The truncated moment problem on some polynomial and rational plane curves

> Aljaž Zalar University of Ljubljana Slovenia

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### Outline

Truncated moment problem on curve p(x, y) = 0

#### 1. Preliminaries

- 2. Main results Using univariate reduction technique
   Concrete solution:

   y<sup>2</sup> = x<sup>3</sup>
   yx<sup>2</sup> = 1
   y(y 1)(y α) = 0.

   LMI based solution:

   y<sup>i</sup> = q(x)
   y<sup>i</sup>x<sup>i</sup> = 1
   y = p(t), x = g(t)
  - Solution in terms of the bound on PSD extensions of M(k):  $\checkmark : y = q(x)$   $\checkmark : y^i x^j = 1$   $\underbrace{\times : y^i = x^j, i, j > 1, gcd(i, j) = 1}_{\text{The pred time concepting}}$ .

The proof gives nonnegative but not sos polynomials.

Bounds on the number of atoms in the minimal measure for the curves above.

#### 3. Proofs

# Bivariate truncated moment problem (TMP)

Let  $k \in \mathbb{N}$  and

$$eta = eta^{(k)} = (eta_{i,j})_{i,j\in\mathbb{Z}_+,i+j\leq k}$$

a bivariate sequence of real numbers of degree k.

 $K \subseteq \mathbb{R}^2$  is a closed subset.

The bivariate truncated moment problem on K (K-TMP): characterize the existence of a positive Borel measure  $\mu$  on  $\mathbb{R}^2$  with support in K, such that

$$eta_{i,j} = \int_{\mathcal{K}} x^i y^j d\mu(x)$$

for  $i, j \in \mathbb{Z}_+$ ,  $i + j \le k$ .

 $\mu$  is called a *K*-representing measure (*K*-RM) of  $\beta$ .

#### Bivariate moment matrix

The moment matrix M(k) associated to  $\beta$  with the rows and columns indexed by  $X^i Y^j$ ,  $i + j \le k$ , in degree-lexicographic order

 $1, X, Y, X^2, XY, Y^2, \dots, X^k, X^{k-1}Y, \dots, Y^k$ 

is defined by where

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#### Necessary conditions for the existence of a RM

▶ To every polynomial  $p := \sum_{i,j} a_{i,j} x^i y^j \in \mathbb{R}[x, y]_k$ , we associate the vector

$$p(X,Y) = \sum_{i,j} a_{i,j} X^{i} Y^{j} = a_{0,0} \cdot \begin{pmatrix} \beta_{0,0} \\ \beta_{1,0} \\ \beta_{0,1} \\ \vdots \\ \beta_{0,k} \end{pmatrix} + a_{1,0} \cdot \begin{pmatrix} \beta_{1,0} \\ \beta_{2,0} \\ \beta_{1,1} \\ \vdots \\ \beta_{1,k} \end{pmatrix} + \dots + a_{0,k} \cdot \begin{pmatrix} \beta_{0,k} \\ \beta_{1,k} \\ \beta_{0,k+1} \\ \vdots \\ \beta_{0,2k} \end{pmatrix}$$

from the column space of the matrix M(k).

► The matrix M(k) is recursively generated (RG) if for  $p, q, pq \in \mathbb{R}[x, y]_k$ 

$$\mathcal{D}(X, Y) = \mathbf{0} \quad \Rightarrow \quad (pq)(X, Y) = \mathbf{0}.$$

Necessary conditions for the existence of a RM

The matrix M(k) satisfies the variety condition (VC) if

 $\operatorname{rank} M(k) \leq \operatorname{card} \mathcal{V},$ 

where

$$\mathcal{V} := igcap_{\substack{g \in \mathbb{R}[x,y] \leq k, \ g(X,Y) = \mathbf{0} \text{ in } \mathcal{M}(k)}} igl( \underbrace{\{(x,y) \in \mathbb{R}^2 \colon g(x,y) = \mathbf{0}\}}_{\mathcal{Z}(g)}.$$

**Proposition** (Curto and Fialkow, 96') If  $\beta^{(2k)}$  has a representing measure  $\mu$ , then

M(k) is positive semidefinite (PSD), RG and satisfies VC.

Theorem (Flat extension theorem, Curto and Fialkow, 96') *TFAE:* 

1.  $\beta^{(2k)}$  admits a (rank M(k))-atomic RM.

2. M(k) is PSD and there is an extension M(k + 1) such that

 $\operatorname{rank} M(k+1) = \operatorname{rank} M(k).$ 

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## Solving the TMP on rational curves

Basic ideas

#### 1. For irreducible curve C:

- Parametrize the curve with one parameter.
- Solve the corresponding univariate TMP.
- 2. For reducible curve C:
  - Study decompositions

$$\beta = \beta^{(1)} + \beta^{(2)},$$

where

 $\beta^{(1)}$ : a moment sequence on one irreducible component of C,  $\beta^{(2)}$ : a moment sequence on the complement of C.

• Apply the solution to the TMP on each summand  $\beta^{(i)}$ , i = 1, 2.

#### Bivariate TMP on p(x, y) = 0 with deg $p \le 3$

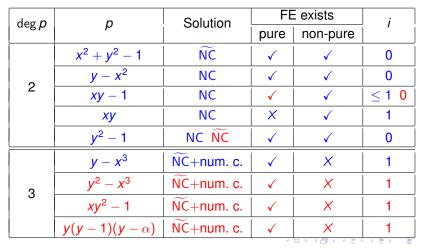
NC = PSD + RG + VC, NC = PSD + RG, num. c.=numerical conditions **pure** ... only relations except coming from *p*, **FE** ... flat extension #**atoms** = rank M(k) + i

## Bivariate TMP on p(x, y) = 0 with deg $p \leq 3$

**NC** = PSD + RG + VC,  $\widetilde{NC}$  = PSD + RG, **num. c.**=numerical conditions **pure** ... only relations except coming from *p*, **FE** ... flat extension #atoms = rank M(k) + **i** 

proved by FE technique

proved by univariate reduction technique



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#### Hankel matrix

Let  $k \in \mathbb{N}$ . For

$$\gamma = (\gamma_0, \dots, \gamma_{2k}) \in \mathbb{R}^{2k+1}$$

we write

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TMP for 
$$p(x, y) = y^2 - x^3$$
  
For  $\beta^{(2k)}$  we define a univariate sequence  
 $\gamma(\mathbf{x}) := (\gamma_0, \mathbf{x}, \underbrace{\gamma_2, \gamma_3, \overbrace{\gamma_4, \dots, \gamma_{6k-2}, \gamma_{6k-1}, \gamma_{6k}}^{\gamma^{(2)}}), \text{ where } \gamma_{2i+3j} = \beta_{i,j}.$ 

Define also  $\gamma^{(3)} = (\gamma_2, \dots, \gamma_{6k-2}).$ 

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Define also  $\gamma^{(3)} = (\gamma_2, \ldots, \gamma_{6k-2}).$ 

#### Existence:

(1)  $\beta$  has a  $\mathcal{Z}(p)$ -RM.  $\Leftrightarrow$ 

(2)  $\beta$  has at most (rank M(k) + 1)-atomic  $\mathcal{Z}(p)$ -RM.  $\in$ 

(3) M(k) is PSD and RG,  $A_{\gamma(1)}$  is PSD and one of the following holds:

- a)  $A_{\gamma^{(1)}}$  is PD and rank (M(k) without column/row  $Y^k) = 3k 1$ .
- b) rank  $A_{\gamma^{(1)}} = \operatorname{rank} A_{\gamma^{(2)}} = \operatorname{rank} A_{\gamma^{(3)}}$ .

#### Uniqueness and cardinality:

- ▶ There is a (rank M(k))-atomic  $\mathcal{Z}(p)$ -RM unless rank M(k) = 3k 1 and  $A_{\gamma^{(1)}}$  is PD.
- The  $\mathcal{Z}(p)$ -RM is unique if rank M(k) < 3k. Otherwise two minimal  $\mathcal{Z}(p)$ -RM exist.

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# Property $(S_{k,m})$

Solution to the TMP based on the size of PSD extensions  $\mathcal{Z}(p) = \{(x,y) \in \mathbb{R}^2 \colon p(x,y) = 0\}$ 

 $\mathcal{Z}(p)$  has property  $(S_{k,m})$  if the following are equivalent:

1.  $\beta^{(2k)}$  has a  $\mathcal{Z}(p)$ -RM.

2. M(k) satisfies p(X, Y) = 0 and admits a PSD extension M(k + m).

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 $\mathcal{Z}(p)$  has property  $(A_{k,m})$  if every  $f \in \mathbb{R}[x, y]_{\leq 2k+2}$  with  $f|_{\mathcal{Z}(p)} > 0$  is of the form

$$f = \sum_i f_i^2 + p \sum_j g_j^2 - p \sum_\ell h_\ell^2;$$

where  $f_i^2$ ,  $pg_j^2$ ,  $ph_\ell^2 \in \mathbb{R}[x, y]_{\leq 2m}$ .

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where  $f_i^2$ ,  $pg_j^2$ ,  $ph_\ell^2 \in \mathbb{R}[x, y]_{\leq 2m}$ .

Theorem (Curto and Fialkow, 08')

$$(A_{k,k+m}) \Rightarrow (S_{k,m})$$
 and  $(S_{k,m}) \Rightarrow (A_{k-1,k+m}).$ 

Bivariate TMP on p(x, y) = 0 with deg  $p \ge 4$ 

proved through property  $(A_{k,m(k)})$  (Fialkow, 11')

proved by univariate reduction technique

LMI ... feasibility problem of a linear matrix inequality

deg <i>p</i>	p	$(S_{k,m})$	т	Solution	# atoms
	y - q(x)	$\checkmark$	$O(k\ell)  \ell-1$	LMI	kℓ
$\ell \geq 4$	$yx^{\ell-1} - 1$	$\checkmark$	<b>O(k</b> ℓ) ℓ	LMI	kℓ
	$y^{j} - x^{\ell}, j > 1$ , irred.	×	×	LMI	kℓ
	$y^j x^{\ell-j} - 1$ , irred.	$\checkmark$	$O(k \max(j, \ell - j))$	LMI	kℓ

p(x, y) = y - q(x) has property  $(S_{k,k+\deg q-1})$ 

Nontrivial: M(k) satisfies Y = q(X) and  $M(k + \ell - 1)$  PSD exists.  $\Rightarrow \beta^{(2k)}$  has a  $\mathcal{Z}(p)$ -RM.

1. Basis of the column space of  $M(k + \ell - 1)$ :

$$Y^{i}X^{j}, \quad i=0,...,k, \ j=0,...,\deg q-1, \ i+j\leq k+\ell-2.$$

2. Relations between moments: Writing  $q(x) = \sum_{i=0}^{\ell} q_i x^i$  we have

$$\beta_{i,j} = q_{\ell}\beta_{i+\ell,j-1} + q_{\ell-1}\beta_{i+\ell-1,j-1} + \ldots + q_{0}\beta_{i,j-1}, \quad i \in \mathbb{Z}_{+}, j \in \mathbb{N}, i+j \le 2(k+\ell-2).$$

3. The corresponding univariate sequence: Let

$$q_{i,j,s} := \begin{cases} \sum_{\substack{0 \le i_1, \dots, i_j \le \ell, \\ i_1 + \dots + i_j = s - i \\ 0, \\ \end{cases}} q_{i_1} q_{i_2} \dots q_{i_j}, & \text{if } i \le s \le i + j\ell, \end{cases}$$

Let

$$\gamma_t = \frac{1}{(q_\ell)^{\lfloor \frac{t}{\ell} \rfloor}} \Big( \beta_{t \mod \ell, \lfloor \frac{t}{\ell} \rfloor} - \sum_{s=0}^{t-1} q_{t \mod \ell, \lfloor \frac{t}{\ell} \rfloor, s} \cdot \gamma_s \Big) \quad t=0, \dots, 2k\ell+2,$$

- 4.  $\beta^{(2k)}$  has a  $\mathcal{Z}(p)$ -RM.  $\Leftrightarrow \gamma^{(2k\ell)} := (\gamma_0, \dots, \gamma_{2k\ell})$  has a  $\mathbb{R}$ -RM.
- 5.  $M(k + \ell 1)$  PSD  $\Rightarrow$   $A_{\gamma^{(2k\ell+2)}}$  PSD, where

$$\gamma^{(2k\ell+2)} := (\gamma_0, \ldots, \gamma_{2k\ell+2}).$$

- 6. Use the solution to the  $\mathbb{R}$ -TMP (Curto and Fialkow, 91'), i.e., TFAE:
  - $\gamma^{(2k\ell)}$  has a  $\mathbb{R}$ -RM.
  - $\gamma^{(2k\ell)}$  has a (rank  $A_{\gamma^{(2k\ell)}}$ )-atomic  $\mathbb{R}$ -RM.
  - $A_{\gamma^{(2k\ell)}}$  has a PSD extension  $A_{\gamma^{(2k\ell+2)}}$ .
- 7. Decrease the number of atoms by 1 in the case rank  $A_{\gamma^{(0,2k\ell)}} = k\ell + 1$ :

This can be achieved by manipulating  $\gamma_{2k\ell-1}$  which does not affect the original sequence  $\beta_{i,j}$ ,  $i, j \in \mathbb{Z}_+, i+j \leq 2k$ .

# LMI based solution for p(x, y) = y - q(x), $q(x) = \sum_{i=0}^{\ell} q_i x^i$ Theorem

TFAE:

- 1.  $\beta^{(2k)}$  has a  $\mathcal{Z}(p)$ –RM.
- 2.  $\beta_{i,j} = \sum_{p=0}^{\ell} q_p \beta_{i+p,j-1}$  for every  $i, j \in \mathbb{Z}_+$ , such that  $i+j \le 2k-\ell+1$  and there exists missing values  $\gamma_i$  in the sequence  $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_{2k\ell+2}$  defined for t from the set

$$\Big\{t\in\mathbb{N}_0\colon t \mod \ell+\Big\lfloor \frac{t}{\ell}\Big\rfloor\leq 2k\Big\},$$

by

$$\gamma_t = \frac{1}{(q_\ell)^{\lfloor \frac{t}{\ell} \rfloor}} \Big( \beta_{t \mod \ell, \lfloor \frac{t}{\ell} \rfloor} - \sum_{s=0}^{t-1} q_{t \mod \ell, \lfloor \frac{t}{\ell} \rfloor, s} \cdot \gamma_s \Big),$$

$$q_{i,j,s} := \begin{cases} \sum_{\substack{0 \le i_1, \dots, i_j \le \ell, \\ i_1 + \dots + i_j = s - i \\ 0, \\ \end{cases}} q_{i_1} q_{i_2} \dots q_{i_j}, & \text{if } i \le s \le i + j\ell, \end{cases}$$

 $A_{\gamma} \succ 0.$ 

such that

Example:  $p(x, y) = y - x^4$ 

$$\gamma_t = \beta_{t \mod t, \left\lfloor \frac{t}{4} 
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ceil}$$

for every t from the set

$$\{t \in \mathbb{Z}_+: t \leq 8k, t \notin \{8k-5, 8k-2, 8k-1\}\}.$$

The matrix  $A_{\gamma}$  is equal to

(	$\gamma_0$	$\gamma_{\rm 1}$	$\gamma_2$	$\gamma_3$					$\gamma_{4k}$	$\gamma_{4k+1}$	
	$\gamma_1$	$\gamma_2$	$\gamma_3$	. · <sup>·</sup>						÷	
	$\gamma_2$	$\gamma_3$	· · <sup>·</sup>					$\gamma_{8k-6}$	$\gamma_{\mathbf{8k-5}}$	$\gamma_{8k-4}$	
	$\gamma_3$						$\gamma_{8k-6}$	$\gamma_{\mathbf{8k-5}}$	$\gamma_{8k-4}$	$\gamma_{8k-3}$	
	÷					$\gamma_{8k-6}$	$\gamma_{\rm 8k-5}$	$\gamma_{8k-4}$	$\gamma_{8k-3}$	$\gamma_{\mathbf{8k-2}}$	
					$\gamma_{8k-6}$	$\gamma_{ m 8k-5}$	$\gamma_{8k-4}$	$\gamma_{8k-3}$	$\gamma_{ m 8k-2}$	$\gamma_{8k-1}$	
	÷			$\gamma_{8k-6}$	$\gamma_{ m 8k-5}$	$\gamma_{8k-4}$	$\gamma_{8k-3}$	$\gamma_{\mathbf{8k-2}}$	$\gamma_{\rm 8k-1}$	$\gamma_{8k}$	
	$\gamma_{4k}$		$\gamma_{8k-6}$	$\gamma_{8k-5}$	$\gamma_{8k-4}$	$\gamma_{8k-3}$	$\gamma_{\mathbf{8k-2}}$	$\gamma_{8k-1}$	$\gamma_{8k}$	$\gamma_{\mathbf{8k+1}}$	
/	$\gamma_{4k+1}$		$\gamma_{\rm 8k-5}$	$\gamma_{8k-4}$	$\gamma_{8k-3}$	$\gamma_{\mathbf{8k-2}}$	$\gamma_{\rm 8k-1}$	$\gamma_{8k}$	$\gamma_{\rm 8k+1}$	$\gamma_{\mathbf{8k+2}}$ /	
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$$p(x,y) = y^{\ell_2} x^{\ell_1} - 1, \operatorname{gcd}(\ell_1,\ell_2) = 1,$$
 has  $(\mathcal{S}_{k,(k+1)\ell_2})$ 

1. Parametrization:  $x = t^{\ell_2}$ ,  $y = t^{-\ell_1}$ .

2. The univariate sequence:  $\beta_{ij} \leftrightarrow \gamma_{i\ell_2 - j\ell_1}$ .

 $\gamma := (\gamma_{-2k\ell_1}, \dots, \gamma_{2k\ell_2})$  has some gaps.

- **3.**  $\beta^{(2k)}$  has a  $\mathcal{Z}(p)$ -RM  $\Leftrightarrow \gamma$  has a  $(\mathbb{R} \setminus \{0\})$ -RM.
- 4. Solution of the strong  $(\mathbb{R} \setminus \{0\})$ -TMP (Z,22'), i.e., TFAE:

γ has a (ℝ \ {0})–RM.
 γ can be extended to the sequence

 $\widetilde{\gamma} := (\gamma_{-2k\ell_1-2}, \dots, \gamma_{2k\ell_2+2})$  without gaps and  $A_{\widetilde{\gamma}}$  is PSD.

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5.  $M(m + \ell)$  PSD for  $\ell$  large enough  $\Rightarrow A_{\widetilde{\gamma}}$  PSD.

 $p(x, y) = y^{\ell_2} - x^{\ell_1}, \ell_2 > \ell_1 > 1$ , irreducible does not have property  $(S_{k,m})$  for every m

- 1. Parametrization:  $x = t^{\ell_2}$ ,  $y = t^{\ell_1}$ .
- 2. The univariate sequence:  $\beta_{ij} \leftrightarrow \gamma_{i\ell_2+j\ell_1}$ .

 $\gamma := \gamma_0, \ldots, \gamma_{2k\ell_2}$  has some gaps.

- **3.**  $\beta^{(2k)}$  has a  $\mathcal{Z}(p)$ -RM  $\Leftrightarrow \gamma$  has a  $\mathbb{R}$ -RM.
- 4. Solution of the  $\mathbb{R}$ -TMP:  $\gamma$  has a  $\mathbb{R}$ -RM  $\Leftrightarrow \gamma$  can be extended to the sequence

 $\gamma^{(2k\ell_2+2)} = (\gamma_0, \dots, \gamma_{2k\ell_2+2}) \quad \text{without gaps and} \quad A_{\gamma^{(2k\ell_2+2)}} \text{ is PSD}.$ 

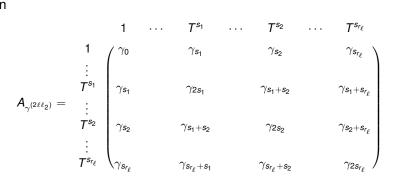
5. One can construct a sequence  $\gamma$  such that  $A_{\gamma}$  is not even partially PSD, but it can be extended with  $\gamma_{2k\ell_2+1}, \gamma_{2k\ell_2+2}, \ldots$  to a matrix such that the submatrices corresponding to matrices M(k + m) are PSD.

► Columns of *M*(ℓ) correspond to columns

$$\mathcal{T}_{\ell} = \{ T^{s} \colon s = a\ell_{1} + b\ell_{2}, \ a, b = 0, \dots, \ell \} = \{ 1, T^{s_{1}}, T^{s_{2}}, \dots, T^{s_{r_{\ell}}} \}$$

of the univariate Hankel matrix  $A_{\gamma^{(2\ell\ell_2)}}$ .

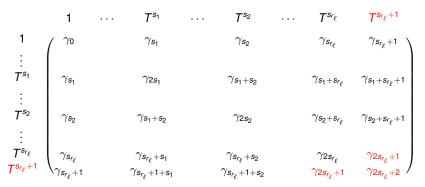
Then



The specified part of  $A_{\gamma^{(2\ell\ell_2)}}$  corresponds to  $M(\ell)|_{\text{rows/columns in the basis}}$ .

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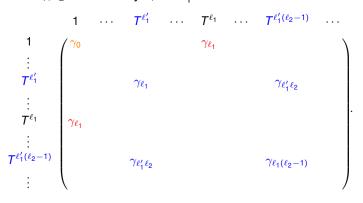
If  $M(k)|_{\text{basis}}$  is PD, then A is PD and it has infinitely many PD extensions:



- γ<sub>2s<sub>r<sub>ℓ</sub>+1</sub> is chosen arbitrarily, while γ<sub>2s<sub>r<sub>ℓ</sub>+2</sub> must be such that the Schur complement is positive.</sub></sub>
- One can continue in this way to determine T<sup>s<sub>rℓ</sub>+2</sup>, T<sup>s<sub>rℓ</sub>+3</sup>,.... On the side of β one gets a sequence of extensions β<sup>(2k)</sup>, β<sup>(2k+2)</sup>, β<sup>(2k+4)</sup>,... such that M(k + 1), M(k + 2),... are PSD.
- So one gets a full sequence  $\beta^{(\infty)}$  with  $M(\infty)$  PSD.

γ can be chosen such that it does not have a measure, even though (A)|<sub>T<sub>k</sub></sub> is PD. Consequently, we will get β with infinitely many extensions but without a measure.

**Case 1:** One of  $\ell_1, \ell_2$  is even. Say  $\ell_1 = 2\ell'_1$ . Then



- 1. Generate any sequence  $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{2s_{r_k}})$  such that  $A_{\gamma}$  is PD.
- 2. You decrease  $\gamma_{\ell_1}$  such that the submatrix  $(A_{\gamma})|_{\{T^{\ell'_1}, T^{\ell'_1(\ell_2-1)}\}}$  is not PSD.
- 3. Since  $\gamma_{\ell_1}$  occurs in  $(A_{\gamma})|_{\mathcal{T}_k}$  only twice at non-diagonal places, you can increase  $\gamma_0$  such that  $(A_{\gamma})|_{\mathcal{T}_k}$  is PD.

#### Nonnegative but not sos polynomial on $\mathcal{Z}(p)$

Let  $(v_1, v_2) \in \mathbb{R}^2$  be the eigenvector of the negative eigenvalue of

$$\begin{pmatrix} \gamma_{\ell_1} & \gamma_{\ell'_1\ell_2} \\ \gamma_{\ell'_1\ell_2} & \gamma_{\ell_1(\ell_2-1)} \end{pmatrix}$$

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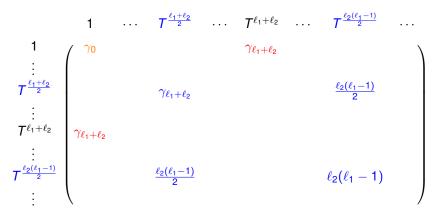
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Then

$$\left(v_{1}t^{\ell_{1}'}+v_{2}t^{\ell_{1}'(\ell_{2}-1)}\right)^{2}=v_{1}^{2}y+2v_{1}v_{2}x^{\ell_{1}'}+v_{2}^{2}y^{\ell_{2}-1}$$

is nonnegative on  $\mathcal{Z}(p)$ , but not sos.

**Case 2:** Both  $\ell_1, \ell_2$  are odd. Then



- 1. Generate any sequence  $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{2s_{r_k}})$  such that  $A_{\gamma}$  is PD.
- 2. You decrease  $\gamma_{\ell_1+\ell_2}$  such that the submatrix  $A|_{\left\{ T^{\ell_1+\ell_2}, T^{\frac{\ell_2(\ell_1-1)}{2}} \right\}}$  is not PSD.

3. Since  $\gamma_{\ell_1+\ell_2}$  occurs in  $(A_{\gamma})|_{\mathcal{T}_k}$  only twice at non-diagonal places, you can increase  $\gamma_0$  such that  $(A_{\gamma})|_{\mathcal{T}_k}$  is PD.

#### Nonnegative but not sos polynomial on $\mathcal{Z}(p)$

Let  $(v_1, v_2) \in \mathbb{R}^2$  be the eigenvector of the negative eigenvalue of

$$\begin{pmatrix} \gamma_{\ell_1+\ell_2} & \gamma_{\frac{\ell_2(\ell_1-1)}{2}} \\ \gamma_{\frac{\ell_2(\ell_1-1)}{2}} & \gamma_{\ell_2(\ell_1-1)} \end{pmatrix}.$$

Then

$$\left(v_{1}t^{\frac{\ell_{1}+\ell_{2}}{2}}+v_{2}t^{\frac{\ell_{2}(\ell_{1}-1)}{2}}\right)^{2}=v_{1}^{2}xy+2v_{1}v_{2}y^{\frac{1+\ell_{2}}{2}}+v_{2}^{2}x^{\ell_{1}-1}$$

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is nonnegative on  $\mathcal{Z}(p)$ , but not sos.

### Thank you for your attention!