A gap between positive polynomials and sums of squares in various settings

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joint work with

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Outline

quantitative estimates on volumes of pos vs sos cones

1. Preliminaries
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     - positive maps vs completely positive maps
     - cross-positive maps vs completely cross-positive maps
     - copositive vs completely positive matrices
   - Converting to polynomials:
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Positive and completely positive maps

Definitions

A linear map

$$\Phi : M_n(\mathbb{R}) \to M_m(\mathbb{R})$$

such that $\Phi(A^T) = \Phi(A)^T$ for all $A \in M_n(\mathbb{R})$, is:

- positive if

$$A \succeq 0 \quad \Rightarrow \quad \phi(A) \succeq 0.$$

- k-positive if

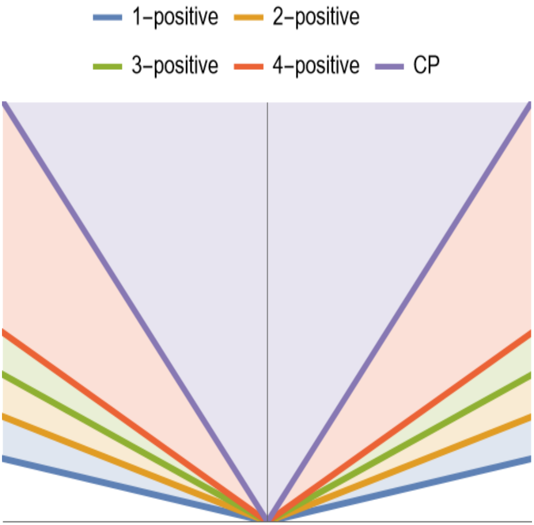
$$\phi_k\left(\begin{pmatrix} A_{11} & \dots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \dots & A_{kk} \end{pmatrix} \right) = \begin{pmatrix} \phi(A_{11}) & \dots & \phi(A_{1k}) \\ \vdots & \ddots & \vdots \\ \phi(A_{k1}) & \dots & \phi(A_{kk}) \end{pmatrix}$$

  is positive.

- completely positive (CP) if it is k-positive for every $k \in \mathbb{N}$.

Positive and completely positive maps

Mental picture



Positive and completely positive maps

Problems and a small sample of existing literature

Problem A.1: Establish asymptotically exact quantitative bounds on the fraction of positive maps that are CP.

Problem A.2: Derive algorithm to produce positive maps that are not CP from random input data.

Small sample of related literature:

- Arveson (2009): Let $n, m \geq 2$. Then the probability p that a positive map $\varphi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is CP satisfies $0 < p < 1$.
- Szarek, Werner, Życzkowski (2008): for the case $m = n$ provide quantitative bounds on p and establish its asymptotic behaviour.
- Collins, Hayden, Nechita (2017): random techniques for constructing k-positive maps that are not $(k+1)$-positive in large dimensions.

Positive maps meet real algebraic geometry

$\mathcal{L}(\mathbb{S}_n, \mathbb{S}_m)$ ... the vector space of all linear maps from $\mathbb{S}_n$ to $\mathbb{S}_m$,

$\mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2}$ ... biforms in $\mathrm{x} = (x_1, \ldots, x_n)$ and $\mathrm{y} = (y_1, \ldots, y_m)$ of bidegree $(2, 2)$

There is a natural bijection

$$\Gamma : \mathcal{L}(\mathbb{S}_n, \mathbb{S}_m) \to \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2},$$
$$\Phi \mapsto p_\Phi(\mathrm{x}, \mathrm{y}) := \mathrm{y}^T \Phi(\mathrm{x}\mathrm{x}^T)\mathrm{y}.$$

Proposition

Let $\Phi : \mathbb{S}_n \to \mathbb{S}_m$ be a linear map. Then:

1. Φ is positive iff p_Φ is nonnegative.
2. Φ is completely positive iff p_Φ is a sum of squares (SOS). (Choi-Kraus theorem)

Corollary

The following probabilities (w.r.t. the corresponding distributions) are equal:

1. *The probability that a positive map $\Phi \in \mathcal{L}(\mathbb{S}_n, \mathbb{S}_m)$ is CP.*
2. *The probability that a nonnegative biform $p_\Phi \in \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2}$ is SOS.*

Cross–positive and completely cross–positive maps

Definitions

A linear map

$$\Phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$$

is:

- cross-positive if
$$\forall U, V \succeq 0 : \langle U, V\rangle = 0 \Rightarrow \langle \phi(U), V\rangle \geq 0.$$
- k-cross-positive if
$$\phi_k\left(\begin{pmatrix} A_{11} & \dots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \dots & A_{kk} \end{pmatrix} \right) = \begin{pmatrix} \phi(A_{11}) & \dots & \phi(A_{1k}) \\ \vdots & \ddots & \vdots \\ \phi(A_{k1}) & \dots & \phi(A_{kk}) \end{pmatrix}$$
is cross–positive.
- completely cross–positive (CCP) if it is k-cross-positive for every $k \in \mathbb{N}$.

Cross–positive and completely cross–positive maps

Problems and a small sample of existing literature

Problem B.1: Establish asymptotically exact quantitative bounds on the fraction of cross–positive maps that are CCP.

Problem B.2: Derive algorithm to produce cross–positive maps that are not CCP from random input data.

Small sample of related literature:

- Schneider, Vidyasagar (1970):
  - $\phi(\cdot)$ is crp if and only if $\exp(t\phi(\cdot))$ is positive for every $t > 0$.
  - Characterized cross–positive maps on polyhedral cones.
- Cuchiero, Filipović, Mayerhofer, Teichmann (2011) established the importance of cross–positive and completely cross-positive maps in math finance.
- Kuzma, Omladič, Šivic, Teichmann (2015) constructed, for the first time, a proper cross–positive map. (Not of the form $X \mapsto \tilde{\phi}(X) + CX + XC^T$, where $\tilde{\phi}$ is positive.)

Cross–positive maps meet RAG

$I \subseteq \mathbb{R}[\mathrm{x}, \mathrm{y}]$ ... the ideal generated by $\mathrm{y}^T\mathrm{x} = \sum_i \mathrm{x}_i \mathrm{y}_i$,

$I_{2,2} \subseteq \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2}$ ... $I_{2,2} = I \cap \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2}$,

$V(I)$ ... the variety $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid y^T x = 0\}$

Let us define

$$\Gamma : \mathcal{L}(M_n, M_n) \to \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2},$$
$$\Phi \mapsto p_\Phi(\mathrm{x}, \mathrm{y}) := \mathrm{y}^T \Phi(\mathrm{x}\mathrm{x}^T)\mathrm{y}.$$

Proposition

Let $\Phi : M_n \to M_n$ be a linear map. Then:

1. Φ is cross–positive iff p_Φ is nonnegative on $V(I)$.
2. If Φ is CCP then p_Φ is a sum of squares modulo I.

Corollary

The probability that a cross–positive map $\Phi \in \mathcal{L}(M_n, M_n)$ is CCP is bounded above by the probability that a nonnegative biform $p_\Phi + I_{2,2} \in \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2}/I_{2,2}$ is SOS. (Here we use compatible distributions.)

Copositive and completely positive matrices

Definitions

$\mathbb{S}_n \dots$ real symmetric $n \times n$ matrices

A matrix

$$A = (a_{ij})_{i,j} \in \mathbb{S}_n$$

is:

- copositive (COP) if $v^T A v \geq 0$ for every $v \in \mathbb{R}^n_{\geq 0}$.
- positive semidefinite (PSD) if $v^T A v \geq 0$ for every $v \in \mathbb{R}^n$.
- nonnegative (NN) if $a_{ij} \geq 0$ for every i, j.
- SPN if $A = P + N$ for some P PSD and N NN.
- doubly nonnegative (DNN) if $A = P \cap N$ for some P PSD and N NN.
- completely positive (CP) if $A = BB^T$ for some $B \in \mathbb{R}^{n \times k}_{\geq 0}$.

Copositive vs completely positive matrices

Problems and a small sample of existing literature

Problem C.1: Establish asymptotically exact quantitative bounds on the fraction of COP matrices that are CP.

Problem C.2: Derive algorithm to produce COP matrices that are not CP.

Small sample of related literature:

- Maxfield, Minc (1962), Hall, Newman (1963): $\mathrm{COP}_n = \mathrm{SPN}_n$ holds only for $n \leq 4$.
- Parrilo (2000): $\mathrm{int}(\mathrm{COP}_n) \subseteq \bigcup_r K_n^{(r)}$, where $(\mathrm{x}^2 = (x_1^2, \ldots, x_n^2))$

$$K_n^{(r)} := \{A \in \mathbb{S}_n \colon (\sum_{i=1}^{n} x_i^2)^r \cdot (\mathrm{x}^2)^T A \mathrm{x}^2 \text{ is a sum of squares of forms}\}.$$

- Dickinson, Dür, Gijben, Hildebrand (2013): $\mathrm{COP}_5 \neq K_5^{(r)}$ for any $r \in \mathbb{N}$.
- Laurent, Schweighofer, Vargas (2022, 23): $\mathrm{COP}_5 = \bigcup_r K_5^{(r)}$ and $\mathrm{COP}_6 \neq \bigcup_r K_6^{(r)}$.

Copositive matrices meet RAG

$\mathbb{R}[\mathrm{x}^2]_{4,e}$ ... forms in $\mathrm{x}^2 = (x_1^2, \ldots, x_n^2)$ of degree 4, i.e., *quartic even forms*.

There is a natural bijection

$$\Gamma : \mathbb{S}_n \to \mathbb{R}[\mathrm{x}]_{4,e}, \quad A \mapsto q_A(\mathrm{x}) := (\mathrm{x}^2)^T A \mathrm{x}^2 = \sum_{i,j=1}^{n} a_{ij} x_i^2 x_j^2.$$

Proposition

Let $A \in \mathbb{S}_n$ be a matrix. Then:

1. A is COP iff q_A is nonnegative. *(q_A ... POS)*
2. A is PSD iff q_A is of the form $\sum_i \left(\sum_j f_{ij} x_j^2\right)^2$. *($q_A$ ... lin-SOS)*
3. A is NN iff q_A has nonnegative coefficients. *(q_A ... NN)*
4. A is SPN iff q_A is of the form $\sum_i \left(\sum_j f_{ij} x_i x_j\right)^2$ (Parrilo, 00') *(q_A ... SOS)*
5. A is DNN iff q_A is ℓ-SOS and NN. *(q_A ... DNN)*
6. A is CP iff q_A is of the form $\sum_i \left(\sum_j f_{ij} x_j^2\right)^2$ with $f_{ij} \geq 0$. *(q_A ... CP)*

Corollary. The gaps between COP/PSD/NN/SPN/DNN/CP matrices correspond to the gaps between POS/ℓ-SOS/NN/SOS/DNN/CP even quartics.

Gap between positive and sos polynomials

$\mathbb{R}[\mathrm{x}]_{2k}$ ... forms in $\mathrm{x} = (x_1, \ldots, x_n)$ of degree $2k$

Theorem (Blekherman, 2006)

For $n \geq 3$ and fixed k the probability p_n that a positive polynomial $f \in \mathbb{R}[\mathrm{x}]_{2k}$ is sum of squares, satisfies

$$\left(C_1 \cdot \frac{1}{n^{(k-1)/2}}\right)^{\dim \mathbb{R}[x]_{2k}-1} \leq p_n \leq \left(C_2 \cdot \frac{1}{n^{(k-1)/2}}\right)^{\dim \mathbb{R}[x]_{2k}-1},$$

where C_1, C_2 are absolute constants.

In particular, for $2k = 4$,

$$p_n \in \Theta\Big(\big(\frac{1}{\sqrt{n}}\big)^{\dim \mathbb{R}[x]_4 - 1}\Big).$$

Solutions to Problems A.1, B.1, C.1

Theorem A.1 [Klep, McCullough, Šivic, Z, 2019]: For $n, m \geq 3$ the probability p_n that a positive map $\Phi : \mathbb{S}_n \to \mathbb{S}_n$ is CP, satisfies

$$p_n \in \Theta\Big(\big(\frac{1}{\sqrt{n}}\big)^d\Big),$$

where $d = \dim\{\Phi \mid \Phi : \mathbb{S}_n \to \mathbb{S}_n \text{ linear map}\} - 1$.

Theorem B.1 [Klep, Šivic, Z, 2024+]: For $n \geq 3$ the probability p_n that a cross–positive map $\Phi : \mathbb{S}_n \to \mathbb{S}_n$ is CCP, satisfies

$$p_n \in O\Big(\big(\frac{1}{\sqrt{n}}\big)^d\Big),$$

where $d = \dim\{\Phi \mid \Phi : \mathbb{S}_n \to \mathbb{S}_n \text{ linear map}\} - 1$.

Theorem C.1 [Klep, Štrekelj, Z, 2024]: For $n > 4$ the probability p_n that a copositive matrix $A \in \mathbb{S}_n$ is CP, satisfies

$$\left(2^{-8} \cdot 3^{-2}\right)^{\dim \mathbb{S}_n - 1} \leq p_n.$$

Solutions to Problems A.2, B.2, C.2

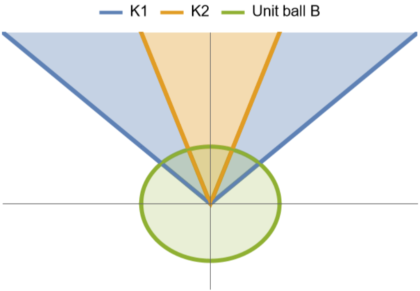
Problem A.2, B.2 *[Klep, McCullough, Šivic, Z, 2019, 2024+]*:

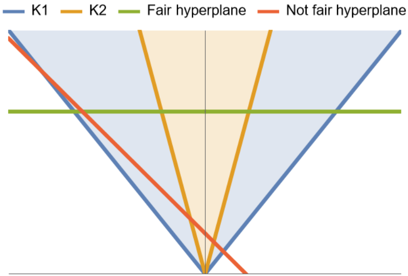
Construction of nonnegative (nonnegative modulo $V(I)$)) biquadratic biforms that are not sums of squares biforms (modulo I) by specializing the algorithm by Blekherman, Smith, Velasco (2016) to produce pos not sos forms on varieties, which are not of minimal degree.

Problem C.2 *[Klep, Štrekelj, Z, 2023+]*:

Free probability inspired construction of $\mathrm{DNN}_n \setminus \mathrm{CP}_n$, $n \geq 5$, matrices. Dually, we obtain matrices from $\mathrm{COP}_n \setminus \mathrm{SPN}_n$.

2. Discussion on volume estimates





Gap between positive and sos polynomials asymptotically not visible in the ball of the ℓ^1 norm

- $\mathbb{R}[\mathrm{x}]_{2k}$ is equipped with the natural L^2 inner product

$$\langle f, g\rangle = \int_{S^{n-1}} fg \, \mathrm{d}\sigma,$$

where and σ is the rotation invariant probability measures on the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

- Let $\| \cdot \|_1$ the ℓ^1 norm on the vector of coefficients, i.e.,

$$\| \sum_{\alpha} a_\alpha \mathrm{x}^\alpha \|_1 = \sum_{\alpha} |a_\alpha|.$$

- E.g., for $k = 2$, due to the equality (and Rogers-Shepard inequality)

$$x_i x_j x_k x_\ell = \frac{1}{2}(x_i x_j + x_k x_\ell)^2 - \frac{1}{2} x_i^2 x_j^2 - \frac{1}{2} x_k^2 x_\ell^2,$$

the volume radii of positive and sos polynomials is the unit ball B_1 of $\| \cdot \|_1$ are bounded by absolute constants.

Blekherman's result on the gap between positive and sos polynomials is in the fair hyperplane of the L^2 norm

- $\mathbb{R}[\mathrm{x}]_{2k}$ is equipped with the natural L^2 inner product

$$\langle f, g \rangle = \int_{S^{n-1}} fg \, \mathrm{d}\sigma,$$

  where and σ is the rotation invariant probability measures on the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

- Volume estimates refer to the sections

$$\mathrm{POS}_{2k} \cap \mathcal{H} \quad \text{and} \quad \mathrm{SOS}_{2k} \cap \mathcal{H},$$

  where $\mathcal{H}$ is the hyperplane of forms with average 1 on S^{n-1}.

3. Proofs

Problem A.1

1. $\mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2}$ is equipped with the natural L^2 inner product

$$\langle f, g\rangle = \int_{S^{n-1}\times S^{m-1}} fg \,\mathrm{d}\sigma = \int_{x\in S^{n-1}} \left(\int_{y\in S^{m-1}} fg \,\mathrm{d}\sigma_2(y)\right) \mathrm{d}\sigma_1(x),$$

   where $\sigma = \sigma_1 \times \sigma_2$ is the product measure of rotation invariant probability measures σ_1, σ_2 on the unit spheres $S^{n-1} \subset \mathbb{R}^n$, $S^{m-1} \subset \mathbb{R}^m$.

2. $\mathcal{H}$ is the affine hyperplane

$$\mathcal{H} = \left\{ f \in \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2} \colon \int_{S^{n-1}\times S^{m-1}} f \,\mathrm{d}\sigma = 1 \right\}.$$

3. $z := \left(\sum_{i=1}^n x_i^2\right)\left(\sum_{j=1}^m y_j^2\right)$ and thus

$$\mathcal{M} = \mathcal{H} - z = \left\{ f \in \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2} \colon \int_{S^{n-1}\times S^{m-1}} f \,\mathrm{d}\sigma = 0 \right\}.$$

4. The estimates of $\mathrm{vrad}(\mathrm{POS} \cap \mathcal{H} - z)$ and $\mathrm{vrad}(\mathrm{SOS} \cap \mathcal{H} - z)$ follow closely Blekherman's proof for $\mathbb{R}[\mathrm{x}]_k$.

Problem B.1

1. Let $T := (S^{n-1} \times S^{n-1}) \cap V(I)$ and equip it with the unique $SO(n)$-invariant measure. T is also known as *the Stiefel manifold* of all 2-frames in $\mathbb{R}^n$.
2. $\mathcal{Q} := \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2}/(I \cap \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2})$ is equipped with the natural L^2 inner product

$$\langle f, g \rangle = \int_T fg \, \mathrm{d}\sigma.$$

3. $\mathcal{H}$ is the affine hyperplane

$$\mathcal{H} = \left\{ f \in \mathcal{Q} \colon \int_T f \, \mathrm{d}\sigma = 1 \right\}.$$

4. $z := \left(\sum_{i=1}^n x_i^2 \right) \left(\sum_{j=1}^n y_j^2 \right)$ and thus

$$\mathcal{M} = \mathcal{H} - z = \left\{ f \in \mathcal{Q} \colon \int_T f \, \mathrm{d}\sigma = 0 \right\}.$$

Problem B.1

5. Only

$$\mathrm{vrad}(\mathrm{SOS} \cap \mathcal{H} - z) \leq (*) \qquad \text{and} \qquad (*) \leq \mathrm{vrad}(\mathrm{POS} \cap \mathcal{H} - z)$$

can be obtained using Blekherman's proof for $\mathbb{R}[\mathrm{x}]_k$, where the main novelty is the following inequality:

Proposition (Reverse Hölder inequality (RHI))

For a bilinear biform $g \in \mathbb{R}[\mathrm{x}, \mathrm{y}]_{1,1}/(I \cap \mathbb{R}[\mathrm{x}, \mathrm{y}]_{1,1})$ we have

$$\left(\int_T g^4 \, \mathrm{d}\sigma\right)^{\frac{1}{4}} = \|g\|_4 \leq \underbrace{\sqrt{6}}_{\substack{\text{Main observation:} \\ \text{independence of } n}} \|g\|_2 = \sqrt{6} \left(\int_T g^2 \, \mathrm{d}\sigma\right)^{\frac{1}{2}}.$$

Idea of the proof:

- Compute the values of the integrals of all bilinear, biquadratic and biquartic monomials.
- Prove RHI separately for symmetric forms g (difficult part: Muirhead inequality used) and antisymmetric ones (easier part: sos type inequality).

Problem C.1

1. $\mathbb{R}[\mathrm{x}]_{4,e}$ is equipped with the natural L^2 inner product

$$\langle f, g \rangle = \int_{S^{n-1}} fg \, \mathrm{d}\sigma,$$

where σ is the rotation invariant probability measures on the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

2. $\mathcal{H}$ is the affine hyperplane of forms from $\mathbb{R}[\mathrm{x}]_{4,e}$ of average 1 on S^{n-1}:

$$\mathcal{H} = \left\{ f \in \mathbb{R}[\mathrm{x}]_{4,e} \colon \int_{S^{n1}} f \, \mathrm{d}\sigma = 1 \right\}.$$

3. $z := \left(\sum_{i=1}^{n} x_i^2 \right)^2$ and thus

$$\mathcal{M} = \mathcal{H} - z = \left\{ f \in \mathbb{R}[\mathrm{x}]_{4,e} \colon \int_{S^{n-1}} f \, \mathrm{d}\sigma = 0 \right\}.$$

4. Let μ be the pushforward of the Lebesgue measure on $\mathbb{R}^{\dim \mathcal{M}}$ to $\mathcal{M}$.

Problem C.1

5. It is crucial to make the following three observations:

Observation 1: $\widetilde{(\mathrm{NN})^*_d} = \widetilde{\mathrm{NN}}$ and $\widetilde{(\mathrm{LF})^*_d} = \widetilde{\mathrm{POS}}$.

Here d stands for the differential/apolar inner product and $*$ for the dual,

$$\mathrm{LF} := \left\{ \mathrm{pr}(f) \in \mathbb{R}[\mathrm{x}]_{4,e} : f = \sum_i f_i^4 \quad \text{for some } f_i \in \mathbb{R}[\mathrm{x}]_1 \right\}$$

and $\mathrm{pr} : \mathbb{R}[\mathrm{x}]_4 \to \mathbb{R}[\mathrm{x}]_{4,e}$ is the projection defined by:

$$\mathrm{pr}\left(\sum_{1\le i\le j\le k\le \ell\le n} a_{ijk\ell} x_i x_j x_k x_\ell \right) = \sum_{1\le i\le j\le n} a_{iijj} x_i^2 x_j^2. \qquad (1)$$

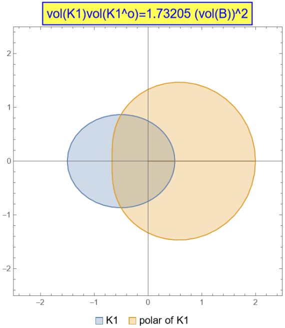
Observation 2: $\widetilde{\mathrm{LF}}$ is ‘central enough’.

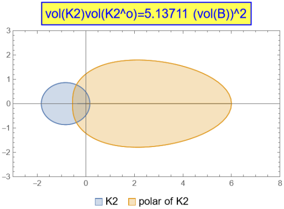
Observation 3: $\widetilde{\mathrm{CP}} \subseteq \widetilde{\mathrm{LF}} \subseteq \widetilde{\mathrm{NN}} \subseteq 4(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}})$.

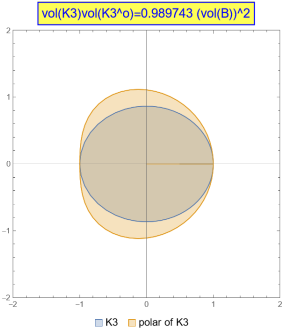
Blaschke-Santaló inequality and its reverse

Geometric picture

K_1 ... the convex hull of the ellipse with a polar equation $r(\varphi) = \frac{3}{4}(1 + \frac{1}{2}\cos\varphi)^{-1}$,
$K_2 = K_1 - (\frac{1}{3}, 0)$, $K_3 = K_1 + (\frac{1}{2}, 0)$,







- The set K_1 is centered in different points on each of the pictures. The first two centers are not close enough to the origin for the BS to hold, while in the third one it is.
- The translation of the body (i.e., Santaló point) so that the BS holds is difficult to determine, unless the body has enough symmetries, fixing only one point which then must be the Santaló one.

Observation 3: $\widetilde{\mathrm{NN}} \subseteq 4(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}})$

Follows from $2ab = (a+b)^2 - a^2 - b^2$

Let $r = \left(\sum_{k=1}^{n} x_k^2\right)^2$. The extreme points of $\widetilde{\mathrm{NN}}$ are of two types:

$$\frac{n(n+2)}{3} x_i^4 - r \qquad \text{and} \qquad n(n+2)x_i^2 x_j^2 - r, \; i \neq j.$$

The first type clearly belong to $\widetilde{\mathrm{CP}}$, while the second type to $4(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}})$:

$$\begin{aligned}
& n(n+2)x_i^2 x_j^2 - r = \\
&= \frac{n(n+2)}{2}\Big((x_i^2 + x_j^2)^2 - x_i^4 - x_j^4\Big) - r \\
&= 4\underbrace{\left(\frac{n(n+2)}{8}(x_i^2 + x_j^2)^2) - r\right)}_{p_1} - \frac{3}{2}\underbrace{\left(\frac{n(n+2)}{3}x_i^4 - r\right)}_{p_2} - \frac{3}{2}\underbrace{\left(\frac{n(n+2)}{3}x_j^4 - r\right)}_{p_3} \\
&= p_1 + \frac{3}{2}(p_1 - p_2) + \frac{3}{2}(p_1 - p_3) \\
&\in \widetilde{\mathrm{CP}} + \frac{3}{2}(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}}) + \frac{3}{2}(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}}) \subseteq 4(\widetilde{\mathrm{CP}} - \widetilde{\mathrm{CP}}).
\end{aligned}$$

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Thank you for your attention!