A gap between positive polynomials and sums of squares in three different settings

Seminar Algebra–Geometrie–Kombinatorik
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Dresden

joint work with

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Outline

quantitative estimates on volumes of pos vs sos cones

1. Preliminaries
   - Problems:
     - positive maps vs completely positive maps
     - cross-positive maps vs completely cross-positive maps
     - copositive vs completely positive matrices
   - Converting to polynomials:
     - pos vs sos biquadratic biforms
     - pos vs sos biquadratic biforms modulo the ideal of all orthonormal 2-frames
     - pos vs sos even quartic forms
2. Discussion on volume estimation
3. Proofs
   - real algebraic geometry
   - asymptotic convex analysis
   - harmonic analysis

1. Preliminaries

Positive and completely positive maps

Definitions

A linear map

$$\Phi : M_n(\mathbb{R}) \to M_m(\mathbb{R})$$

such that $\Phi(A^T) = \Phi(A)^T$ for all $A \in M_n(\mathbb{R})$, is:

- positive if

$$A \succeq 0 \quad \Rightarrow \quad \phi(A) \succeq 0.$$

- k-positive if

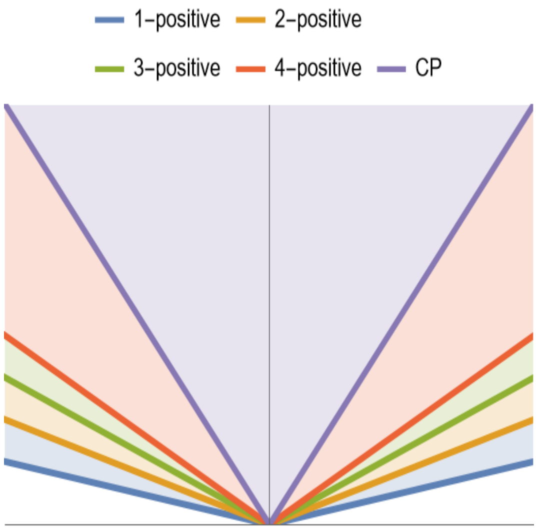
$$\phi_k\left(\begin{pmatrix} A_{11} & \dots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \dots & A_{kk} \end{pmatrix}\right) = \begin{pmatrix} \phi(A_{11}) & \dots & \phi(A_{1k}) \\ \vdots & \ddots & \vdots \\ \phi(A_{k1}) & \dots & \phi(A_{kk}) \end{pmatrix}$$

is positive.

- completely positive (CP) if it is k-positive for every $k \in \mathbb{N}$.

Positive and completely positive maps

Mental picture



Positive and completely positive maps

Problems and a small sample of existing literature

Problem A.1: Establish asymptotically exact quantitative bounds on the fraction of positive maps that are CP.

Problem A.2: Derive algorithm to produce positive maps that are not CP from random input data.

Positive and completely positive maps

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Problem A.1: Establish asymptotically exact quantitative bounds on the fraction of positive maps that are CP.

Problem A.2: Derive algorithm to produce positive maps that are not CP from random input data.

- Arveson (2009): Let $n, m \geq 2$. Then the probability p that a positive map $\varphi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is CP satisfies $0 < p < 1$.
- Szarek, Werner, Życzkowski (2008): for the case $m = n$ provide quantitative bounds on p and establish its asymptotic behaviour.
- Collins, Hayden, Nechita (2017): random techniques for constructing k-positive maps that are not $(k+1)$-positive in large dimensions.

Positive maps meet real algebraic geometry

$\mathcal{L}(\mathbb{S}_n, \mathbb{S}_m)$ ... the vector space of all linear maps from $\mathbb{S}_n$ to $\mathbb{S}_m$,

$\mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2}$ ... biforms in $\mathrm{x} = (x_1, \ldots, x_n)$ and $\mathrm{y} = (y_1, \ldots, y_m)$ of bidegree $(2, 2)$

There is a natural bijection

$$\Gamma : \mathcal{L}(\mathbb{S}_n, \mathbb{S}_m) \to \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2},$$
$$\Phi \mapsto p_\Phi(\mathrm{x}, \mathrm{y}) := \mathrm{y}^T \Phi(\mathrm{x}\mathrm{x}^T)\mathrm{y}.$$

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Proposition

Let $\Phi : \mathbb{S}_n \to \mathbb{S}_m$ be a linear map. Then:

1. Φ is positive iff p_Φ is nonnegative.
2. Φ is completely positive iff p_Φ is a sum of squares (SOS). (Choi-Kraus theorem)

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Corollary

The following probabilities (w.r.t. the corresponding distributions) are equal:

1. *The probability that a positive map $\Phi \in \mathcal{L}(\mathbb{S}_n, \mathbb{S}_m)$ is CP.*
2. *The probability that a nonnegative biform $p_\Phi \in \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2}$ is SOS.*

Cross–positive and completely cross–positive maps

A linear map

$$\Phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$$

is:

- cross-positive if
$$\forall U, V \succeq 0 : \langle U, V \rangle = 0 \Rightarrow \langle \phi(U), V \rangle \geq 0.$$
- k-cross-positive if
$$\phi_k\left(\begin{pmatrix} A_{11} & \dots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \dots & A_{kk} \end{pmatrix} \right) = \begin{pmatrix} \phi(A_{11}) & \dots & \phi(A_{1k}) \\ \vdots & \ddots & \vdots \\ \phi(A_{k1}) & \dots & \phi(A_{kk}) \end{pmatrix}$$
is cross–positive.
- completely cross–positive (CCP) if it is k-cross-positive for every $k \in \mathbb{N}$.

Cross–positive and completely cross–positive maps

Problems and a small sample of existing literature

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Problem B.2: Derive algorithm to produce cross–positive maps that are not CCP from random input data.

Cross–positive and completely cross–positive maps

Problem B.1: Establish asymptotically exact quantitative bounds on the fraction of cross–positive maps that are CCP.

Problem B.2: Derive algorithm to produce cross–positive maps that are not CCP from random input data.

- Schneider, Vidyasagar (1970):
  - $\phi(\cdot)$ is crp if and only if $\exp(t\phi(\cdot))$ is positive for every $t > 0$.
  - Characterized cross–positive maps on polyhedral cones.
- Cuchiero, Filipović, Mayerhofer, Teichmann (2011) established the importance of cross–positive and completely cross-positive maps in math finance.
- Kuzma, Omladič, Šivic, Teichmann (2015) constructed, for the first time, a proper cross–positive map. (Not of the form $X \mapsto \tilde{\phi}(X) + CX + XC^T$, where $\tilde{\phi}$ is positive.)

Cross–positive maps meet RAG

$$I \subseteq \mathbb{R}[\mathrm{x}, \mathrm{y}] \quad \ldots \quad \text{the ideal generated by } \mathrm{y}^T\mathrm{x} = \sum_i \mathrm{x}_i\mathrm{y}_i,$$

$$I_{2,2} \subseteq \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2} \quad \ldots \quad I_{2,2} = I \cap \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2},$$

$$V(I) \quad \ldots \quad \text{the variety } \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid y^T x = 0\}$$

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Proposition

Let $\Phi : \mathbb{S}_n \to \mathbb{S}_n$ be a linear map. Then:

1. Φ is cross–positive iff p_Φ is nonnegative on $V(I)$.
2. Φ is CCP iff p_Φ is a sum of squares modulo I.

Corollary

The following probabilities (w.r.t. the corresponding distributions) are equal:

1. *The probability that a cross–positive map $\Phi \in \mathcal{L}(\mathbb{S}_n, \mathbb{S}_n)$ is CCP.*
2. *The probability that a nonnegative biform $p_\Phi + I_{2,2} \in \mathbb{R}[\mathrm{x},\mathrm{y}]_{2,2}/I_{2,2}$ is SOS.*

Copositive and completely positive matrices

Definitions

$\mathbb{S}_n$ ... real symmetric $n \times n$ matrices

A matrix

$$A = (a_{ij})_{i,j} \in \mathbb{S}_n$$

is:

- positive semidefinite (PSD) if $v^T A v \geq 0$ for every $v \in \mathbb{R}^n$.

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- copositive (COP) if $v^T A v \geq 0$ for every $v \in \mathbb{R}^n_{\geq 0}$.
- positive semidefinite (PSD) if $v^T A v \geq 0$ for every $v \in \mathbb{R}^n$.

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- completely positive (CP) if $A = BB^T$ for some $B \in \mathbb{R}^{n \times k}_{\geq 0}$.

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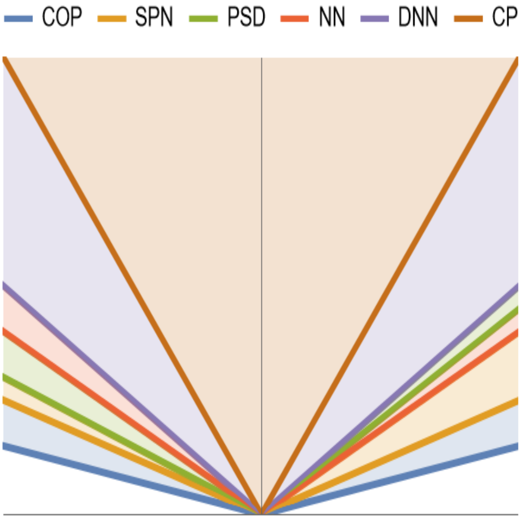
- positive semidefinite (PSD) if $v^T A v \geq 0$ for every $v \in \mathbb{R}^n$.
- nonnegative (NN) if $a_{ij} \geq 0$ for every i, j.
- SPN if $A = P + N$ for some P PSD and N NN.
- doubly nonnegative (DNN) if $A = P \cap N$ for some P PSD and N NN.
- completely positive (CP) if $A = BB^T$ for some $B \in \mathbb{R}^{n \times k}_{\geq 0}$.

Copositive and completely positive matrices

Mental picture



Copositive vs completely positive matrices

Problems and a small sample of existing literature

Problem C.1: Establish asymptotically exact quantitative bounds on the fraction of COP matrices that are CP.

Problem C.2: Derive algorithm to produce COP matrices that are not CP.

Copositive vs completely positive matrices

Problems and a small sample of existing literature

Problem C.1: Establish asymptotically exact quantitative bounds on the fraction of COP matrices that are CP.

Problem C.2: Derive algorithm to produce COP matrices that are not CP.

- Maxfield, Minc (1962), Hall, Newman (1963): $\mathrm{COP}_n = \mathrm{SPN}_n$ holds only for $n \leq 4$.
- Parrilo (2000): $\mathrm{int}(\mathrm{COP}_n) \subseteq \bigcup_r K_n^{(r)}$, where $(\mathrm{x}^2 = (x_1^2, \ldots, x_n^2))$

$$K_n^{(r)} := \{A \in \mathbb{S}_n : (\sum_{i=1}^{n} x_i^2)^r \cdot (\mathrm{x}^2)^T A \mathrm{x}^2 \text{ is a sum of squares of forms}\}.$$

- Dickinson, Dür, Gijben, Hildebrand (2013): $\mathrm{COP}_5 \neq K_5^{(r)}$ for any $r \in \mathbb{N}$.
- Laurent, Schweighofer, Vargas (2022, 23+): $\mathrm{COP}_5 = \bigcup_r K_5^{(r)}$ and $\mathrm{COP}_6 \neq \bigcup_r K_6^{(r)}$.

Copositive matrices meet RAG

$\mathbb{R}[\mathrm{x}^2]_{4,e}$ ... forms in $\mathrm{x}^2 = (x_1^2, \ldots, x_n^2)$ of degree 4, i.e., *quartic even forms*.

There is a natural bijection

$$\Gamma : \mathbb{S}_n \to \mathbb{R}[\mathrm{x}]_{4,e}, \quad A \mapsto q_A(\mathrm{x}) := (\mathrm{x}^2)^T A \mathrm{x}^2 = \sum_{i,j=1}^{n} a_{ij} x_i^2 x_j^2.$$

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Proposition

Let $A \in \mathbb{S}_n$ be a matrix. Then:

1. A is COP iff q_A is nonnegative. *(q_A ... POS)*
2. A is PSD iff q_A is of the form $\sum_i \left(\sum_j f_{ij} x_j^2\right)^2$. *($q_A$ ... lin-SOS)*

6. A is CP iff q_A is of the form $\sum_i \left(\sum_j f_{ij} x_j^2\right)^2$ with $f_{ij} \geq 0$. *(q_A ... CP)*

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2. A is PSD iff q_A is of the form $\sum_i \left(\sum_j f_{ij} x_j^2\right)^2$. *($q_A$ ... lin-SOS)*
3. A is NN iff q_A has nonnegative coefficients. *(q_A ... NN)*
4. A is SPN iff q_A is of the form $\sum_i \left(\sum_j f_{ij} x_i x_j\right)^2$ (Parrilo, 00') *(q_A ... SOS)*
5. A is DNN iff q_A is ℓ-SOS and NN. *(q_A ... DNN)*
6. A is CP iff q_A is of the form $\sum_i \left(\sum_j f_{ij} x_j^2\right)^2$ with $f_{ij} \geq 0$. *(q_A ... CP)*

Corollary. The gaps between COP/PSD/NN/SPN/DNN/CP matrices correspond to the gaps between POS/ℓ-SOS/NN/SOS/DNN/CP even quartics.

Gap between positive and sos polynomials

$\mathbb{R}[\mathrm{x}]_{2k}$ ... forms in $\mathrm{x} = (x_1, \ldots, x_n)$ of degree $2k$

Theorem (Blekherman, 2006)

For $n \geq 3$ and fixed k the probability p_n that a positive polynomial $f \in \mathbb{R}[\mathrm{x}]_{2k}$ is sum of squares, satisfies

$$\left(C_1 \cdot \frac{1}{n^{(k-1)/2}}\right)^{\dim \mathbb{R}[x]_{2k}-1} \leq p_n \leq \left(C_2 \cdot \frac{1}{n^{(k-1)/2}}\right)^{\dim \mathbb{R}[x]_{2k}-1},$$

where C_1, C_2 are absolute constants.

In particular, for $2k = 4$,

$$p_n \in \Theta\left(\left(\frac{1}{\sqrt{n}}\right)^{\dim \mathbb{R}[x]_4 - 1}\right).$$

Solutions to Problems A.1, B.1, C.1

Theorem A.1 [Klep, McCullough, Šivic, Z, 2019]: For $n, m \geq 3$ the probability $p_{n,m}$ that a positive map $\Phi : \mathbb{S}_n \to \mathbb{S}_m$ is CP, satisfies

$$\left(\frac{3\sqrt{3}}{2^{10}\sqrt{2}} \cdot \frac{1}{\sqrt{\min(m,n)}}\right)^d \leq p_{n,m} \leq \left(\frac{2^{12} \cdot 5^2 \cdot 6^{\frac{1}{2}} 10^{\frac{2}{9}}}{3^3} \cdot \frac{1}{\sqrt{\min(m,n)}}\right)^d,$$

where $d = \dim\{\Phi \mid \Phi : \mathbb{S}_n \to \mathbb{S}_m \text{ linear map}\} - 1$.

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Theorem B.1 [Klep, Šivic, Z, 2024+]: For $n \geq 3$ the probability p_n that a cross–positive map $\Phi : \mathbb{S}_n \to \mathbb{S}_n$ is CCP, satisfies

$$p_n \leq \left(\frac{2^5 \cdot 2^{\frac{1}{2}} \cdot 5^2 \cdot 10^{\frac{2}{9}}}{3^{\frac{3}{2}}} \cdot \frac{1}{\sqrt{n}}\right)^d,$$

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Theorem C.1 *[Klep, Štrekelj, Z, 2023+]*: For $n > 4$ the probability p_n that a copositive matrix $A \in \mathbb{S}_n$ is CP, satisfies

$$\left(2^{-8} \cdot 3^{-2}\right)^{\dim \mathbb{S}_n - 1} \leq p_n.$$

Solutions to Problems A.2, B.2, C.2

Problem A.2, B.2 *[Klep, McCullough, Šivic, Z, 2019, 2024+]*:

Construction of nonnegative (nonnegative modulo $V(I)$)) biquadratic biforms that are not sums of squares biforms (modulo I) by specializing the algorithm by Blekherman, Smith, Velasco (2016) to produce pos not sos forms on varieties, which are not of minimal degree.

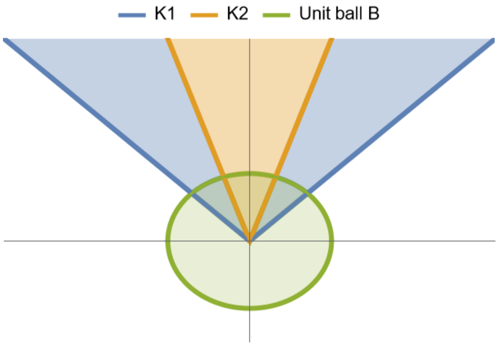
Problem C.2 *[Klep, Štrekelj, Z, 2023+]*:

Free probability inspired construction of $\mathrm{DNN}_n \setminus \mathrm{CP}_n$, $n \geq 5$, matrices. Dually, we obtain matrices from $\mathrm{COP}_n \setminus \mathrm{SPN}_n$.

2. Discussion on volume estimates

Cones in question

Intersect with a unit ball in some metric



- **Goal:** Compare the sizes of the intersections $K_1 \cap B$ and $K_2 \cap B$.

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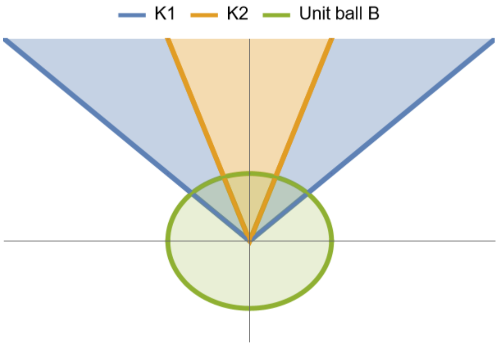

- Goal: Compare the sizes of the intersections $K_1 \cap B$ and $K_2 \cap B$.
- Beware 1: Size estimates might differ according to the choice of the measure.
- Beware 2: Equipping the ambient vector space V with the pushforward of the Lebesgue measure is independent of the isomorphism $\phi : V \to \mathbb{R}^{\dim V}$ only if ϕ is a Hilbert space isomorphism (V being a normed spaces is not enough).
- Beware 3: Size estimates might differ according to the choice of the inner product and for balls in different metrics.

Volume radius

Proper measure of the asymptotic sizes of a sequence of compact sets

The volume radius $\mathrm{vrad}(C)$ of a compact set $C \subseteq \mathbb{R}^n$, equipped with an inner product $\langle \cdot, \cdot \rangle$ and a measure μ, is

$$\mathrm{vrad}(C) = \left(\frac{\mathrm{Vol}(C)}{\mathrm{Vol}(B)} \right)^{1/n},$$

where B is the unit ball in $\langle \cdot, \cdot \rangle$.

- Since we are concerned with the asymptotic behavior as n goes to infinity, we need to eliminate the dimension effect when dilating K by some factor c.
- A dilation multiplies the volume of C by c^n, but a more appropriate effect would be multiplication by c.

Gap between positive and sos polynomials asymptotically not visible in the ball of the ℓ^1 norm

- $\mathbb{R}[\mathrm{x}]_{2k}$ is equipped with the natural L^2 inner product

$$\langle f, g\rangle = \int_{S^{n-1}} fg \, \mathrm{d}\sigma,$$

  where and σ is the rotation invariant probability measures on the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

- Let $\|\cdot\|_1$ the ℓ^1 norm on the vector of coefficients, i.e.,

$$\|\sum_\alpha a_\alpha \mathrm{x}^\alpha\|_1 = \sum_\alpha |a_\alpha|.$$

- E.g., for $k = 2$, due to the equality (and Rogers-Shepard inequality)

$$x_i x_j x_k x_\ell = \frac{1}{2}(x_i x_j + x_k x_\ell)^2 - \frac{1}{2}x_i^2 x_j^2 - \frac{1}{2}x_k^2 x_\ell^2,$$

  the volume radii of positive and sos polynomials in the unit ball B_1 of $\|\cdot\|_1$ are bounded by absolute constants.

Blekherman's result on the gap between positive and sos polynomials refers to the unit ball in the L^2 norm

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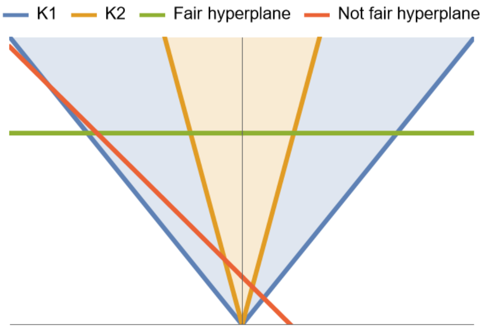
- Let B_2 be the unit ball in the L^2 norm.
- Direct volume estimates for the sections $\mathrm{POS}_{2k} \cap B_2$ and $\mathrm{SOS}_{2k} \cap B_2$ are difficult to obtain.
- Instead, it is natural to compare POS_{2k} and SOS_{2k} when intersected with some affine hyperplane.

Choice of the affine hyperplane for comparison of the cones



1. In case the cones share a unique line of symmetry, it is natural to take the hyperplane whose normal is this line of symmetry.
2. Under the action $O \cdot f(\mathrm{x}) := f(O^{-1}\mathrm{x})$ for $O \in O(n)$, POS_{2k} and SOS_{2k} are invariant, while $\alpha(x_1^2 + \ldots + x_n^2)^2$, $\alpha \in \mathbb{R}$, are the only fixed points.
3. So the hyperplane with the normal $(x_1^2 + \ldots + x_n^2)^2$ is the 'fairest' choice.

A general procedure to obtain the volume estimates

Inputs:

- A convex cone K in a finite-dimensional inner product space V.
- A norm $\|\cdot\|$ w.r.t. which the size of K is to be estimated.

Output: Quantitative bounds on the size of K.

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1. Equip V with a pushforward measure of the Lebesgue measure.
2. Try to estimate $\mathtt{vrad}(K \cap B)$, where B is the unit ball of $\|\cdot\|$. If this is achieved, you are done. Otherwise go to step 3.
3. Choose a fair affine hyperplane $\mathcal{H}$: ...such that $K' = K \cap \mathcal{H}$ is bounded.
4. Translate $\mathcal{H}$ to a hyperplane $\mathcal{M}$.
5. Equip $\mathcal{M}$ with a pushforward measure of the Lebesgue measure and estimate $\mathtt{vrad}(K \cap \mathcal{H})$ in $\mathcal{M}$.

3. Proofs

Procedure applied to Problem A.1

1. $\mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2}$ is equipped with the natural L^2 inner product

$$\langle f, g\rangle = \int_{S^{n-1}\times S^{m-1}} fg \,\mathrm{d}\sigma = \int_{x\in S^{n-1}} \left(\int_{y\in S^{m-1}} fg \,\mathrm{d}\sigma_2(y)\right) \mathrm{d}\sigma_1(x),$$

   where $\sigma = \sigma_1 \times \sigma_2$ is the product measure of rotation invariant probability measures σ_1, σ_2 on the unit spheres $S^{n-1} \subset \mathbb{R}^n$, $S^{m-1} \subset \mathbb{R}^m$.
2. $\mathcal{H}$ is the affine hyperplane

$$\mathcal{H} = \left\{ f \in \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2} \colon \int_{S^{n-1}\times S^{m-1}} f \,\mathrm{d}\sigma = 1 \right\}.$$

3. $z := \left(\sum_{i=1}^n x_i^2\right)\left(\sum_{j=1}^m y_j^2\right)$ and thus

$$\mathcal{M} = \mathcal{H} - z = \left\{ f \in \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2} \colon \int_{S^{n-1}\times S^{m-1}} f \,\mathrm{d}\sigma = 0 \right\}.$$

4. The estimates of $\mathrm{vrad}(\mathrm{POS} \cap \mathcal{H} - z)$ and $\mathrm{vrad}(\mathrm{SOS} \cap \mathcal{H} - z)$ follow closely Blekherman's proof for $\mathbb{R}[\mathrm{x}]_k$.

Upper bound for $\widetilde{\mathrm{SOS}} := \mathrm{SOS} \cap \mathcal{H} - z$

1. The support function $L_{\widetilde{\mathrm{SOS}}}$ of $\widetilde{\mathrm{SOS}}$ by

$$L_{\widetilde{\mathrm{SOS}}} : \mathcal{M} \to \mathbb{R}, \quad L_{\widetilde{\mathrm{SOS}}}(f) = \max_{g \in \widetilde{\mathrm{SOS}}} \langle f, g \rangle .$$

2. By Urysohn's inequality applied to $\widetilde{\mathrm{SOS}}$ we have

$$\left(\frac{\mathrm{Vol}\,\widetilde{\mathrm{SOS}}}{\mathrm{Vol}\,B_{\mathcal{M}}} \right)^{\frac{1}{D_{\mathcal{M}}}} \leq \int_{S_{\mathcal{M}}} L_{\widetilde{\mathrm{SOS}}}(f)\, \mathrm{d}\widetilde{\mu}(f),$$

   where $S_{\mathcal{M}}$ is the unit sphere in $\mathcal{M}$.

3. The extreme points of $\widetilde{\mathrm{SOS}}$ are of the form

$$g^2 - (\sum_{i=1}^{n} x_i^2)(\sum_{j=1}^{m} y_j^2) \text{ where } g \in \mathcal{U} := \mathbb{R}[\mathrm{x}, \mathrm{y}]_{1,1} \text{ and } \int_{S^{n-1} \times S^{m-1}} g^2 \mathrm{d}\sigma = 1.$$

4. So

$$\left(\frac{\mathrm{Vol}\,\widetilde{\mathrm{SOS}}}{\mathrm{Vol}\,B_{\mathcal{M}}} \right)^{\frac{1}{D_{\mathcal{M}}}} \leq \int_{S_{\mathcal{M}}} \overbrace{\max_{g \in S_{\mathcal{U}}} |\langle f, g^2 \rangle|}^{H_f(g^2)} .$$

Upper bound for $\widetilde{\text{SOS}} := \text{SOS} \cap \mathcal{H} - z$

5. Further on, ($D_{\mathcal{U}}$ dimension of $\mathcal{U}$, $\widehat{\mu}$ normalized Leb. measure on $\mathcal{U}$, $\text{pr}_{\mathcal{M}}$ orthogonal projection into $\mathcal{M}$)

$$\left(\frac{\text{Vol}\,\widetilde{\text{SOS}}}{\text{Vol}\,B_{\mathcal{M}}}\right)^{\frac{1}{D_{\mathcal{M}}}} \leq 2\sqrt{3}\int_{S_{\mathcal{M}}}\left(\int_{S_{\mathcal{U}}}\langle f, g^2\rangle^{2D_{\mathcal{U}}}\,\mathrm{d}\widehat{\mu}(g)\right)^{\frac{1}{2D_{\mathcal{U}}}}\mathrm{d}\widetilde{\mu}(f)$$

$$\leq 2\sqrt{3}\left(\int_{S_{\mathcal{U}}}\int_{S_{\mathcal{M}}}\langle f, g^2\rangle^{2D_{\mathcal{U}}}\,\mathrm{d}\widetilde{\mu}(f)\mathrm{d}\widehat{\mu}(g)\right)^{\frac{1}{2D_{\mathcal{U}}}}$$

$$\leq 2\sqrt{3}\underbrace{\left\|g^2\right\|_2}_{\substack{\text{reverse Hölder}\\ \text{inequality (RHI):}\\ \leq 4^4}}\left(\int_{S_{\mathcal{U}}}\underbrace{\int_{S_{\mathcal{M}}}\left\langle f, \frac{\text{pr}_{\mathcal{M}}(g^2)}{\|\text{pr}_{\mathcal{M}}(g^2)\|_2}\right\rangle^{2D_{\mathcal{U}}}\mathrm{d}\widetilde{\mu}(f)}_{\leq\left(\sqrt{\frac{2D_{\mathcal{U}}}{D_{\mathcal{M}}}}\right)^{2D_{\mathcal{U}}}}\mathrm{d}\widehat{\mu}(g)\right)^{\frac{1}{2D_{\mathcal{U}}}}$$

Proposition (RHI, Duoandikoetxea, 1987)

Let σ be a normalized Lebesgue measure on S^{n-1}. If $g \in \mathbb{R}[\mathrm{x}]_k$ and $2 < p < \infty$, then

$$\left(\int_{S^{n-1}} g^p\,\mathrm{d}\sigma\right)^{\frac{1}{p}} = \|g\|_p \leq \underbrace{p^{k/2}}_{\substack{\text{Main observation:}\\ \text{independence of } n}}\|g\|_2 = p^{k/2}\left(\int_{S^{n-1}} g^2\,\mathrm{d}\sigma\right)^{\frac{1}{2}}.$$

Procedure applied to Problem B.1

1. Let $T := (S^{n-1} \times S^{n-1}) \cap V(I)$ and equip it with the unique $SO(n)$-invariant measure. T is also known as *the Stiefel manifold* of all 2-frames in $\mathbb{R}^n$.
2. $\mathcal{Q} := \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2}/(I \cap \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2})$ is equipped with the natural L^2 inner product
$$\langle f, g \rangle = \int_T fg \, \mathrm{d}\sigma.$$
3. $\mathcal{H}$ is the affine hyperplane
$$\mathcal{H} = \left\{ f \in \mathcal{Q} \colon \int_T f \, \mathrm{d}\sigma = 1 \right\}.$$
4. $z := \left(\sum_{i=1}^n x_i^2 \right) \left(\sum_{j=1}^n y_j^2 \right)$ and thus
$$\mathcal{M} = \mathcal{H} - z = \left\{ f \in \mathcal{Q} \colon \int_T f \, \mathrm{d}\sigma = 0 \right\}.$$

Procedure applied to our Problem B.1

5. Only

$$\mathrm{vrad}(\mathrm{SOS} \cap \mathcal{H} - z) \leq (*) \qquad \text{and} \qquad (*) \leq \mathrm{vrad}(\mathrm{POS} \cap \mathcal{H} - z)$$

can be obtained using Blekherman's proof for $\mathbb{R}[\mathrm{x}]_k$, where the main novelty is the following inequality:

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Proposition (Reverse Hölder inequality (RHI))

For a bilinear biform $g \in \mathbb{R}[\mathrm{x}, \mathrm{y}]_{1,1}/(I \cap \mathbb{R}[\mathrm{x}, \mathrm{y}]_{1,1})$ we have

$$\left(\int_T g^4 \, \mathrm{d}\sigma\right)^{\frac{1}{4}} = \|g\|_4 \leq \underbrace{\sqrt{6}}_{\text{Main observation: independence of } n} \|g\|_2 = \sqrt{6} \left(\int_T g^2 \, \mathrm{d}\sigma\right)^{\frac{1}{2}}.$$

Idea of the proof:

- Compute the values of the integrals of all bilinear, biquadratic and biquartic monomials.
- Prove RHI separately for symmetric forms g (difficult part: Muirhead inequality used) and antisymmetric ones (easier part: sos type inequality).

Integrals w.r.t. σ on T

1. We write $\underline{\phi} = (\phi_1, \phi_2, \dots, \phi_{n-1})$, $\phi_1, \dots, \phi_{n-2} \in [0, \pi]$, $\phi_{n-1} \in [0, 2\pi]$. We first define an orthogonal basis of $\mathbb{R}^n$ in spherical coordinates:

$$x(\underline{\phi}) = \begin{pmatrix} x_1(\underline{\phi}) \\ x_2(\underline{\phi}) \\ x_3(\underline{\phi}) \\ \vdots \\ x_{n-1}(\underline{\phi}) \\ x_n(\underline{\phi}) \end{pmatrix} = \begin{pmatrix} \cos(\phi_1), \\ \sin(\phi_1)\cos(\phi_2), \\ \sin(\phi_1)\sin(\phi_2)\cos(\phi_3), \\ \vdots \\ \sin(\phi_1)\cdots\sin(\phi_{n-2})\cos(\phi_{n-1}), \\ \sin(\phi_1)\cdots\sin(\phi_{n-2})\sin(\phi_{n-1}) \end{pmatrix},$$

$$e_2(\underline{\phi}) = \frac{dx(\underline{\phi})}{d\phi_1}, \quad e_3(\underline{\phi}) = \frac{1}{\sin(\phi_1)}\frac{dx(\underline{\phi})}{d\phi_2}, \quad e_4(\underline{\phi}) = \frac{1}{\sin(\phi_1)\sin(\phi_2)}\frac{dx(\underline{\phi})}{d\phi_3}, \dots$$

$$e_n(\underline{\phi}) = \frac{1}{\sin(\phi_1)\cdots\sin(\phi_{n-2})}\frac{dx(\underline{\phi})}{d\phi_{n-1}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\sin(\phi_{n-1}) \\ \cos(\phi_{n-1}) \end{pmatrix}.$$

Integrals w.r.t. σ on T

2. Let $\underline{\psi} = (\psi_1, \psi_2, \ldots, \psi_{n-2})$, $\psi_1, \ldots, \psi_{n-3} \in [0, \pi]$, $\psi_{n-2} \in [0, 2\pi]$. Spherical coordinates over the orthonormal set $\{e_2(\underline{\phi}), e_3(\underline{\phi}), \ldots, e_n(\underline{\phi})\}$ are the following:

$$\begin{pmatrix} y_1(\underline{\phi}, \underline{\psi}) \\ y_2(\underline{\phi}, \underline{\psi}) \\ y_3(\underline{\phi}, \underline{\psi}) \\ y_4(\underline{\phi}, \underline{\psi}) \\ \vdots \\ y_{n-1}(\underline{\phi}, \underline{\psi}) \\ y_n(\underline{\phi}, \underline{\psi}) \end{pmatrix} = \begin{pmatrix} e_2(\underline{\phi}) & e_3(\underline{\phi}) & \cdots & e_n(\underline{\phi}) \end{pmatrix} \begin{pmatrix} \cos(\psi_1) \\ \sin(\psi_1)\cos(\psi_2) \\ \sin(\psi_1)\sin(\psi_2)\cos(\psi_3) \\ \sin(\psi_1)\sin(\psi_2)\sin(\psi_3)\cos(\phi_4) \\ \vdots \\ \sin(\psi_1)\cdots\sin(\psi_{n-3})\cos(\psi_{n-2}) \\ \sin(\psi_1)\cdots\sin(\psi_{n-2}) \end{pmatrix}.$$

In particular,

$$\begin{aligned} y_1(\underline{\phi}, \underline{\psi}) &= -\sin(\phi_1)\cos(\psi_1), \\ y_2(\underline{\phi}, \underline{\psi}) &= \cos(\phi_1)\cos(\phi_2)\cos(\psi_1) - \sin(\phi_2)\sin(\psi_1)\cos(\psi_2), \\ y_3(\underline{\phi}, \underline{\psi}) &= \cos(\phi_1)\sin(\phi_2)\cos(\phi_3)\cos(\psi_1) + \cos(\phi_2)\cos(\phi_3)\sin(\psi_1)\cos(\psi_2) \\ &\qquad - \sin(\phi_3)\sin(\psi_1)\sin(\psi_2)\cos(\psi_3), \\ y_4(\underline{\phi}, \underline{\psi}) &= \cos(\phi_1)\sin(\phi_2)\sin(\phi_3)\cos(\phi_4)\cos(\psi_1) + \cos(\phi_2)\sin(\phi_3)\cos(\phi_4)\sin(\psi_1)\cos(\psi_2) \\ &\quad + \cos(\phi_3)\cos(\phi_4)\sin(\psi_1)\sin(\psi_2)\cos(\psi_3) - \sin(\phi_4)\sin(\psi_1)\sin(\psi_2)\sin(\psi_3)\cos(\psi_4). \end{aligned}$$

Integrals w.r.t. σ on T

3. In this new coordinate system T is parametrized by

$$(\underline{\phi}, \underline{\psi}) \mapsto (x(\underline{\phi}), y(\underline{\phi}, \underline{\psi})), \quad \text{where } (\underline{\phi}, \underline{\psi}) \in \big([0,\pi]^{n-2} \times [0,2\pi]\big) \times \big([0,\pi]^{n-3} \times [0,2\pi]\big).$$

We have

$$\int_T g(x,y)\,\mathrm{d}\sigma = \underbrace{\int_0^\pi \cdots \int_0^\pi}_{n-2} \int_0^{2\pi} \underbrace{\int_0^\pi \cdots \int_0^\pi}_{n-3} \int_0^{2\pi} g(x(\underline{\phi}), y(\underline{\phi}, \underline{\psi}))\, V_n(\underline{\phi}, \underline{\psi})\,\mathrm{d}\underline{\phi}\mathrm{d}\underline{\psi},$$

where

$$V_n(\underline{\phi}, \underline{\psi}) = \underbrace{\frac{1}{S_n} \prod_{i=1}^{n-2} \sin(\phi_i)^{n-1-i}}_{\substack{\text{usual Jacobian of} \\ n\text{-spherical coordinates}}} \cdot \underbrace{\frac{1}{S_{n-1}} \prod_{i=1}^{n-3} \sin(\psi_i)^{n-2-i}}_{\substack{\text{usual Jacobian of} \\ (n-1)\text{-spherical coordinates}}},$$

with

$$S_n = \underbrace{\int_0^\pi \cdots \int_0^\pi}_{n-2} \int_0^{2\pi} \prod_{i=1}^{n-2} \sin(\phi_i)^{n-1-i}\,\mathrm{d}\underline{\phi},$$

$$S_{n-1} = \underbrace{\int_0^\pi \cdots \int_0^\pi}_{n-3} \int_0^{2\pi} \prod_{i=1}^{n-3} \sin(\psi_i)^{n-2-i}\,\mathrm{d}\underline{\psi}.$$

RHI for symmetric g

1. WLOG:

$$g(\mathrm{x},\mathrm{y}) = d_1\mathrm{x}_1\mathrm{y}_1 + d_2\mathrm{x}_2\mathrm{y}_2 + \ldots + d_n\mathrm{x}_n\mathrm{y}_n, \quad d_i \in \mathbb{R}.$$

2. RHI equivalent to:

$$(n-3)\Big(\sum_{i<j} d_i^2 d_j^2 (n-2) - 2 \sum_{\substack{i,j,k \\ \text{pairw.diff,} \\ j<k}} d_i^2 d_j d_k\Big) + 12 \sum_{i<j<k<l} d_i d_j d_k d_l \geq 0.$$

3. Induction on n starting with $n = 3$ and noticing that the inequality is invariant under

$$(d_1, \ldots, d_n) \mapsto (d_1 + a, \ldots, d_n + a), \quad \text{where } a \in \mathbb{R}.$$

   WLOG:

$$d_1 \geq d_2 \geq \ldots \geq d_n \geq d_{n+1} = 0.$$

RHI for symmetric g

4. $n \mapsto n+1$:

$$(n-2)\Big(\sum_{i<j\leq n} d_i^2 d_j^2 (n-1) - 2 \sum_{\substack{i,j,k\leq n \\ \text{pairw.diff,} \\ j<k}} d_i^2 d_j d_k\Big) + 12 \sum_{i<j<k<l\leq n} d_i d_j d_k d_l \geq 0.$$

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Equivalently

$$\underbrace{(n-3)\Big(\sum_{i<j\leq n} d_i^2 d_j^2 (n-2) - 2 \sum_{\substack{i,j,k\leq n\\ \text{pairw.diff},\\ j<k}} d_i^2 d_j d_k\Big) + 12 \sum_{i<j<k<l\leq n} d_i d_j d_k d_l}_{\geq 0 \text{ by the induction hypothesis}}$$

$$+2 \underbrace{\Big(\sum_{i<j\leq n} d_i^2 d_j^2 (n-2) - \sum_{\substack{i,j,k\leq n\\ \text{pairw.diff},\\ j<k}} d_i^2 d_j d_k\Big)}_{\geq 0 \text{ by Muirhead's inequality for } (2,2,0,\ldots,0)\succ(2,1,1,0,\ldots,0)} \geq 0.$$

Procedure applied to Problem C.1

1. $\mathbb{R}[\mathrm{x}]_{4,e}$ is equipped with the natural L^2 inner product

$$\langle f, g \rangle = \int_{S^{n-1}} fg \, \mathrm{d}\sigma,$$

   where σ is the rotation invariant probability measures on the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

2. $\mathcal{H}$ is the affine hyperplane of forms from $\mathbb{R}[\mathrm{x}]_{4,e}$ of average 1 on S^{n-1}:

$$\mathcal{H} = \left\{ f \in \mathbb{R}[\mathrm{x}]_{4,e} \colon \int_{S^{n-1}} f \, \mathrm{d}\sigma = 1 \right\}.$$

3. $z := \left(\sum_{i=1}^{n} x_i^2 \right)^2$ and thus

$$\mathcal{M} = \mathcal{H} - z = \left\{ f \in \mathbb{R}[\mathrm{x}]_{4,e} \colon \int_{S^{n-1}} f \, \mathrm{d}\sigma = 0 \right\}.$$

4. Let μ be the pushforward of the Lebesgue measure on $\mathbb{R}^{\dim \mathcal{M}}$ to $\mathcal{M}$.

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Here d stands for the differential inner product and $*$ for the dual,

$$\mathrm{LF} := \left\{ \mathrm{pr}(f) \in \mathbb{R}[\mathrm{x}]_{4,e} : f = \sum_i f_i^4 \quad \text{for some } f_i \in \mathbb{R}[\mathrm{x}]_1 \right\}$$

and $\mathrm{pr} : \mathbb{R}[\mathrm{x}]_4 \to \mathbb{R}[\mathrm{x}]_{4,e}$ is the projection defined by:

$$\mathrm{pr}\left(\sum_{1 \le i \le j \le k \le \ell \le n} a_{ijk\ell} x_i x_j x_k x_\ell \right) = \sum_{1 \le i \le j \le n} a_{iijj} x_i^2 x_j^2 . \tag{1}$$

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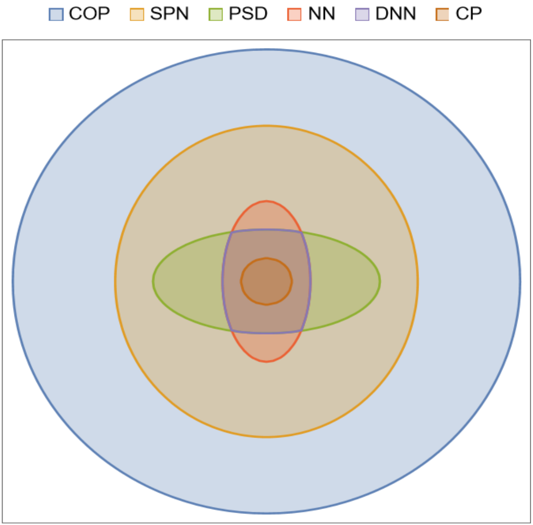
Observation 3: $\widetilde{\mathrm{LF}} \subseteq \widetilde{\mathrm{NN}} \subseteq 4(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}})$.

Cones in question

Compact bases of the cones



Perspective: Use results of real algebraic geometry, convex analysis and harmonic analysis to estimate the volumes from both sides.

Blaschke-Santaló inequality and its reverse

$\langle \cdot, \cdot \rangle$	...	the inner product on $\mathbb{R}^n$
B	...	the unit ball w.r.t. $\langle \cdot, \cdot \rangle$
K	...	a bounded convex set with a non-empty interior in $\mathbb{R}^n$
K°	...	the polar dual of a set $K \subseteq \mathbb{R}^n$:

$$K^\circ = \{y \in \mathbb{R}^n \colon \langle x, y \rangle \le 1 \quad \forall x \in K\}$$

Theorem (Bourgain, Milman, '87, Kuperberg, 2008; Blaschke, 1917, Santaló, 49')

If K is 'central enough', then

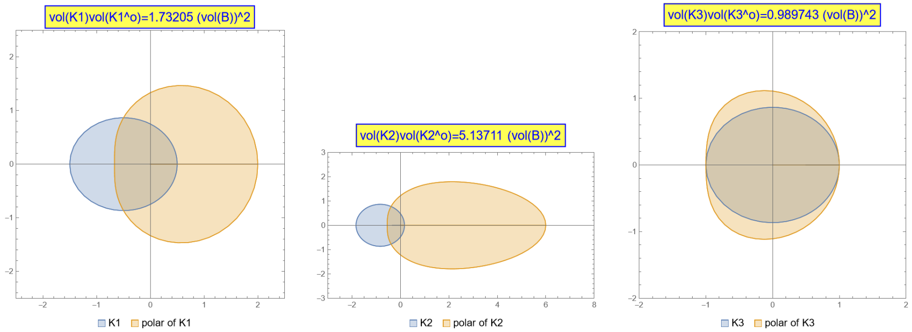
$$4^{-n}(\mathrm{Vol}(B))^2 \le \mathrm{Vol}(K)\,\mathrm{Vol}(K^\circ) \le (\mathrm{Vol}(B))^2,$$

Remark: The left inequality holds also without the centrality assumption, but with the origin in the interior.

Blaschke-Santaló inequality and its reverse

Geometric picture

K_1 ... the convex hull of the ellipse with a polar equation $r(\varphi) = \frac{3}{4}(1 + \frac{1}{2}\cos\varphi)^{-1}$,
$K_2 = K_1 - (\frac{1}{3}, 0)$, $K_3 = K_1 + (\frac{1}{2}, 0)$,



- The set K_1 is centered in different points on each of the pictures. The first two centers are not close enough to the origin for the BS to hold, while in the third one it is.
- The translation of the body (i.e., Santaló point) so that the BS holds is difficult to determine, unless the body has enough symmetries, fixing only one point which then must be the Santaló one.

The differential (also apolar) inner product

From Observation 1

For

$$f(\mathrm{x}) = \sum_{1 \leq i,j,k,\ell \leq n} a_{ijk\ell} x_i x_j x_k x_\ell \in \mathbb{R}[\mathrm{x}]_4$$

the differential operator $D_f : \mathbb{R}[\mathrm{x}]_4 \to \mathbb{R}$ is defined by

$$D_f(g) = \sum_{1 \leq i,j,k,\ell \leq n} a_{ijk\ell} \frac{\partial^4 g}{\partial x_i \partial x_j \partial x_k \partial x_\ell}.$$

The differential inner product on $\mathbb{R}[\mathrm{x}]_4$ is given by

$$\langle f, g \rangle_d = D_f(g).$$

Blaschke-Santaló inequality and its reverse in $\langle \cdot, \cdot \rangle_d$

For a cone $K \subseteq \mathbb{R}[\mathrm{x}]_{4,e}$ let K_d^* be its dual in $\langle \cdot, \cdot \rangle_d$:

$$K_d^* = \{f \in \mathbb{R}[\mathrm{x}]_{4,e} \colon \langle f, g \rangle_d \geq 0 \quad \forall g \in K\}$$

Theorem (BS$_d$ inequality and its reverse; Blekherman, 06')

Let K be any of the cones from Problem C.1. Then

$$\frac{1}{2n^2} \underbrace{\leq}_{n \geq 5} \frac{2}{(n+4)(n+6)} \leq \mathrm{vrad}(\widetilde{K})\,\mathrm{vrad}(\widetilde{K_d^*}).$$

Moreover, if $\widetilde{K}$ is 'central enough', then

$$\mathrm{vrad}(\widetilde{K})\,\mathrm{vrad}(\widetilde{K_d^*}) \leq \left(\frac{8}{(n+4)(n+6)}\right)^{1-\frac{2n-1}{n^2+n-1}} \underbrace{\leq}_{n \geq 5} \frac{9}{n^2}.$$

The proof uses representation theory, i.e., $\mathrm{SO}(n)$ acting on $\mathbb{R}[\mathrm{x}]_{4,e}$ by rotation of coordinates.

Observation 3: $\widetilde{\mathsf{NN}} \subseteq 4(\widetilde{\mathsf{CP}} - \widetilde{\mathsf{CP}})$

Follows from $2ab = (a+b)^2 - a^2 - b^2$

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Let $r = \left(\sum_{k=1}^{n} x_k^2\right)^2$. The extreme points of $\widetilde{\mathrm{NN}}$ are of two types:

$$\frac{n(n+2)}{3} x_i^4 - r \qquad \text{and} \qquad n(n+2)x_i^2 x_j^2 - r, \;\; i \neq j.$$

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Let $r = \left(\sum_{k=1}^{n} x_k^2\right)^2$. The extreme points of $\widetilde{\mathrm{NN}}$ are of two types:

$$\frac{n(n+2)}{3} x_i^4 - r \qquad \text{and} \qquad n(n+2)x_i^2 x_j^2 - r, \; i \neq j.$$

The first type clearly belong to $\widetilde{\mathrm{CP}}$, while the second type to $4(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}})$:

$$\begin{aligned}
& n(n+2)x_i^2 x_j^2 - r = \\
&= \frac{n(n+2)}{2}\left((x_i^2 + x_j^2)^2 - x_i^4 - x_j^4\right) - r \\
&= 4\underbrace{\left(\frac{n(n+2)}{8}(x_i^2 + x_j^2)^2) - r\right)}_{p_1} - \frac{3}{2}\underbrace{\left(\frac{n(n+2)}{3}x_i^4 - r\right)}_{p_2} - \frac{3}{2}\underbrace{\left(\frac{n(n+2)}{3}x_j^4 - r\right)}_{p_3} \\
&= p_1 + \frac{3}{2}(p_1 - p_2) + \frac{3}{2}(p_1 - p_3) \\
&\in \widetilde{\mathrm{CP}} + \frac{3}{2}(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}}) + \frac{3}{2}(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}}) \subseteq 4(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}}).
\end{aligned}$$

Roger's-Shepard inequality

Crucial for Observation 3 to be applicable

K ... a bounded convex set with a non-empty interior in $\mathbb{R}^n$

The difference body $\mathrm{Diff}(K)$ of K is defined by

$$\mathrm{Diff}(K) := K - K.$$

Theorem (Roger's-Shepard inequality, 1957)

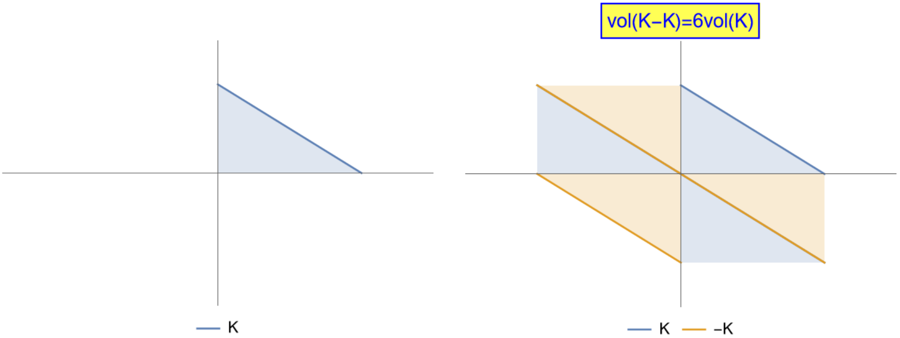
$$\mathrm{Vol}(\mathrm{Diff}(K)) \leq \binom{2n}{n} \mathrm{Vol}(K)$$

Hence,

$$\mathrm{vrad}(\mathrm{Diff}(K)) \leq 4\,\mathrm{vrad}(K).$$

Roger's-Shepard inequality

Geometric picture



Remark: Working with Diff K instead of K is one of the crucial steps to obtain our volume estimates for the problem of copositive matrices.

Proof of the gap for Problem C.1

Theorem For all $K \in \mathcal{C} := \{\mathsf{POS}, \mathsf{SOS}, \mathsf{NN}, \mathsf{PSD}, \mathsf{DNN}, \mathsf{LF}, \mathsf{CP}\}$ we have that

$$\mathrm{vrad}(\widetilde{K}) = \Theta(n^{-1}). \tag{2}$$

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$$\mathrm{vrad}(\widetilde{K}) = \Theta(n^{-1}). \tag{2}$$

Proof:

1. By $\widetilde{(\mathrm{NN})^*_d} = \widetilde{\mathrm{NN}}$ and the reverse BS_d inequality:

$$\frac{1}{2n^2} \leq \left(\mathrm{vrad}(\widetilde{\mathrm{NN}})\right)^2.$$

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2. By $\widetilde{\mathrm{CP}} \subseteq \widetilde{\mathrm{NN}} \subseteq 4(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}})$ and the RS inequality:

$$\frac{1}{16\sqrt{2}n} \leq \frac{1}{16}\mathrm{vrad}(\widetilde{\mathrm{NN}}) \leq \mathrm{vrad}(\widetilde{\mathrm{CP}}), \tag{3}$$

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$$\frac{1}{16\sqrt{2}n} \leq \frac{1}{16}\mathrm{vrad}(\widetilde{\mathrm{NN}}) \leq \mathrm{vrad}(\widetilde{\mathrm{CP}}), \tag{3}$$

3. By $\widetilde{(\mathrm{LF})_d^*} = \widetilde{\mathrm{POS}}$ and the BS_d inequality:

$$\mathrm{vrad}(\widetilde{\mathrm{POS}}) \leq \frac{9}{n^2}(\mathrm{vrad}(\widetilde{\mathrm{LF}}))^{-1} \leq \frac{9}{n^2}(\mathrm{vrad}(\widetilde{\mathrm{CP}}))^{-1} \leq 2^4 \cdot 3^2 \frac{1}{n}. \tag{4}$$

Proof of the gap for Problem C.1

Theorem For all $K \in \mathcal{C} := \{\mathrm{POS}, \mathrm{SOS}, \mathrm{NN}, \mathrm{PSD}, \mathrm{DNN}, \mathrm{LF}, \mathrm{CP}\}$ we have that

$$\mathrm{vrad}(\widetilde{K}) = \Theta(n^{-1}). \tag{2}$$

Proof:

1. By $\widetilde{(\mathrm{NN})^*_d} = \widetilde{\mathrm{NN}}$ and the reverse BS_d inequality:
$$\frac{1}{2n^2} \leq \left(\mathrm{vrad}(\widetilde{\mathrm{NN}})\right)^2.$$
2. By $\widetilde{\mathrm{CP}} \subseteq \widetilde{\mathrm{NN}} \subseteq 4(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}})$ and the RS inequality:
$$\frac{1}{16\sqrt{2}n} \leq \frac{1}{16}\mathrm{vrad}(\widetilde{\mathrm{NN}}) \leq \mathrm{vrad}(\widetilde{\mathrm{CP}}), \tag{3}$$
3. By $\widetilde{(\mathrm{LF})^*_d} = \widetilde{\mathrm{POS}}$ and the BS_d inequality:
$$\mathrm{vrad}(\widetilde{\mathrm{POS}}) \leq \frac{9}{n^2}(\mathrm{vrad}(\widetilde{\mathrm{LF}}))^{-1} \leq \frac{9}{n^2}(\mathrm{vrad}(\widetilde{\mathrm{CP}}))^{-1} \leq 2^4 \cdot 3^2 \frac{1}{n}. \tag{4}$$
4. Now by observing that
$$\mathrm{CP} \subseteq K \subseteq \mathrm{POS},$$
the inequalities (3) and (4) imply that for all cones $K \in \mathcal{C}$ the statement (2) holds. ∎

4. Algorithms and Examples

A.2. and B.2. (Cross)-Positive but not (Cross)-CP maps

Positive polynomials that are not SOS

Algorithm by Blekherman, Smith, Velasco, 2013

1. The setting:

   $X \subseteq \mathbb{P}^n \ldots$ a nondegenerate (not contained in a hyperplane),

   $\ldots$ totally-real (real points $X(\mathbb{R})$ are Zariski dense),

   $\ldots$ irreducible variety,

   $\ldots$ $\deg(X) > \operatorname{codim}(X) + 1$,

   $R = \mathbb{R}[x_0, \ldots, x_n]/I(X) \ldots$ the coordinate ring of X.

2. Step 1:
   - Choose linear forms $h_1, \ldots, h_{\dim(X)}$ intersecting in $\deg(X)$ distinct points with at least $\operatorname{codim}(X) + 1$ real and smooth ones, $p_1, \ldots, p_{\operatorname{codim}(X)+1}$.
   - Choose a linear form h_0 vanishing in $p_1, \ldots, p_{\operatorname{codim}(X)}$, but not in $p_{\operatorname{codim}(X)+1}$.
   - Let $I = \langle h_0, \ldots, h_m \rangle$.
3. Step 2: Choose a quadratic form $f \in R \setminus I^2$ vanishing of order > 1 in $p_1, \ldots, p_{\operatorname{codim}(X)}$.
4. Step 3: For $\delta > 0$ small enough, $\delta f + h_0^2 + \ldots + h_m^2$ is nonnegative on X but not SOS.

Positive but not sos biquadratic biforms

Algorithm

1. The setting:

$$X = \sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) \subseteq \mathbb{P}^{nm-1}, \quad \sigma_{n,m} \text{ Segre embedding}$$

$$\sigma_{n,m} : ([x_1 : \ldots : x_n], [y_1 : \ldots : y_m]) \mapsto [x_1 y_1 : x_1 y_2 : \ldots : x_n y_m],$$

$$\mathrm{z} = (z_{11}, z_{12}, \ldots, z_{1m}, \ldots, z_{nm}),$$

$I_{n,m} \ldots$ the ideal generated by 2×2 minors of $(z_{ij})_{i,j}$,

$\sigma^{\#}_{n,m} : \mathbb{C}[\mathrm{z}]/I_{n,m} \to \mathbb{C}[\mathrm{x}, \mathrm{y}], \quad \sigma^{\#}_{n,m}(z_{ij} + I_{n,m}) = x_i y_j$ ring homomorphism,

$\dim(X) = n + m - 2, \ \mathrm{codim}(X) = (n-1)(m-1).$

2. Step 1:
   - Choose $\mathrm{codim}(X) + 1$ random points $x^{(i)} \in \mathbb{R}^n$, $y^{(i)} \in \mathbb{R}^m$ and compute $z^{(i)} = x^{(i)} \otimes y^{(i)} \in \mathbb{R}^{nm}$.
   - Choose $\dim(X) = n + m - 2$ random vectors $v_1, \ldots v_{\dim(X)} \in \mathbb{R}^{nm}$ from the kernel of the matrix

     $$\begin{pmatrix} z^{(1)} & \ldots & z^{(\mathrm{codim}(X)+1)} \end{pmatrix}^*$$

     and define

     $$h_j(\mathrm{z}) = v_j^* \cdot \mathrm{z} \in \mathbb{R}[\mathrm{z}] \quad \text{for } j = 1, \ldots, \dim(X).$$

   - Let $I = \langle h_0, \ldots, h_{\dim(X)} \rangle$.

Positive but not sos biquadratic biforms

Algorithm

3. Step 2:
   3.1 Let $g_1(\mathrm{z}), \ldots, g_{\binom{n}{2}\binom{m}{2}}(\mathrm{z})$ be the generators of the ideal $I_{n,m}$. For each $i = 1, \ldots, \mathrm{codim}(X)$ compute a basis $\{w_1^{(i)}, \ldots, w_{\dim(X)+1}^{(i)}\} \subseteq \mathbb{R}^{nm}$ of the kernel of the matrix
   $$\begin{pmatrix} \nabla g_1(z^{(i)}) & \cdots & \nabla g_{\binom{n}{2}\binom{m}{2}}(z^{(i)}) \end{pmatrix}^* .$$
   3.2 Choose a random vector $v \in \mathbb{R}^{n^2 m^2}$ from the intersection of the kernels of the matrices
   $$\begin{pmatrix} z^{(i)} \otimes w_1^{(i)} & \cdots & z^{(i)} \otimes w_{\dim(X)+1}^{(i)} \end{pmatrix}^* \quad \text{for } i = 1, \ldots, \mathrm{codim}(X)$$
   with the kernels of the matrices
   $$\left(\mathrm{e}_i \otimes \mathrm{e}_j - \mathrm{e}_j \otimes \mathrm{e}_i\right)^* \quad \text{for } 1 \leq i < j \leq nm$$
   and define
   $$f(\mathrm{z}) = v^* \cdot (\mathrm{z} \otimes \mathrm{z}) \in \mathbb{R}[\mathrm{z}]/I_{n,m}.$$
4. Step 3: Calculate the greatest $\delta_0 > 0$ such that $\delta_0 f + \sum_{i=0}^{\mathrm{codim}(X)} h_i^2$ is nonnegative on $V_{\mathbb{R}}(I_{n,m})$. Then
   $$(\delta f + \sum_i h_i^2)(\mathrm{z}) \in \mathrm{POS} \setminus \mathrm{SOS} \quad \text{for every } 0 < \delta < \delta_0.$$

Positive but not sos biquadratic biforms

Example

$$p_\Phi(x,y) = 104x_1^2y_1^2 + 283x_1^2y_2^2 + 18x_1^2y_3^2 - 310x_1^2y_1y_2 + 18x_1^2y_1y_3 + 4x_1^2y_2y_3 +$$
$$310x_1x_2y_1^2 - 18x_1x_3y_1^2 - 16x_1x_2y_2^2 + 52x_1x_3y_2^2 + 4x_1x_2y_3^2 - 26x_1x_3y_3^2$$
$$- 610x_1x_2y_1y_2 - 44x_1x_3y_1y_2 + 36x_1x_2y_1y_3 - 200x_1x_3y_1y_3 - 44x_1x_2y_2y_3$$
$$+ 322x_1x_3y_2y_3 + 285x_2^2y_1^2 + 16x_3^2y_1^2 + 4x_2x_3y_1^2 + 63x_2^2y_2^2 + 9x_3^2y_2^2 + 20x_2x_3y_2^2$$
$$+ 7x_2^2y_3^2 + 125x_3^2y_3^2 - 20x_2x_3y_3^2 + 16x_2^2y_1y_2 + 4x_3^2y_1y_2 - 60x_2x_3y_1y_2$$
$$+ 52x_2^2y_1y_3 + 26x_3^2y_1y_3 - 330x_2x_3y_1y_3 - 20x_2^2y_2y_3 + 20x_3^2y_2y_3 - 100x_2x_3y_2y_3.$$

Positive but not CP map

Example $\Phi : \mathbb{S}_3 \to \mathbb{S}_3$

$$\Phi(E_{11}) = \begin{bmatrix} 104 & -155 & 9 \\ -155 & 283 & 2 \\ 9 & 2 & 18 \end{bmatrix}, \quad \Phi(E_{22}) = \begin{bmatrix} 285 & 8 & 26 \\ 8 & 63 & -10 \\ 26 & -10 & 7 \end{bmatrix},$$

$$\Phi(E_{33}) = \begin{bmatrix} 16 & 2 & 13 \\ 2 & 9 & 10 \\ 13 & 10 & 125 \end{bmatrix}, \quad \Phi(E_{12} + E_{21}) = \begin{bmatrix} 310 & -305 & 18 \\ -305 & -16 & -22 \\ 18 & -22 & 4 \end{bmatrix},$$

$$\Phi(E_{13}+E_{31}) = \begin{bmatrix} -18 & -22 & -100 \\ -22 & 52 & 161 \\ -100 & 161 & -26 \end{bmatrix}, \quad \Phi(E_{23}+E_{32}) = \begin{bmatrix} 4 & -30 & -165 \\ -30 & 20 & -50 \\ -165 & -50 & -20 \end{bmatrix}.$$

C.2. Exceptional DNN and exceptional COP matrices

DNN matrices that are not CP of size $n \geq 5$

1. The setting:

   $L^2[0,1] \dots$ an ambient space,

   $\mathcal{B} := \{1\} \cup \{\sqrt{2}\cos(2k\pi)\colon k \in \mathbb{N}\} \cup \{\sqrt{2}\sin(2k\pi)\colon k \in \mathbb{N}\} \dots$ a basis,

   $M_f : L^2[0,1] \to L^2[0,1],\ M_f(g) = fg \dots$ the multiplication operator.

DNN matrices that are not CP of size $n \geq 5$

Algorithm

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   $M_f : L^2[0,1] \to L^2[0,1],\ M_f(g) = fg \ldots$ the multiplication operator.

2. The idea: Find a closed infinite dimensional subspace $\mathcal{H}$ and $f \in \mathcal{H}$ such that

   $$\boxed{M_f^{\mathcal{H}} := P_{\mathcal{H}} M_f P_{\mathcal{H}}}$$

   has all finite principal submatrices DNN but not CP, where $P_{\mathcal{H}} : L^2[0,1] \to \mathcal{H}$ is the orthogonal projection onto $\mathcal{H}$.

DNN matrices that are not CP of size $n \geq 5$

Algorithm

1. The setting:

   $L^2[0,1]\ldots$ an ambient space,

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2. The idea: Find a closed infinite dimensional subspace $\mathcal{H}$ and $f \in \mathcal{H}$ such that

   $$M_f^{\mathcal{H}} := P_{\mathcal{H}} M_f P_{\mathcal{H}}$$

   has all finite principal submatrices DNN but not CP, where $P_{\mathcal{H}} : L^2[0,1] \to \mathcal{H}$ is the orthogonal projection onto $\mathcal{H}$.

3. Choice of $\mathcal{H}$ and $f \in \mathcal{H}$:

   $\mathcal{H} \subseteq L^2[0,1]\ldots$ a closed subspace spanned by $\cos(2k\pi), k \in \mathbb{N}_0$,

   $$f = 1 + 2\sum_{k=1}^{m} a_k \cos(2k\pi), \quad m \in \mathbb{N},$$

DNN matrices that are not CP of size $n \geq 5$

4. Certificates:
   4.1 NN: $a_1 \geq 0, \ldots, a_m \geq 0$.
   4.2 PSD: $f = \sum_i h_i^2$.
   4.3 Not CP:

$\mathcal{H}_n \ldots$ a subspace spanned by $1, \cos(2\pi), \ldots, \cos(2(n-1)\pi)$,

$P_n : \mathcal{H} \to \mathcal{H}_n \ldots$ the orthogonal projection onto $\mathcal{H}_n$,

$A^{(n)} := P_n M_f^{\mathcal{H}} P_n$,

$$H = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix} \in \mathrm{COP} \setminus \mathrm{SPN},$$

(Horn matrix; Hall, Newman, 1963)

We demand

$$\langle A^{(5)}, H \rangle < 0,$$

with $\langle \cdot, \cdot \rangle$ the usual Frobenius inner product on symmetric matrices.

DNN matrices that are not CP of size $n \geq 5$

Justification of the certificates

1. NN is certified by the following equation:

$$\int_0^1 \cos(2j\pi x)\cos(2k\pi x)\cos(2\ell\pi x)dx = \begin{cases} \frac{1}{2}, & \text{if } j = \ell, k = 0, \\ \frac{1}{4}, & \text{if } k \neq 0 \text{ and } j \in \{\ell + k, \ell - k\}, \\ 0, & \text{otherwise.} \end{cases}$$

In particular,

$$A^{(5)} = \begin{pmatrix} 1 & \sqrt{2}a_1 & \sqrt{2}a_2 & \sqrt{2}a_3 & \sqrt{2}a_4 \\ \sqrt{2}a_1 & a_2 + 1 & a_1 + a_3 & a_2 + a_4 & a_3 + a_5 \\ \sqrt{2}a_2 & a_1 + a_3 & a_4 + 1 & a_1 + a_5 & a_2 + a_6 \\ \sqrt{2}a_3 & a_2 + a_4 & a_1 + a_5 & 1 + a_6 & a_1 \\ \sqrt{2}a_4 & a_3 + a_5 & a_2 + a_6 & a_1 & 1 \end{pmatrix}.$$

2. PSD is certified by

$$M_f^{\mathcal{H}} = \sum_i \left(M_{h_i}^{\mathcal{H}}\right)^2 = \sum_i M_{h_i}^{\mathcal{H}} \left(M_{h_i}^{\mathcal{H}}\right)^*.$$

3. Not CP is certified by

$$\mathrm{COP}^* = \mathrm{CP} \quad \text{(in the Frobenius inner product).}$$

DNN matrices that are not CP of size $n \geq 5$

The feasibility semidefinite program (SDP) implements the algorithm above:

$$\text{tr}(A^{(5)}H) = -\frac{1}{20},$$

$$f = v^{\mathsf{T}}Bv \quad \text{with} \quad B \succeq 0 \text{ of size } 4 \times 4,$$

$$a_i \geq 0, \quad i = 1, \ldots, 6,$$

where

$$v^T = \begin{pmatrix} 1 & \cos(2\pi x) & \cos(4\pi x) & \cos(6\pi x) \end{pmatrix}.$$

DNN matrices that are not CP of size $n \geq 5$

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where

$$v^T = \begin{pmatrix} 1 & \cos(2\pi x) & \cos(4\pi x) & \cos(6\pi x) \end{pmatrix}.$$

Solving this SDP, we get

$$A^{(5)} = \begin{pmatrix} 1 & \frac{16\sqrt{2}}{27} & \frac{\sqrt{2}}{123} & \frac{1}{147\sqrt{2}} & \frac{5\sqrt{2}}{21} \\ \frac{16\sqrt{2}}{27} & \frac{124}{123} & \frac{1577}{2646} & \frac{212}{861} & \frac{1205}{8526} \\ \frac{\sqrt{2}}{123} & \frac{1577}{2646} & \frac{26}{21} & \frac{572}{783} & \frac{1777340\sqrt{2}-2413803}{3254580} \\ \frac{1}{147\sqrt{2}} & \frac{212}{861} & \frac{572}{783} & \frac{1777340\sqrt{2}+814317}{3254580} & \frac{16}{27} \\ \frac{5\sqrt{2}}{21} & \frac{1205}{8526} & \frac{1777340\sqrt{2}-2413803}{3254580} & \frac{16}{27} & 1 \end{pmatrix}.$$

COP matrices that are not SPN of size $n \geq 5$

Let $A^{(n)}$ be a DNN not CP matrix. To obtain a matrix $C \in \text{COP} \setminus \text{SPN}$ of size $n \times n$ we demand

$$\langle A^{(n)}, C \rangle < 0, \tag{5}$$

$$\Big(\sum_{i=1}^{n} x_i^2\Big)^k \big((\mathrm{x}^2)^T C \mathrm{x}^2\big) \quad \text{is SOS for some } k \in \mathbb{N}. \tag{6}$$

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Let $A^{(n)}$ be a DNN not CP matrix. To obtain a matrix $C \in \mathrm{COP} \setminus \mathrm{SPN}$ of size $n \times n$ we demand

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(5) certifies C is not SPN due to

$$\mathrm{SPN}^* = \mathrm{DNN} \quad \text{(in the Frobenius inner product)},$$

while (6) certifies C is COP.

COP matrices that are not SPN of size $n \geq 5$

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(5) certifies C is not SPN due to

$$\text{SPN}^* = \text{DNN} \quad \text{(in the Frobenius inner product)},$$

while (6) certifies C is COP. This is again a feasibility SDP. Using $A^{(5)}$ as above we obtain (with $\langle A^{(5)}, C \rangle = -\frac{1}{10}$ and $k = 1$)

$$C = \begin{pmatrix} 17 & -\frac{91}{5} & \frac{33}{2} & \frac{38}{3} & -\frac{36}{5} \\ -\frac{91}{5} & \frac{59}{3} & -\frac{53}{4} & 8 & \frac{33}{4} \\ \frac{33}{2} & -\frac{53}{4} & \frac{39}{4} & -\frac{13}{2} & 8 \\ \frac{38}{3} & 8 & -\frac{13}{2} & \frac{16}{3} & -\frac{13}{3} \\ -\frac{36}{5} & \frac{33}{4} & 8 & -\frac{13}{3} & \frac{1373628701}{353935575} \end{pmatrix}.$$

Thank you for your attention!