Matrix Fejér-Riesz theorem with gaps

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Notation

R - the ring of complex polynomials $\mathbb{C}[x]$ ($x^* = \overline{x} = x$) or complex Laurent polynomials $\mathbb{C}[z, \frac{1}{z}]$ ($z^* = \overline{z} = \frac{1}{z}$)

$M_n(R)$ - matrix polynomials ($F^* = \overline{F}^T$)

$H_n(R)$ - hermitian matrix polynomials ($F^* = F$)

$\sum M_n(R)^2$ - sums of hermitian matrix squares, i.e. finite sums of the form

$$\sum A_i^* A_i, \quad \text{where } A_i \in M_n(R)$$

Matrix Fejér-Riesz theorem

Theorem (Fejér-Riesz theorem on $\mathbb{T}$)

Let

$$A(z) = \sum_{m=-N}^{N} A_m z^m \in M_n\Big(\mathbb{C}[z, \frac{1}{z}]\Big)$$

be a $n \times n$ matrix Laurent polynomial, such that $A(z)$ is positive semidefinite for every

$$z \in \mathbb{T} := \{z \in \mathbb{C} : \ |z| = 1\}.$$

Then there exists a matrix polynomial

$$B(z) = \sum_{m=0}^{N} B_m z^m \in M_n(\mathbb{C}[z]),$$

such that

$$A(z) = B(z)^* B(z).$$

Matrix Fejér-Riesz theorem

Theorem (Fejér-Riesz theorem on $\mathbb{R}$)

Let

$$F(x) = \sum_{m=0}^{2N} F_m x^m \in M_n(\mathbb{C}[x])$$

be a $n \times n$ matrix polynomial, such that $F(x)$ is positive semidefinite for every $x \in \mathbb{R}$. Then there exists a matrix polynomial

$$G(x) = \sum_{m=0}^{N} G_m x^m \in M_n(\mathbb{C}[x]),$$

such that

$$F(x) = G(x)^* G(x).$$

Many proofs of the matrix Fejér-Riesz theorem

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Main problem

Problem

1. *Characterize univariate matrix Laurent polynomials, which are positive semidefinite on a union of points and arcs in $\mathbb{T}$.*
2. *Characterize univariate matrix polynomials, which are positive semidefinite on a union of points and intervals (not necessarily bounded) in $\mathbb{R}$.*

Semialgebraic set and preordering

A *basic closed semialgebraic set* $K_S \subseteq \mathbb{R}$ associated to a finite subset

$$S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$$

is given by

$$K := K_S = \{x \in \mathbb{R} \colon g_j(x) \geq 0,\ j = 1, \ldots, s\}.$$

We define the *n-th matrix preordering* T_S^n by

$$T_S^n \quad := \quad \Big\{ \sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e \colon \sigma_e \in \sum M_n(\mathbb{C}[x])^2 \text{ for all } e \in \{0,1\}^s \Big\},$$

where $e = (e_1, \ldots, e_s)$ and $\underline{g}^e$ stands for $g_1^{e_1} \cdots g_s^{e_s}$.

Saturated preordering

Let $\mathrm{Pos}^n_{\succeq 0}(K_S)$ be the set of all $n \times n$ hermitian matrix polynomials, which are positive semidefinite on K_S, i.e.,

$$F \in \mathrm{Pos}^n_{\succeq 0}(K_S) \quad \Leftrightarrow \quad F(x) \succeq 0 \quad \forall x \in K_S.$$

Matrix preordering T^n_S is *saturated* if $T^n_S = \mathrm{Pos}^n_{\succeq 0}(K_S)$.

Saturated matrix preordering T^n_S is *boundedly saturated*, if every $F \in \mathrm{Pos}^n_{\succeq 0}(K_S)$ is of the form

$$F = \sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e,$$

where

$$\deg(\sigma_e \underline{g}^e) \leq \deg(F)$$

holds for every $e \in \{0,1\}^s$.

Natural description and scalar saturated preorderings

Let $K \subseteq \mathbb{R}$ be a basic closed semialgebraic set.

A set $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ is the *natural description* of K, if it satisfies the following conditions:

(a) If K has the least element a, then $x - a \in S$.

(b) If K has the greatest element a, then $a - x \in S$.

(c) For every $a \neq b \in K$, if $(a, b) \cap K = \emptyset$, then $(x - a)(x - b) \in S$.

(d) These are the only elements of S.

Theorem (Kuhlmann, Marshall, 2002)

If S is the natural description of K, then the preordering T_S^1 is boundedly saturated.

Matricial saturated preorderings

Theorem (Gohberg, Krein, 1958)

For $K = \mathbb{R}$, $T_{\emptyset}^{n}$ is boundedly saturated for every $n \in \mathbb{N}$.

Theorem (Dette, Studden, 2002)

For $K = K_{\{x,1-x\}} = [0,1]$, $T_{\{x,1-x\}}^{n}$ is boundedly saturated for every $n \in \mathbb{N}$.

Theorem (Schmüdgen, Savchuk, 2012)

For $K = K_{\{x\}} = [0,\infty)$, $T_{\{x\}}^{n}$ is boundedly saturated for every $n \in \mathbb{N}$.

Matricial saturated preorderings

Theorem (Compact Nichtnegativstellensatz; Z., 2016)

Let K be a compact semialgebraic set with the natural description S. Then T_S^n is saturated for every $n \in \mathbb{N}$.

Theorem (Non-compact Nichtnegativstellensatz; Z., 2016)

Suppose K be an unbounded basic closed semialgebraic set in $\mathbb{R}$ and S its natural description. Then, for a hermitian $F \in M_n(\mathbb{C}[x])$, the following are equivalent:

1. $F \in Pos^n_{\succeq 0}(K)$.
2. $(1+x^2)^k F \in T_S^n$ for some $k \in \mathbb{N} \cup \{0\}$.

Classification of non-compact sets K

K	T_S^n saturated
an unbounded interval	Yes
a union of an unbounded interval and an isolated point	?
a union of an unbounded interval and m isolated points with $m \geq 2$	No
a union of two unbounded intervals	Yes
a union of two unbounded intervals and an isolated point	?
a union of two unbounded intervals and m isolated points with $m \geq 2$	No
includes a bounded and an unbounded interval	No

Classification of non-compact sets K

Theorem (Union of an interval and point; Sun, Z., 2025)

Let $K = \{a\} \cup [b, c]$, $a, b, c \in \mathbb{R}$, $a < b < c$. Then $T^n_{\{x-a,(x-a)(x-b),c-x\}}$ is boundedly saturated for every $n \in \mathbb{N}$.

K	T^n_S sat.
an unbounded interval	Yes
a union of an unbounded interval and an isolated point	*Yes*
a union of an unbounded interval and m isolated points with $m \geq 2$	No
a union of two unbounded intervals	Yes
a union of two unbounded intervals and an isolated point	*Yes*
a union of two unbounded intervals and m isolated points with $m \geq 2$	No
includes a bounded and an unbounded interval	No

Proof of Compact Nichtnegativstellensatz

Proposition

Suppose K is a non-empty basic closed semialgebraic set in $\mathbb{R}$ and S a natural description of K. Then for every $F \in Pos^n_{\succeq 0}(K)$ and every $w \in \mathbb{C}$ there exists $h \in \mathbb{R}[x]$, such that $h(w) \neq 0$ and

$$h^2 F \in T^n_S.$$

Proof of Proposition.

The proof is by induction of the size of matrix polynomials n. We write

$$F(x) = p(x)^m G(x),$$

where

$$p(x) = \begin{cases} x - w, & w \in \mathbb{R} \\ (x-w)(x-\overline{w}), & w \notin \mathbb{R} \end{cases}, \quad m \in \mathbb{Z}_+, \; G(w) \neq 0.$$

Proof of Compact Nichtnegativstellensatz

Proof of Proposition.

Writing

$$G := \begin{bmatrix} a & \beta \\ \beta^* & C \end{bmatrix} \in \begin{bmatrix} \mathbb{R}[x] & M_{1,n-1}(\mathbb{C}[x]) \\ M_{n-1,1}(\mathbb{C}[x]) & H_{n-1}(\mathbb{C}[x]) \end{bmatrix},$$

it holds that

$$a^4 \cdot G = \begin{bmatrix} a & 0 \\ \beta^* & aI_{n-1} \end{bmatrix} \begin{bmatrix} a^3 & 0 \\ 0 & a(aC - \beta^*\beta) \end{bmatrix} \begin{bmatrix} a & \beta \\ 0 & aI_{n-1} \end{bmatrix},$$

$$\begin{bmatrix} a^3 & 0 \\ 0 & a(aC - \beta^*\beta) \end{bmatrix} = \begin{bmatrix} a & 0 \\ -\beta^* & aI_{n-1} \end{bmatrix} \cdot G \cdot \begin{bmatrix} a & -\beta \\ 0 & aI_{n-1} \end{bmatrix}.$$

Proof of Compact Nichtnegativstellensatz

Proof of Proposition.

WLOG: $a(w) \neq 0$ (otherwise use a permutation).

$$a^4 F = \begin{bmatrix} a & 0 \\ \beta^* & aI_{n-1} \end{bmatrix} \begin{bmatrix} d & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} a & \beta \\ 0 & aI_{n-1} \end{bmatrix},$$

$$\begin{bmatrix} d & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} a & 0 \\ -\beta^* & aI_{n-1} \end{bmatrix} F \begin{bmatrix} a & -\beta \\ 0 & aI_{n-1} \end{bmatrix},$$

where $d = p^m a^3 \in \mathbb{R}[x]$ and $D = p^m (aC - \beta^* \beta) \in H_{n-1}(\mathbb{C}[x])$. By the induction hypothesis, there exists $h_1 \in \mathbb{R}[x]$ with $h_1(w) \neq 0$, such that

$$h_1^2 D \in T_S^{n-1}.$$

Together with $h_1^2 d \in T_S^1$, it follows that

$$(a^2 h_1)^2 F \in T_S^n.$$

□

Getting rid of the denominator

To conclude the proof we need the following:

Proposition (Scheiderer, 2006)

Suppose R is a commutative ring with 1 and $\mathbb{Q} \subseteq R$. Let

$$\Phi : R \to C(K, \mathbb{R})$$

be a ring homomorphism, where K is a topological space which is compact and Hausdorff, and $\Phi(R)$ separates points in K. Suppose $f_1, \ldots, f_k \in R$ are such that

$$\langle f_1, \ldots, f_k \rangle = R \quad \textit{and} \quad \Phi(f_j) \geq 0, \ j = 1, \ldots, k.$$

Then there exist $s_1, \ldots, s_k \in R$ such that

$$s_1 f_1 + \ldots + s_k f_k = 1 \quad \textit{and} \quad \Phi(s_j) > 0, \ j = 1, \ldots, k.$$

Proof of Compact Nichtnegativstellensatz

We have

$$I := \langle h^2 \colon h \in \mathbb{R}[x], h^2 F \in T_S^n \rangle \underbrace{=}_{\text{“}h^2F-\text{proposition”}} \mathbb{R}[x].$$

By Scheiderer's result, there exist $s_1, \ldots, s_k \in \mathrm{Pos}^1_{\succ 0}(K)$ and $h_1, \ldots, h_k \in I$, such that

$$\sum_{j=1}^{k} s_j h_j^2 = 1.$$

Hence,

$$F = 1 \cdot F = \sum_{j=1}^{k} \underbrace{s_j}_{\in T_S^1} \underbrace{h_j^2 F}_{\in T_S^n} \in T_S^n,$$

which concludes the proof.

Counterexample for non-compact case

Example

The matrix polynomial

$$F(x) := \begin{bmatrix} x+2 & \sqrt{6} \\ \sqrt{6} & x^2 - 2x + 3 \end{bmatrix}$$

is positive semidefinite on $K := [-1, 0] \cup [1, \infty)$, but $F \notin T_S^2$, where S is the natural description of K.

Proof.

All the principal minors of F, i.e. $x+2$, $x^2 - 2x + 3$ and $\det(F) = x^3 - x$ are non-negative on K.
Suppose

$$F(x) = \sigma_0 + \sigma_1(x+1) + \sigma_2 x(x-1) + \sigma_3(x+1)x(x-1), \qquad (*)$$

where $\sigma_i \in \sum M_2(\mathbb{C}[x])^2$.

Counterexample for non-compact case

Proof.

After comparing degrees of both sides we conclude that $\sigma_3 = 0$, $\deg(\sigma_0) \leq 2$, $\deg(\sigma_1) = 0$, $\deg(\sigma_2) = 0$ and observing the monomial x^2 on both sides, it follows that $\sigma_2 = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$ for some $c \in [0, 1]$.

$(*)$ is equivalent to

$$F(x) - \sigma_2 x(x-1) = \sigma_0 + \sigma_1(x+1).$$

The right-hand side is positive semidefinite on $[-1, \infty)$. But the determinant of the left-hand side is

$$q(x) := -(-1+x)x(-1+2c+(-1+c)x).$$

Since $q \not\equiv 0$ and q cannot have double zeroes at $x = 0$ and $x = 1$, it is not non-negative on $[-1, \infty)$. Contradiction. □

Union of an interval and a point

Theorem (Sun, Z., 2025)

Let $K = \{a\} \cup [b, c]$, $a, b, c \in \mathbb{R}$, $a < b < c$. If $F \in Pos^n_{\succeq 0}(K)$ and:

$$\deg F = 2m, m \in \mathbb{N}, \quad \textit{then}$$

$$F(x) = \underbrace{F_0(x)}_{degree \leq \deg F} + \underbrace{(x-a)(x-b)F_1(x)}_{degree \leq \deg F} + \underbrace{(x-a)(c-x)F_2(x)}_{degree \leq \deg F},$$

$$F_i \in \sum M_n(\mathbb{R}[x])^2.$$

$$\deg F = 2m-1, m \in \mathbb{N}, \quad \textit{then}$$

$$F(x) = \underbrace{(x-a)F_0(x)}_{degree \leq \deg F} + \underbrace{(c-x)F_1(x)}_{degree \leq \deg F} + \underbrace{(x-a)^2(x-b)F_2(x)}_{degree \leq \deg F} +$$

$$+ \underbrace{(x-a)(x-b)(c-x)F_3(x)}_{degree \leq \deg F}, \quad F_i \in \sum M_n(\mathbb{R}[x])^2.$$

Proof is done on the dual side by solving the corresponding truncated matrix moment problem.

Positive matrix measures

Let $K \subseteq \mathbb{R}$ be a closed set and $\mathrm{Bor}(K)$ the Borel σ-algebra of K. We call

$$\mu := (\mu_{ij})_{i,j=1}^n : \mathrm{Bor}(K) \to \mathbb{S}_n$$

a $n \times n$ *positive Borel matrix-valued measure* supported on K if:

1. $\mu_{ij} : \mathrm{Bor}(K) \to \mathbb{R}$ is a real measure for every $i, j = 1, \ldots, n$ and
2. $\mu(\Delta) \succeq 0$ for every $\Delta \in \mathrm{Bor}(K)$.

Let $\tau := \mathrm{tr}(\mu) = \sum_{i=1}^n \mu_{ii}$ denote the *trace measure*. A polynomial $f \in \mathbb{R}[x]_{\leq k}$ is μ-integrable if $f \in L^1(\tau)$. We define its integral by

$$\int_K f \, d\mu = \Big(\int_K f \, d\mu_{ij} \Big)_{i,j=1}^n .$$

Truncated matrix-valued moment problem

Let $k, n \in \mathbb{N}$. Given a linear mapping

$$L : \mathbb{R}[x]_{\leq k} \to \mathbb{S}_n,$$

the *truncated matrix-valued moment problem* supported on K asks to characterize the existence of a $\mathbb{S}_n$-valued positive matrix measure μ such that

$$L(f) = \int_K f \, d\mu \quad \text{for every} \quad f \in \mathbb{R}[x]_{\leq k}.$$

Equivalently, one can define L by a sequence of its values on monomials x^i, $i = 0, \ldots, k$, which we denote by $\Gamma_i := L(x^i)$. We write

$$\Gamma := (\Gamma_0, \Gamma_1, \ldots, \Gamma_k) \in \mathbb{S}_n^{k+1}.$$

Univariate Compact Matricial Truncated Riesz-Haviland

Proposition

Let $n, k \in \mathbb{N}$, $\Gamma = (\Gamma_0, \ldots, \Gamma_k) \in \mathbb{S}_n^{k+1}$ *and* K *a compact set. The following statements are equivalent:*

1. Γ *has a positive matrix measure supported on* K.
2. $\sum_{i=0}^{k} A_i x^i \in Pos^n_{\succeq 0}(K)$ *implies that* $\sum_{i=0}^{k} \mathrm{tr}(\Gamma_i A_i) \geq 0$.

$$\sum_{i=0}^{k} t^i A_i \succeq 0 \quad \text{for all } t \in K$$

$$\iff \sum_{i=0}^{k} t^i a^t A_i a \geq 0 \quad \text{for all } a \in \mathbb{R}^n \text{ and } t \in K$$

$$\iff \sum_{i=0}^{k} \mathrm{tr}\left(A_i\, t^i aa^t\right) \geq 0 \quad \text{for all } a \in \mathbb{R}^n \text{ and } t \in K$$

$$\iff \sum_{i=0}^{k} \mathrm{tr}(A_i \Gamma_i) \geq 0 \quad \text{for all moment sequences } (\Gamma_0, \ldots, \Gamma_k).$$

Moment matrix

For $m, k \in \mathbb{N}$, $m \leq \frac{k}{2}$ we denote by

$$\mathcal{M}_m = \left(\Gamma_{i+j-2}\right)_{i,j=1}^{m+1} = \begin{bmatrix} \Gamma_0 & \Gamma_1 & \Gamma_2 & \cdots & \Gamma_m \\ \Gamma_1 & \Gamma_2 & & & \Gamma_{m+1} \\ \Gamma_2 & & & & \vdots \\ \vdots & & & & \Gamma_{2m-1} \\ \Gamma_m & \Gamma_{m+1} & \cdots & \Gamma_{2m-1} & \Gamma_{2m} \end{bmatrix}$$

the *m–th truncated moment matrix.*

Localizing moment matrices

Fix $f \in \mathbb{R}[x]_{\leq k}$ and write

$$\Gamma_i^{(f)} := L(fx^i).$$

An *f–localizing ℓ-th truncated moment matrix $\mathcal{H}_f$* is

$$\mathcal{H}_f(\ell) := \left(\Gamma_{i+j-2}^{(f)}\right)_{i,j=1}^{\ell+1} = \begin{bmatrix} \Gamma_0^{(f)} & \Gamma_1^{(f)} & \Gamma_2^{(f)} & \cdots & \Gamma_\ell^{(f)} \\ \Gamma_1^{(f)} & \Gamma_2^{(f)} & & & \Gamma_{\ell+1}^{(f)} \\ \Gamma_2^{(f)} & & & & \vdots \\ \vdots & & & & \Gamma_{2\ell-1}^{(f)} \\ \Gamma_\ell^{(f)} & \Gamma_{\ell+1}^{(f)} & \cdots & \Gamma_{2\ell-1}^{(f)} & \Gamma_{2\ell}^{(f)} \end{bmatrix}.$$

The Flat Extension Theorem

Theorem

Let $k, s, n \in \mathbb{N}$, $K = K_S$ *be a closed nonempty semialgebraic set such that, where* $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$, *and* $\Gamma = (\Gamma_0, \Gamma_1, \ldots, \Gamma_{2k}) \in \mathbb{S}_n^{2k+1}$ *be a given sequence. Then the following statements are equivalent:*

1. *The following statemets hold:*
   1. $\mathcal{M}_k \succeq 0$.
   2. $\mathcal{H}_{g_j} \succeq 0$.
   3. $\operatorname{rank} \mathcal{M}_{k-v} = \operatorname{rank} \mathcal{M}_k$, *where* $v := \max(\max_j \lceil \deg g_j/2 \rceil, 1)$.
2. Γ *has a* $(\operatorname{rank} \mathcal{M}_{k-v})$*–atomic positive measure* μ *with* $\operatorname{supp} \mu \subseteq K$.

The moment problem: a union of an interval and a point

Theorem

Let $k, n \in \mathbb{N}$, $a, b, c \in \mathbb{R}$, $a < b < c$,

$$K = K_{\{x-a,(x-a)(x-b),c-x\}} = \{a\} \cup [b, c],$$

and $\Gamma = (\Gamma_0, \Gamma_1, \ldots, \Gamma_k) \in \mathbb{S}_n^{k+1}$. *Then the following facts are equivalent:*

1. *There exists a* K*–representing matrix measure for* Γ*.*
2. *There exists a finitely–atomic* K*–representing matrix measure for* Γ*.*
3. *One of the following statements holds:*
   1. $k = 2m$ *for some* $m \in \mathbb{N}$ *and*
   $$\mathcal{M}_m \succeq 0, \;\; \mathcal{H}_{(x-a)(x-b)}(m-1) \succeq 0 \;\; \text{and} \;\; \mathcal{H}_{(x-a)(c-x)}(m-1) \succeq 0.$$
   2. $k = 2m+1$ *for some* $m \in \mathbb{N}$ *and*
   $$\mathcal{H}_{x-a}(m),\; \mathcal{H}_{c-x}(m),\; \mathcal{H}_{(x-a)^2(x-b)}(m-1),\; \mathcal{H}_{(x-a)(x-b)(c-x)}(m-1) \succeq 0.$$

Sketch of the proof

The nontrivial implication is (3) $\Rightarrow$ (2). WLOG: $a = 0$, $b = 1$ and $c > 1$. Assume that $k = 2m$, $m \in \mathbb{N}$.

Note that Γ_0 only appears in $\mathcal{M}_m$.

Let us replace Γ_0 by the smallest $\widetilde{\Gamma}_0$ such that $\widetilde{\mathcal{M}}_m \succeq 0$, where $\widetilde{\mathcal{M}}_\ell$ is the moment matrix corresponding to

$$\widetilde{\Gamma} = (\widetilde{\Gamma}_0, \Gamma_1, \ldots, \Gamma_{2\ell}), \quad 1 \leq \ell \leq m.$$

Namely, using Schur complements,

$$\widetilde{\Gamma}_0 = \begin{bmatrix} \Gamma_1 & \cdots & \Gamma_m \end{bmatrix} (\mathcal{H}_{x^2}(m-1))^\dagger \begin{bmatrix} \Gamma_1 & \cdots & \Gamma_m \end{bmatrix}^T$$

and

$$\operatorname{rank} \widetilde{\mathcal{M}}_m = \operatorname{rank} \mathcal{H}_{x^2}(m-1).$$

It turns out that

$$\operatorname{rank} \widetilde{\mathcal{M}}_m = \operatorname{rank} \widetilde{\mathcal{M}}_{m-1}.$$

Sketch of the proof

By the Flat Extension Theorem, $\widetilde{\Gamma}$ has a K–representing matrix measure of the form

$$\sum_{i=1}^{r} c_i c_i^T \delta_{d_i},$$

where $r = \operatorname{rank} \widetilde{\mathcal{M}}_m$, $c_i \in \mathbb{R}^n$ and $d_i \in \mathbb{R}$. Then

$$\sum_{i=1}^{r} c_i c_i^T \delta_{d_i} + (\Gamma_0 - \widetilde{\Gamma}_0)\delta_0$$

is a $(\operatorname{rank} \mathcal{M}_m)$–atomic K–representing matrix-valued measure for Γ. $\square$

Corollary: Nichtnegativstellensatz

Namely, assume that $k = 2m$. Note that

$$\mathcal{M}_m \succeq 0$$

$$\Leftrightarrow \langle \mathcal{M}_m, B \rangle \geq 0 \text{ for every } B \in \mathbb{S}^{\succeq 0}_{(m+1)n}$$

$$\Leftrightarrow \langle \mathcal{M}_m, \widetilde{B}\widetilde{B}^T \rangle \geq 0 \text{ for every } \widetilde{B} = (\widetilde{B}_i)_{i=0}^m \in (M_n(\mathbb{R}))^{m+1}$$

$$\Leftrightarrow \sum_{i,j=0}^{m} \mathrm{tr}(\widetilde{B}_i^T \Gamma_{i+j} \widetilde{B}_j) \geq 0 \text{ for every } \widetilde{B} = (\widetilde{B}_i)_{i=0}^m \in (M_n(\mathbb{R}))^{m+1}$$

$$\Leftrightarrow \sum_{i,j=0}^{m} \mathrm{tr}(\Gamma_{i+j} \widetilde{B}_j \widetilde{B}_i^T) \geq 0 \text{ for every } \widetilde{B} = (\widetilde{B}_i)_{i=0}^m \in (M_n(\mathbb{R}))^{m+1}$$

$$\Leftrightarrow \sum_{\ell=0}^{k} \mathrm{tr}(\Gamma_\ell A_\ell) \geq 0 \text{ for every } \sum_{i=0}^{k} A_i x^i = \Big(\sum_{j=0}^{m} \widetilde{B}_j x^j\Big)\Big(\sum_{j=0}^{m} \widetilde{B}_j x^j\Big)^T \in M_n(\mathbb{R}[x]_{\leq k})$$

$$\Leftrightarrow \sum_{\ell=0}^{k} \mathrm{tr}(\Gamma_\ell A_\ell) \geq 0 \text{ for every } \sum_{i=0}^{k} A_i x^i \in \sum M_n(\mathbb{R}[x])^2.$$

Corollary: Nichtnegativstellensatz

Similarly, for

$$f := c_2x^2 + c_1x + c_0 \in \{(x-a)(x-b), (x-a)(c-x)\},$$

we have that

$$\begin{aligned}
&\mathcal{H}_f(m-1) \succeq 0 \\
\Leftrightarrow\ & \langle \mathcal{H}_f(m-1, C\rangle \geq 0 \ \text{ for every } C \in \mathbb{S}_{mn}^{\succeq 0} \\
\Leftrightarrow\ & \langle \mathcal{H}_f(m-1), \widetilde{C}^T\widetilde{C}\rangle \geq 0 \ \text{ for every } \widetilde{C} = (\widetilde{C}_i)_{i=0}^{m-1} \in (M_n(\mathbb{R}))^m \\
\Leftrightarrow\ & \sum_{\ell=0}^{k-2} \operatorname{tr}(\Gamma_\ell^{(f)} A_\ell) \geq 0 \ \text{ for every } \sum_{i=0}^{k-2} A_i x^i \in \sum M_n(\mathbb{R}[x])^2 \\
\Leftrightarrow\ & \sum_{\ell=0}^{k-2} \operatorname{tr}\Big((\Gamma_{\ell+2}c_2 + \Gamma_{\ell+1}c_1 + \Gamma_\ell c_0)A_k\Big) \geq 0 \ \text{ for every } \sum_{\ell=0}^{k-2} A_i x^i \in \sum M_n(\mathbb{R}[x])^2 \\
\Leftrightarrow\ & \sum_{\ell=0}^{k} \operatorname{tr}(\Gamma_\ell \widetilde{A}_\ell) \geq 0 \ \text{ for every } \sum_{i=0}^{k} \widetilde{A}_i x^i = f\Big(\sum_{i=0}^{k-2} A_i x^i\Big) \ \text{ with} \\
& \qquad \sum_{i=0}^{k-2} A_i x^i \in \sum M_n(\mathbb{R}[x])^2.
\end{aligned}$$

Corollary: Nichtnegativstellensatz

$$\sum_{\ell=0}^{k} \mathrm{tr}(\Gamma_\ell A_\ell) \geq 0 \quad \text{for every } \sum_{i=0}^{k} A_i x^i \in \mathrm{Pos}^n_{\succeq 0}(\{a\} \cup [b,c])$$

$$\Leftrightarrow \quad \sum_{\ell=0}^{k} \mathrm{tr}(\Gamma_\ell A_\ell) \geq 0 \quad \text{for every } \sum_{\ell=0}^{k} A_\ell x^\ell \in \underbrace{QM^n_{\{(x-a)(x-b),(x-a)(c-x)\}}}_{QM^n_S}.$$

Since QM^n_S is closed, it follows that

$$\mathrm{Pos}^n_{\succeq 0}(\{a\} \cup [b,c]) = QM^n_S.$$

Indeed, otherwise there is $\sum_{\ell=0}^{k} \widetilde{A}_\ell x^\ell \in \mathrm{Pos}^n_{\succeq 0}(\{a\} \cup [b,c])$, which is not contained in QM^n_S. By the Hahn-Banach theorem there is $\widetilde{\Gamma} := (\widetilde{\Gamma}_0, \ldots, \widetilde{\Gamma}_n)$ such that $\sum_{\ell=0}^{k} \mathrm{tr}(\widetilde{\Gamma}_\ell \widetilde{A}_\ell) < 0$ and $\sum_{\ell=0}^{k} \mathrm{tr}(\widetilde{\Gamma}_\ell A_\ell) \geq 0$ for every $\sum_{i=0}^{k} A_i x^i \in QM^n_S$. Contradiction.

Open problems

Problem

Solve the matrix-valued truncated moment problem for K a finite union of closed intervals in $\mathbb{R}$.

Problem (Savchuk, Schmüdgen, 2012)

Characterize positive semidefinite matrix polynomials on

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

or equivalently solve the corresponding truncated matrix moment problem.

References

- with S. Sun: Matrix Fejér-Riesz type theorem for a union of an interval and a point, to appear in J. Pure Appl. Algebra.
- Z.: Matrix Fejér-Riesz theorem with gaps, J. Pure Appl. Algebra 220 (2016) 2533-2548.

Thank you for your attention!