# Maximal Signed Volume for $k$-increasing Quasi-Copulas

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# Outline

1. Quasi-copulas and $k$-increasingness

   $k$-dimensional sections and volumes of boxes

2. Maximal signed volumes

   Definitions of $\mathrm{MNV}(d,k)$ and $\mathrm{MPV}(d,k)$

3. Main results

   Exact formulas, extremal boxes, and realizing quasi-copulas

4. Examples and asymptotic behavior

   Dependence on $d$ and $k$, including the case $k=1$

5. Proof strategy

   Symmetrization, finite differences, and primal-dual linear programming

# Sections of a quasi-copula

A mapping $Q : [0,1]^d \to [0,1]$ is a $d$-variate quasi-copula if it is coordinatewise nondecreasing and

$$i)\ Q(x) = 0 \quad \text{if some } x_i = 0, \quad ii)\ Q(1,\ldots,1,x_i,1,\ldots,1) = x_i, \quad iii)\ |Q(x) - Q(y)| \le \sum_{i=1}^{d} |x_i - y_i|.$$

$k$-dimensional section of $Q$:

1. Choose a $k$-element set $A \subseteq \{1,\ldots,d\}$.
2. The coordinates indexed by $A$ are allowed to vary.
3. The coordinates of $A^c$ are fixed at some $a \in [0,1]^d$.
4. The resulting restriction of $Q$ is denoted by $Q_{a,A}$.

J.J. Arias-García, R. Mesiar, B. De Baets, *The unwalked path between quasi-copulas and copulas: Stepping stones in higher dimensions*, International Journal of Approximate Reasoning **80** (2017), 89–99.

$Q$ is $k$-increasing if every section $Q_{a,A}$ has nonnegative volume on every $k$-box $R$:

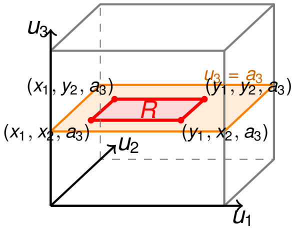
$$V_{Q_{a,A}}(R) = \sum_{u \in \mathrm{Vert}(R)} (-1)^{S_R(u)} Q_{a,A}(u) \ge 0.$$

Here $S_R(u)$ is the number of coordinates of $u$ that are lower endpoints of $R$.

# A 2-dimensional section in dimension 3

Fix $u_3 = a_3$. Then

$$Q_{a,\{1,2\}}(u_1, u_2) = Q(u_1, u_2, a_3), \qquad R = [x_1, y_1] \times [x_2, y_2] \times \{a_3\}.$$




Here

$$x_1 \leq y_1, \qquad x_2 \leq y_2.$$

The signed volume of $R$ is

$$\begin{aligned} V_Q(R) = Q(y_1, y_2, a_3) \\ - Q(x_1, y_2, a_3) \\ - Q(y_1, x_2, a_3) \\ + Q(x_1, x_2, a_3). \end{aligned}$$

The section is 2-increasing if

$$V_Q(R) \geq 0$$

for every rectangle $R$.

# Main result: maximal negative volume

Let $d, k \in \mathbb{N}$ with $2 \leq k \leq d$.

$\mathrm{MNV}(d,k)$ is the minimum of $V_Q(\mathcal{B})$, where:

- $\mathcal{B}$ ranges over all $d$-boxes in $[0,1]^d$ and
- $Q$ ranges over all $k$-increasing $d$-variate quasi-copulas.

## Theorem (Omladič, Vuk, Z., 2026b)

*We have*

$$\mathrm{MNV}(d,k) = \begin{cases} 0, & k = d, \\ -\dfrac{1}{\binom{d-1}{k-1}} \max\limits_{\substack{0 \leq r \leq d-k \\ r \equiv d-k-1 \pmod 2}} \dbinom{d-1}{r}, & k < d. \end{cases}$$

Asymptotically, for a fixed $k$, the growth is exponential in $d$:

$$\mathrm{MNV}(d,k) \sim -\frac{(k-1)!}{\sqrt{2\pi}} \frac{2^d}{d^{\,k-\frac{1}{2}}}, \qquad d \to \infty,$$

# Realization of the extremal value

## Theorem (Omladič, Vuk, Z., 2026b)

*Assume $2 \le k < d$, and let $r_0$ be as on the previous slide.*

*Extremal box*

$$a = \frac{k-1}{d-1-r_0}, \qquad \mathcal{B} = [a,1]^d.$$

*For $z \in \mathrm{Vert}(\mathcal{B})$, set*

$$S(z) = \mathrm{card}\{j : z_j = a\}.$$

*Thus, $S(z)$ counts the lower coordinates of $z$.*

*Vertex values*

*Define*

$$Q(z) = \frac{\binom{d-S(z)-r_0-1}{k-1}}{\binom{d-1-r_0}{k-1}}, \qquad z \in \mathrm{Vert}(\mathcal{B}).$$

*Vertices with the same number of lower coordinates receive the same value.*

*We use $\binom{n}{m} = 0$ whenever $n < m$.*

*These vertex values extend to a $k$-increasing quasi-copula $Q : [0,1]^d \to [0,1]$ satisfying*

$$\boxed{V_Q(\mathcal{B}) = \mathrm{MNV}(d,k).}$$

# Main result: maximal positive volume

Let $d, k \in \mathbb{N}$ with $2 \leq k \leq d$.

$\mathrm{MPV}(d, k)$ is the maximum of $V_Q(\mathcal{B})$, where:

- $\mathcal{B}$ ranges over all $d$-boxes in $[0, 1]^d$,
- $Q$ ranges over all $k$-increasing $d$-variate quasi-copulas.

## Theorem (Omladič, Vuk, Z., 2026b)

*We have*

$$\mathrm{MPV}(d, k) = \frac{1}{\binom{d-1}{k-1}} \max_{\substack{0 \leq r \leq d-k \\ r \equiv d-k \pmod 2}} \binom{d-1}{r}.$$

*In particular,*

$$\mathrm{MPV}(d, d) = 1.$$

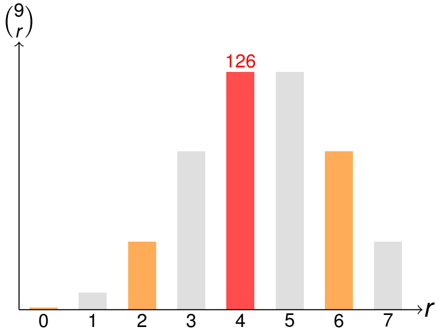
For every fixed $k \geq 2$, the growth is exponential in $d$:

$$\mathrm{MPV}(d, k) \sim \frac{(k-1)!}{\sqrt{2\pi}} \frac{2^d}{d^{k-\frac{1}{2}}}, \qquad d \to \infty.$$

# Example: choosing the optimal index

Let $d = 10$ and $k = 3$. Then

$$0 \le r \le d - k = 7, \qquad r \equiv d - k - 1 \equiv 0 \pmod 2.$$




The admissible indices are

$$r \in \{0, 2, 4, 6\}.$$

The closest one to the peak of the binomial coefficients is

$$r_0 = 4.$$

Hence

$$\mathrm{MNV}(10, 3) = -\frac{\binom{9}{4}}{\binom{9}{2}} = -\frac{7}{2}.$$

# Example: the truncated regime

If $d - k \leq \left\lfloor \frac{d-1}{2} \right\rfloor$, the formula simplifies to

$$\boxed{\mathrm{MNV}(d, k) = -\frac{d-k}{k}}.$$

For $d = 10$ and $k = 6$,

$$d - k = 4 \leq \left\lfloor \frac{d-1}{2} \right\rfloor, \qquad r_0 = d - k - 1 = 3.$$

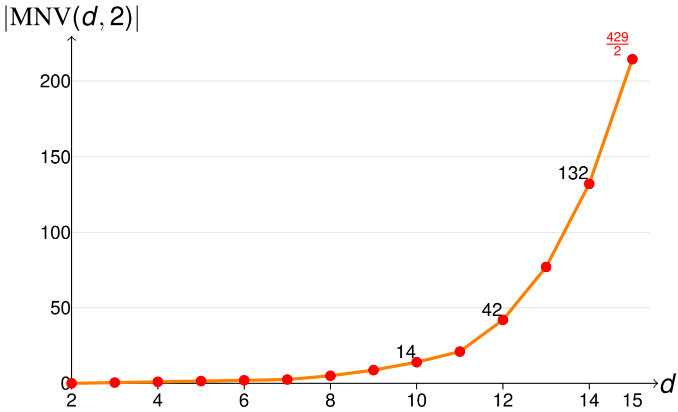
Therefore

$$\mathrm{MNV}(10, 6) = -\frac{\binom{9}{3}}{\binom{9}{5}} = -\frac{84}{126} = -\frac{2}{3}.$$

**Remark:** Although the maximal positive volume equals 1, the negative value $\mathrm{MNV}(10, 6) < 0$ shows that 6-increasing quasi-copulas are not necessarily copulas.

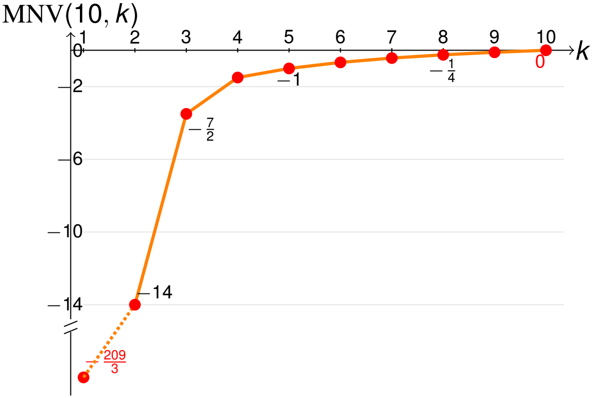
# Growth of the maximal negative volume

Values of $\left|\mathrm{MNV}(d,2)\right|$ for $2 \leq d \leq 15$:




For fixed $k = 2$, the optimal index approaches the peak of $\binom{d-1}{r}$, and the magnitude of the negative volume grows rapidly with the dimension.

# Effect of increasing $k$ for fixed dimension




For $d = 10$:

$$\mathrm{MNV}(10,1) = -\frac{209}{3},$$
$$\mathrm{MNV}(10,2) = -14,$$
$$\mathrm{MNV}(10,3) = -\frac{7}{2},$$
$$\mathrm{MNV}(10,5) = -1,$$
$$\mathrm{MNV}(10,8) = -\frac{1}{4},$$
$$\mathrm{MNV}(10,10) = 0.$$

As $k$ increases, the maximal negative volume approaches zero.

$$\boxed{\mathrm{MNV}(d,d) = 0}$$

# Behavior for fixed $d$ and increasing $k$

The classes of $k$-increasing quasi-copulas become smaller as $k$ increases. Consequently,

$$\mathrm{MNV}(d,k) \le \mathrm{MNV}(d,k+1) \le 0,$$

so $|\mathrm{MNV}(d,k)|$ decreases with $k$.

## Small values of $k$

For

$$2 \le k \le \left\lfloor \frac{d}{2} \right\rfloor,$$

the maximizing binomial coefficient lies at or near the center:

$$|\mathrm{MNV}(d,k)| \approx \frac{\binom{d-1}{\lfloor (d-1)/2 \rfloor}}{\binom{d-1}{k-1}}.$$

Hence

$$\frac{|\mathrm{MNV}(d,k+1)|}{|\mathrm{MNV}(d,k)|} \approx \frac{k}{d-k}.$$

The magnitude initially decreases rapidly.

## Large values of $k$

For

$$k \ge \left\lfloor \frac{d}{2} \right\rfloor + 1,$$

the formula simplifies exactly to

$$\mathrm{MNV}(d,k) = -\frac{d-k}{k}.$$

Writing $k = d - s$ gives

$$\mathrm{MNV}(d,d-s) = -\frac{s}{d-s} \sim -\frac{s}{d} \quad (s \ll d).$$

Thus, near $k = d$, the magnitude decreases approximately linearly with the gap $d - k$.

# Previous result: the case $k = 1$

## Theorem (Omladič, Vuk, Z., 2026a)

*Let $d \geq 7$ and $m = \lfloor d/2 \rfloor$.*

*Binomial coefficients*

*Set $c_0 = 0$ and, for $1 \leq i \leq m$,*

$$c_i = \begin{cases} \binom{d-1}{i-1}, & d \text{ even}, \\ \binom{d-1}{2\lfloor (i-1)/2 \rfloor + 1}, & d \text{ odd}. \end{cases}$$

*Optimization*

*Define*

$$w_{i,-} = \frac{\sum_{j=0}^{i-1} c_{m-j} - 1}{i+1}, \qquad 1 \leq i \leq m,$$

*and choose the first maximizer*

$$i_0 = \min\left( \arg\max_{1 \leq i \leq m} w_{i,-} \right).$$

*Then*

$$\boxed{\mathrm{MNV}(d,1) = -\max_{1 \leq i \leq m} w_{i,-} = -w_{i_0,-}.}$$

*An extremal box is*

$$\boxed{\mathcal{B} = \left[ \frac{i_0}{i_0+1}, 1 \right]^d.}$$

The dimensions $2 \leq d \leq 6$ are treated separately.

# The cases $k = 1$ and $k \geq 2$

After symmetry reduction, let $q_i$ be the common value at vertices with $i$ upper coordinates, and let $h = b - a$. Define

$$\delta_i := \delta_i^{(1)} = q_i - q_{i-1}, \qquad \delta_i^{(j)} = \delta_i^{(j-1)} - \delta_{i-1}^{(j-1)} \quad (j \geq 2).$$

| | $k = 1$ | $k \geq 2$ |
|---|---|---|
| Constraints | The increments are independent: $$0 \leq \delta_i \leq h.$$ | Higher differences satisfy $$\delta_i^{(j)} \geq 0, \qquad 1 \leq j \leq k.$$ In particular, $0 \leq \delta_1 \leq \cdots \leq \delta_d \leq h$. |
| Geometry | For $h = 1$, the feasible increments form the cube $[0,1]^d$, with $$2^d$$ binary vertices. | For $k = 2$, they form a simplex with vertices $$00\cdots0011\cdots11.$$ Higher $k$ gives a simplicial structure in the higher differences. |
| Dual | The $d$ upper bounds remain separate, producing dual variables $$l_1, \ldots, l_d.$$ | Most constraints become redundant; the reduced dual has only four variables. |
| Extremizer | Several increments may be active. The answer involves partial binomial sums and a stopping index $i_0$. | Only one $k$th difference is active. The problem reduces to maximizing one binomial coefficient. |

For $k \geq 2$, the higher-difference constraints remove most of the combinatorial freedom.

# Sketch of the proof

# Proof idea I: reduction to vertex values

Fix

$$\mathcal{B} = \prod_{i=1}^{d} [a_i, b_i].$$

For $\mathbb{I} = (\mathbb{I}_1, \ldots, \mathbb{I}_d) \in \{0,1\}^d$, define

$$(x_{\mathbb{I}})_j = a_j + (b_j - a_j)\mathbb{I}_j, \qquad q_{\mathbb{I}} := Q(x_{\mathbb{I}}).$$

The volume depends only on the $2^d$ vertex values:

$$V_Q(\mathcal{B}) = \sum_{\mathbb{I} \in \{0,1\}^d} (-1)^{d - \|\mathbb{I}\|_1} q_{\mathbb{I}}.$$

The problem becomes a finite linear program:

$$\begin{aligned}
0 \le q_{\mathbb{J}} - q_{\mathbb{I}} \le b_\ell - a_\ell \qquad & \text{if } \mathbb{I} \prec_\ell \mathbb{J}, \\
V_Q(F) \ge 0 \qquad & \text{for every } j\text{-face } F,\ 1 \le j \le k, \\
\max\left\{0, \sum_{j=1}^{d} (x_{\mathbb{I}})_j - d + 1\right\} \le q_{\mathbb{I}} \le \min_j (x_{\mathbb{I}})_j \qquad & \text{for every } \mathbb{I}.
\end{aligned}$$

Here $\mathbb{I} \prec_\ell \mathbb{J}$ means that $\mathbb{I}$ and $\mathbb{J}$ differ only in coordinate $\ell$, with $\mathbb{I}_\ell = 0$ and $\mathbb{J}_\ell = 1$.

# Proof idea II: extension from the grid

For the optimizer constructed below, we will have

$$b_1 = \cdots = b_d = 1.$$

Prescribe $q_{\mathbb{I}}$ on

$$D = \prod_{i=1}^{d} \{a_i, 1\},$$

and set $Q = 0$ whenever some coordinate is zero.
The grid partitions $[0, 1]^d$ into boxes

$$\mathcal{B}_{\mathbb{I}} = \prod_{j=1}^{d} \left[\mathbb{I}_j a_j,\ a_j + \mathbb{I}_j(1 - a_j)\right], \qquad \mathbb{I} \in \{0, 1\}^d.$$

Define a constant density on each box:

$$\rho\Big|_{\mathrm{int}(\mathcal{B}_{\mathbb{I}})} = \frac{V_Q(\mathcal{B}_{\mathbb{I}})}{\mathrm{vol}(\mathcal{B}_{\mathbb{I}})}.$$

Then extend the vertex data by

$$Q(x) = \int_0^{x_1} \cdots \int_0^{x_d} \rho(t)\, dt_d \cdots dt_1.$$

The face constraints and multilinearity imply that this extension is a $k$-increasing quasi-copula.

# Proof idea III: symmetry and finite differences

Averaging over all coordinate permutations gives an optimal symmetric solution:

$$a_i = a, \qquad b_i = b, \qquad q_{\mathbb{I}} = q_{\|\mathbb{I}\|_1}.$$

Hence only $d+1$ vertex values remain, and

$$V_Q(\mathcal{B}) = \sum_{i=0}^{d} (-1)^{d-i} \binom{d}{i} q_i.$$

Define the successive backward differences

$$\delta_i^{(1)} = q_i - q_{i-1}, \qquad \delta_i^{(j)} = \delta_i^{(j-1)} - \delta_{i-1}^{(j-1)}.$$

Then the increasing conditions reduce to

$$\delta_i^{(j)} \geq 0 \qquad (1 \leq j \leq k,\ j \leq i \leq d).$$

After eliminating the redundant variables, the objective is

$$V_Q(\mathcal{B}) = \sum_{i=k}^{d} (-1)^{d+i} \binom{d-k}{i-k} \delta_i^{(k)}.$$

# Proof idea IV: the symmetric primal program

Variables:

$$x_i := \delta_i^{(\min\{i,k\})}, \qquad i = 1, \ldots, d.$$

$$\mathrm{MNV}(d,k) = \min \sum_{i=k}^{d} (-1)^{d+i} \binom{d-k}{i-k} x_i$$

subject to

$$\begin{aligned} b &\le 1, \\ a - b + \sum_{i=1}^{d} \alpha_i^{(k)} x_i &\le 0, \\ -a + q_0 + \sum_{i=1}^{d-1} \beta_i^{(k)} x_i &\le 0, \\ db - q_0 - \sum_{i=1}^{d} \gamma_i^{(k)} x_i &\le d-1, \\ a, b, q_0, x_1, \ldots, x_d &\ge 0. \end{aligned}$$

The coefficients $\alpha_i^{(k)},\ \beta_i^{(k)},\ \gamma_i^{(k)}$ have explicit formulas in terms of binomial coefficients.

The four inequalities give the four nonnegative dual variables $y_1, y_2, y_3, y_4$.

# Proof idea V: the explicit dual program

$$\underset{y_1, y_2, y_3, y_4}{\text{maximize}} \quad -y_1 - (d-1)y_4$$

subject to

$$\begin{aligned}
y_1 - y_2 + dy_4 &\geq 0, \\
y_2 - y_3 &\geq 0, \\
y_3 - y_4 &\geq 0, \\
y_2 + (d-1)y_3 - dy_4 &\geq 0,
\end{aligned}$$

$$\alpha_i^{(k)} y_2 + \beta_i^{(k)} y_3 - \gamma_i^{(k)} y_4 \geq 0, \qquad 2 \leq i \leq k-1,$$

$$\alpha_i^{(k)} y_2 + \beta_i^{(k)} y_3 - \gamma_i^{(k)} y_4 \geq (-1)^{d+1-i}\binom{d-k}{i-k}, \quad k \leq i \leq d,$$

$$y_1, y_2, y_3, y_4 \geq 0.$$

Here $\alpha_i^{(k)}, \beta_i^{(k)}, \gamma_i^{(k)}$ are the coefficients from the primal program.

By strong duality, the optimal value equals $\mathrm{MNV}(d, k)$.

# Proof idea VI: solving the dual

Choose

$$r_0 \in \operatorname*{arg\,max}_{\substack{0 \le r \le d-k \\ r \equiv d-k-1 \pmod 2}} \binom{d-1}{r},$$

and set

$$W^* := \frac{\binom{d-1}{r_0}}{\binom{d-1}{k-1}}.$$

For $i = k + r$, we use

$$\gamma_{k+r}^{(k)} = \binom{d-r-1}{k-1}, \qquad \frac{\binom{d-k}{r}}{\gamma_{k+r}^{(k)}} = \frac{\binom{d-1}{r}}{\binom{d-1}{k-1}}.$$

Hence $W^*$ satisfies all the indexed dual inequalities.

A concrete optimal dual solution is

$$\boxed{(y_1^*, y_2^*, y_3^*, y_4^*) = (W^*, W^*, W^*, 0).}$$

Its objective value is

$$-y_1^* - (d-1)y_4^* = -W^*.$$

Therefore

$$\boxed{\mathrm{MNV}(d, k) = -W^*.}$$

# Proof idea VII: an optimal primal solution

Let

$$i_0 := k + r_0, \qquad G := \gamma_{i_0}^{(k)} = \binom{d-1-r_0}{k-1}.$$

Complementary slackness gives an optimal primal solution with

$$\boxed{\begin{array}{l} b^* = 1, \qquad a^* = \dfrac{k-1}{d-1-r_0}, \qquad q_0^* = 0, \\[2ex] x_{i_0}^* = \dfrac{1}{G}, \qquad x_i^* = 0 \quad (i \neq i_0). \end{array}}$$

Thus only one $k$th difference is active. Solving the corresponding finite-difference equation gives

$$q_i^* = \frac{\binom{i-r_0-1}{k-1}}{\binom{d-1-r_0}{k-1}}, \qquad 0 \le i \le d,$$

where $\binom{n}{m} = 0$ for $n < m$.
For

$$\mathcal{B} = [a^*, 1]^d, \qquad Q(z) = q^*_{d-S(z)}, \quad z \in \mathrm{Vert}(\mathcal{B}),$$

these vertex values extend to a $k$-increasing quasi-copula and

$$\boxed{V_Q(\mathcal{B}) = -W^* = \mathrm{MNV}(d, k).}$$

# Future work

1. Investigate extreme volumes for ultramodular and modular quasi-copulas.
2. Characterize the extreme points of the convex set of $k$-increasing quasi-copulas.

The maximal-volume quasi-copula constructed in the paper is symmetric and therefore is not an extreme point.

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Thank you for your attention!