Operator Fejér–Riesz Theorems with Gaps and Optimal Degree Growth

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Notation

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$.

$$\mathcal{B}(\mathcal{H})[x] \quad \text{operator polynomials,} \qquad x^* = x.$$

The involution acts on the coefficients by taking adjoints:

$$\left(\sum_k A_k x^k\right)^* = \sum_k A_k^* x^k.$$

A polynomial F is hermitian if $F^* = F$.

Matrix polynomials correspond to $\mathcal{H} = \mathbb{C}^n$.

Fejér–Riesz theorem on $\mathbb{R}$

Theorem

Let $\mathcal{H}$ be a complex Hilbert space, let $N \in \mathbb{N} \cup \{0\}$, and let

$$F(x) = \sum_{m=0}^{2N} F_m x^m \in \mathcal{B}(\mathcal{H})[x]$$

be hermitian. If

$$F(x) \succeq 0 \qquad \text{for every } x \in \mathbb{R},$$

then there exists an operator polynomial

$$G(x) = \sum_{m=0}^{N} G_m x^m \in \mathcal{B}(\mathcal{H})[x]$$

such that

$$F(x) = G(x)^* G(x).$$

Fejér–Riesz: proofs and a bivariate extension

- M. Rosenblum (1968). *Vectorial Toeplitz operators and the Fejér–Riesz theorem*, J. Math. Anal. Appl. **23**, 139–147.
- M. A. Dritschel (2004). *On factorization of trigonometric polynomials*, Integral Equations Operator Theory **49**, 11–42.
- M. A. Dritschel and H. J. Woerdeman (2005). *Outer factorizations in one and several variables*, Trans. Amer. Math. Soc. **357**, 4661–4679.
- M. A. Dritschel, I. Klep, S. McCullough and J. Volčič (2026). *Addendum to "Factoring non-negative operator valued trigonometric polynomials in two variables"*, arXiv:2608.23073.
  Bivariate matrix case: every positive semidefinite matrix trigonometric polynomial on the two-torus is P^*P for a rectangular analytic matrix polynomial P. No general degree bound exists.

Main problems

Let $K \subseteq \mathbb{R}$ be a closed semialgebraic set and $\mathcal{H}$ a complex Hilbert space.

Thus K is a finite union of points and closed intervals, possibly unbounded.

Problem

1. *Find algebraic positivity certificates for hermitian operator polynomials* $F \in \mathcal{B}(\mathcal{H})[x]$ *satisfying*

$$F(x) \succeq 0 \qquad \text{for every } x \in K.$$

2. *Can the certificate degrees be bounded in terms of* $\deg F$ *and a fixed description of* K*, independently of* $\mathcal{H}$*?*
3. *How does the geometry of* K *affect the degree bounds?*

Semialgebraic set, quadratic module and preordering

Let $S = \{g_1, \ldots, g_s\} \subseteq \mathbb{R}[x]$, put $g_0 = 1$, and set

$$K_S = \{x \in \mathbb{R} : g_j(x) \geq 0,\ j = 1, \ldots, s\}.$$

For a complex Hilbert space $\mathcal{H}$, write

$$\mathrm{SOS}_{\mathcal{H}} = \{A^*A : A \in \mathcal{B}(\mathcal{H})[x]\}.$$

The operator quadratic module and operator preordering are

$$\mathcal{M}_{\mathcal{H}}(S) = \left\{ \sum_{j=0}^{s} g_j \Sigma_j : \Sigma_j \in \mathrm{SOS}_{\mathcal{H}} \right\},$$

$$\mathcal{T}_{\mathcal{H}}(S) = \left\{ \sum_{e \in \{0,1\}^s} g^e \Sigma_e : \Sigma_e \in \mathrm{SOS}_{\mathcal{H}} \right\},$$

where $g^e = g_1^{e_1} \cdots g_s^{e_s}$.

Saturation and stability of quadratic modules

The set of positive operator polynomials on K is

$$\mathrm{Pos}_{\mathcal{H}}(K) = \{F \in \mathcal{B}(\mathcal{H})[x] : F = F^*,\ F(x) \succeq 0\ \forall x \in K\}.$$

For a certificate $F = \sum_{j=0}^{s} g_j \Sigma_j$, its weighted degree is $\max_j \deg(g_j \Sigma_j)$.

1. $\mathcal{M}_{\mathcal{H}}(S)$ is saturated if $\mathcal{M}_{\mathcal{H}}(S) = \mathrm{Pos}_{\mathcal{H}}(K_S)$.

2. A saturated module is stable if, for every d, there is $N(d)$ such that every $F \in \mathrm{Pos}_{\mathcal{H}}(K_S)$ with $\deg F \leq d$ has a certificate of weighted degree at most $N(d)$. The bound may depend on S and $\mathcal{H}$, but not on F.

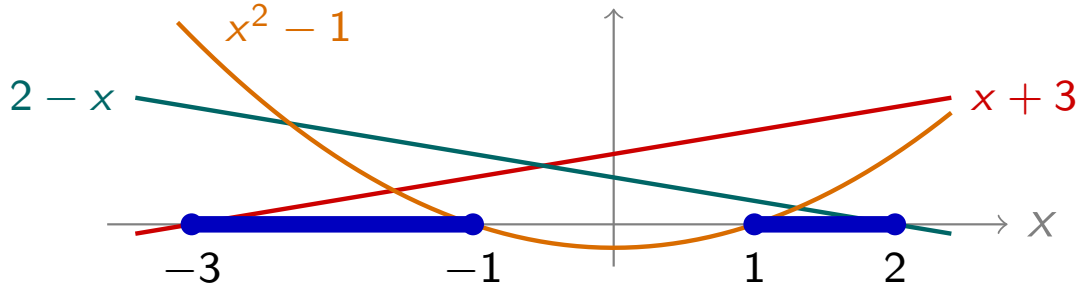
3. It is perfectly stable if one can take $N(d) = d$, i.e.,

$$\deg(g_j \Sigma_j) \leq \deg F \qquad (j = 0, \ldots, s).$$

Natural description and scalar preorderings

For nonempty closed semialgebraic $K \subseteq \mathbb{R}$, its natural description consists precisely of:

- $x - a$ if $a = \min K$, and $b - x$ if $b = \max K$;
- $(x - a)(x - b)$ for each gap (a, b) with $a, b \in K$, $a < b$.



$$K = [-3, -1] \cup [1, 2], \qquad S = \{x + 3,\ (x + 1)(x - 1),\ 2 - x\}.$$

Theorem (Kuhlmann–Marshall; Kuhlmann–Marshall–Schwartz)[1,2]

If S is the natural description of K, then the scalar preordering $\mathcal{T}_{\mathbb{C}}(S)$ is perfectly stable: every $f \geq 0$ on K has a certificate with $\deg(g^e \Sigma_e) \leq \deg f$.

[1] S. Kuhlmann and M. Marshall, *Positivity, sums of squares and the multi-dimensional moment problem*, Trans. Amer. Math. Soc. **354** (2002), 4285–4301.

[2] S. Kuhlmann, M. Marshall and N. Schwartz, *Positivity, sums of squares and the multi-dimensional moment problem II*, Adv. Geom. **5** (2005), 583–606.

Saturated descriptions of compact sets

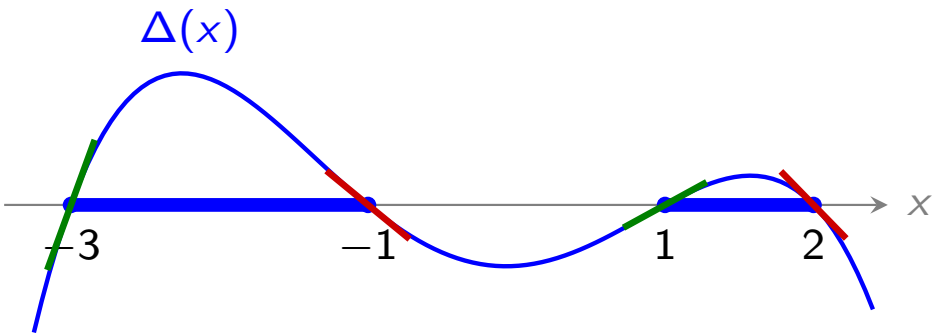
A finite $S \subseteq \mathbb{R}[x]$ is a saturated description of K if $K = K_S$ and its scalar quadratic module contains every $f \geq 0$ on K.

Theorem (Scheiderer, 2003, 2005)[1,2]

Let $K = K_S = \bigcup_{j=1}^{\ell} [a_j, b_j]$ be nonempty and compact, with disjoint components (possibly $a_j = b_j$). Then S is saturated if and only if, for every j,

$$\begin{aligned} &\text{left endpoint:} && \exists g \in S: && g(a_j) = 0, && g'(a_j) > 0, \\ &\text{right endpoint:} && \exists h \in S: && h(b_j) = 0, && h'(b_j) < 0. \end{aligned}$$

$$K = [-3,-1] \cup [1,2] = \{\Delta \geq 0\}, \qquad \Delta = -(x+3)(x+1)(x-1)(x-2).$$



left endpoints: $\Delta' > 0$

right endpoints: $\Delta' < 0$

$\{\Delta\}$ is saturated.

[1] C. Scheiderer, *Sums of squares on real algebraic curves*, Math. Z. **245** (2003), 725–760, Thm. 5.17.

[2] C. Scheiderer, *Distinguished representations of non-negative polynomials*, J. Algebra **289** (2005), 558–573, Cor. 4.4.

Perfect stability for operator polynomials

Let $\mathcal{H}$ be a complex Hilbert space.

Theorem[1,2,3,4]

The natural operator preorderings for the following sets are perfectly stable:

K	*Natural description* S	*Preordering*
$\mathbb{R}$	$\emptyset$	$\mathcal{T}_{\mathcal{H}}(\emptyset)$
$[0,\infty)$	$\{x\}$	$\mathcal{T}_{\mathcal{H}}(\{x\})$
$[0,1]$	$\{x, 1-x\}$	$\mathcal{T}_{\mathcal{H}}(\{x, 1-x\})$

[1] **Operator case:** M. Rosenblum, *Vectorial Toeplitz operators and the Fejér–Riesz theorem*, J. Math. Anal. Appl. **23** (1968), 139–147.

[2] **Operator case:** J. Cimprič and A. Z., *Moment problems for operator polynomials*, J. Math. Anal. Appl. **401** (2013), 307–316, Proposition 3.

[3] **Matricial only:** H. Dette and W. J. Studden, *Matrix measures, moment spaces and Favard's theorem for the interval* $[0, 1]$ *and* $[0, \infty)$, Linear Algebra Appl. **345** (2002), 169–193.

[4] **Matricial only:** Y. Savchuk and K. Schmüdgen, *Positivstellensätze for algebras of matrices*, Linear Algebra Appl. **436** (2012), 758–788.

Saturation of matrix quadratic modules

Let $K \subseteq \mathbb{R}$ be nonempty, closed and semialgebraic, and let $n \in \mathbb{N}$.

Theorem (Z., 2016)[1]

Compact K: exact certificates. *If S is a saturated description of K, then*

$$\mathcal{M}_{\mathbb{C}^n}(S) \text{ is saturated.}$$

Unbounded K: certificates with a denominator. *If S is the natural description of K, then for every hermitian $F \in M_n(\mathbb{C}[x])$,*

$$F \succeq 0 \text{ on } K \quad \Longleftrightarrow \quad (1+x^2)^k F \in \mathcal{T}_{\mathbb{C}^n}(S) \quad \text{for some } k \geq 0.$$

Which unbounded sets admit certificates without a denominator?

[1] A. Z., *Matrix Fejér–Riesz theorem with gaps*, J. Pure Appl. Algebra **220** (2016), 2533–2548.

Unbounded sets: the classification in 2016

For $n = 1$, natural scalar preorderings are always perfectly stable.

Let S be the natural description and $n \geq 2$.

An obstruction

If K has both a nondegenerate bounded interval component and an unbounded interval component, then $\mathcal{T}_{\mathbb{C}^n}(S)$ is not saturated.

The entries indicate whether $\mathcal{T}_{\mathbb{C}^n}(S)$ is saturated.

	Number of isolated points		
Interval components of K	0	1	≥ 2
One unbounded interval	Yes	Open	No
Two unbounded intervals	Yes	Open	No

The completed matricial classification

Theorem (Sun–Z., 2026)[1]

For $K = \{a\} \cup [b, c]$, where $a < b < c$, the natural matrix preordering is perfectly stable for every $n \in \mathbb{N}$.

This resolves the remaining cases with one isolated point.

	Number of isolated points		
Interval components of K	0	1	≥ 2
One unbounded interval	Yes	Yes	No
Two unbounded intervals	Yes	Yes	No

Here $n \geq 2$ and S is natural. A nondegenerate bounded interval component together with an unbounded one still prevents saturation.

[1] S. Sun and A. Z., *Matrix Fejér–Riesz type theorem for a union of an interval and a point*, J. Pure Appl. Algebra **230** (2026), 108173.

Compact Nichtnegativstellensatz: three steps

Let K be nonempty and compact, and S a saturated description. Recall the equality

$$\mathrm{Pos}_{\mathbb{C}^n}(K) = \mathcal{M}_{\mathbb{C}^n}(S).$$

For $F \in \mathrm{Pos}_{\mathbb{C}^n}(K)$, the proof has three steps.

1. Find a denominator avoiding any prescribed point. We use LDU decomposition of matrix polynomials.[1] For every $w \in \mathbb{C}$, find $h \in \mathbb{R}[x]$ such that
$$h(w) \neq 0, \qquad h^2 F \in \mathcal{M}_{\mathbb{C}^n}(S).$$

2. Apply Scheiderer's results.[2] Choose finitely many denominators $h_1, \ldots, h_r$ with no common complex zero. These results give $s_j \in \mathbb{R}[x]$ with
$$s_j > 0 \text{ on } K, \qquad \sum_{j=1}^{r} s_j h_j^2 = 1.$$

3. Remove the denominators. By scalar saturation, $s_j g_k \in \mathcal{M}_{\mathbb{C}}(S)$ for all j, k. Thus
$$F = \sum_{j=1}^{r} s_j \underbrace{h_j^2 F}_{\in \mathcal{M}_{\mathbb{C}^n}(S)} \in \mathcal{M}_{\mathbb{C}^n}(S).$$

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[1] K. Schmüdgen, *Noncommutative real algebraic geometry—some basic concepts and first ideas*, in *Emerging Applications of Algebraic Geometry*, IMA Vol. Math. Appl. **149**, Springer, 2009, 325–350.

[2] C. Scheiderer, *Sums of squares on real algebraic surfaces*, Manuscr. Math. **119** (2006), 395–410, Proposition 2.7.

An interval and a point: the proof idea

Normalize $K = \{0\} \cup [1, c]$, $c > 1$. For $F \in \mathrm{Pos}_{\mathbb{C}^n}(K)$ of degree $2m$, seek

$$F = \Sigma_0 + x(x-1)\Sigma_1 + x(c-x)\Sigma_2, \quad \deg(\text{each summand}) \leq 2m. \tag{1}$$

1. Pass to the dual moment problem. For hermitian matrices $\Gamma_0, \ldots, \Gamma_{2m}$, set

$$L_\Gamma\left(\sum_k F_k x^k\right) = \sum_k \mathrm{tr}(F_k \Gamma_k).$$

2. Translate positivity into matrix inequalities. Nonnegativity of L_Γ on certificates (1) is equivalent to

$$\begin{aligned} &\text{(i)} && M_m(\Gamma) \succeq 0, \\ &\text{(ii)} && H_{x(x-1)}(m-1) \succeq 0, \\ &\text{(iii)} && H_{x(c-x)}(m-1) \succeq 0. \end{aligned}$$

   The **moment** and **localizing matrices** are

$$M_m(\Gamma) = (\Gamma_{i+j})_{i,j=0}^m, \qquad H_g(r) = \left(\sum_k g_k \Gamma_{i+j+k}\right)_{i,j=0}^r,$$

   where $g(x) = \sum_k g_k x^k$.

3. Construct a representing matrix measure on K. Duality and closedness of the cone of certificates of weighted degree at most $2m$ give the certificate (1).

Counterexample for an unbounded set

Example

For $K = [-1, 0] \cup [1, \infty)$ and its natural description $S = \{x + 1, x(x - 1)\}$, the matrix polynomial

$$F(x) = \begin{bmatrix} x + 2 & \sqrt{6} \\ \sqrt{6} & x^2 - 2x + 3 \end{bmatrix}$$

is positive semidefinite on K, but $F \notin \mathcal{T}_{\mathbb{C}^2}(S)$.

Proof idea.

The principal minors $x + 2$, $x^2 - 2x + 3$ and $\det F = x^3 - x$ are nonnegative on K. Membership in $\mathcal{T}_{\mathbb{C}^2}(S)$ would require

$$F = \sigma_0 + (x + 1)\sigma_1 + x(x - 1)\sigma_2 + (x + 1)x(x - 1)\sigma_3,$$

with matrix sums of hermitian squares σ_i. A case analysis shows that F cannot have this form. □

Counterexamples for general unbounded sets

Theorem (Z., 2016)

Let $K \subseteq \mathbb{R}$ be unbounded, closed and semialgebraic. For every $n \geq 2$ and every finite $S \subseteq \mathbb{R}[x]$ with $K_S = K$,

$$\mathrm{Pos}_{\mathbb{C}^n}(K) \setminus \mathcal{T}_{\mathbb{C}^n}(S) \neq \emptyset$$

in either of the following cases:

1. *K has both a nondegenerate bounded interval component and an unbounded interval component;*
2. *K is a union of one or two unbounded interval components and at least two isolated points.*

Thus counterexamples exist for every unbounded configuration outside the few saturated cases in the classification.

Changing the finite polynomial description does not remove this obstruction.

Stability of the operator quadratic module

From saturation to degree bounds

Operator polynomials $F \in \mathcal{B}(\mathcal{H})[x]$, with arbitrary complex Hilbert space $\mathcal{H}$, on a nonempty compact semialgebraic set $K \subseteq \mathbb{R}$. Let $S = \{g_1, \ldots, g_s\}$ be a saturated description of K and $g_0 = 1$.

Problem

1. *Is $\mathcal{M}_{\mathcal{H}}(S)$ saturated?*
2. *Is $\mathcal{M}_{\mathcal{H}}(S)$ stable?*
3. *If $\mathcal{M}_{\mathcal{H}}(S)$ is stable, how do its degree bounds depend on the geometry of K?*

Operator Fejér–Riesz theorem with gaps

Theorem A (Jindal–Z., 2026)[1]

Let $S \subseteq \mathbb{R}[x]$ be finite, with K_S nonempty and compact. Assume that $\mathcal{M}_{\mathbb{C}}(S)$ is saturated.

For every complex Hilbert space $\mathcal{H}$,

$$\mathcal{M}_{\mathcal{H}}(S) \text{ is saturated and stable.}$$

Moreover, there is an integer $c_S \geq 0$, depending only on S, such that

$$N(d) \leq d + c_S \qquad (d \geq 0).$$

Here $N(d)$ is a bound such that every $F \in \mathrm{Pos}_{\mathcal{H}}(K_S)$ with $\deg F \leq d$ admits a certificate of weighted degree at most $N(d)$.

The same c_S works for every $\mathcal{H}$.

[1] A. Jindal and A. Z., *Operator Fejér–Riesz Theorems with Gaps and Degree Bounds*, https://arxiv.org/abs/2611.02967.

Reduction to quadratic polynomials

For $S = \{g_1, \ldots, g_s\}$ and $g_0 = 1$, define

$$\mathcal{M}_{\mathcal{H}}(S)_{\leq N} = \left\{ \sum_{j=0}^{s} g_j \Sigma_j : \quad \begin{array}{l} \Sigma_j \in \mathrm{SOS}_{\mathcal{H}}, \\ \deg(g_j \Sigma_j) \leq N \quad (0 \leq j \leq s) \end{array} \right\}.$$

Assume K_S is compact with nonempty interior.

Quadratic reduction

Suppose one integer $N \geq 2$ works for every complex Hilbert space $\mathcal{E}$:

$$\deg L \leq 2, \quad L \succeq 0 \text{ on } K_S \quad \Longrightarrow \quad L \in \mathcal{M}_{\mathcal{E}}(S)_{\leq N}.$$

Then, for every $d \geq 1$,

$$\deg F \leq d, \quad F \succeq 0 \text{ on } K_S \quad \Longrightarrow \quad F \in \mathcal{M}_{\mathcal{H}}(S)_{\leq d+N-1}.$$

For $d \geq 3$, the enlarged space is $\mathcal{E} = \mathcal{H}^{\oplus \lceil d/2 \rceil}$, the direct sum of $\lceil d/2 \rceil$ copies of $\mathcal{H}$. For $d = 0$, take the square root of the positive constant operator.

This reduction is not essential for proving stability, but it simplifies the proof technically. It is useful for proving that the bound $N(d)$ has slope 1.

Quadratic reduction: three proof steps

Write $F(x) = \sum_{k=0}^{d} F_k x^k$, with $F_k = 0$ for $k > d$. For $d \geq 3$, put $r = \lceil d/2 \rceil - 1$, $\mathcal{E} = \mathcal{H}^{\oplus(r+1)}$ and

$$V(x) = (I_{\mathcal{H}}, xI_{\mathcal{H}}, \ldots, x^r I_{\mathcal{H}})^T.$$

Here $I_{\mathcal{H}}$ and $I_{\mathcal{E}}$ denote the identity operators on $\mathcal{H}$ and $\mathcal{E}$, respectively.

1. Lift to a quadratic polynomial. One possible choice is

$$A(x) = \operatorname{diag}(F_{2j} + xF_{2j+1})_{j=0}^{r} + x^2 \operatorname{diag}(0, \ldots, 0, F_{2r+2}).$$

For $B(x)(h_0, \ldots, h_r) = (h_1 - xh_0, \ldots, h_r - xh_{r-1})$,

$$V^* A V = F, \qquad BV = 0, \qquad \ker B(x) = \operatorname{ran} V(x).$$

2. Make the lift positive. For every $\delta > 0$, compactness gives $t_\delta > 0$ with

$$L_\delta = A + \delta I_{\mathcal{E}} + t_\delta B^* B \succeq 0 \quad \text{on } K_S.$$

3. Certify, compress and take a limit. Apply the quadratic bound and compress by V:

$$F + \delta \left(\sum_{k=0}^{r} x^{2k} \right) I_{\mathcal{H}} \in \mathcal{M}_{\mathcal{H}}(S)_{\leq N+2r}.$$

Closedness in coefficientwise operator norm allows $\delta \downarrow 0$.

Key idea: one parameter for all intervals

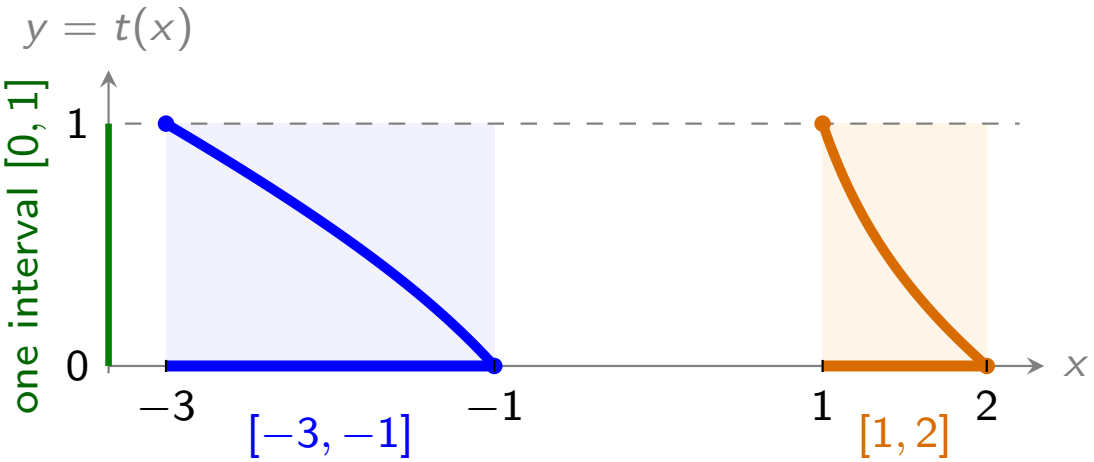
We want to parametrize $K = \bigcup_{j=1}^{\ell} [a_j, b_j]$ by one parameter $y \in [0,1]$, using the same rational function on every component.

Why the obvious diagonal choice does not work here

$$\widetilde{T}(y) = (1-y)\,\mathrm{diag}(b_j) + y\,\mathrm{diag}(a_j), \qquad y = \frac{b_j - x}{b_j - a_j}.$$

It covers the intervals, but the inverse depends on j: there is no common rational substitution for transferring the certificate back.

The successful rank-one construction



$$t(x) = \frac{(x+1)(x-2)}{1-3x}$$

$$t(a_j) = 1, \quad t(b_j) = 0$$

Each component covers $[0,1]$.

Key idea: a rank-one symmetric pencil

Let $K = \bigcup_{j=1}^{\ell}[a_j, b_j]$, where $a_1 < b_1 < \cdots < a_\ell < b_\ell$. Set

$$P(x) = \prod_{j=1}^{\ell}(x - a_j), \qquad Q(x) = \prod_{j=1}^{\ell}(x - b_j),$$

A rank-one symmetric pencil

The **converse to rank-one eigenvalue interlacing**[1] gives the following construction:

$$v = (v_1, \ldots, v_\ell)^T, \qquad v_j = \sqrt{\frac{P(b_j)}{Q'(b_j)}} \quad (1 \le j \le \ell),$$

$$\boxed{T(y) = \operatorname{diag}(b_1, \ldots, b_\ell) - yvv^T.}$$

Its ordered eigenvalues satisfy

$$\lambda_j(T(0)) = b_j, \qquad \lambda_j(T(1)) = a_j,$$

$$a_j \le \lambda_j(T(y)) \le b_j \qquad (0 \le y \le 1).$$

[1] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge University Press, 2013, Theorem 4.3.26.

One rational substitution for every interval

Put $R = Q - P$ and $\Delta = -PQ$. Then $K = \{\Delta \geq 0\}$ and $\mathcal{M}_{\mathbb{C}}(\{\Delta\})$ is saturated.

$$\det(xI_\ell - T(y)) = (1-y)Q(x) + yP(x) = Q(x) - yR(x).$$

For $x \in K$,

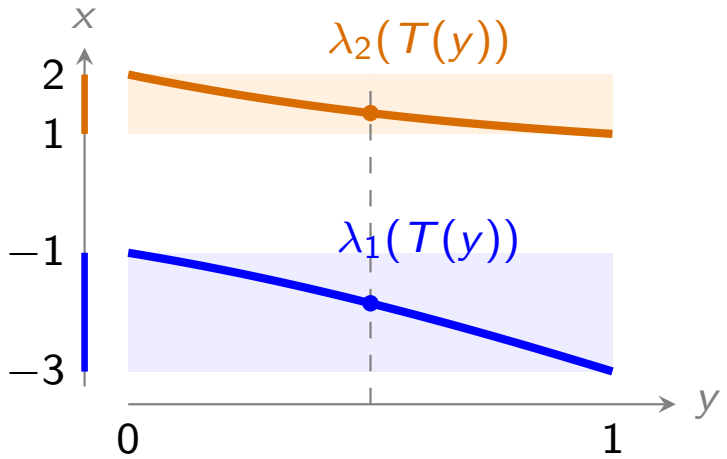
$$x \in \sigma(T(y)) \iff y = t(x) = \frac{Q(x)}{R(x)}.$$

The same inverse works on every interval:

$$t(a_j) = 1, \qquad t(b_j) = 0,$$

$$t([a_j, b_j]) = [0,1], \qquad \bigcup_{0 \leq y \leq 1} \sigma(T(y)) = K.$$

$K = [-3,-1] \cup [1,2]$



$t(\lambda_j(T(y))) = y$

For $F(x) = \sum_{k=0}^{2} F_k x^k$, put $F(T(y)) = \sum_{k=0}^{2} F_k \otimes T(y)^k$. Then

$$F \succeq 0 \text{ on } K \iff F(T(y)) \succeq 0 \text{ on } [0,1].$$

Quadratic F stays quadratic in y: apply the interval Fejér–Riesz theorem.

Transfer back to K: what must C satisfy?

For $D = \mathrm{diag}(b_j)$ and $t = Q/R$, choose

$$\boxed{u(x) = \mathrm{adj}(xI - D)v}, \qquad u_j(x) = v_j \prod_{i \neq j} (x - b_i).$$

Its components have no common real zero, so $w = u^* u > 0$ on $\mathbb{R}$. For $R(x) \neq 0$, we have

$$T(t(x))u(x) = xu(x),$$

and hence

$$(I_{\mathcal{H}} \otimes u(x))^* F(T(t(x)))(I_{\mathcal{H}} \otimes u(x)) = w(x)F(x).$$

Direct substitution in the interval certificate gives $R^2 wF$. R has zeros in the gaps. Modify the lift by a polynomial congruence:

$$G(y) = (I_{\mathcal{H}} \otimes C(y))^* F(T(y))(I_{\mathcal{H}} \otimes C(y)).$$

1. Preserve recovery: require $C(t(x))u(x) = \beta(x)u(x)$. Then
$$(I_{\mathcal{H}} \otimes u(x))^* G(t(x))(I_{\mathcal{H}} \otimes u(x)) = |\beta(x)|^2 w(x)F(x).$$
2. Preserve quadratic degree: require $\deg G \leq 2$ for every quadratic F. An arbitrary affine C can give degree 4, requiring R^4 to clear denominators.
3. Remove the zeros: with $\deg G \leq 2$, the multiplier is $R^2|\beta|^2 w = |R\beta|^2 w$. We want $R\beta$ to be a polynomial with no real zeros.

Transfer back to K: Lemma selects C

Lemma

For $\ell \geq 2$, the recovery and quadratic-degree requirements are equivalent to

$$CT = TC, \qquad C \text{ and } TC \text{ are affine,}$$

and, equivalently, to

$$C(y) = q(T(y))R(T(y))^{-1}, \qquad q \in \mathbb{C}[x], \quad \deg q \leq \ell - 1.$$

Here $\beta(x) = q(x)/R(x)$, so clearing denominators gives $|q|^2 wF$. Choose $q = 1$:

$$(I_{\mathcal{H}} \otimes u(x))^* G(t(x))(I_{\mathcal{H}} \otimes u(x)) = \frac{w(x)}{R(x)^2} F(x).$$

The lemma ensures that C is polynomial and G is quadratic. Thus

$$G(y) = A(y)^* A(y) + y(1-y)B^* B, \qquad A(y) = A_0 + yA_1,$$

with B constant. Substitute $t = Q/R$, compress, and multiply by R^2:

$$\boxed{wF = U^* U + \Delta V^* V}.$$

$$U(x) = (R(x)A_0 + Q(x)A_1)(I_{\mathcal{H}} \otimes u(x)), \qquad \deg U \leq 2\ell - 1,$$
$$V(x) = B(I_{\mathcal{H}} \otimes u(x)), \qquad \deg V \leq \ell - 1.$$

The weighted degree is at most $4\ell - 2$, independently of $\mathcal{H}$.

Remove the multiplier with controlled degree

1. Fix a scalar identity depending only on K. Scalar saturation gives $\sigma_0, \sigma_1 \in \mathrm{SOS}_{\mathbb{C}}$ with
$$\Delta = \sigma_0 + w\Delta\sigma_1.$$

2. Use positivity on the whole real line. Together with $wF = U^*U + \Delta V^*V$, this gives
$$w\big(F - \Delta\sigma_1 V^*V\big) = U^*U + \sigma_0 V^*V \succeq 0.$$
Since $w > 0$, the real-line operator Fejér–Riesz theorem applies.

3. Obtain the quadratic certificate.
$$F = H^*H + \Delta\sigma_1 V^*V, \qquad F \in \mathcal{M}_{\mathcal{H}}(\{\Delta\})_{\leq \beta_K},$$
where $\beta_K = 4\ell - 2 + \deg\sigma_1$ depends only on K.

Adjoining an isolated point: Douglas factorization

Suppose $K_0 = \{\Delta \geq 0\}$ and

$$F = A^*A + \Delta B^*B.$$

Fix $\xi \notin K_0$, put $\delta = -\Delta(\xi) > 0$, and assume $F(\xi) \succeq 0$.

1. Positivity at the point compares the factors.

   $$\delta B(\xi)^* B(\xi) \preceq A(\xi)^* A(\xi).$$

   Douglas' lemma gives a contraction Γ with $\Gamma A(\xi) = \sqrt{\delta}\, B(\xi)$.

2. Match the values and divide out $x - \xi$.

   $$W = \Gamma A/\sqrt{\delta}, \qquad E = \frac{B - W}{x - \xi}, \qquad L = (I - \Gamma^*\Gamma)^{1/2} A.$$

   Since $W(\xi) = B(\xi)$, E is a polynomial, with $\deg E \leq \max\{\deg A, \deg B\} - 1$.

3. Encode the extra point by a fixed matrix.

   $$G_\xi = \begin{pmatrix} \delta + \Delta & (x-\xi)\Delta \\ (x-\xi)\Delta & (x-\xi)^2\Delta \end{pmatrix}, \qquad V = \begin{pmatrix} W \\ E \end{pmatrix}.$$

$$F = L^*L + V^*(G_\xi \otimes I_{\mathcal{H}})V, \qquad \{G_\xi \succeq 0\} = K_0 \cup \{\xi\}.$$

Several isolated points: the role of ψ_i

Let $K = K_0 \cup \{\xi_1, \dots, \xi_r\}$, $K_0 = \{\Delta \geq 0\}$, with a saturated description S.

1. Separate the isolated points. Choose $n_i > \max_{x \in K_0} |x - \xi_i|$ and set

$$\boxed{\psi_i(x) = \left(\prod_{j \neq i} \frac{x - \xi_j}{\xi_i - \xi_j} \right) \left(1 - \frac{(x - \xi_i)^2}{n_i^2} \right)^N .}$$

   Uniform convergence to 0 on K_0 gives, for sufficiently large N,

$$\psi_i(\xi_j) = \delta_{ij}, \qquad \sum_{i=1}^r \psi_i(x)^2 \leq 1 \quad (x \in K_0).$$

2. Make every local matrix positive on all of K. Put

$$a = 1 - \sum_{i=1}^r \psi_i^2, \qquad \Psi_i = \psi_i^2 G_{\xi_i}.$$

3. Choose fixed scalar and matrix certificates. Therefore

$$a,\ a\Delta \in \mathcal{M}_{\mathbb{C}}(S), \qquad \Psi_i \in \mathcal{M}_{\mathbb{C}^2}(S).$$

   These certificates depend only on S, not on F or $\mathcal{H}$.

Combining the factorizations: conclusion of Theorem A

Start with $F = A^*A + \Delta B^*B$. At each ξ_i, Douglas factorization gives

$$F = L_i^* L_i + V_i^*(G_{\xi_i} \otimes I_{\mathcal{H}})V_i.$$

1. Multiply by ψ_i^2 and add. Since $1 = a + \sum_i \psi_i^2$ and $\Psi_i = \psi_i^2 G_{\xi_i}$,

$$\begin{aligned} F &= aF + \textstyle\sum_i \psi_i^2 F \\ &= aA^*A + (a\Delta)B^*B + \textstyle\sum_i (\psi_i L_i)^*(\psi_i L_i) + \sum_i V_i^*(\Psi_i \otimes I_{\mathcal{H}})V_i. \end{aligned}$$

   Insert the fixed certificates for a, $a\Delta$ and Ψ_i. Every term then belongs to $\mathcal{M}_{\mathcal{H}}(S)$.

2. Bound the additional degree cost. The fixed certificate degrees and $2\deg\psi_i$ have a common bound η_S. Thus an initial weighted degree bound n becomes $n + \eta_S$. For ℓ interval components in K_0, the quadratic bound is

$$\boxed{N_S = 4\ell - 2 + \deg\sigma_1 + \eta_S.}$$

   The scalar degree is where the interval geometry enters.

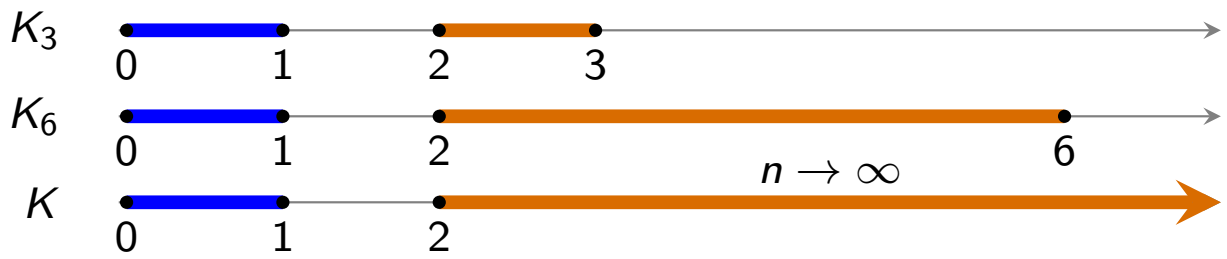
3. Apply quadratic reduction. For general F, the bound is $d + c_S$, with $c_S = N_S - 1$.

If K is finite, Lagrange interpolation and fixed certificates for $\pm\prod_{\xi\in K}(x-\xi)$ give the bound directly.

Geometry matters: an endpoint tends to infinity

Translate the earlier unbounded counterexample:

$$F(x) = \begin{pmatrix} x+1 & \sqrt{6} \\ \sqrt{6} & x^2 - 4x + 6 \end{pmatrix}, \qquad K_n = [0,1] \cup [2,n] \nearrow K = [0,1] \cup [2,\infty).$$



Set $q = (x-1)(x-2)$ and $S_n = \{x, q, 1 - x/n\}$. Although $F \succeq 0$ on every K_n,

$$\min\{k : F \in \mathcal{T}_{\mathbb{C}^2}(S_n)_{\leq k}\} \longrightarrow \infty.$$

The limiting set admits no certificate for this F.

Why the certificate degrees must diverge

1. Assume a common certificate degree bound k. Along $n_j \to \infty$,
$$F = \sum_{a,b,c\in\{0,1\}} x^a q^b (1 - x/n_j)^c \Sigma_{j,abc}, \qquad \deg \Sigma_{j,abc} \le k.$$

2. Bound the multipliers on a fixed interval. On $I = [1/4, 3/4]$, all eight scalar weights are $\ge \delta > 0$ uniformly in j. Since every summand is positive semidefinite,
$$0 \preceq \Sigma_{j,abc}(x) \preceq \delta^{-1} F(x) \qquad (x \in I).$$
For fixed degree, a bound on I bounds all coefficients.

3. Pass to a coefficientwise convergent subsequence.
$$\Sigma_{j,abc} \longrightarrow \Sigma_{abc} \in \mathrm{SOS}_{\mathbb{C}^2}, \qquad 1 - x/n_j \longrightarrow 1.$$
The limiting identity is
$$F = \sum_{a,b\in\{0,1\}} x^a q^b (\Sigma_{ab0} + \Sigma_{ab1}) \in \mathcal{T}_{\mathbb{C}^2}(\{x, q\}),$$
contradicting the unbounded counterexample.

Can degrees diverge even when all sets stay inside $[0, 1]$?

A shrinking interval forces degree growth

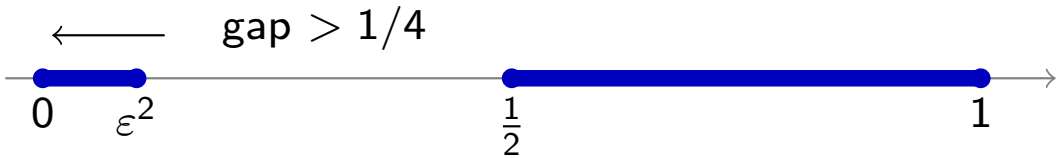
Theorem B (Jindal–Z., 2026)

For $0 < \varepsilon < 1/2$, *let*

$$K_\varepsilon = [0, \varepsilon^2] \cup [1/2, 1], \qquad S_\varepsilon = \{x, 1-x, (x-\varepsilon^2)(x-1/2)\}.$$

There are quadratic 2×2 *matrix polynomials* $F_\varepsilon \succeq 0$ *on* K_ε *such that*

$$\min\{k : F_\varepsilon \in \mathcal{T}_{\mathbb{C}^2}(S_\varepsilon)_{\leq k}\} \longrightarrow \infty \qquad (\varepsilon \downarrow 0).$$



All sets lie in $[0, 1]$. Only the first interval collapses. Even for natural matrix preorderings, no universal additive constant exists.

Uniform degree bounds under geometric constraints

Theorem C (Jindal–Z., 2026)

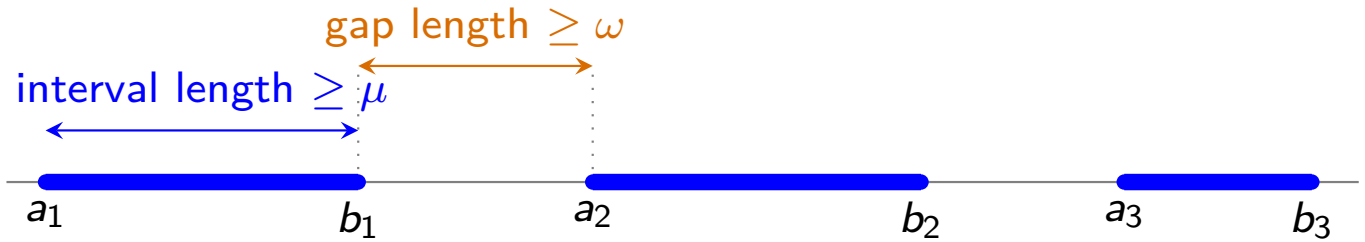
Fix a compact interval J and $\mu, \omega > 0$. Consider

$$K = \bigcup_{j=1}^{\ell} [a_j, b_j] \subseteq J, \qquad b_j - a_j \geq \mu, \quad a_{j+1} - b_j \geq \omega.$$

There is $C = C(J, \mu, \omega)$ such that, for every complex Hilbert space $\mathcal{H}$ and hermitian F with $\deg F \leq d$,

$$F \succeq 0 \text{ on } K \quad \Longleftrightarrow \quad F \in \mathcal{M}_{\mathcal{H}}(\{\Delta_K\})_{\leq d+C},$$

where $\Delta_K = -\prod_j (x - a_j)(x - b_j)$.



One constant works for all these sets and all Hilbert spaces. The same bound holds over $\mathbb{R}$.

Geometric uniformity: three proof steps

1. Compactness of the endpoint parameters. Since $\ell\mu + (\ell - 1)\omega \leq |J|$, only finitely many values of ℓ occur. For each ℓ, the admissible endpoint tuples form a compact set.

2. Scalar certificates persist under perturbations. Construct identities

$$1 = w_K \sigma_K + \Delta_K \tau_K$$

with multipliers in the interiors of fixed-degree SOS cones. They persist when the endpoints change slightly.

3. Take a finite cover and apply quadratic reduction. Compactness gives uniform scalar degrees, then a uniform quadratic operator bound, and finally the bound $d + C$.

Answers to the three questions

1. Which positivity certificates exist? On compact semialgebraic subsets of $\mathbb{R}$, scalar saturation extends to operator quadratic modules, including isolated points.

2. How large must the degrees be? For fixed S, the bound is $d + c_S$, uniformly in $\mathcal{H}$. The linear slope 1 is optimal.

3. How does geometry affect the bound? Collapsing intervals can force unbounded degree even inside $[0, 1]$. A fixed enclosing interval and positive lower bounds on interval and gap lengths give a uniform additive constant.

Thank you for your attention!