\underline{z}}]$ ... operator Laurent polynomials.

2. Noncommutative Fejér-Riesz theorems

- matrix version for $\mathbb{T}$ and $\mathbb{R}$
- operator version for $\mathbb{T}$
- operator version for $\mathbb{T}^d$

Fejér-Riesz theorem

matrix version for $\mathbb{T}$, 1958

$\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$

Let

$$A(\mathrm{z}) = \sum_{m=-N}^{N} A_m \mathrm{z}^m \in M_n(\mathbb{C}[\mathrm{z}, \frac{1}{\mathrm{z}}])$$

be such that $A(\mathrm{z})^* = A(\mathrm{z})$ and $A(z) \succeq 0$ for $z \in \mathbb{T}$. Then there exists

$$B(\mathrm{z}) = \sum_{m=0}^{N} B_m \mathrm{z}^m \in M_n(\mathbb{C}[\mathrm{z}]),$$

such that

$$A(\mathrm{z}) = B(\mathrm{z})^* B(\mathrm{z}).$$

Moreover, if

$$\det A(\mathrm{z}) = b(\mathrm{z})^* b(\mathrm{z}), \quad b(\mathrm{z}) \in \mathbb{C}[\mathrm{z}],$$

then there exists $B(\mathrm{z})$ additionally satisfying $\det B(\mathrm{z}) = b(\mathrm{z})$.

Fejér-Riesz theorem matrix version for $\mathbb{R}$

Let

$$F(\mathrm{x}) = \sum_{m=0}^{2N} F_m \mathrm{x}^m \in M_n(\mathbb{C}[\mathrm{x}])$$

be such that $F(\mathrm{x})^* = F(\mathrm{x})$ and $F(x) \succeq 0$ for $x \in \mathbb{R}$. Then there exists

$$G(\mathrm{x}) = \sum_{m=0}^{N} G_m \mathrm{x}^m \in M_n(\mathbb{C}[\mathrm{x}])$$

such that

$$F(\mathrm{x}) = G(\mathrm{x})^* G(\mathrm{x}).$$

Moreover, if

$$\det F(\mathrm{x}) = g(\mathrm{x})^* g(\mathrm{x}), \quad g(\mathrm{x}) \in \mathbb{C}[\mathrm{x}],$$

then there exists $g(\mathrm{x})$ additionally satisfying $\det G(\mathrm{x}) = g(\mathrm{x})$.

Equivalence of the $\mathbb{T}$-Fejér-Riesz and $\mathbb{R}$-Fejér-Riesz

For $z_0 \in \mathbb{T}$ and $w_0 \in \mathbb{C} \setminus \mathbb{R}$ let

$$\lambda_{z_0,w_0} : \mathbb{R} \cup \{\infty\} \to \mathbb{T}, \quad \lambda_{z_0,w_0}(\mathrm{x}) := z_0 \frac{\mathrm{x} - w_0}{\mathrm{x} - \overline{w_0}}$$

be the Möbius transformations that map $\mathbb{R} \cup \{\infty\}$ bijectively into $\mathbb{T}$. Then

$$\lambda^{-1}_{z_0,w_0} : \mathbb{T} \to \mathbb{R} \cup \{\infty\}, \quad \lambda^{-1}_{z_0,w_0}(\mathrm{z}) = \frac{\mathrm{z}\,\overline{w_0} - z_0 w_0}{\mathrm{z} - z_0}.$$

Let $F(\mathrm{x}) \in M_n(\mathbb{F}[\mathrm{x}])$ and

$$\Lambda_{z_0,w_0,F}(\mathrm{z}) := ((\mathrm{z} - z_0)^*(\mathrm{z} - z_0))^{\lceil \frac{\deg(F)}{2} \rceil} \cdot F\left(\lambda^{-1}_{z_0,w_0}(\mathrm{z})\right) \in M_n\left(\mathbb{F}\left[\mathrm{z}, \frac{1}{\mathrm{z}}\right]\right),$$

where $\lceil \cdot \rceil$ is the ceiling function. We also have

$$F(\mathrm{x}) = \left(\frac{(\mathrm{x} - \overline{w_0})(\mathrm{x} - w_0)}{4 \cdot \Im(w_0)^2}\right)^{\lceil \frac{\deg(F)}{2} \rceil} \Lambda_{z_0,w_0,F}(\lambda_{z_0,w_0}(\mathrm{x})), \qquad (*)$$

where $\Im(a)$ is the imaginary part of $a \in \mathbb{C}$.

Equivalence of the $\mathbb{T}$-Fejér-Riesz and $\mathbb{R}$-Fejér-Riesz

Note that

- $F(x) \succeq 0$ for every $x \in \mathbb{R}$. $\quad \Leftrightarrow \quad$ $\Lambda_{z_0,w_0,F}(z) \succeq 0$ for every $z \in \mathbb{T}$.
- If $\deg F = 2k$, then $\deg \Lambda_{z_0,w_0,F} = k$.

If $\Lambda_{z_0,w_0,F}(\mathrm{z}) = B(\mathrm{z})^* B(\mathrm{z})$, then

$$F(\mathrm{x}) = \left(\frac{(\mathrm{x} - \overline{w_0})(\mathrm{x} - w_0)}{4 \cdot \Im(w_0)^2} \right)^{\frac{\deg(F)}{2}} B(\lambda_{z_0,w_0}(\mathrm{x}))^* B(\lambda_{z_0,w_0}(\mathrm{x}))$$

$$= \underbrace{\left(\left(\frac{\mathrm{x} - w_0}{2 \cdot \Im(w_0)} \right)^{\frac{\deg(F)}{2}} B(\lambda_{z_0,w_0}(\mathrm{x})) \right)^*}_{G(\mathrm{x})^*} \underbrace{\left(\left(\frac{\mathrm{x} - \overline{w_0}}{2 \cdot \Im(w_0)} \right)^{\frac{\deg(F)}{2}} B(\lambda_{z_0,w_0}(\mathrm{x})) \right)}_{G(\mathrm{x})}.$$

The other direction ($\mathbb{R}$-Fejér-Riesz version implies $\mathbb{T}$–Fejér-Riesz version) is analogous.

Many proofs of the matrix Fejér-Riesz theorem

With the moreover part.

- I. T. Gohberg, M. G. Krein, A system of integral equation on a semiaxis with kernels depending on different arguments. Uspekhi matemat. nauk 13 (1958) 3–72.
- M. Rosenblatt, A multidimensional prediction problem, Ark. Mat. vol. 3 (1958) 407–424.
- V. A. Jakubovič, Factorization of symmetric matrix polynomials, Dokl. Akad. Nauk 194 (1970) 532–535.
- V. M. Popov, Hyperstability of control systems, Springer-Verlag, Berlin, 1973.
- D. Z. Djoković, Hermitian matrices over polynomial rings. J. Algebra 43 (1976) 359–374.
- M. D. Choi, T. Y. Lam, B. Reznick, Real zeros of positive semidefinite forms I, Math. Z. 171 (1980) 1–26.
- I. Gohberg, P. Lancaster, L. Rodman, Matrix polynomials, Computer Science and Applied Mathematics. Academic Press, Inc., New York-London, 1982.
- A. N. Malyshev, Factorization of matrix polynomials, Sibirsk. Mat. Zh. 23 (1982) 136–146.
- C. Hanselka, M. Schweighofer, Positive semidefinite matrix polynomials, unpublished.

Main technique of the proof from Popov, Hyperstability of control systems, Springer-Verlag, Berlin, 1973. Book in control theory, the proof is in Appendix B.

'Massaging' (using only elementary linear algebraic techniques) the Smith normal form for $\mathrm{z}^{\deg A}A(\mathrm{z})$:

$$\mathrm{z}^{\deg A}A(\mathrm{z}) = \underbrace{P(\mathrm{z})}_{\substack{\in M_n(\mathbb{F}[\mathrm{z}]),\\ \text{invertible},\\ \det\in\mathbb{F}\setminus\{0\}}} \cdot \underbrace{\begin{pmatrix} q_1(\mathrm{z}) & 0 & \dots & 0 \\ 0 & q_2(\mathrm{z}) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0\dots & q_n(\mathrm{z}) \end{pmatrix}}_{q_i(\mathrm{z}) \text{ divides } q_{i+1}(z)} \cdot \underbrace{R(\mathrm{z})}_{\substack{\in M_n(\mathbb{F}[\mathrm{z}]),\\ \text{invertible},\\ \det\in\mathbb{F}\setminus\{0\}}} .$$

Fejér-Riesz theorem

operator version for $\mathbb{T}$

Let

$$A(\mathrm{z}) = \sum_{m=-N}^{N} A_m \mathrm{z}^m \in B(\mathcal{H})[\mathrm{z}, \frac{1}{\mathrm{z}}]$$

be such that $A(\mathrm{z})^* = A(\mathrm{z})$ and $A(z) \succeq 0$ for $z \in \mathbb{T}$. Then there exists

$$B(\mathrm{z}) = \sum_{m=0}^{N} B_m \mathrm{z}^m \in B(\mathcal{H})[\mathrm{z}],$$

such that

$$A(\mathrm{z}) = B(\mathrm{z})^* B(\mathrm{z}).$$

- M. Rosenblum, Vectorial Toeplitz operators and Fejér-Riesz theorem, J. Math. Anal. Appl. 23 (1968) 139–147. Technique: Toeplitz operators and Lowdenslager criterion.
- M.A. Dritschel, On factorization of trigonometric polynomials, Integral Equations Operator Theory 49 (2004), 11–42. Technique: Schur complements - version 1.
- M.A. Dritschel and H.J. Woerdeman, Outer factorizations in one and several variables, Trans. Amer. Math. Soc. 357 (2005) 4661–4679. Technique: Schur complements - version 2.

Main technique of the proof from M.A. Dritschel and H.J. Woerdeman, Outer factorizations in one and several variables, Trans. Amer. Math. Soc. 357 (2005) 4661–4679.

Let $\mathcal{K} \leq \mathcal{H}$ a closed subspace.

$$0 \leq T := \begin{matrix} & \mathcal{K} & \mathcal{K}^\perp \\ \begin{matrix} \mathcal{K} \\ \mathcal{K}^\perp \end{matrix} & \left[\begin{matrix} A \\ B \end{matrix} \right. & \left. \begin{matrix} B^* \\ C \end{matrix} \right] \end{matrix} \in B(\mathcal{H}).$$

$0 \leq S := \mathcal{S}(T, \mathcal{K}) \in B(\mathcal{K})$ is the Schur complement of T supported on $\mathcal{K}$ if

$$\begin{pmatrix} A - S & B^* \\ B & C \end{pmatrix} \geq 0 \text{ and } 0 \leq \widetilde{S} \in B(\mathcal{K}), \begin{pmatrix} A - \widetilde{S} & B^* \\ B & C \end{pmatrix} \geq 0 \text{ implies that } \widetilde{S} \leq S.$$

Main two properties of Schur complements in the proof:

- **Factorization result:** If
  $$S(T, \mathcal{K}) = P^* P \quad \text{and} \quad C = R^* R$$
  for some $P \in B(\mathcal{K})$, $R \in B(\mathcal{K}^\perp)$, then there is unique $X \in B(\mathcal{K}, \mathcal{K}^\perp)$ such that
  $$T = \begin{pmatrix} P^* & X^* \\ 0 & R^* \end{pmatrix} \begin{pmatrix} P & 0 \\ X & R \end{pmatrix} \quad \text{and} \quad \operatorname{Ran} X \subseteq \overline{\operatorname{Ran}} R. \tag{1}$$

- **Inheritance property**: Let $\mathcal{K}_1 \leq \mathcal{K}_2$ be closed subspaces of $\mathcal{H}$. Then
$$\mathcal{S}(T, \mathcal{K}_1) = \mathcal{S}(\mathcal{S}(T, K_2), \mathcal{K}_1).$$

Main steps in the proof:

1. Setting $A_m = 0$ for $|m| > N$, let
$$T_A = \begin{pmatrix} A_0 & A_{-1} & A_{-2} & \cdots \\ A_1 & A_0 & A_{-1} & \ddots \\ A_2 & A_1 & A_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \in B(\oplus_{i \in \mathbb{Z}_+} \mathcal{H}).$$
2. $A(z) \geq 0$ for every $z \in \mathbb{T}$. $\Leftrightarrow$ $T_A \geq 0$.
3. Construct decomposition (using factorization result & inheritance property)
$$T_A = \begin{pmatrix} B_0 & 0 & 0 & \cdots \\ B_1 & B_0 & 0 & \ddots \\ B_2 & B_1 & B_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}^* \begin{pmatrix} B_0 & 0 & 0 & \cdots \\ B_1 & B_0 & 0 & \ddots \\ B_2 & B_1 & B_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Multivariate Fejér-Riesz theorem

Operator version on $\mathbb{T}^d$

Let

$$A(\underline{z}) = \sum_{\text{finite}} A_{\underline{\alpha}} \underline{z}^{\underline{\alpha}} \in B(\mathcal{H})\left[\underline{z}, \frac{1}{\underline{z}}\right]$$

such that $A(\underline{z})^* = A(\underline{z})$ and $A(\underline{z}) \succeq \delta I_{\mathcal{H}}$ for $\underline{z} \in \mathbb{T}^d$ and some $\delta > 0$. Then there exists

$$B_j(\underline{z}) = \sum_{\text{finite}} B_{j,\underline{\alpha}} \underline{z}^{\underline{\alpha}} \in B(\mathcal{H})[\underline{z}],$$

such that

$$A(\underline{z}) = \sum_{\text{finite}} B_j(\underline{z})^* B_j(\underline{z}).$$

M.A. Dritschel, On factorization of trigonometric polynomials. Integral Equ. Oper. Theory 49(1), 11–42 (2004) For $d = 2$.

M.A. Dritschel and J. Rovnyak, The operator Fejér-Riesz theorem. In: A glimpse at Hilbert pace Operators, vol. 207, pp 223–254. Oper. Theory Adv. Appl., Birkhäuser Verlag, Basel (2010) For any d.

M. Bakonyi, H.J. Woerdeman, Matrix Completions, Moments, and Sums of Hermitian Squares, Princeton University Press, Princeton, 2011.

Main steps of the proof M. Bakonyi, H.J. Woerdeman, Matrix Completions, Moments, and Sums of Hermitian Squares, Princeton University Press, Princeton, 2011.

1. Let $\underline{n} := (n_1, n_2, \dots, n_d) \in \mathbb{N}_0^d$ and

$$\ell_{\underline{n}}(\underline{z}) = \left(\underline{z}^{\underline{\alpha}} I_{\mathcal{H}}\right)_{\underline{\alpha} \leq \underline{n}}.$$

   where the inequality is a coordinate-wise one. Let $|\underline{n}| = \prod_{j=1}^{d}(n_j + 1)$. Clearly,

$$\exists \underline{n} \ \ \exists Q \in B(\mathcal{H}^{|n|}) : \quad A(\underline{z}) = \ell_{\underline{n}}(\underline{z})^* \cdot Q \cdot \ell_{\underline{n}}(\underline{z}).$$

2. If we show that there is $Q \geq 0$, then $Q = Q_1^* Q_1$ and

$$A = (Q_1 \ell_{\underline{n}}(\underline{z}))^* (Q_1 \ell_{\underline{n}}(\underline{z})).$$

3. From $A(\underline{z}) \succeq \delta I_{\mathcal{H}}$ for $\underline{z} \in \mathbb{T}^d$ it follows that

$$(A_{\underline{\alpha}-\underline{\beta}})_{\underline{\alpha},\underline{\beta} \leq \underline{n}} \geq \delta I_{\mathcal{H}^{|n|}}.$$

4. Then

$$(\ell_{\underline{n}}(\underline{z}))^* \cdot (A_{\underline{\alpha}-\underline{\beta}})_{\underline{\alpha},\underline{\beta}\le\underline{n}} \cdot \ell_{\underline{n}}(\underline{z}) = \sum_{-|\underline{n}|\le\underline{\alpha}\le|\underline{n}|} \prod_{i=1}^{d}(n_i + 1 - |\alpha_i|) \cdot A_{\underline{\alpha}}\underline{z}^{\underline{\alpha}}$$

$$= |\underline{n}| \cdot \sum_{-\underline{N}\le\underline{\alpha}\le\underline{n}} \underbrace{\frac{\prod_{i=1}^{d}(n_i + 1 - |\alpha_i|)}{|\underline{n}|}}_{\mu_{\underline{\alpha},\underline{n}}} \cdot A_{\underline{\alpha}}\underline{z}^{\underline{\alpha}}$$

5. For $\underline{n}$ large enough

$$\sum_{-\underline{n}\le\underline{\alpha}\le\underline{n}} \left\| 1 - \frac{1}{\mu_{\underline{\alpha},\underline{n}}} \right\| \|A_{\underline{\alpha}}\| < \delta.$$

6. For $\widetilde{A}(\underline{z}) := \sum_{\underline{\alpha}} \frac{1}{\mu_{\underline{\alpha},\underline{n}}} A_{\underline{\alpha}}\underline{z}^{\underline{\alpha}}$ we have

$$(\widetilde{A}_{\alpha-\beta})_{\underline{\alpha},\underline{\beta}\le\underline{n}} \ge 0,$$

whence

$$\frac{1}{\underline{n}}(\ell_{\underline{n}}(\underline{z}))^* \cdot (\widetilde{A}_{\underline{\alpha}-\underline{\beta}})_{\underline{\alpha},\underline{\beta}\le\underline{n}} \cdot \ell_{\underline{n}}(\underline{z}) = \sum_{-\underline{n}\le\underline{\alpha}\le\underline{n}} A_{\underline{\alpha}}\underline{z}^{\underline{\alpha}}.$$

3. Matrix versions of classical theorems of real algebra

- matrix Artin
- matrix Schmüdgen
- matrix Putinar
- matrix Krivine-Stengle

Matrix version of Artin's theorem

$\sum M_n(\mathbb{R}[\underline{x}])^2 \ldots$ the set of all finite sums $\sum_i G_i(\underline{x})^* G_i(\underline{x})$ where $G_i(\underline{x}) \in M_n(\mathbb{R}[\underline{x}])$.

$S_n(\mathbb{R}[\underline{x}]) \ldots n \times n$ symmetric matrix polynomials $(G(\underline{x})^T = G(\underline{x}))$.

$F(\underline{x}) \in S_n(\mathbb{R}[\underline{x}])$

$$F(\underline{x}) \succeq 0 \text{ for every } \underline{x} \in \mathbb{R}^d. \quad \Rightarrow \quad p^2 F \in \sum M_n(\mathbb{R}[\underline{x}])^2 \text{ for some } 0 \neq p(\underline{x}) \in \mathbb{R}[\underline{x}].$$

- D. Gondard, P. Ribenboim: Le 17e probléme de Hilbert pour les matrices, Bull. Sci. Math. (2) 98 (1974) 49–56.
- D.Ž. Djokovič: Positive semi-definite matrices as sums of squares, Linear Algebra Appl. 14 (1976) 37–40,
- C. Procesi, M. Schacher: A non-commutative real Nullstellensatz and Hilbert's 17th problem, Ann. of Math. (2) 104 (1976) 395–406.
- C.J. Hillar, J. Nie: An elementary and constructive solution to Hilbert's 17th problem for matrices, Proc. Amer. Math. Soc. 136 (2008) 73–76
- K. Schmüdgen, Noncommutative real algebraic geometry - some basic concepts and first ideas. in: Emerging applications of algebraic geometry, IMA Vol. Math. Appl., 149, Springer, New York, 2009, pp. 325–350.

Main steps of the proof

K. Schmüdgen, Noncommutative real algebraic geometry - some basic concepts and first ideas.

LDU decomposition of a matrix ($a \in \mathbb{R} \setminus \{0\}$, $C \in M_{n-1}(\mathbb{R})$):

$$M = \begin{pmatrix} a & \beta \\ \beta^T & C \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{a}\beta^T & I \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & C - \frac{1}{a}\beta^T\beta \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{a}\beta \\ 0 & I \end{pmatrix},$$

$$\begin{pmatrix} a & 0 \\ 0 & C - \frac{1}{a}\beta^T\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{a}\beta^T & I \end{pmatrix} \cdot M \cdot \begin{pmatrix} 1 & -\frac{1}{a}\beta \\ 0 & I \end{pmatrix},$$

For

$$F(\underline{x}) = \begin{pmatrix} a & \beta \\ \beta^* & C \end{pmatrix} \in M_n(\mathbb{R}[\underline{x}]), \text{ where } a \in \mathbb{R}[\underline{x}],\ C = C^* \in M_{n-1}(\mathbb{R}[\underline{x}])$$

it holds

$$a^4 \cdot F = \begin{pmatrix} a & 0 \\ \beta^* & aI_{n-1} \end{pmatrix} \begin{pmatrix} a^3 & 0 \\ 0 & a(aC - \beta^*\beta) \end{pmatrix} \begin{pmatrix} a & \beta \\ 0 & aI_{n-1} \end{pmatrix},$$

$$\begin{pmatrix} a^3 & 0 \\ 0 & a(aC - \beta^*\beta) \end{pmatrix} = \begin{pmatrix} a & 0 \\ -\beta^* & aI_{n-1} \end{pmatrix} \cdot F \cdot \begin{pmatrix} a & -\beta \\ 0 & aI_{n-1} \end{pmatrix}.$$

Lemma

Let

$$G = (g_{k\ell})_{k\ell} \in M_n(\mathbb{R}[\underline{x}])$$

be a matrix polynomial. For every $k, \ell \in \mathbb{N}$ *satisfying* $1 \le k \le \ell \le n$ *there exists an orthogonal matrix* $U_{k\ell} \in M_n(\mathbb{R})$ *such that*

$$U_{k\ell} G U_{k\ell}^* = \begin{bmatrix} p_{k\ell} & * \\ * & * \end{bmatrix},$$

where

$$p_{k\ell} = \begin{cases} g_{k\ell}, & \text{for } 1 \le k = \ell \le n, \\ g_{k\ell} + \frac{1}{2}(g_{kk} + g_{\ell\ell}), & \text{for } 1 \le k < \ell \le n. \end{cases}$$

Proposition

For $0 \neq F(\underline{x}) \in S_n(\mathbb{R}[\underline{x}])$ there exist finitely many diagonal matrices $D_l \in M_n(\mathbb{R}[\underline{x}])$, matrices $X_{+,l}, X_{-,l} \in M_n(\mathbb{R}[\underline{x}])$ and $z_l \in \sum \mathbb{R}[\underline{x}]^2$ such that

$$D_l = X_{-,l} F X_{-,l}^T, \quad z_l F = X_{+,l} D_l X_{+,l}^T.$$

$$F(\underline{x}) \succeq 0 \quad \Leftrightarrow \quad D_l(\underline{x}) \succeq 0 \text{ for all } l.$$

The proof of the **matrix Artin's theorem**:

Applying both lemmas inductively it is possible diagonalize F:

$$c(\underline{x})^2 \cdot F(\underline{x}) = G(\underline{x})^* \cdot \underbrace{\mathrm{diag}(d_1(\underline{x}), \ldots, d_n(\underline{x}))}_{D(\underline{x})} \cdot G(\underline{x})$$

$$F(\underline{x}) \succeq 0 \;\Leftrightarrow\; D(\underline{x}) \succeq 0.$$

Finally, apply the **scalar version of Artin's theorem** for each $d_i(\underline{x})$.

Matrix quadratic module and matrix preordering

$I_n \ldots n \times n$ identity matrix.

A set $M \subseteq S_n(\mathbb{R}[\underline{x}])$ is a matrix quadratic module if

$$I_n \in M, \quad M + M \subseteq M, \quad A^T M A \subseteq M \text{ for every } A \in M_n(\mathbb{R}[\underline{x}]).$$

A set $T \subseteq S_n(\mathbb{R}[\underline{x}])$ is a matrix preordering if it is a matrix quadratic module and

$$T \cap \mathbb{R}[\underline{x}] \cdot I_n \quad \text{is closed under multiplication.}$$

Matrix quadratic module and matrix preordering

$\mathcal{G} := \{G_1(\underline{x}), \ldots, G_m(\underline{x})\} \subset S_n(\mathbb{R}[\underline{x}])$

$K_{\mathcal{G}} := \{\underline{x} \in \mathbb{R}^d \colon G_1(\underline{x}) \succeq 0, \ldots . G_m(\underline{x}) \succeq 0\}$

Matrix quadratic module $M_{\mathcal{G}}$ generated by $\mathcal{G}$:

$$M_{\mathcal{G}} := \Big\{F + \sum_{\text{finite}} F_i^T G_{j_i} F_i \colon F \in \sum M_n(\mathbb{R}[\underline{x}])^2, F_i \in M_n(\mathbb{R}[\underline{x}]),\ G_{j_i} \in \mathcal{G}\Big\}.$$

$\mathcal{G}' = \big\{v^T G v \colon G \in \mathcal{G}, v \in \mathbb{R}[\underline{x}]^n\big\} \subseteq \mathbb{R}[\underline{x}]$.

$\prod \mathcal{G}' \ldots$ the set of all finite products of elements from $\mathcal{G}'$.

Lemma

A preordering $T_{\mathcal{G}}$ generated by $\mathcal{G}$ is equal to

$$M_{\mathcal{G} \cup (\prod \mathcal{G}' \cdot I_n)}.$$

Matrix version of Schmüdgen's theorem

$\mathcal{G} := \{G_1(\underline{\mathrm{x}}), \ldots, G_m(\underline{\mathrm{x}})\} \subset S_n(\mathbb{R}[\mathrm{x}])$

$K_{\mathcal{G}} := \{\underline{x} \in \mathbb{R}^d : G_1(\underline{x}) \succeq 0, \ldots . G_m(\underline{x}) \succeq 0\}$ is compact.

$F \in S_n(\mathbb{R}[\underline{\mathrm{x}}])$

$$F(\underline{x}) \succ 0 \text{ for every } \underline{x} \in K_{\mathcal{G}} \quad \Rightarrow \quad F \in T_{\mathcal{G}}.$$

J. Cimprič, A. Zalar, Moment problems for operator polynomials. J. Math. Anal. Appl. 401 (2013) 307–316.

By reduction to matrix Putinar's theorem.

Alternative proof

1. Using identities from the proof of matrix Artin's theorem we can replace $G_1, \ldots, G_m$ with diagonal matrices

$$D_1 = \operatorname{diag}(d_{11}, \ldots, d_{n1}), \ldots, D_t = \operatorname{diag}(d_{1t}, \ldots, d_{nt}) \in \mathcal{M}_n(\mathbb{R}[\underline{x}].$$

2. Choose $\underline{z} \in \mathbb{C}^d$ and find $\varepsilon_{\underline{z}} > 0$ such that

$$F - \varepsilon_{\underline{z}} I_n \succ 0 \quad \text{on } K_{\mathcal{G}} \quad \text{and} \quad \operatorname{rank}\big((F - \varepsilon_{\underline{z}} I_n)(\underline{z})\big) = n.$$

By a small adaptation of the diagonalization procedure for $F - \varepsilon_{\underline{z}} I_n$ we have

$$\underbrace{b_{\underline{z}}^4}_{\substack{b_{\underline{z}} \in \mathbb{R}[\underline{x}], \\ b_{\underline{z}} > 0 \text{ on } K_{\mathcal{G}}, \\ b_{\underline{z}}(\underline{z}) \neq 0}} (F - \varepsilon_{\underline{z}} I_n) = X_{\underline{z}}^t \underbrace{D_{\underline{z}}}_{\substack{D_{\underline{z}} \succ 0 \text{ on } K_{\mathcal{G}}, \\ D_{\underline{z}} \text{ diagonal}}} X_{\underline{z}},$$

$X_{\underline{z}} \in M_n(\mathbb{R}[\underline{x}])$,

3. Let

$$I = \langle b_{\underline{z}}^4 \colon \underline{z} \in \mathbb{C}^d \rangle \subseteq \mathbb{R}[\underline{x}]$$

be the ideal generated by $b_{\underline{z}}^4$. Since $b_{\underline{z}}(\underline{z}) \neq 0$, we have that

$$I = \mathbb{R}[\underline{x}].$$

Proposition

$R \dots$ a commutative ring with 1 and $\mathbb{Q} \subseteq R$

$K \dots$ a topological space which is compact and Hausdorff.

Let

$$\Phi : R \to C(K, \mathbb{R})$$

be a ring homomorphism, such that $\Phi(R)$ separates points in K.

Suppose $f_1, \dots, f_k \in R$ are such that

$$\langle f_1, \dots, f_k \rangle = R \quad \text{and} \quad \Phi(f_j) \geq 0, \quad j = 1, \dots, k.$$

Then there exist $s_1, \dots, s_k \in R$ such that

$$s_1 f_1 + \dots + s_k f_k = 1 \quad \textit{and} \quad \Phi(s_j) > 0, \quad j = 1, \dots, k.$$

C. Scheiderer, Sums of squares on real algebraic surfaces, Manuscr. Math. 119 (2006) 395–410.

S. Kuhlmann, M. Marshall, N. Schwartz, Positivity, sums of squares and the multidimensional moment problemII, Adv. Geom. 5 (2005) 583–607. For $k = 2$.

Back to the proof of the matrix Schmüdgen's theorem.

4. Let $R := \mathbb{R}[\underline{x}]$ and

$$\Phi : R \to C(K_{\mathcal{G}}, \mathbb{R}), \quad \Phi(f) = f|_{K_{\mathcal{G}}}.$$

By Proposition, there exist

$$s_1, \ldots, s_k \in \mathbb{R}[\underline{x}], \quad s_j > 0 \text{ on } K_{\mathcal{G}}$$

and

$$b^4_{\underline{z}_1}, \ldots, b^4_{\underline{z}_k} \in I \text{ such that } \sum_{j=1}^{k} s_j b^4_{\underline{z}_j} = 1.$$

Hence

$$F = \sum_{j=1}^{k} s_j b^4_{\underline{z}_j} F = \sum_{j} \left(\varepsilon_{\underline{z}_j} s_j b^4_{\underline{z}_j} + X^t_{\underline{z}_j} s_j D_{\underline{z}_j} X_{\underline{z}_j} \right) \in T_{\mathcal{G}}$$

where a **scalar Schmüdgen's theorem** was used for the last inclusion.

Matrix version of Putinar's theorem

$\mathcal{G} := \{G_1(\underline{x}), \ldots, G_m(\underline{x})\} \subset S_n(\mathbb{R}[\mathrm{x}])$

$K_{\mathcal{G}} := \{\underline{x} \in \mathbb{R}^d \colon G_1(\underline{x}) \succeq 0, \ldots . G_m(\underline{x}) \succeq 0\}$ is compact.

$M_{\mathcal{G}}$ is archimedean: $\exists N \in \mathbb{N}$ such that $N - \sum_i x_i^2 \in M_{\mathcal{G}}$.

$F \in S_n(\mathbb{R}[\underline{\mathrm{x}}])$

$$F(\underline{x}) \succ 0 \text{ for every } \underline{x} \in K_{\mathcal{G}} \quad \Rightarrow \quad F \in M_{\mathcal{G}}.$$

C.W. Scherer, C.W.J. Hol, Matrix sum-of-squares relaxations for robust semi-definite programs. Math. Program. 107 (2006) 189–211.

I. Klep, M. Schweighofer: Pure states, positive matrix polynomials and sums of Hermitian squares, Indiana Univ. Math. J. 59 (2010) 857–874.

The proof of the matrix Schmüdgen's theorem presented above works also for the matrix Putinar's theorem by using scalar Putinar's theorem instead of Schmüdgen's in the last step.

Matrix version of Krivine-Stengle's theorem

$\mathcal{G} := \{G_1(\underline{x}), \ldots, G_m(\underline{x})\} \subset S_n(\mathbb{R}[\mathrm{x}])$

$F \in S_n(\mathbb{R}[\underline{x}])$

Then:

1. $K_{\mathcal{G}} = \emptyset \;\Leftrightarrow\; -I_n \in T_{\mathcal{G}}$.
2. $F(\underline{x}) \succ 0 \;\forall \underline{x} \in K_{\mathcal{G}} \quad\Leftrightarrow\quad FB = I_n + B'$ for some $B \in T_{\mathcal{G}} \cap (\mathbb{R}[x] \cdot I_n),\; B' \in T_{\mathcal{G}}$.
3. $F(\underline{x}) \succeq 0 \;\forall \underline{x} \in K_{\mathcal{G}} \quad\Leftrightarrow\quad FB = BF = F^{2k} + B'$ for some $B, B' \in T_{\mathcal{G}}$ and $k \in \mathbb{N}$.
4. $F(\underline{x}) = 0 \;\forall \underline{x} \in K_{\mathcal{G}} \quad\Leftrightarrow\quad -F^{2k} \in T_{\mathcal{G}}$ for some $k \in \mathbb{N}$.

J. Cimprič, Strict positivstellensätze for matrix polynomials with scalar constraints, Linear algebra appl. 434 (2011), 1879–1883.

J. Cimprič, Real algebraic geometry for matrices over commutative rings, J. Algebra 359 (2012), 89–103.

4. Matrix versions of Nichtnegativstellensätze in $\mathbb{R}$

Cases $K = [0,1]$, $K = [0,\infty)$ $\quad F \in S_n(\mathbb{R}[\mathrm{x}])$

$F(x) \succeq 0$ for every $x \in [0,1] \quad \Rightarrow$

$$F(\mathrm{x}) = \underbrace{F_0(\mathrm{x})}_{\text{degree}\leq \deg F} + \underbrace{\mathrm{x}F_1(\mathrm{x})}_{\text{degree}\leq \deg F} + \underbrace{(1-\mathrm{x})F_2(\mathrm{x})}_{\text{degree}\leq \deg F} + \underbrace{\mathrm{x}(1-\mathrm{x})F_3(\mathrm{x})}_{\text{degree}\leq \deg F},$$

$$F_i \in \sum M_n(\mathbb{R}[\mathrm{x}])^2.$$

$G(x) \succeq 0$ for every $x \in [0,\infty) \quad \Rightarrow$

$$F(\mathrm{x}) = \underbrace{G_0(\mathrm{x})}_{\text{degree}\leq \deg G} + \underbrace{\mathrm{x}G_1(\mathrm{x})}_{\text{degree}\leq \deg G}, \quad G_i \in \sum M_n(\mathbb{R}[\mathrm{x}])^2.$$

Compactly: $F \in T_{\{x,1-x\}}$, $G \in T_{\{x\}}$ with the degrees best possible.

- H. Dette and W. J. Studden, Matrix measures, moment spaces and Favard's theorem for the interval $[0,1]$ and $[0,\infty)$, Linear Algebra Appl. **345** (2002), 169–193. The limit argument for the case $[0,\infty)$ suspicious.
- Y. Savchuk and K. Schmüdgen K., Positivstellensätze for algebras of matrices, Linear Algebra Appl. **436** (2012), 758–788.
- J. Cimprič and A. Zalar, Moment problems for operator polynomials, J. Math. Anal. Appl. **401** (2013), 307–316. The operator case.

Main steps of the proof of the $[0, \infty)$ case

1. Define $G(\mathrm{x}) := F(\mathrm{x}^2)$. Note that $G(x) \succeq 0$ on $\mathbb{R}$.

2. By the Fejér-Riesz theorem:

$$\begin{aligned} G(\mathrm{x}) &= H_1(\mathrm{x})^* H_1(\mathrm{x}) + H_2(\mathrm{x})^* H_2(\mathrm{x}) \\ &= \sum_{j=1}^{2} \left(H_{j,1}(\mathrm{x}^2) + \mathrm{x} H_{j,2}(\mathrm{x}^2)\right)^* \left(H_{j,1}(\mathrm{x}^2) + \mathrm{x} H_{j,2}(\mathrm{x}^2)\right) \\ &= \sum_{j=1}^{2} \left(H_{j,1}(\mathrm{x}^2)^* H_{j,1}(\mathrm{x}^2) + \mathrm{x}^2 H_{j,2}(\mathrm{x}^2)^* H_{j,2}(\mathrm{x}^2)\right). \end{aligned}$$

3. So

$$F(\mathrm{x}) = \sum_{j=1}^{2} \Big(H_{j,1}(\mathrm{x})^* H_{j,1}(\mathrm{x}) + \mathrm{x} H_{j,2}(\mathrm{x})^* H_{j,2}(\mathrm{x})\Big).$$

For the bounded interval case one can assume that the interval is $[-1, 1]$. Then by substitution $x = \cos\varphi$, $G(\varphi) := F(\cos\varphi) = \sum_{i=-n}^{n} G_k e^{ik\varphi} \succeq 0$ for $\varphi \in \mathbb{R}$.

Case $K = \{a\} \cup [b, c]$, $a < b < c$ $\quad F \in S_n(\mathbb{R}[x])$, $F \succeq 0$ on K

With Shengding Sun, in preparation.

$F \in T_{\{x-a,(x-a)(x-b),(c-x)\}}$ with the degrees best possible.

$\deg F = 2m, m \in \mathbb{N} \quad \Rightarrow$

$$F(x) = \underbrace{F_0(x)}_{\text{degree} \le \deg F} + \underbrace{(x-a)(x-b)F_1(x)}_{\text{degree} \le \deg F} + \underbrace{(x-a)(c-x)F_2(x)}_{\text{degree} \le \deg F} + \underbrace{(x-a)^2(x-b)(c-x)F_3(x)}_{\text{degree} \le \deg F}, \quad F_i \in \sum M_n(\mathbb{R}[x])^2.$$

$\deg F = 2m-1, m \in \mathbb{N} \quad \Rightarrow$

$$F(x) = \underbrace{(x-a)F_0(x)}_{\text{degree} \le \deg F} + \underbrace{(c-x)F_1(x)}_{\text{degree} \le \deg F} + \underbrace{(x-a)^2(x-b)F_2(x)}_{\text{degree} \le \deg F} + \underbrace{(x-a)(x-b)(c-x)F_3(x)}_{\text{degree} \le \deg F}, \quad F_i \in \sum M_n(\mathbb{R}[x])^2.$$

Proof is done on the dual side by solving the corresponding truncated matrix moment problem.

Compact semialgebraic set K which is not connected and not of the form $[a, b] \cup \{c\}$ or $\{a, b\}$ or $\{a, b, c\}$

$S = \{g_1, \dots, g_s\} \subseteq \mathbb{R}[x]$ finite set with $K_S = K$.

The matrix preordering T_S^n is saturated if it contains every $F \in S_n(\mathbb{R}[\mathrm{x}])$ such that $F \succeq 0$ on K.

The matrix preordering T_S^n is strongly boundedly saturated is saturated and every $F \in T_S^n$ has a representation of the form $\sum_{\underline{\alpha} \in \{0,1\}^s} \underbrace{G_{\underline{\alpha}}^* G_{\underline{\alpha}} \cdot \underline{g}^{\underline{\alpha}}}_{\text{degree} \leq \deg F}$.

Proposition (Let K be as in the title.)

There does not exist a finite set $S \subseteq \mathbb{R}[\mathrm{x}]$ with a strongly boudedly saturated preordering T_S^n for $n > 1$.

Natural description of a closed semialgebraic set $K \subseteq \mathbb{R}$

$K = \{x \in \mathbb{R} \colon h_1(x) \geq 0, \dots, h_l(x) \geq 0\}$ for some $h_i \in \mathbb{R}[\mathrm{x}]$.

A set $S = \{g_1, \dots, g_s\} \subset \mathbb{R}[x]$ is the natural description of K, if it satisfies the following conditions:

(a) If K has the least element a, then $x - a \in S$.

(b) If K has the greatest element b, then $b - x \in S$.

(c) For every $a \neq b \in K$, if $(a, b) \cap K = \emptyset$, then $(x - a)(x - b) \in S$.

(d) These are the only elements of S.

Theorem

$f(x) \geq 0$ for every $x \in K$. $\Rightarrow$ $f \in T_S$.

Moreover, the degrees are the best possible, i.e., the degree of each summand $t_i \in T_S$ in $f = \sum_i t_i$ is bounded by the degree of f.

- S. Kuhlmann, M. Marshall, Positivity, sums of squares and the multidimensional moment problem, Trans. Amer. Math. Soc. 354 (2002) 4285–4301.
- S. Kuhlmann, M. Marshall, N. Schwartz, Positivity, sums of squares and the multidimensional moment problem II, Adv. Geom. 5 (2005) 583–607.

Saturated descriptions of a compact semialgebraic set

$K \subseteq \mathbb{R}$ $\quad K = \bigcup_{j=1}^{m} [x_j, y_j] \subseteq \mathbb{R}$

A set $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ with $K = K_S$ is the saturated description of K, if it satisfies the following conditions:

(a) For every left endpoint x_j there exists $k \in \{1, \ldots, s\}$, such that $g_k(x_j) = 0$ and $g_k'(x_j) > 0$.

(b) For every right endpoint y_j there exists $k \in \{1, \ldots, s\}$, such that $g_k(y_j) = 0$ and $g_k'(y_j) < 0$.

Theorem

$f(x) \geq 0$ for every $x \in K$. $\quad \Rightarrow \quad f \in M_S$.

- C. Scheiderer, Sums of squares on real algebraic curves, Math. Z. 245 (2003) 725–760.
- C. Scheiderer, Distinguished representatitons of non-negative polynomials, J. Algebra 289 (2005) 558–573.

Compact univariate matrix Nichtnegativstellensatz

$K = \bigcup_{j=1}^{m} [x_j, y_j] \subseteq \mathbb{R}$ a compact set.

$S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ a saturated description of K.

$$M_S^n = \{A_0 + \sum_{i=1}^{s} A_i g_i : A_j \in \sum M_n(\mathbb{R}[x])^2\}$$

$F \in S_n(\mathbb{R}[x])$

$$F(x) \succeq 0 \text{ for every } x \in K. \quad \Rightarrow \quad F \in M_S^n.$$

A. Zalar, A matrix Fejér-Riesz theorem with gaps, J. Pure Appl. Algebra **220** (2016), 2533–2548.

The technique is as in the proof of the matrix Schmüdgen's theorem presented above without the substraction of εI_n part. Due to univariate situation it is possible to construct denominators avoiding chosen complex point z.

Noncompact univariate matrix Nichtnegativstellensatz

$K \subseteq \mathbb{R}$ closed semiaglebraic set, S a natural description of K

For $F \in S_n(\mathbb{R}[x])$, the following are equivalent:

1. $F(x) \succeq 0$ for every $x \in K$.
2. For every point $w \in \mathbb{C} \setminus K$ there exists $k_w \in \mathbb{N} \cup \{0\}$ such that
$$((\mathrm{x} - w)(\mathrm{x} - \overline{w}))^{k_w} \cdot F \in M_S^n.$$
3. There exists $k \in \mathbb{N} \cup \{0\}$ such that
$$(1 + \mathrm{x}^2)^k \cdot F \in M_S^n.$$
4. For every natural number $p \in \mathbb{N}$ there exists a polynomial $h \in \mathbb{R}[\mathrm{x}]$, $h > 0$ on $\mathbb{R}$, and a matrix polynomial $G \in M_S^n$ such that
$$hF = F^{2p} + G \in M_S^n.$$

A. Zalar, Contributions to a noncommutative real algebraic geometry, PhD thesis, 2017, http://www.matknjiz.si/doktorati/2017/Zalar-14521-29.pdf.

Main steps of the proof of 2.

Fix $z_0 \in \mathbb{T}$, $w_0 \in \mathbb{C} \setminus \mathbb{R}$ and define $\lambda_{z_0,w_0} : \mathbb{R} \cup \{\infty\} \to \mathbb{T}$ by

$$\lambda_{z_0,w_0}(\mathrm{x}) := z_0 \frac{\mathrm{x} - w_0}{\mathrm{x} - \overline{w_0}}, \quad \lambda^{-1}_{z_0,w_0}(\mathrm{z}) = \frac{\mathrm{z}\overline{w_0} - z_0 w_0}{\mathrm{z} - z_0}.$$

For $F(\mathrm{x}) \in M_n(\mathbb{R}[\mathrm{x}])$ we have

$$\Lambda_{z_0,w_0,F}(\mathrm{z}) := ((\mathrm{z} - z_0)^*(\mathrm{z} - z_0))^{\lceil \frac{\deg(F)}{2} \rceil} \cdot F\left(\lambda^{-1}_{z_0,w_0}(\mathrm{z})\right) \in M_n\left(\mathbb{R}\left[\mathrm{z}, \frac{1}{\mathrm{z}}\right]\right)$$

and

$$F(\mathrm{x}) = \left(\frac{(\mathrm{x} - \overline{w_0})(\mathrm{x} - w_0)}{4 \cdot \Im(w_0)^2}\right)^{\lceil \frac{\deg(F)}{2} \rceil} \Lambda_{z_0,w_0,F}(\lambda_{z_0,w_0}(\mathrm{x})),$$

Let

$$\mathscr{K}_{z_0,w_0} := \overline{\lambda_{z_0,w_0}(K)} \subseteq \mathbb{T}.$$

Claim 1. $\Lambda_{1,w,F}(\mathrm{z}) \succeq$ on $\mathscr{K}_{1,w}$.

Claim 2. By the $\mathbb{T}$-version of the Compact Positivstellesatz

$$\Lambda_{1,w,F}(\mathrm{z}) = \sum_{i=0}^{s} A_i^* A_i \cdot \Lambda_{1,w,g_i}$$

where each $A_i \in M_n(\mathbb{R}[\mathrm{z}, \frac{1}{\mathrm{z}}])$.

For

$$k_w = \max_{i=0,\ldots,s} \left\{ \deg(A_i^* A_i) + \deg(\Lambda_{1,w,g_i}) - \left\lceil \frac{\deg(F)}{2} \right\rceil \right\}$$

we have

$$\begin{aligned}
\left(\frac{|\mathrm{x} - w|}{2 \cdot \Im(w)}\right)^{2k_w} \cdot F(\mathrm{x}) &= \left(\frac{|\mathrm{x} - w|}{2 \cdot \Im(w)}\right)^{2k_w + 2\left\lceil \frac{\deg(F)}{2} \right\rceil} \cdot \Lambda_{1,w,F}(\lambda_{1,w}(x)) \\
&= \left(\frac{|\mathrm{x} - w|}{2 \cdot \Im(w)}\right)^{2k_w + 2\left\lceil \frac{\deg(F)}{2} \right\rceil} \cdot \left(\sum_{i=0}^{s} A_i^* A_i \cdot \Lambda_{1,w,g_i}\right)(\lambda_{1,w}(\mathrm{x})) \\
&= \sum_{i=0}^{s} \left(\left(\frac{|x - w|}{2 \cdot \Im(w)}\right)^{2k_w + 2\left\lceil \frac{\deg(F)}{2} \right\rceil} \cdot \left((A_i^* A_i) \cdot \Lambda_{1,w,g_i}\right)(\lambda_{1,w}(\mathrm{x})) \right) \\
&= \sum_{i=0}^{s} \left(\frac{|x - w|}{2 \cdot \Im(w)}\right)^{2k_w + 2\left\lceil \frac{\deg(F)}{2} \right\rceil - 2\left\lceil \frac{\deg(g_i)}{2} \right\rceil} \cdot (A_i^* A_i)(\lambda_{1,w}(\mathrm{x})) \cdot g_i(\mathrm{x}) \in M_S^n,
\end{aligned}$$

Thank you for your attention!