The truncated moment problem on some polynomial and rational plane curves

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Outline

Truncated moment problem on curve $p(x, y) = 0$

1. Preliminaries

2. Main results — Using univariate reduction technique

   - Concrete solution:
     $\circ\ y^2 = x^3 \qquad \circ\ yx^2 = 1 \qquad \circ\ y(y-1)(y-\alpha) = 0.$
   - LMI based solution:
     $\circ\ y^i = q(x) \qquad \circ\ y^i x^j = 1 \qquad \circ\ y = p(t), x = g(t)$
   - Solution in terms of the bound on PSD extensions of $M(k)$:
     $\checkmark:\ y = q(x) \qquad \checkmark:\ y^i x^j = 1 \qquad \underbrace{\times:\ y^i = x^j, i, j > 1, \gcd(i,j) = 1}_{\text{The proof gives nonnegative but not sos polynomials.}}.$
   - Bounds on the number of atoms in the minimal measure for the curves above.

3. Proofs

Bivariate truncated moment problem (TMP)

Question

Let $k \in \mathbb{N}$ and

$$\beta = \beta^{(k)} = (\beta_{i,j})_{i,j \in \mathbb{Z}_+, i+j \leq k}$$

a bivariate sequence of real numbers of degree k.

$K \subseteq \mathbb{R}^2$ is a closed subset.

The bivariate truncated moment problem on K (K–TMP): characterize the existence of a positive Borel measure μ on $\mathbb{R}^2$ with support in K, such that

$$\beta_{i,j} = \int_K x^i y^j d\mu(x)$$

for $i, j \in \mathbb{Z}_+$, $i + j \leq k$.

μ is called a K–representing measure (K–RM) of β.

Bivariate moment matrix

The moment matrix $M(k)$ associated to β with the rows and columns indexed by $X^iY^j,\ i+j\le k,$ in degree-lexicographic order

$$1, X, Y, X^2, XY, Y^2, \ldots, X^k, X^{k-1}Y, \ldots, Y^k$$

is defined by where

$$M(k) := \begin{array}{c} \\ 1 \\ X \\ Y \\ \vdots \\ X^{i_1}Y^{j_1} \\ \vdots \\ Y^k \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & X & Y & \cdots & X^{i_2}Y^{j_2} & \cdots & Y^k \end{array} \\ \left[\begin{array}{ccccccc} \beta_{0,0} & \beta_{1,0} & \beta_{0,1} & \cdots & \beta_{i_2,j_2} & \cdots & \beta_{0,k} \\ \beta_{1,0} & \beta_{2,0} & \beta_{1,1} & \cdots & \beta_{i_2+1,j_2} & \cdots & \beta_{1,k} \\ \beta_{0,1} & \beta_{1,1} & \beta_{0,2} & \cdots & \beta_{i_2,j_2+1} & \cdots & \beta_{0,k+1} \\ \vdots & & & \ddots & & & \vdots \\ \beta_{i_1,j_1} & \beta_{i_1+1,j_1} & \beta_{i_1,j_1+1} & \cdots & \beta_{i_1+i_2,j_1+j_2} & \cdots & \beta_{i_1,j_1+k} \\ \vdots & & & & \vdots & \ddots & \vdots \\ \beta_{0,k} & \beta_{1,k} & \beta_{0,k+1} & \cdots & \beta_{i_2,j_2+k} & \cdots & \beta_{0,2k} \end{array}\right] \end{array}$$

Necessary conditions for the existence of a RM

- To every polynomial $p := \sum_{i,j} a_{i,j} x^i y^j \in \mathbb{R}[x,y]_k$, we associate the vector

$$p(X,Y) = \sum_{i,j} a_{i,j} X^i Y^j = a_{0,0} \cdot \overset{1}{\begin{pmatrix} \beta_{0,0} \\ \beta_{1,0} \\ \beta_{0,1} \\ \vdots \\ \beta_{0,k} \end{pmatrix}} + a_{1,0} \cdot \overset{X}{\begin{pmatrix} \beta_{1,0} \\ \beta_{2,0} \\ \beta_{1,1} \\ \vdots \\ \beta_{1,k} \end{pmatrix}} + \cdots + a_{0,k} \cdot \overset{Y^k}{\begin{pmatrix} \beta_{0,k} \\ \beta_{1,k} \\ \beta_{0,k+1} \\ \vdots \\ \beta_{0,2k} \end{pmatrix}}$$

  from the column space of the matrix $M(k)$.

- The matrix $M(k)$ is recursively generated (RG) if for $p, q, pq \in \mathbb{R}[x,y]_k$

$$p(X,Y) = \mathbf{0} \quad \Rightarrow \quad (pq)(X,Y) = \mathbf{0}.$$

Necessary conditions for the existence of a RM

- The matrix $M(k)$ satisfies the variety condition (VC) if

$$\text{rank}\, M(k) \leq \text{card}\, \mathcal{V},$$

where

$$\mathcal{V} := \bigcap_{\substack{g \in \mathbb{R}[x,y]_{\leq k}, \\ g(X,Y)=\mathbf{0} \text{ in } M(k)}} \underbrace{\left\{(x,y) \in \mathbb{R}^2 \colon g(x,y) = 0\right\}}_{\mathcal{Z}(g)}.$$

Proposition (Curto and Fialkow, 96')

If $\beta^{(2k)}$ has a representing measure μ, then

$M(k)$ is positive semidefinite (PSD), RG and satisfies VC.

Sufficient condition for the existence of a RM

Theorem (Flat extension theorem, Curto and Fialkow, 96')

TFAE:

1. $\beta^{(2k)}$ *admits a* $(\operatorname{rank} M(k))$*–atomic RM*.
2. $M(k)$ *is PSD and there is an extension* $M(k+1)$ *such that*

$$\operatorname{rank} M(k+1) = \operatorname{rank} M(k).$$

Solving the TMP on rational curves

Basic ideas

1. For irreducible curve $\mathcal{C}$:
   - Parametrize the curve with one parameter.
   - Solve the corresponding univariate TMP.

2. For reducible curve $\mathcal{C}$:
   - Study decompositions

     $$\beta = \beta^{(1)} + \beta^{(2)},$$

     where

     $\beta^{(1)}$: a moment sequence on one irreducible component of $\mathcal{C}$,

     $\beta^{(2)}$: a moment sequence on the complement of $\mathcal{C}$.

   - Apply the solution to the TMP on each summand $\beta^{(i)}$, $i = 1, 2$.

Bivariate TMP on $p(x,y)=0$ with $\deg p \leq 3$

NC $=$ PSD $+$ RG $+$ VC, $\widetilde{\textbf{NC}}$ $=$ PSD $+$ RG, **num. c.**=numerical conditions

pure ... only relations except coming from p, **FE** ... flat extension

$\#$**atoms** $= \operatorname{rank} M(k) +$ **i**

Bivariate TMP on $p(x,y)=0$ with $\deg p \leq 3$

NC $=$ PSD $+$ RG $+$ VC, $\quad$ $\widetilde{\textbf{NC}}$ $=$ PSD $+$ RG, $\quad$ **num. c.**=numerical conditions

pure ... only relations except coming from p, $\quad$ **FE** ... flat extension

$\#$**atoms** $=$ rank $M(k)+\mathbf{i}$

proved by FE technique $\quad$ proved by univariate reduction technique

$\deg p$	p	Solution	FE exists		i
			pure	non-pure	
2	x^2+y^2-1	$\widetilde{\mathrm{NC}}$	✓	✓	0
	$y-x^2$	NC	✓	✓	0
	$xy-1$	NC	✓	✓	≤ 1 0
	xy	NC	X	✓	1
	y^2-1	NC $\widetilde{\mathrm{NC}}$	✓	✓	0
3	$y-x^3$	$\widetilde{\mathrm{NC}}$+num. c.	✓	X	1
	y^2-x^3	$\widetilde{\mathrm{NC}}$+num. c.	✓	X	1
	xy^2-1	$\widetilde{\mathrm{NC}}$+num. c.	✓	X	1
	$y(y-1)(y-\alpha)$	$\widetilde{\mathrm{NC}}$+num. c.	✓	X	1

Hankel matrix

Let $k \in \mathbb{N}$. For

$$\gamma = (\gamma_0, \dots, \gamma_{2k}) \in \mathbb{R}^{2k+1}$$

we write

$$A_\gamma = \begin{array}{c} \\ 1 \\ T \\ T^2 \\ \vdots \\ T^k \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & T & T^2 & \cdots & T^k \end{array} \\ \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \iddots & \iddots & \gamma_{k+1} \\ \gamma_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \gamma_{2k-1} \\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k-1} & \gamma_{2k} \end{pmatrix} \end{array}$$

TMP for $p(x,y) = y^2 - x^3$

For $\beta^{(2k)}$ we define a univariate sequence

$$\gamma(\mathbf{x}) := (\gamma_0, \mathbf{x}, \underbrace{\gamma_2, \gamma_3, \overbrace{\gamma_4, \ldots, \gamma_{6k-2}, \gamma_{6k-1}, \gamma_{6k}}^{\gamma^{(2)}}}_{\gamma^{(1)}}), \text{ where } \gamma_{2i+3j} = \beta_{i,j}.$$

Define also $\gamma^{(3)} = (\gamma_2, \ldots, \gamma_{6k-2})$.

TMP for $p(x,y) = y^2 - x^3$

For $\beta^{(2k)}$ we define a univariate sequence

$$\gamma(\mathbf{x}) := (\gamma_0, \mathbf{x}, \underbrace{\gamma_2, \gamma_3, \overbrace{\gamma_4, \ldots, \gamma_{6k-2}, \gamma_{6k-1}, \gamma_{6k}}^{\gamma^{(2)}}}_{\gamma^{(1)}}), \text{ where } \gamma_{2i+3j} = \beta_{i,j}.$$

Define also $\gamma^{(3)} = (\gamma_2, \ldots, \gamma_{6k-2})$.

Existence:

(1) β has a $\mathcal{Z}(p)$–RM. $\Leftrightarrow$

(2) β has at most $(\operatorname{rank} M(k) + 1)$–atomic $\mathcal{Z}(p)$–RM. $\Leftrightarrow$

(3) $M(k)$ is PSD and RG, $A_{\gamma^{(1)}}$ is PSD and one of the following holds:

- a) $A_{\gamma^{(1)}}$ is PD and $\operatorname{rank}\left(M(k) \text{ without column/row } Y^k\right) = 3k - 1$.
- b) $\operatorname{rank} A_{\gamma^{(1)}} = \operatorname{rank} A_{\gamma^{(2)}} = \operatorname{rank} A_{\gamma^{(3)}}$.

Uniqueness and cardinality:

- There is a $(\operatorname{rank} M(k))$–atomic $\mathcal{Z}(p)$–RM unless $\operatorname{rank} M(k) = 3k - 1$ and $A_{\gamma^{(1)}}$ is PD.
- The $\mathcal{Z}(p)$–RM is unique if $\operatorname{rank} M(k) < 3k$. Otherwise two minimal $\mathcal{Z}(p)$–RM exist.

Property $(S_{k,m})$

Solution to the TMP based on the size of PSD extensions

$\mathcal{Z}(p) = \{(x, y) \in \mathbb{R}^2 \colon p(x, y) = 0\}$

$\mathcal{Z}(p)$ has property $(S_{k,m})$ if the following are equivalent:

1. $\beta^{(2k)}$ has a $\mathcal{Z}(p)$–RM.
2. $M(k)$ satisfies $p(X, Y) = 0$ and admits a PSD extension $M(k+m)$.

Property $(S_{k,m})$

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$\mathcal{Z}(p)$ has property $(A_{k,m})$ if every $f \in \mathbb{R}[x,y]_{\leq 2k+2}$ with $f|_{\mathcal{Z}(p)} > 0$ is of the form

$$f = \sum_i f_i^2 + p\sum_j g_j^2 - p\sum_\ell h_\ell^2,$$

where $f_i^2, pg_j^2, ph_\ell^2 \in \mathbb{R}[x,y]_{\leq 2m}$.

Property $(S_{k,m})$

Solution to the TMP based on the size of PSD extensions

$\mathcal{Z}(p) = \{(x, y) \in \mathbb{R}^2 \colon p(x, y) = 0\}$

$\mathcal{Z}(p)$ has property $(S_{k,m})$ if the following are equivalent:

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where $f_i^2, pg_j^2, ph_\ell^2 \in \mathbb{R}[x, y]_{\leq 2m}$.

Theorem (Curto and Fialkow, 08')

$$(A_{k,k+m}) \Rightarrow (S_{k,m}) \quad \textit{and} \quad (S_{k,m}) \Rightarrow (A_{k-1,k+m}).$$

Bivariate TMP on $p(x,y)=0$ with $\deg p \geq 4$

proved through property $(A_{k,m(k)})$ (Fialkow,11')

proved by univariate reduction technique

LMI . . . feasibility problem of a linear matrix inequality

$\deg p$	p	$(S_{k,m})$	m	Solution	# atoms
$\ell \geq 4$	$y - q(x)$	✓	$O(k\ell)$ $\ell - 1$	LMI	$k\ell$
	$yx^{\ell-1} - 1$	✓	$O(k\ell)$ ℓ	LMI	$k\ell$
	$y^j - x^\ell$, $j > 1$, irred.	✗	✗	LMI	$k\ell$
	$y^j x^{\ell-j} - 1$, irred.	✓	$O(k\max(j,\ell-j))$	LMI	$k\ell$

$p(x,y)=y-q(x)$ has property $(S_{k,k+\deg q-1})$

Nontrivial: $M(k)$ satisties $Y=q(X)$ and $M(k+\ell-1)$ PSD exists. $\Rightarrow$ $\beta^{(2k)}$ has a $\mathcal{Z}(p)$-RM.

1. Basis of the column space of $M(k+\ell-1)$:

$$Y^iX^j, \quad {\scriptstyle i=0,\ldots,k,\; j=0,\ldots,\deg q-1,\; i+j\leq k+\ell-2.}$$

2. Relations between moments: Writing $q(x)=\sum_{i=0}^{\ell} q_i x^i$ we have

$$\beta_{i,j}=q_\ell\beta_{i+\ell,j-1}+q_{\ell-1}\beta_{i+\ell-1,j-1}+\ldots+q_0\beta_{i,j-1}, \quad {\scriptstyle i\in\mathbb{Z}_+,\; j\in\mathbb{N},\; i+j\leq 2(k+\ell-2).}$$

3. The corresponding univariate sequence: Let

$$q_{i,j,s}:=\begin{cases} \displaystyle\sum_{\substack{0\leq i_1,\ldots,i_j\leq \ell,\\ i_1+\ldots+i_j=s-i}} q_{i_1}q_{i_2}\ldots q_{i_j}, & \text{if } i\leq s\leq i+j\ell,\\ 0, & \text{otherwise.}\end{cases}$$

Let

$$\gamma_t=\frac{1}{(q_\ell)^{\lfloor \frac{t}{\ell}\rfloor}}\Big(\beta_{t \bmod \ell,\lfloor \frac{t}{\ell}\rfloor}-\sum_{s=0}^{t-1} q_{t \bmod \ell,\lfloor \frac{t}{\ell}\rfloor,s}\cdot\gamma_s\Big) \quad {\scriptstyle t=0,\ldots,2k\ell+2,}$$

4. $\beta^{(2k)}$ has a $\mathcal{Z}(p)$–RM. $\Leftrightarrow$ $\gamma^{(2k\ell)} := (\gamma_0, \ldots, \gamma_{2k\ell})$ has a $\mathbb{R}$–RM.

5. $M(k+\ell-1)$ PSD $\Rightarrow$ $A_{\gamma^{(2k\ell+2)}}$ PSD, where

$$\gamma^{(2k\ell+2)} := (\gamma_0, \ldots, \gamma_{2k\ell+2}).$$

6. Use the solution to the $\mathbb{R}$–TMP (Curto and Fialkow, 91'), i.e., TFAE:
   - $\gamma^{(2k\ell)}$ has a $\mathbb{R}$-RM.
   - $\gamma^{(2k\ell)}$ has a (rank $A_{\gamma^{(2k\ell)}}$)–atomic $\mathbb{R}$–RM.
   - $A_{\gamma^{(2k\ell)}}$ has a PSD extension $A_{\gamma^{(2k\ell+2)}}$.

7. Decrease the number of atoms by 1 in the case rank $A_{\gamma^{(0,2k\ell)}} = k\ell + 1$:
   This can be achieved by manipulating $\gamma_{2k\ell-1}$ which does not affect the original sequence $\beta_{i,j}$, $i, j \in \mathbb{Z}_+$, $i + j \leq 2k$.

LMI based solution for $p(x,y) = y - q(x)$, $q(x)=\sum_{i=0}^{\ell} q_i x^i$

Theorem

TFAE:

1. $\beta^{(2k)}$ *has a* $\mathcal{Z}(p)$*–RM.*
2. $\beta_{i,j} = \sum_{p=0}^{\ell} q_p \beta_{i+p,j-1}$ *for every* $i,j \in \mathbb{Z}_+$*, such that* $i+j \leq 2k-\ell+1$ *and there exists missing values* γ_i *in the sequence* $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_{2k\ell+2}$ *defined for* t *from the set*

$$\left\{t \in \mathbb{N}_0 \colon t \bmod \ell + \left\lfloor \frac{t}{\ell} \right\rfloor \leq 2k\right\},$$

by

$$\gamma_t = \frac{1}{(q_\ell)^{\lfloor \frac{t}{\ell} \rfloor}} \left(\beta_{t \bmod \ell, \lfloor \frac{t}{\ell} \rfloor} - \sum_{s=0}^{t-1} q_{t \bmod \ell, \lfloor \frac{t}{\ell} \rfloor, s} \cdot \gamma_s \right),$$

$$q_{i,j,s} := \begin{cases} \displaystyle\sum_{\substack{0 \leq i_1, \ldots, i_j \leq \ell, \\ i_1 + \ldots + i_j = s-i}} q_{i_1} q_{i_2} \ldots q_{i_j}, & \text{if } i \leq s \leq i + j\ell, \\ 0, & \text{otherwise.} \end{cases}$$

such that

$$A_\gamma \succeq 0.$$

Example: $p(x, y) = y - x^4$

$$\gamma_t = \beta_{t \bmod t, \left\lfloor \frac{t}{4} \right\rfloor}$$

for every t from the set

$$\{t \in \mathbb{Z}_+ : \ t \le 8k, \ t \notin \{8k-5, 8k-2, 8k-1\}\}.$$

The matrix A_γ is equal to

$$\left(\begin{array}{ccccccccc|c}
\gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \cdots & & & \cdots & \gamma_{4k} & \gamma_{4k+1} \\
\gamma_1 & \gamma_2 & \gamma_3 & \iddots & & & & & & \vdots \\
\gamma_2 & \gamma_3 & \iddots & & & & & \gamma_{8k-6} & \boldsymbol{\gamma_{8k-5}} & \gamma_{8k-4} \\
\gamma_3 & \iddots & & & & & \gamma_{8k-6} & \boldsymbol{\gamma_{8k-5}} & \gamma_{8k-4} & \gamma_{8k-3} \\
\vdots & & & & & \gamma_{8k-6} & \boldsymbol{\gamma_{8k-5}} & \gamma_{8k-4} & \gamma_{8k-3} & \boldsymbol{\gamma_{8k-2}} \\
 & & & & \gamma_{8k-6} & \boldsymbol{\gamma_{8k-5}} & \gamma_{8k-4} & \gamma_{8k-3} & \boldsymbol{\gamma_{8k-2}} & \boldsymbol{\gamma_{8k-1}} \\
\vdots & & & \gamma_{8k-6} & \boldsymbol{\gamma_{8k-5}} & \gamma_{8k-4} & \gamma_{8k-3} & \boldsymbol{\gamma_{8k-2}} & \boldsymbol{\gamma_{8k-1}} & \gamma_{8k} \\
\gamma_{4k} & & \gamma_{8k-6} & \boldsymbol{\gamma_{8k-5}} & \gamma_{8k-4} & \gamma_{8k-3} & \boldsymbol{\gamma_{8k-2}} & \boldsymbol{\gamma_{8k-1}} & \gamma_{8k} & \boldsymbol{\gamma_{8k+1}} \\
\hline
\gamma_{4k+1} & \cdots & \boldsymbol{\gamma_{8k-5}} & \gamma_{8k-4} & \gamma_{8k-3} & \boldsymbol{\gamma_{8k-2}} & \boldsymbol{\gamma_{8k-1}} & \gamma_{8k} & \boldsymbol{\gamma_{8k+1}} & \boldsymbol{\gamma_{8k+2}}
\end{array}\right)$$

$p(x,y) = y^{\ell_2}x^{\ell_1} - 1$, $\gcd(\ell_1,\ell_2) = 1$, has $(S_{k,(k+1)\ell_2})$

1. Parametrization: $x = t^{\ell_2}$, $y = t^{-\ell_1}$.
2. The univariate sequence: $\beta_{ij} \leftrightarrow \gamma_{i\ell_2 - j\ell_1}$.

   $\gamma := (\gamma_{-2k\ell_1}, \ldots, \gamma_{2k\ell_2})$ has some gaps.

3. $\beta^{(2k)}$ has a $\mathcal{Z}(p)$–RM $\quad\Leftrightarrow\quad$ γ has a $(\mathbb{R}\setminus\{0\})$–RM.
4. Solution of the strong $(\mathbb{R}\setminus\{0\})$-TMP (Z,22'), i.e., TFAE:
   - γ has a $(\mathbb{R}\setminus\{0\})$–RM.
   - γ can be extended to the sequence

     $\widetilde{\gamma} := (\gamma_{-2k\ell_1-2}, \ldots, \gamma_{2k\ell_2+2})$ without gaps and $A_{\widetilde{\gamma}}$ is PSD.

5. $M(m+\ell)$ PSD for ℓ large enough $\quad\Rightarrow\quad$ $A_{\widetilde{\gamma}}$ PSD.

$p(x,y) = y^{\ell_2} - x^{\ell_1}$, $\ell_2 > \ell_1 > 1$, irreducible does not have property $(S_{k,m})$ for every m

1. Parametrization: $x = t^{\ell_2}$, $y = t^{\ell_1}$.
2. The univariate sequence: $\beta_{ij} \;\leftrightarrow\; \gamma_{i\ell_2 + j\ell_1}$.

   $\gamma := \gamma_0, \ldots, \gamma_{2k\ell_2}$ has some gaps.

3. $\beta^{(2k)}$ has a $\mathcal{Z}(p)$–RM $\quad\Leftrightarrow\quad$ γ has a $\mathbb{R}$–RM.
4. Solution of the $\mathbb{R}$-TMP: γ has a $\mathbb{R}$–RM $\Leftrightarrow$ γ can be extended to the sequence

   $$\gamma^{(2k\ell_2+2)} = (\gamma_0, \ldots, \gamma_{2k\ell_2+2}) \quad \text{without gaps and} \quad A_{\gamma^{(2k\ell_2+2)}} \text{ is PSD.}$$

5. One can construct a sequence γ such that A_γ is not even partially PSD, but it can be extended with $\gamma_{2k\ell_2+1}, \gamma_{2k\ell_2+2}, \ldots$ to a matrix such that the submatrices corresponding to matrices $M(k+m)$ are PSD.

- Columns of $M(\ell)$ correspond to columns

$$\mathcal{T}_\ell = \{T^s : s = a\ell_1 + b\ell_2,\ a, b = 0, \ldots, \ell\} = \{1, T^{s_1}, T^{s_2}, \ldots, T^{s_{r_\ell}}\}$$

of the univariate Hankel matrix $A_{\gamma^{(2\ell\ell_2)}}$.

- Then

$$A_{\gamma^{(2\ell\ell_2)}} = \begin{array}{c} 1 \\ \vdots \\ T^{s_1} \\ \vdots \\ T^{s_2} \\ \vdots \\ T^{s_{r_\ell}} \end{array} \overset{\begin{array}{ccccccc} 1 & \cdots & T^{s_1} & \cdots & T^{s_2} & \cdots & T^{s_{r_\ell}} \end{array}}{\begin{pmatrix} \gamma_0 & \gamma_{s_1} & \gamma_{s_2} & \gamma_{s_{r_\ell}} \\ \gamma_{s_1} & \gamma_{2s_1} & \gamma_{s_1+s_2} & \gamma_{s_1+s_{r_\ell}} \\ \gamma_{s_2} & \gamma_{s_1+s_2} & \gamma_{2s_2} & \gamma_{s_2+s_{r_\ell}} \\ \gamma_{s_{r_\ell}} & \gamma_{s_{r_\ell}+s_1} & \gamma_{s_{r_\ell}+s_2} & \gamma_{2s_{r_\ell}} \end{pmatrix}}$$

The specified part of $A_{\gamma^{(2\ell\ell_2)}}$ corresponds to $M(\ell)|_{\text{rows/columns in the basis}}$.

If $M(k)|_{\text{basis}}$ is PD, then A is PD and it has infinitely many PD extensions:

$$
\begin{array}{c}
\\
1 \\ \vdots \\ T^{s_1} \\ \vdots \\ T^{s_2} \\ \vdots \\ T^{s_{r_\ell}} \\ \color{red}{T^{s_{r_\ell}+1}}
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc}
1 & \cdots & T^{s_1} & \cdots & T^{s_2} & \cdots & T^{s_{r_\ell}} & \color{red}{T^{s_{r_\ell}+1}}
\end{array}
\\
\begin{pmatrix}
\gamma_0 & \gamma_{s_1} & \gamma_{s_2} & \gamma_{s_{r_\ell}} & \gamma_{s_{r_\ell}+1} \\
\gamma_{s_1} & \gamma_{2s_1} & \gamma_{s_1+s_2} & \gamma_{s_1+s_{r_\ell}} & \gamma_{s_1+s_{r_\ell}+1} \\
\gamma_{s_2} & \gamma_{s_1+s_2} & \gamma_{2s_2} & \gamma_{s_2+s_{r_\ell}} & \gamma_{s_2+s_{r_\ell}+1} \\
\gamma_{s_{r_\ell}} & \gamma_{s_{r_\ell}+s_1} & \gamma_{s_{r_\ell}+s_2} & \gamma_{2s_{r_\ell}} & \color{red}{\gamma_{2s_{r_\ell}+1}} \\
\gamma_{s_{r_\ell}+1} & \gamma_{s_{r_\ell}+1+s_1} & \gamma_{s_{r_\ell}+1+s_2} & \color{red}{\gamma_{2s_{r_\ell}+1}} & \color{red}{\gamma_{2s_{r_\ell}+2}}
\end{pmatrix}
\end{array}
$$

- $\gamma_{2s_{r_\ell}+1}$ is chosen arbitrarily, while $\gamma_{2s_{r_\ell}+2}$ must be such that the Schur complement is positive.
- One can continue in this way to determine $T^{s_{r_\ell}+2}, T^{s_{r_\ell}+3}, \ldots$. On the side of β one gets a sequence of extensions $\beta^{(2k)}, \beta^{(2k+2)}, \beta^{(2k+4)}, \ldots$ such that $M(k+1), M(k+2), \ldots$ are PSD.
- So one gets a full sequence $\beta^{(\infty)}$ with $M(\infty)$ PSD.

- γ can be chosen such that it does not have a measure, even though $(A)|_{\mathcal{T}_k}$ is PD. Consequently, we will get β with infinitely many extensions but without a measure.

Case 1: One of ℓ_1, ℓ_2 is even. Say $\ell_1 = 2\ell_1'$. Then

$$\begin{array}{c} \\ 1 \\ \vdots \\ T^{\ell_1'} \\ \vdots \\ T^{\ell_1} \\ \vdots \\ T^{\ell_1'(\ell_2-1)} \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & \dots & T^{\ell_1'} & \dots & T^{\ell_1} & \dots & T^{\ell_1'(\ell_2-1)} & \dots \end{array} \\ \begin{pmatrix} \gamma_0 & & & & \gamma_{\ell_1} & & & \\ & & \gamma_{\ell_1} & & & & \gamma_{\ell_1'\ell_2} & \\ \gamma_{\ell_1} & & & & & & & \\ & & \gamma_{\ell_1'\ell_2} & & & & \gamma_{\ell_1(\ell_2-1)} & \\ & & & & & & & \end{pmatrix} \end{array}.$$

1. Generate any sequence $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{2s_{r_k}})$ such that A_γ is PD.
2. You decrease γ_{ℓ_1} such that the submatrix $(A_\gamma)|_{\{T^{\ell_1'}, T^{\ell_1'(\ell_2-1)}\}}$ is not PSD.
3. Since γ_{ℓ_1} occurs in $(A_\gamma)|_{\mathcal{T}_k}$ only twice at non-diagonal places, you can increase γ_0 such that $(A_\gamma)|_{\mathcal{T}_k}$ is PD.

Nonnegative but not sos polynomial on $\mathcal{Z}(p)$

Let $(v_1, v_2) \in \mathbb{R}^2$ be the eigenvector of the negative eigenvalue of

$$\begin{pmatrix} \gamma_{\ell_1} & \gamma_{\ell_1'\ell_2} \\ \gamma_{\ell_1'\ell_2} & \gamma_{\ell_1(\ell_2-1)} \end{pmatrix}.$$

Then

$$\left(v_1 t^{\ell_1'} + v_2 t^{\ell_1'(\ell_2-1)}\right)^2 = v_1^2 y + 2 v_1 v_2 x^{\ell_1'} + v_2^2 y^{\ell_2-1}$$

is nonnegative on $\mathcal{Z}(p)$, but not sos.

Case 2: Both ℓ_1, ℓ_2 are odd. Then

$$
\begin{array}{c|ccccccc}
 & 1 & \cdots & T^{\frac{\ell_1+\ell_2}{2}} & \cdots & T^{\ell_1+\ell_2} & \cdots & T^{\frac{\ell_2(\ell_1-1)}{2}} \quad \cdots \\
\hline
1 & \gamma_0 & & & & \gamma_{\ell_1+\ell_2} & & \\
\vdots & & & & & & & \\
T^{\frac{\ell_1+\ell_2}{2}} & & & \gamma_{\ell_1+\ell_2} & & & & \frac{\ell_2(\ell_1-1)}{2} \\
\vdots & & & & & & & \\
T^{\ell_1+\ell_2} & \gamma_{\ell_1+\ell_2} & & & & & & \\
\vdots & & & & & & & \\
T^{\frac{\ell_2(\ell_1-1)}{2}} & & & \frac{\ell_2(\ell_1-1)}{2} & & & & \ell_2(\ell_1-1) \\
\vdots & & & & & & &
\end{array}
$$

1. Generate any sequence $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{2s_{r_k}})$ such that A_γ is PD.
2. You decrease $\gamma_{\ell_1+\ell_2}$ such that the submatrix $A|_{\{T^{\ell_1+\ell_2}, T^{\frac{\ell_2(\ell_1-1)}{2}}\}}$ is not PSD.
3. Since $\gamma_{\ell_1+\ell_2}$ occurs in $(A_\gamma)|_{\mathcal{T}_k}$ only twice at non-diagonal places, you can increase γ_0 such that $(A_\gamma)|_{\mathcal{T}_k}$ is PD.

Nonnegative but not sos polynomial on $\mathcal{Z}(p)$

Let $(v_1, v_2) \in \mathbb{R}^2$ be the eigenvector of the negative eigenvalue of

$$\begin{pmatrix} \gamma_{\ell_1+\ell_2} & \gamma_{\frac{\ell_2(\ell_1-1)}{2}} \\ \gamma_{\frac{\ell_2(\ell_1-1)}{2}} & \gamma_{\ell_2(\ell_1-1)} \end{pmatrix}.$$

Then

$$\left(v_1 t^{\frac{\ell_1+\ell_2}{2}} + v_2 t^{\frac{\ell_2(\ell_1-1)}{2}}\right)^2 = v_1^2 xy + 2v_1 v_2 y^{\frac{1+\ell_2}{2}} + v_2^2 x^{\ell_1-1}$$

is nonnegative on $\mathcal{Z}(p)$, but not sos.

Thank you for your attention!