

Matricial Gaussian Quadrature Rules

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SIAM OP 2026

Edinburgh, UK, June 2026

joint work with

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1. Preliminaries

- ▶ Matrix measures
- ▶ Truncated matrix moment problem
- ▶ Truncated matrix Hamburger moment problem
- ▶ Main goal: description of matricial Gaussian quadrature rules

Matrix-valued measures

$p \in \mathbb{N}$

$\mathbb{S}_p(\mathbb{R})$: real symmetric $p \times p$ matrices

$\mathbb{S}_p^{\geq 0}(\mathbb{R})$: positive semidefinite real symmetric $p \times p$ matrices

$\mathcal{B}(\mathbb{R})$: the Borel σ -algebra of \mathbb{R}

A mapping

$$\mu = (\mu_{ij})_{i,j=1}^p : \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{S}_p(\mathbb{R})$$

is a **positive matrix-valued measure (POVM)** if

1. each $\mu_{ij} : \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{R}$ is a **real measure**,
2. $\mu(T) \in \mathbb{S}_p^{\geq 0}(\mathbb{R})$ for every $T \in \mathcal{B}(\mathbb{R})$.

Let μ be POVM and let $\tau := \text{tr}(\mu) = \sum_{i=1}^p \mu_{ii}$ be its **trace measure**. A measurable mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ is **μ -integrable** if $f \in L^1(\tau)$. We define

$$\int_{\mathbb{R}} f d\mu = \left(\int_{\mathbb{R}} f d\mu_{ij} \right)_{i,j=1}^p.$$

Finitely atomic matrix-valued measures

A POVM μ is **finitely atomic** if

$$\mu = \sum_{j=1}^{\ell} \delta_{x_j} A_j,$$

for some $\ell \in \mathbb{N}$, with distinct $x_j \in \mathbb{R}$ and $A_j \in \mathbb{S}_p^{\geq 0}(\mathbb{R})$.

- ▶ x_j are the **atoms** of μ .
- ▶ $A_j = \mu(\{x_j\})$ is the **mass** of μ at x_j .
- ▶ $\text{mult}_{\mu}(x_j) := \text{rank}(A_j)$ is the **multiplicity** of x_j in μ .

We say that μ is **r -atomic** if

$$\sum_{j=1}^{\ell} \text{rank}(A_j) = r.$$

Truncated matrix Hamburger moment problem

$$p \in \mathbb{N}, n \in \mathbb{N}_0$$

$\mathbb{R}[x]_{\leq n}$: real univariate polynomials of degree at most n .

Truncated matrix Hamburger moment problem (TMHMP): Given a linear mapping

$$L : \mathbb{R}[x]_{\leq n} \rightarrow \mathbb{S}_p(\mathbb{R}),$$

characterize the existence of a POVM μ on \mathbb{R} such that

$$L(f) = \int_{\mathbb{R}} f d\mu \quad \text{for all } f \in \mathbb{R}[x]_{\leq n}.$$

The POVM μ is called an \mathbb{R} -representing measure for L .

Moment matrix

$L : \mathbb{R}[x]_{\leq 2n} \rightarrow \mathbb{S}_p(\mathbb{R})$ a linear operator

Define the *truncated moment sequence*

$$\mathcal{S} := (S_0, S_1, \dots, S_{2n}) \in (\mathbb{S}_p(\mathbb{R}))^{2n+1}, \quad S_i := L(x^i).$$

The n -th *truncated moment matrix* of L (equivalently, of \mathcal{S}) is

$$M(n) := (S_{i+j-2})_{i,j=1}^{n+1} = \begin{matrix} & & & & 1 & x & x^2 & \dots & x^n \\ & & & & S_0 & S_1 & S_2 & \dots & S_n \\ & & & x & S_1 & S_2 & \ddots & \ddots & S_{n+1} \\ & & x^2 & S_2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ & \vdots & & \vdots & \ddots & \ddots & \ddots & \ddots & S_{2n-1} \\ & x^n & & S_n & S_{n+1} & \dots & S_{2n-1} & S_{2n} \end{matrix}.$$

Solution to the matricial Hamburger moment problem

Theorem (Bakonyi & Woerdeman, 2011). Let $n, p \in \mathbb{N}$ and let

$$L : \mathbb{R}[x]_{\leq 2n} \rightarrow \mathbb{S}_p(\mathbb{R})$$

be a linear operator with truncated moment matrix $M(n)$. Then the following statements are equivalent:

1. There exists a **representing matrix measure** for L .
2. There exists a **(rank $M(n)$)-atomic representing matrix measure** for L .
3. $M(n)$ is **positive semidefinite** and

$$\mathcal{C} \begin{pmatrix} S_{n+1} \\ S_{n+2} \\ \vdots \\ S_{2n} \end{pmatrix} \subseteq \mathcal{C}(M(n-1)).$$

$\mathcal{C}(X)$ denotes the **column space** of X (the linear span of its columns).

In particular, any **(rank $M(n)$)-atomic** POVM provided by the theorem yields a **matricial Gaussian quadrature rule for L** , i.e., a finitely atomic POVM representing L with minimal total rank.

2. Problem and Results

- ▶ Main goal: description of matricial Gaussian quadrature rules
- ▶ Scalar Gaussian quadrature rules: several prescribed nodes
- ▶ Matricial Gaussian quadrature rules: nonsingular and singular cases
- ▶ Solution of a strong truncated Hamburger moment problem

Main problem

Let

$$L : \mathbb{R}[x]_{\leq 2n} \rightarrow \mathbb{S}_p(\mathbb{R})$$

be a linear mapping that admits an \mathbb{R} -representing measure. Given

$$t \in \mathbb{R} \quad \text{and} \quad m \in \mathbb{N}_0,$$

characterize when there exists a **minimal** \mathbb{R} -representing measure μ for L such that

$$\text{mult}_{\mu}(t) = m.$$

Remark.

1. This problem is important for the truncated univariate **rational matrix moment problem**.
2. In the scalar case ($p = 1$), it suffices to consider positive definite moment matrices: for the semidefinite case, the minimal representing measure is unique.
3. In the matricial setting, even when $M(n) \succeq 0$, representing measures may be non-unique, since the last block column can still increase the rank.

Scalar setting: 1 prescribed atom

Gaussian formulas

Theorem (Blekhman, Kummer, Riener, Schweighofer & Vinzant, 2020). Let $n \in \mathbb{N}$ and let $\gamma^{(2n)} = (\gamma_0, \gamma_1, \dots, \gamma_{2n}) \in \mathbb{R}^{2n+1}$ be such that

$$\mathcal{H}_{\gamma^{(2n)}} := (\gamma_{i+j-2})_{i,j=1}^{n+1} \text{ is positive definite.}$$

Fix $t \in \mathbb{R}$. The following statements are equivalent:

1. There exists a $(n+1)$ -atomic \mathbb{R} -representing measure for γ with one atom equal to t .
2. The matrix

$$\mathcal{H}_{x-t}(n-1) := (\gamma_{i+j-1} - t\gamma_{i+j-2})_{i,j=1}^n$$

is invertible.

Moreover, if (1),(2) hold, then the remaining n atoms (besides t) are precisely the real roots of

$$g(v) = 0, \quad g(v) := \det(G(t, v)),$$

where

$$G(u, v) := \mathcal{H}_{(x-u)(x-v)}(n-1) = (\gamma_{i+j})_{i,j=1}^n - (u+v)(\gamma_{i+j-1})_{i,j=1}^n + uv(\gamma_{i+j-2})_{i,j=1}^n$$

Scalar setting: Several prescribed atom

Gauss-Kronrod rules

Theorem (Nailwal, Z., 24). Let $k, n \in \mathbb{N}$ and $t_1, \dots, t_k \in \mathbb{R}$ be distinct. Let $d = k + 2n - 1$. Assume that $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_d) \in \mathbb{R}^{d+1}$ is such that

$$\mathcal{H}_\gamma := \left(\gamma_{i+j-2} \right)_{i,j=1}^{\lfloor d/2 \rfloor + 1} \text{ is positive definite.}$$

The following statements are equivalent:

1. There exists a $(k + n)$ -atomic \mathbb{R} -representing measure for γ with k atoms equal to t_1, \dots, t_k .
2. The following conditions hold:
 - 2.1 The matrix

$$\mathcal{H}_{\prod_{l=1}^k (x-t_l)}(n-1) := \left(L_\gamma \left(\prod_{l=1}^k (x-t_l) x^{i+j-2} \right) \right)_{i,j=1}^n$$

is invertible. Here $L_\gamma : \mathbb{R}[x]_{\leq d} \rightarrow \mathbb{R}$ is the functional of $\gamma^{(d)}$.

2.2 “Positive extension condition” holds. (Next slide)

“Positive extension condition”.

Notation:

$$e_i = \sum_{1 \leq j_1 < \dots < j_i \leq k} t_{j_1} \cdots t_{j_i}, \quad f(x) = \prod_{i=1}^k (x - t_i) = \sum_{i=0}^k (-1)^i e_i x^{k-i}.$$

Compute:

- $$\begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_{n-1} \end{pmatrix} = \mathcal{H}_f(n-1)^{-1} \left(\sum_{i=0}^k (-1)^i e_i v_{k+n-i}^{(n-1)} \right), \quad v_i^{(j)} := (\gamma_{i+r-1})_{r=1}^{j+1}.$$
- $$g(x) := x^n - \sum_{i=0}^{n-1} \lambda_i x^i, \quad h(x) := f(x)g(x) = x^{k+n} - \sum_{i=0}^{k+n-1} \varphi_i x^i.$$

Extension:

$$\tilde{\gamma}_u := \begin{cases} \gamma_u, & 0 \leq u \leq d, \\ \sum_{i=1}^{k+n} \varphi_{k+d-i} \tilde{\gamma}_{u-i}, & u > d. \end{cases}$$

Positive extension condition:

$\mathcal{H}_{k+d-1}(\tilde{\gamma})$ is positive definite.

Remark. Other atoms in the measure are the zeros of the polynomial $g(x)$.

Numerical example

Let

$$\gamma = (\gamma_0, \dots, \gamma_9) = (1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880), \quad \gamma_i = i!,$$

and prescribe the atoms

$$t_1 = 1, \quad t_2 = 11 \quad \Rightarrow \quad k = 2, \quad n = 8.$$

We ask whether there exists a 6-atomic \mathbb{R} -representing measure for γ with t_1, t_2 in its support.

Set

$$f(x) = (x - 1)(x - 11) = x^2 - 12x + 11.$$

It turns out that

$$\mathcal{H}_f(3) = \begin{pmatrix} 1 & -7 & -26 & -102 \\ -7 & -26 & -102 & -456 \\ -26 & -102 & -456 & -2280 \\ -102 & -456 & -2280 & -12240 \end{pmatrix}.$$

is invertible.

The last column of $\mathcal{H}_f(4)$, restricted to the first 4 rows, is

$$v = \begin{pmatrix} -456 \\ -2280 \\ -12240 \\ 65520 \end{pmatrix} \Rightarrow \mathcal{H}_f(3)^{-1} v = \begin{pmatrix} \frac{95824}{1601} \\ -\frac{753912}{1601} \\ \frac{1476768}{1601} \\ -\frac{220344}{1601} \end{pmatrix} =: \lambda.$$

Define

$$g(x) := x^4 - (x^3, x^2, x, 1)\lambda = x^4 - \frac{95824}{1601}x^3 + \frac{753912}{1601}x^2 - \frac{1476768}{1601}x + \frac{220344}{1601}.$$

Now define

$$\begin{aligned} h(x) &= f(x)g(x) = x^6 - \sum_{i=0}^5 \varphi_i x^i, \\ &= x^6 - \frac{115036}{1601}x^5 + \frac{1921411}{1601}x^4 - \frac{11577776}{1601}x^3 + \frac{26234592}{1601}x^2 - \frac{18888576}{1601}x + \frac{2423784}{1601}. \end{aligned}$$

The induced candidate for the next moment is

$$\gamma_{10} = \sum_{i=1}^6 \varphi_{6-i} \gamma_{10-i} = \frac{5944515264}{1601}.$$

It turns out that the extended moment matrix M_5 is positive definite and hence a 6-atomic representing measure exists.

Its atoms are the prescribed points

$$t_1 = 1, \quad t_2 = 11,$$

together with the zeros of g , namely

$$t_3 \approx 0.16, \quad t_4 \approx 2.81, \quad t_5 \approx 5.91, \quad t_6 \approx 50.97.$$

Matricial setting: related literature

- ▶ Stable numerical algorithms for computing the atoms and masses of a \mathbb{R} -representing measure for a linear operator $L : \mathbb{R}[x]_{\leq 2n+1} \rightarrow \mathbb{S}_p(\mathbb{R})$.
Durán & Defez, '02, Durán & Lopez-Rodriguez, '96, Dette & Studden, '03
- ▶ Description of the set of all possible masses at a fixed $t \in \mathbb{R}$ over all \mathbb{R} -representing measures for a linear operator L and in particular, determining the maximal mass.
Fritzsche, Kirstein & Mädler, '24
- ▶ Characterizations of the set of possible atoms in a representing measure and possible masses for a linear operator in a multivariate setting.
Mädler & Schmüdgen, '24

Matricial setting: nonsingular case, 1 prescribed atom

Theorem (Zobovič, Z., 2025). Let $n, p \in \mathbb{N}$ and let

$$L : \mathbb{R}[x]_{\leq 2n} \rightarrow \mathbb{S}_p(\mathbb{R})$$

be a linear mapping such that its moment matrix

$M(n)$ is positive definite.

Fix $t \in \mathbb{R}$ and $m \in \mathbb{N}_0$. Then the following statements are equivalent:

1. There exists a **minimal** \mathbb{R} -representing measure μ for L such that $\text{mult}_\mu(t) = m$.
2. $m \leq \text{rank } \mathcal{H}_{x-t}(n-1) - (n-1)p$.

Here $\mathcal{H}_{x-t}(n-1)$ is the $(x-t)$ -localizing moment matrix,

$$\mathcal{H}_{x-t}(n-1) := \left(S_{i+j-1} - tS_{i+j-2} \right)_{i,j=1}^n, \quad \text{where } S_k := L(x^k).$$

Numerical example

- ▶ Let $p = 2$, $n = 1$, and $L : \mathbb{R}[x]_{\leq 2} \rightarrow \mathbb{S}_2(\mathbb{R})$ with

$$S_0 = L(1) = \begin{pmatrix} 18 & 10 \\ 10 & 7 \end{pmatrix}, \quad S_1 = L(x) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad S_2 = L(x^2) = \begin{pmatrix} 50 & 26 \\ 26 & 14 \end{pmatrix}.$$

- ▶ The block moment matrix is

$$M(1) = \begin{pmatrix} S_0 & S_1 \\ S_1 & S_2 \end{pmatrix} = \begin{pmatrix} 18 & 10 & 2 & 2 \\ 10 & 7 & 2 & 2 \\ 2 & 2 & 50 & 26 \\ 2 & 2 & 26 & 14 \end{pmatrix} \succ 0.$$

- ▶ Choose $t := 0$. Then

$$\mathcal{H} := \mathcal{H}_x(0) = (S_1) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad \text{rank } \mathcal{H} = 1.$$

- ▶ Hence

$$\text{rank } \mathcal{H} - (n-1)p = 1 - (1-1) \cdot 2 = 1.$$

- ▶ By the theorem, for each $m \in \{0, 1\}$ there exists a minimal \mathbb{R} -representing measure μ for L such that

$$\text{mult}_\mu 0 = m.$$

Numerical example: construction of μ with $\text{mult}_\mu 0 = 1$



$$(\mathcal{H} \quad K) := (\mathcal{H} \quad S_2) = \left(\begin{array}{cc|cc} 2 & 2 & 50 & 26 \\ 2 & 2 & 26 & 14 \end{array} \right), \quad \text{rank}(\mathcal{H} \quad K) = 2.$$

► Write the columns of K as

$$K = (K_1 \quad K_2) = \left(\begin{array}{cc} 50 & 26 \\ 26 & 14 \end{array} \right) \Rightarrow K_2 = (\mathcal{H} \quad K_1) \underbrace{\begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}}_J.$$

► We will define

$$S_3 = \begin{pmatrix} Z_1 & Z_2 \\ Z_2^T & Z_3 \end{pmatrix} \in S_2(\mathbb{R}),$$

such that

$$\text{rank} \begin{pmatrix} \mathcal{H} & K_1 \\ K_1^T & Z_1 \end{pmatrix} = \text{rank} \begin{pmatrix} \mathcal{H} \\ K_1^T \end{pmatrix} + (p - m), \quad \text{and} \quad \begin{pmatrix} Z_2 \\ Z_3 \end{pmatrix} = \begin{pmatrix} K_1^T & Z_1 \\ K_2^T & Z_2 \end{pmatrix} J.$$

Numerical example: construction of μ with $\text{mult}_\mu 0 = 1$

► For $Z_1 = (98)$ we get $S_3 = \begin{pmatrix} 98 & 50 \\ 50 & 26 \end{pmatrix}$.

► Compute

$$\begin{pmatrix} H_0 \\ H_1 \end{pmatrix} = M(1)^{-1} \begin{pmatrix} S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 3 & \frac{3}{2} \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 6 & \frac{5}{2} \\ -8 & -3 \end{pmatrix} \end{pmatrix},$$

so that the matrix polynomial

$$H(x) := x^2 I_2 - xH_1 - H_0$$

is a block column relation for $M(2)$, i.e.,

$$M(2) \begin{pmatrix} H_0 & H_1 & I_2 \end{pmatrix}^T = 0 \quad \text{and} \quad \det H(x) = x(x-3)(x-1)(x+1).$$

► By [Kimsey & Trachana, 22', Mädler & Schmüdgen, 24'], the atoms of the representing measure μ are zeroes of $\det H(x)$:

$$\begin{aligned} (t_1, \text{mult}_\mu t_1) &= (0, 1), & (t_2, \text{mult}_\mu t_2) &= (3, 1), \\ (t_3, \text{mult}_\mu t_3) &= (1, 1), & (t_4, \text{mult}_\mu t_4) &= (-1, 1). \end{aligned}$$

► The masses are obtained using Vandermonde system:

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 12 & 6 \\ 6 & 3 \end{pmatrix}.$$

Matricial setting: singular case, 1 prescribed atom

Theorem (Zobovič, Z., 2025). Let $n, p \in \mathbb{N}$ and let

$$L : \mathbb{R}[x]_{\leq 2n} \rightarrow \mathbb{S}_p(\mathbb{R})$$

be a linear mapping admitting a **representing measure** such that its moment matrix

$$M(n) \text{ is positive semidefinite.}$$

Fix $t \in \mathbb{R}$ and $m \in \mathbb{N} \cup \{0\}$. Then the following statements are equivalent:

1. There exists a **minimal representing matrix measure** μ for L such that **mult** $_{\mu} t = m$.
2. $\text{card } A \leq m \leq \text{card}(A \cup B)$.

Let $T_i := L((x - t)^i)$ for $i \in [0; 2n]$ and

$$\mathcal{M}_1 := (T_{i+j-2})_{i,j=1}^{n+1} = \begin{pmatrix} \mathbf{y}_1 & \cdots & \mathbf{y}_{(n+1)p} \end{pmatrix},$$

$$\mathcal{M}_2 := (T_{i+j-2})_{\substack{i=1, \dots, n, \\ j=1, \dots, n+2}} = \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_{(n+2)p} \end{pmatrix}$$

Let

$$V_j := \text{span}\{\mathbf{y}_1, \dots, \mathbf{y}_j\}, \quad V_j^+ := \text{span}\{\mathbf{y}_{p+1}, \dots, \mathbf{y}_j\}, \quad W_j^+ := \text{span}\{\mathbf{u}_{p+1}, \dots, \mathbf{u}_j\},$$

and

$$A := \left\{ i \in [p] : \exists k \in [n] \text{ such that } \mathbf{y}_{(k-1)p+i} \notin V_{(k-1)p+i-1} \text{ and } \mathbf{y}_{kp+i} \in V_{kp+i-1}^+ \right\},$$

$$B := \left\{ i \in [p] : \mathbf{y}_{np+i} \notin V_{np+i-1} \text{ and } \mathbf{u}_{(n+1)p+i} \in W_{(n+1)p+i-1}^+ \right\}.$$

Sets A and B : numerical example

$$\left(\begin{array}{c|c} T_0 & T_3 \\ T_1 & T_4 \\ T_2 & ? \end{array} \right) = \left(\begin{array}{cccccccc|ccc} 4 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & -6 & 0 & 0 \\ 0 & 2 & -1 & 0 & 2 & -2 & 0 & 4 & -4 & 0 & 8 & -8 \\ 0 & -1 & 2 & 0 & 0 & 4 & 0 & 0 & -4 & 8 & 0 & 16 \\ 0 & 0 & 0 & 6 & 0 & 0 & -6 & 0 & 0 & 18 & 0 & 0 \\ 0 & 2 & -2 & 0 & 4 & -4 & 0 & 8 & -8 & 0 & 16 & -16 \\ 0 & -2 & 4 & 0 & -4 & 8 & 0 & -8 & 16 & 0 & -16 & 32 \\ 6 & 0 & 0 & -6 & 0 & 0 & 18 & 0 & 0 & ? & ? & ? \\ 0 & 4 & -4 & 0 & 8 & -8 & 0 & 16 & -16 & ? & ? & ? \\ 0 & -4 & 8 & 0 & -8 & 16 & 0 & -16 & 32 & ? & ? & ? \end{array} \right)$$

$$\text{rank } M(2) = 6, \quad \mathbf{y}_6 = 2\mathbf{y}_3, \quad \mathbf{y}_9 = 2\mathbf{y}_6, \quad \mathbf{y}_8 = 2\mathbf{y}_5,$$

So

$$A = \{2\}$$

and due to

$$\begin{pmatrix} -6 \\ 0 \\ 0 \\ 18 \\ 0 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \\ 0 \\ -6 \\ 0 \\ 0 \end{pmatrix} \Rightarrow B = \{1\}.$$

Strong matricial Hamburger moment problem

Corollary of special case: $\text{mult}_\mu 0 = 0$

Corollary (Simonov, 2006). Let $n_1, n_2, p \in \mathbb{N}$ and $S_i \in \mathbb{S}_p(\mathbb{R})$ for $i \in [-2n_1; 2n_2]$. Then the following statements are equivalent:

1. There exists a **measure** μ such that $S_i = \int_{\mathbb{R}} x^i d\mu$ for $i \in [-2n_1; 2n_2]$.
2. There exists a **finitely atomic measure** μ such that $S_i = \int_{\mathbb{R}} x^i d\mu$ for $i \in [-2n_1; 2n_2]$.
3. There exists a **minimal representing measure** μ such that

$$S_i = \int_{\mathbb{R}} x^i d\mu \quad \text{for } i \in [-2n_1; 2n_2].$$

4. $(S_{i+j})_{i,j=-n_1}^{n_2}$ is **positive semidefinite** and $\ker(S_{i+j})_{i,j=-n_1}^{n_2-1} = \ker(S_{i+j})_{i,j=-n_1+1}^{n_2}$.
5. $(S_{i+j})_{i,j=-n_1}^{n_2}$ is **positive semidefinite** and for every sequence $\{v_k\}_{k=0}^{n_1+n_2-1} \subset \mathbb{R}^p$

$$\sum_{i,j=-n_1}^{n_2-1} v_{j+n_1}^T S_{i+j} v_{i+n_1} = 0 \quad \text{if and only if} \quad \sum_{i,j=-n_1}^{n_2-1} v_{j+n_1}^T S_{i+j+2} v_{i+n_1} = 0.$$

Open problems

1. When there exists a minimal representing measure for L avoiding finitely many prescribed points? (Rational matricial moment problem)
2. Which tuples $((t_1, m_1), \dots, (t_k, m_k))$ are realizable for L , i.e., there exists a minimal measure μ such that $\text{mult}_\mu(t_i) = m_i$.
3. When can a degree $k + 2n - 1$ sequence $\{S_i\}_{i=0}^{k+2n-1}$ be represented by $(k + n)$ -atomic matricial measure containing given k atoms (allowing repetition)? (Gauss-Kronrod formulas)

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Thank you for your attention!