Positive polynomials and the truncated moment problem on plane cubic curves

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SAFA 2025

joint work with

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1. Preliminaries

Bivariate truncated moment problem (TMP)

Question

Let $k \in \mathbb{N}$ and

$$\beta = \beta^{(k)} = (\beta_{i,j})_{i,j \in \mathbb{Z}_+, i+j \leq k}$$

a bivariate sequence of real numbers of degree k.

$K \subseteq \mathbb{R}^2$ is a closed subset.

The bivariate truncated moment problem on K (K–TMP): characterize the existence of a positive Borel measure μ on $\mathbb{R}^2$ with support in K, such that

$$\beta_{i,j} = \int_K x^i y^j d\mu(x)$$

for $i, j \in \mathbb{Z}_+$, $i + j \leq k$.

μ is called a K–representing measure (K–RM) of β.

Bivariate moment matrix

The moment matrix $M(k)$ associated to β with the rows and columns indexed by $X^iY^j,\ i+j\le k,$ in degree-lexicographic order

$$1, X, Y, X^2, XY, Y^2, \ldots, X^k, X^{k-1}Y, \ldots, Y^k$$

is defined by where

$$M(k) := \begin{array}{c} \\ 1 \\ X \\ Y \\ \vdots \\ X^{i_1}Y^{j_1} \\ \vdots \\ Y^k \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & X & Y & \cdots & X^{i_2}Y^{j_2} & \cdots & Y^k \end{array} \\ \left[\begin{array}{ccccccc} \beta_{0,0} & \beta_{1,0} & \beta_{0,1} & \cdots & \beta_{i_2,j_2} & \cdots & \beta_{0,k} \\ \beta_{1,0} & \beta_{2,0} & \beta_{1,1} & \cdots & \beta_{i_2+1,j_2} & \cdots & \beta_{1,k} \\ \beta_{0,1} & \beta_{1,1} & \beta_{0,2} & \cdots & \beta_{i_2,j_2+1} & \cdots & \beta_{0,k+1} \\ \vdots & & & \ddots & & & \vdots \\ \beta_{i_1,j_1} & \beta_{i_1+1,j_1} & \beta_{i_1,j_1+1} & \cdots & \beta_{i_1+i_2,j_1+j_2} & \cdots & \beta_{i_1,j_1+k} \\ \vdots & & & & \vdots & \ddots & \vdots \\ \beta_{0,k} & \beta_{1,k} & \beta_{0,k+1} & \cdots & \beta_{i_2,j_2+k} & \cdots & \beta_{0,2k} \end{array}\right] \end{array}$$

Necessary conditions for the existence of a RM

- To every polynomial $p := \sum_{i,j} a_{i,j} x^i y^j \in \mathbb{R}[x,y]_k$, we associate the vector

$$p(X,Y) = \sum_{i,j} a_{i,j} X^i Y^j = a_{0,0} \cdot \underset{1}{\begin{pmatrix} \beta_{0,0} \\ \beta_{1,0} \\ \beta_{0,1} \\ \vdots \\ \beta_{0,k} \end{pmatrix}} + a_{1,0} \cdot \underset{X}{\begin{pmatrix} \beta_{1,0} \\ \beta_{2,0} \\ \beta_{1,1} \\ \vdots \\ \beta_{1,k} \end{pmatrix}} + \cdots + a_{0,k} \cdot \underset{Y^k}{\begin{pmatrix} \beta_{0,k} \\ \beta_{1,k} \\ \beta_{0,k+1} \\ \vdots \\ \beta_{0,2k} \end{pmatrix}}$$

  from the column space of the matrix $M(k)$.

- The matrix $M(k)$ is recursively generated (RG) if for $p, q, pq \in \mathbb{R}[x,y]_k$

$$p(X,Y) = \mathbf{0} \quad \Rightarrow \quad (pq)(X,Y) = \mathbf{0}.$$

Necessary conditions for the existence of a RM

- The matrix $M(k)$ satisfies the variety condition (VC) if

$$\operatorname{rank} M(k) \le \operatorname{card} \mathcal{V},$$

where

$$\mathcal{V} := \bigcap_{\substack{g \in \mathbb{R}[x,y]_{\le k}, \\ g(X,Y)=\mathbf{0} \text{ in } M(k)}} \underbrace{\left\{(x,y) \in \mathbb{R}^2 \colon g(x,y) = 0\right\}}_{\mathcal{Z}(g)}.$$

Proposition (Curto and Fialkow, 96')

If $\beta^{(2k)}$ has a representing measure μ, then

$M(k)$ is positive semidefinite (PSD), RG and satisfies VC.

Sufficient condition for the existence of a RM

Theorem (Flat extension theorem, Curto and Fialkow, 96')

TFAE:

1. $\beta^{(2k)}$ *admits a* $(\operatorname{rank} M(k))$*–atomic RM.*
2. $M(k)$ *is PSD and there is an extension* $M(k+1)$ *such that*

$$\operatorname{rank} M(k+1) = \operatorname{rank} M(k).$$

Type of solutions to the K–TMP

Constructive solution

A representing measure is explicitly constructed. The most desired solution.

Concrete solution

This is the solution in terms of explicit numerical conditions on β.

Solution based on feasibility of a LMI

If an explicit solution does not exist, then we are satisfied with a LMI based solution with bounded sizes of LMIs.

Known constructive/concrete solutions

1. **Quadratic TMP, i.e.** $\beta = \beta^{(2)}$**:** Completely solved. Curto & Fialkow, '96
2. **Cubic TMP, i.e.** $\beta = \beta^{(3)}$**:** Completely solved. Kimsey, '14, Curto & Yoo, '18
3. **Quartic TMP, i.e.** $\beta = \beta^{(4)}$**:** Completely solved.
   $M(2)$ singular: Curto & Fialkow, '02
   $M(2)$ nonsingular: Fialkow & Nie, '10, Curto & Yoo, '16
4. **Quintic TMP, i.e.** $\beta = \beta^{(5)}$**:** Completely solved. El Azhar, Harrat, Idrissi, Zerouali, '19
5. **Sextic TMP, i.e.** $\beta = \beta^{(6)}$**:** Partially solved.
   - Extremal case - $\text{rank}\, M(3) = \text{card}\, \mathcal{V}$ Curto & Fialkow & Möller, '05
   - On variety $y = x^3$ Fialkow, '11
   - $\text{rank}\, M(3) \in \{7, 8\}$ Curto, Yoo, '14, '15
   - On special cases of reducible varieties Yoo, '17
   - $M(3)$ invertible Fialkow, '17, Fialkow & Blekherman, '20
6. **TMP on quadratic curves:** Completely solved. Curto & Fialkow, '02, '04, '05, '14
7. **TMP on cubic curves, i.e.** $\beta = \beta^{(2k)}$**:** Cases solved.
   - Infinite variety: $y = x^3$, $y^2 = x^3$, $xy^2 = 1$, $y(y-1)(y-2) = 0$ Fialkow, '11, Z. '21,'22,'23
   - Finite variety: $z^3 = itz + u\bar{z}$, $t, u \in \mathbb{R}$ Curto, Yoo '14, '15
8. **Bounds on the number of atoms:** Riener & Schweighofer, '18, di Dio & Schmüdgen, '18, di Dio & Kummer '21, Z. '24, Riener & Texteira Turatti, '25

2. Solving the TMP for $y^2 = x^3 + ax + b$ using the flat extension theorem

with A. Bhardwaj,
Non-negative Polynomials, Sums of Squares & the Moment Problem,
PhD Thesis, Australian National University, 2020.

TMP for $p(x,y) = y^2 - x^3 - ax - b$

$k \geq 3$, $\beta := \{\beta_{ij}\}_{i,j\in\mathbb{Z}_+, i+j\leq 2k}$, analysis of the existence of a flat extension

$$M(k+1) = \begin{pmatrix} M(k) & B(k+1) \\ (B(k+1))^T & C(k+1) \end{pmatrix}$$

of $M(k)$ following Fialkow's $p(x,y) = y - x^3$ approach:

1. The block $B(k+1)$ restricted to rows of degree k is of the form :

$$\begin{array}{c} \\ X^k \\ X^{k-1}Y \\ \vdots \\ \vdots \\ X^2Y^{k-2} \\ XY^{k-1} \\ Y^k \end{array}
\begin{array}{c} \begin{array}{ccccccc} X^{k+1} & X^kY & \cdots & \cdots & X^2Y^{k-1} & XY^k & Y^{k+1} \end{array} \\
\begin{pmatrix}
\beta_{2k+1,0} & \beta_{2k,1} & \cdots & \cdots & \beta_{k+2,k-1} & \beta_{k+1,k} & \beta_{k,k+1} \\
\beta_{2k,1} & \beta_{2k-1,2} & \iddots & \iddots & \beta_{k+1,k} & \beta_{k,k+1} & \beta_{k-1,k+2} \\
\vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\
\vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\
\beta_{k+3,k-2} & \beta_{k+2,k-1} & \iddots & \iddots & \iddots & \iddots & \theta \\
\beta_{k+2,k-1} & \beta_{k+1,k} & \iddots & \iddots & \iddots & \theta & \phi \\
\beta_{k+1,k} & \beta_{k,k+1} & \cdots & \cdots & \theta & \phi & \psi
\end{pmatrix} \end{array},$$

where

$$\beta_{i,2k+1-i} = \beta_{i-3,2k+3-i} - a\beta_{i-2,2k+1-i} - b\beta_{i-3,2k+1-i} \quad \text{for } 3 \leq i \leq 2k+1$$

and θ, ϕ, ψ are arbitrary.

2.

$$C(k+1) := (B(k+1))^T M(k)^\dagger B(k+1)$$

$$= \begin{array}{c} \\ \cdots \\ X^3Y^{k-2} \\ X^2Y^{k-1} \\ XY^k \\ Y^{k+1} \end{array} \begin{array}{c} \begin{array}{ccccc} \cdots & X^3Y^{k-2} & X^2Y^{k-1} & XY^n & Y^{k+1} \end{array} \\ \begin{bmatrix} \ddots & \vdots & & & \vdots \\ \cdots & C_{k-1,k-1} & C_{k,k-1} & C_{k+1,k-1} & C_{k+2,k-1} \\ & C_{k,k-1} & C_{k,k} & C_{k+1,k} & C_{k+2,k} \\ & C_{k+1,k-1} & C_{k+1,k} & C_{k+1,k+1} & C_{k+2,k+1} \\ \cdots & C_{k+2,k-1} & C_{k+2,k} & C_{k+2,k+1} & C_{k+2,k+2} \end{bmatrix} \end{array}$$

has a moment structure iff:

$$C_{k,k} = C_{k+1,k-1},$$

$$\phi = f_2\theta^2 + f_1\theta + f_0$$

$$C_{k+1,k} = C_{k+2,k-1},$$

$$\psi = j_{11}\phi\theta + j_{10}\phi + j_{02}\theta^2 + j_{01}\theta + j_{00}$$

$$C_{k+1,k+1} = C_{k+2,k}$$

$$k_{101}\psi\theta + k_{100}\psi + k_{011}\phi\theta + k_{010}\phi + k_{002}\theta^2 + k_{001}\theta + k_{000} =$$

$$\ell_{20}\phi^2 + \ell_{11}\phi\theta + \ell_{10}\phi + \ell_{02}\theta^2 + \ell_{01}\theta + \ell_{00}$$

2. $C(k+1) := (B(k+1))^T M(k)^\dagger B(k+1)$ has a moment structure iff:

$$C_{k,k} = C_{k+1,k-1},$$
$$\phi = f_2\theta^2 + f_1\theta + f_0$$
$$C_{k+1,k} = C_{k+2,k-1},$$
$$\psi = j_{11}\phi\theta + j_{10}\phi + j_{02}\theta^2 + j_{01}\theta + j_{00}$$
$$C_{k+1,k+1} = C_{k+2,k}$$
$$k_{101}\psi\theta + k_{100}\psi + k_{011}\phi\theta + k_{010}\phi + k_{002}\theta^2 + k_{001}\theta + k_{000} =$$
$$\ell_{20}\phi^2 + \ell_{11}\phi\theta + \ell_{10}\phi + \ell_{02}\theta^2 + \ell_{01}\theta + \ell_{00}$$

3. A short computation shows that the last equation is of the form

$$\alpha_2\theta^2 + \alpha_1\theta + \alpha_0 = 0$$

and a flat extension $M(k+1)$ exists iff it has a real root θ.

TMP for $p(x, y) = y^2 - x^3 - ax - b$

There are cases with a measure but without flat extension.

Generating $M(3)$ with 10 atoms $(x_i, y_i), (x_i, -y_i)$ where

$$x_i = \frac{1}{i}, \quad y_i = \sqrt{x_i^3 - \frac{524287}{262144}x_i + 1}, \quad i = 1, \dots, 5,$$

$M(3)$ is of rank 9 having a column relation

$$p(X, Y) = Y^2 - X^3 + \frac{524287}{262144}X - 1 = 0.$$

A flat extension $M(4)$ does not exist, since in

$$\alpha_2\theta^2 + \alpha_1\theta + \alpha_0 = 0$$

α_2, α_0 are rationals of the same sign, $\alpha_1 = 0$ and hence a real solution θ does not exist.

TMP for $p(x, y) = y^2 - x^3 - ax - b$

Theorem (Bhardwaj, Z)

Assume $M(k) \succeq 0$ and there are no other column relations besides the ones obtained from p by RG. The following statements are equivalent:

1. *L has a $(\operatorname{rank} M(k))$–atomic $\mathcal{Z}(p)$–representing measure.*
2. *Quadratic polynomial $\mathcal{Q}(\theta)$, completely determined by β, has a real root.*

Using a recent result (2024+) by Baldi, Blekherman and Sinn on the number of atoms in a minimal measure, this result solves the TMP in case $\mathcal{Z}(p)$ has one connected component and the homogenization of $p(x, y)$ determines a projectively smooth curve.

3. Solving the TMP for

$$y = x^3$$

using the univariate reduction technique

Z.: *The truncated Hamburger moment problems with gaps in the index set*, Integ. Equ. Oper. Theory 93 (2021).

Univariate reduction technique

Let $\beta^{(2k)}$ be a sequence with $M(k)$ satisfying the column relation $Y = X^3$.

Every atom must be of the form (t, t^3) for some $t \in \mathbb{R}$. So $\beta_{i,j}$ corresponds to the moment of z^{i+3j}.

As i, j run over $0, 1, \ldots, 2k$ such that $i + j \leq 2k$, the sum $i + 3j$ runs over the set

$$\{0, 1, \ldots, 6k - 2, 6k\}.$$

The problem is equivalent to the **truncated Hamburger moment problem (THMP) with a gap** γ_{6k-1}, i.e., does there exist $x \in \mathbb{R}$ such that

$$(\gamma_0, \gamma_1, \ldots, \gamma_{6k-2}, x, \gamma_{6k})$$

admits a measure μ on $\mathbb{R}$, i.e., $\gamma_i = \int_{\mathbb{R}} x^i d\mu$ for each i. This is a PSD matrix completion problem with constraints.

Matrix completion result

Proposition

Let

$$A(?) := \begin{bmatrix} A_1 & a & b \\ a^T & \alpha & ? \\ b^T & ? & \beta \end{bmatrix} = \begin{bmatrix} A_1 & a & * \\ a^T & \alpha & * \\ * & * & * \end{bmatrix} = \begin{bmatrix} A_1 & * & b \\ * & * & * \\ b^T & * & \beta \end{bmatrix}$$

be a $n \times n$ matrix, where A_1 is a symmetric matrix, $a, b \in \mathbb{R}^{n-2}$ are vectors, $\alpha, \beta \in \mathbb{R}$ real numbers and x is a variable. Let A_2 and A_3 be the colored submatrices of $A(x)$ and

$$x_\pm := b^T A_1^\dagger a \pm \sqrt{(A_2/A_1)(A_3/A_1)} \in \mathbb{R},$$

where $A_2/A_1 = \alpha - a^T A^\dagger a$ and $A_3/A_1 = \beta - b^T B^\dagger b$. Then:

1. $A(x_0)$ is PSD if and only if A_2, A_3 are PSD and $x_0 \in [x_-, x_+]$.
2. $$\operatorname{rank} A(x_0) = \max\left\{\operatorname{rank} A_2, \operatorname{rank} A_3\right\} + \begin{cases} 0, & \text{for } x_0 \in \{x_-, x_+\}, \\ 1, & \text{for } x_0 \in (x_-, x_+). \end{cases}$$

Notation - Hankel matrix

Let $k \in \mathbb{N}$. For $\gamma = (\gamma_0, \ldots, \gamma_{2k}) \in \mathbb{R}^{2k+1}$ we define the corresponding Hankel matrix as

$$A_\gamma := [\gamma_{i+j}]_{i,j=0}^{k} = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \iddots & \iddots & \gamma_{k+1} \\ \gamma_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \gamma_{2k-1} \\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k-1} & \gamma_{2k} \end{pmatrix}.$$

We use

$$A_\gamma(m)$$

to denote the restriction of A to the first m rows and columns.

THMP of degree $2k$ with a gap γ_{2k-1}

Theorem

Let $k > 1$ and $\gamma(x) := (\gamma_0, \gamma_1, \ldots, \gamma_{2k-2}, x, \gamma_{2k}),$

be a sequence, where x is a variable, $\gamma^{(1)} = (\gamma_0, \gamma_1, \ldots, \gamma_{2k-2})$, $\gamma^{(2)} = (\gamma_0, \gamma_1, \ldots, \gamma_{2k-4})$ *with the moment matrix*

$$A_{\gamma(x)} = \left[\begin{array}{c|c} A_{\gamma^{(1)}} & \begin{matrix} v \\ x \end{matrix} \\ \hline \begin{matrix} v^T & x \end{matrix} & \gamma_{2k} \end{array}\right] = \left[\begin{array}{cc|c} A_{\gamma^{(2)}} & u & v \\ u^T & \gamma_{2k-2} & x \\ \hline v^T & x & \gamma_{2k} \end{array}\right],$$

where $v = (\gamma_k, \ldots, \gamma_{2k-2})$ *and* $u = (\gamma_{k-1}, \ldots, \gamma_{2k-3})$. *TFAE:*

1. *There exists $x_0 \in \mathbb{R}$ and a RM for $\gamma(x_0)$.*
2. $A_{\gamma^{(1)}}$ *and* $\begin{bmatrix} A_{\gamma^{(2)}} & v \\ v^T & \gamma_{2k} \end{bmatrix}$ *are PSD and one of the following conditions is true:*
   a) $A_{\gamma^{(1)}}$ *is PD.*
   b) $\operatorname{rank} A_{\gamma^{(2)}} = \operatorname{rank} A_{\gamma^{(1)}} = \operatorname{rank} \begin{bmatrix} A_{\gamma^{(2)}} & v \\ v^T & \gamma_{2k} \end{bmatrix}.$

4. Solving the TMP for plane cubics using positivity certificates

M. Kummer, Z.:
Positive polynomials and the truncated moment problem on plane cubics, 2025,
arXiv preprint `https://arxiv.org/abs/2508.13850`

Reformulation of the TMP

In the language of linear functionals

Let $k \in \mathbb{N}$ and

$$L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$$

a linear functional.

$C \subseteq \mathbb{R}^2$ is a plane cubic.

The bivariate truncated moment problem on C (C–TMP): characterize the existence of a positive Borel measure μ on $\mathbb{R}^2$ with support in C, such that

$$L(f) = \int_C f \, d\mu$$

for $i, j \in \mathbb{Z}_+$, $i + j \leq k$.

If μ exists, it is called a C–representing measure (C–RM) of L and L is called a C–moment functional.

Classification of plane cubics

Up to invertible affine change of coordinates

Irreducible cases:

$$(I)\ y = p(x), \quad (II)\ xy = p(x), \quad (III)\ y^2 = p(x),$$
$$(IV)\ xy^2 + ay = p(x),$$

where $p(x) = bx^3 + cx^2 + dx + e$.

Reducible cases:

$(i)\ y(ay + x^2 + y^2),\ a \neq 0, \quad (ii)\ y(1 + ay - x^2 - y^2),\ |a| > 2,$
$(iii)\ y(1 + ay - x^2 - y^2),\ (iv)\ y(y - x^2),\ (v)\ y(x - y^2),$
$(vi) y(1 + y + x^2),\ (vii) y(1 + y - x^2),\ (viii)\ y(1 - xy),$
$(ix)\ y(x + y + axy),\ a \neq 0,\ (x)\ y(ay + x^2 - y^2),\ a \neq 0,$
$(xi)\ y(1 + ay + x^2 - y^2),\ (xii)\ y(1 + ay - x^2 + y^2),$
$(xiii)\ y(a + y)(b + y),\ a, b \neq 0, a \neq b,\ (xiv)\ y(x - y)(x + y),$
$(xv)\ yx(y + 1),\ (xvi)\ y(1 - x + y)(1 + x + y),$

Some definitions

$C = \mathcal{Z}(P)$ a plane cubic, $I = \langle P \rangle \subseteq \mathbb{R}[x, y]$ an ideal generated by P,

$L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ a linear functional

$\mathbb{R}[C] = \mathbb{R}[x, y]/I$	a coordinate ring of C
$\mathbb{R}[C]_{\leq m}$	an image of $\mathbb{R}[x, y]_{\leq m}$ under the restriction map $f \mapsto f\|_C$
$Q(\mathbb{R}[C])$	a quotient ring of $\mathbb{R}[C]$
$L_C : \mathbb{R}[C]_{\leq 2k} \to \mathbb{R}$	an induced functional
$\ker \overline{L}_C$	the kernel of the bil. form $\overline{L}_C : \mathbb{R}[C]_{\leq k} \times \mathbb{R}[C]_{\leq k} \to \mathbb{R}$ induced by L_C
$\mathrm{POS}_{2k}(C)$	a set of all $p \in \mathbb{R}[C]_{\leq 2k}$ with $p(x) \geq 0$ for $x \in C$
V	a finite–dimensional vector space in $Q(\mathbb{R}[C])$
f	an element of $\mathbb{R}[C]$
U	a vector space generated by $\{gh \colon g, h \in V\}$
U_f	a vector space generated by $\{fgh \colon g, h \in V\}$

Assume that $U_f \subseteq \mathbb{R}[C]_{\leq k}$. Then the functional

$$L_{C,V,f} : U \to \mathbb{R}, \quad L_{C,V,f}(g) := L_C(fg)$$

if well–defined and called a (V, f)**–localizing functional of** L_C.

Some definitions

Assume $V_f \subseteq \mathbb{R}[C]_{\leq k}$.

L_C is **strictly positive** if $L_C(p) > 0$ for every $0 \neq p \in \mathrm{POS}_{2k}(C)$.

Theorem (di Dio, Schmüdgen, 2018)

Every strictly positive functional L_C is a C–moment functional.

Checking positivity is difficult.

But checking square positivity is simple.

L_C is **strictly square positive** if $L_C(g^2) > 0$ for every $0 \neq g \in \mathbb{R}[C]_{\leq k}$.

L_C is (V, f)**–locally strictly square positive** if $L_{C,V,f}(g^2) > 0$ for every $g \in V$.

Solution to the TMP on plane cubics - part 1

Assume $V_f \subseteq \mathbb{R}[C]_{\leq k}$.

Assume C is irreducible or C is reducible without non-real intersection points.

Theorem (Kummer, Z., 25+)

There exists $f \in Q(\mathbb{R}[C])$ such that for every $k \in \mathbb{N}$ there is a vector subspace $V^{(k)} \subseteq Q(\mathbb{R}[C])$ of dimension $3k$ so that the following holds: Let

$$L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$$

be a linear functional with $\ker \overline{L} = I_{\leq k}$ and $\ker \overline{L}_{C, V^{(k)}, f} = \{0\}$. Then the following are equivalent:

1. *L_C is strictly positive.*
2. *L_C is strictly square positive and $(V^{(k)}, f)$–locally strictly square positive.*

Solution to the TMP on plane cubics - part 2

Assume that C is reducible with non-real intersection points, defined by

$$P(x,y) = P_1(x,y)P_2(x,y), \quad \deg P_1 = 1,\ \deg P_2 = 2.$$

Theorem (Kummer, Z., 25+)

Let

$$L : \mathbb{R}[x,y]_{\leq 2k} \to \mathbb{R}$$

be a linear functional with $\ker \overline{L} = I_{\leq k}$, $\ker \overline{L}_{C,\mathbb{R}[C]_{\leq k-1},P_1} = \{0\}$ *and* $\ker \overline{L}_{C,\mathbb{R}[C]_{\leq k-1},P_2} = \{0\}$. *Then the following are equivalent:*

1. L_C *is strictly positive.*
2. L_C *is strictly square positive,* $(\mathbb{R}[C]_{\leq k-1}, \chi_1 P_1)$*–locally strictly square positive and* $(\mathbb{R}[C]_{\leq k-1}, \chi_2 P_2)$*–locally strictly square positive,*

where

$$\chi_1 = \begin{cases} 1, & \text{if } P_1 \text{ is nonnegative on } \mathcal{Z}(P_2), \\ -1, & \text{if } P_1 \text{ is nonpositive on } \mathcal{Z}(P_2), \\ 0, & \text{if } P_1 \text{ changes sign on } \mathcal{Z}(P_2), \end{cases}$$

$$\chi_2 = \begin{cases} 1, & \text{if } P_2 \text{ is nonnegative on } \mathcal{Z}(P_1), \\ -1, & \text{if } P_2 \text{ is nonpositive on } \mathcal{Z}(P_1). \end{cases}$$

Specifying $V^{(k)}$ and f for irreducible cases

$C = \mathcal{Z}(P)$, $\mathcal{B}_k$ is a basis for $\mathbb{R}[C]_{\leq k}$, $\mathcal{B}_{V^{(k)}}$ is a basis for $V^{(k)}$,

$$\Phi_1(p(x,y)) := p(t^2, t^3 - t),$$
$$\Phi_2(p(x,y)) := p(t^2 + 1, t^3 + t).$$

P	$\mathcal{B}_k$	$\mathcal{B}_{V^{(k)}}$	f
$y^2 - x(x-a)(x-b)$, $a, b \in \mathbb{R}$, $0 < a < b$	$\{1, x, y, \dots, x^2y^{i-2}, xy^{i-1}, y^i, \dots x^2y^{k-2}, xy^{k-1}, y^k\}$	$\mathcal{B}_k \setminus \{y^k\} \cup \{\frac{y}{x}\}$	x
$y^2 - x(x^2 + c)$, $c \in (0, \infty)$	$\{1, x, y, \dots, x^2y^{i-2}, xy^{i-1}, y^i, \dots x^2y^{k-2}, xy^{k-1}, y^k\}$	$\mathcal{B}_k \setminus \{y^k\} \cup \{\frac{y}{x}\}$	x
$y^2 - x^3$	$\{1, x, y, \dots, x^2y^{i-2}, xy^{i-1}, y^i, \dots x^2y^{k-2}, xy^{k-1}, y^k\}$	$\mathcal{B}_k \setminus \{1\} \cup \{\frac{y}{x}\}$	1
$y^2 - x(x-1)^2$	$\Phi_1^{-1}(\{1, t^2 - 1, t^3 - t, \dots, t^{k-1} - t^{k-3}, t^k - t^{k-2}\})$	$\mathcal{B}_k \setminus \{1\} \cup \{\frac{y}{x-1}\}$	1
$y^2 - x^2(x-1)$	$\Phi_2^{-1}(\{1, t^2 + 1, t^3 + t, \dots, t^{k-1} + t^{k-3}, t^k + t^{k-2}\})$	$\mathcal{B}_k \setminus \{1\} \cup \{\frac{y}{x}\}$	1
$yx - c(x)$, c of degree 3, $c(0) \neq 0$	$\{1, x, y, \dots, x^2y^{i-2}, xy^{i-1}, y^i, \dots x^2y^{k-2}, xy^{k-1}, y^k\}$	$\mathcal{B}_k \setminus \{y^k\} \cup \{y^k - 2[x^{2k}]\}$	1
$xy^2 + ax - by - c$, $a, b, c \in \mathbb{R}$, $c \neq 0$ or $ab \neq 0$	$\{x^k, x^{k-1}, x^{k-1}y, \dots, x, xy, 1, y, \dots, y^k\}$	$\mathcal{B}_k \setminus \{x^k\} \cup \{x^k y\}$	1

Specifying $V^{(k)}$ and f for reducible cases

$C = \mathcal{Z}(P)$, $\mathcal{B}_k$ is a basis for $\mathbb{R}[C]_{\leq k}$, $\mathcal{B}_{V^{(k)}}$ is a basis for $V^{(k)}$, f is always 1

P	$\mathcal{B}_k$	$\mathcal{B}_{V^{(k)}}$
$y(ay + x^2 + y^2)$, $a \in \mathbb{R} \setminus \{0\}$	$\{1, x, y, \ldots, x^j, x^{j-1}y, x^{j-2}y^2, \ldots x^k, x^{k-1}y, x^{k-2}y^2\}$	$\mathcal{B}_k \setminus \{1\} \cup \{\frac{ay+x^2+y^2}{x}\}$
$y(1 + ay - x^2 - y^2)$, $a \in \mathbb{R}$	$\{1, x-1, x^2-1, \ldots, x^{k-2}(x^2-1), y, yx, \ldots, yx^{k-1}, y^2, \ldots, y^2x^{k-2}\}$	$\mathcal{B}_k \setminus \{1\} \cup \{1 - 2\frac{1+ay-x^2-y^2}{1-x^2}\}$
$y(x - y^2)$	$\{1, x, \ldots, x^k, y, y^2, yx, y^2x, \ldots, yx^j, y^2x^j, \ldots, y^2x^{k-2}, yx^{k-1}\}$	$\mathcal{B}_k \setminus \{x^k\} \cup \{x^k - 2y^2x^{k-1}\}$
$y(1 + y - x^2)$	$1, x-1, x^2-1, \ldots, x^{k-2}(x^2-1), y, yx, y^2, y^2x \ldots, y^{k-1}x, y^k\}$	$\mathcal{B}_k \setminus \{x^k\} \cup \{1 - x - 2\frac{1+y-x^2}{1+x}\}$
$y(x - y)(x + y)$	$\{1, x, y, x^2, xy, y^2, \ldots x^k, x^{k-1}y, y^k\}$	$\mathcal{B}_k \setminus \{1\} \cup \{\frac{x^2-y^2}{x}]\}$
$yx(y + 1)$	$\{1, x, y, x^2, xy, y^2, \ldots x^k, x^{k-1}y, y^k\}$	$\mathcal{B}_k \setminus \{x^k\} \cup \{x^k + 2yx^k\}$

Main method in proofs

$C = \mathcal{Z}(P)$, $P = \prod_{i=1}^r P_i$ with P_i irreducible

Theorem (Baldi, Blekherman, Sinn, 24+ & Kummer, 24+)

Assume that the restriction of $Q \in \mathbb{R}[x, y]_{\leq 2d}$ to C generates an extreme ray of $\mathrm{POS}_{2d}(C)$. Denote $Q^h(x, y, z) = z^{2d} \cdot Q(\frac{x}{z}, \frac{y}{z})$.

Irreducible C: *The set*

$$\{x \in \mathbb{P}^2 \mid Q^h(x) = P^h(x) = 0\}$$

consists only of real points.

Reducible C: *Let S be the set of indices $i \in \{1, \ldots, r\}$ for which Q is divisible by P_i. Then, for every $j \in \{1, \ldots, r\} \setminus S$, the set*

$$\{x \in \mathbb{P}^2 \mid Q^h(x) = P_j^h(x) = 0 \textit{ and } P_i^h(x) \neq 0 \textit{ for all } i \in S\}$$

consists only of real points.

Positivstellensatz

$V^{(k)}$ and f appearing in the tables above also appear in the following Positivstellensatz.

Theorem

There are $f \in \mathbb{R}[C]$ and a finite–dimensional vector space $V^{(k)}$ in $Q(\mathbb{R}[C])$ with $V_f^{(k)} \subseteq \mathbb{R}[C]_{\leq k}$ such that the following are equivalent:

1. $p \in \mathrm{POS}_{2k}(C)$.
2. *There exist finitely many $g_i \in \mathbb{R}[C]_{\leq k}$ and $h_j \in V^{(k)}$ such that* $p = \sum_i g_i^2 + f \sum_j h_j^2$.

TMP for $y^2 - x(x-a)(x-b) = 0$, $a, b \in \mathbb{R}, 0 < a < b$

A C–degree function $\deg_C$:

$$\deg_C(x^i y^j) = 2i + 3j \quad \text{including negative } i, j.$$

A basis $\mathcal{B}_k$ for $\mathbb{R}[C]_{\leq k}$ and $\mathcal{B}_{V^{(k)}}$ for $V^{(k)}$:

$\mathcal{B}_k$	1	x	y	$\dots$	x^2y^{i-2}	xy^{i-1}	y^i	$\dots$	x^2y^{k-2}	xy^{k-1}	y^k
$\deg_C$	0	2	3	$\dots$	$3i-2$	$3i-1$	$3i-2$	$\dots$	$3k-2$	$3k-1$	$3k/1$
$\mathcal{B}_{V^{(k)}}$	1	x	y	$\dots$	x^2y^{i-2}	xy^{i-1}	y^i	$\dots$	x^2y^{k-2}	xy^{k-1}	$\frac{y}{x}$

Theorem

Let $p \in \mathrm{POS}_{2k}(C)$. Then there exist finitely many $g_i \in \mathbb{R}[C]_{\leq k}$ and $h_j \in V^{(k)}$ such that $p = \sum_i g_i^2 + x \sum_j h_j^2$.

Sketch of the proof:

- Let $u \in \mathrm{POS}_{2k}(C)$ be an extreme ray and $u^h(x,y,z) = z^{2k} u(\frac{x}{z}, \frac{y}{z})$ a homogenization of u.
- Then u^h has only real zeroes P_i, $i = 1, \dots, 3k$, of the form $P_i = [x_i : y_i : 1]$, $x_i, y_i \in \mathbb{R}$ or $P_i = [0 : 1 : 0]$, each of multiplicity 2.
- Known fact: $P := P_1 \oplus \dots \oplus P_{3k}$ is a 2–torsion point in the group law of C.
- If P is the point at infinity $O := [0 : 1 : 0]$, then $u^h = (u_1^h)^2$ for some $u_1^h \in \mathbb{R}[x,y,z]_{\leq k}$ and $u = u_1^2$ is a square of $u_1(x,y) = u_1^h(x,y,1) \in \mathbb{R}[C]_{\leq k}$.
- Otherwise $P = [0 : 0 : 1]$ and $xzu^h = (u_2^h)^2$ for some $u_2^h \in \mathbb{R}[x,y,z]_{\leq k+1}$. Then $u = x(\frac{u_2}{x})^2$, where $u_2 = u_2^h(x,y,1)$. Considering $\deg_C$ of both sides, u_2 cannot contain 1, y^{k+1} or xy^k.

TMP for $y^2 - x(x-a)(x-b) = 0$, $a, b \in \mathbb{R}, 0 < a < b$

Example: $2k = 6$, $\beta_{ij} = L(x^i y^j)$

L_C strict square positivity and V_x–local strict square positivity are equivalent to positive definiteness of the following matrices:

$$
\begin{array}{c}
 \\ 1 \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \\ X^2Y \\ XY^2 \\ Y^3
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc}
1 & X & Y & X^2 & XY & Y^2 & X^2Y & XY^2 & Y^3
\end{array} \\
\begin{bmatrix}
\beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} \\
\beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} \\
\beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} \\
\beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{23} \\
\beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{14} \\
\beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} & \beta_{23} & \beta_{14} & \beta_{05} \\
\beta_{21} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{23} & \beta_{42} & \beta_{33} & \beta_{24} \\
\beta_{12} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{14} & \beta_{33} & \beta_{24} & \beta_{15} \\
\beta_{03} & \beta_{13} & \beta_{04} & \beta_{23} & \beta_{14} & \beta_{05} & \beta_{24} & \beta_{15} & \beta_{06}
\end{bmatrix}
\end{array},
$$

$$
\begin{array}{c}
 \\ 1 \\ Y/X \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \\ X^2Y \\ XY^2
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc}
X & Y & X^2 & XY & X^3 & X^2Y & XY^2 & X^3Y & X^2Y^2
\end{array} \\
\begin{bmatrix}
\beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} \\
\beta_{01} & L((x-a)(x-b)) & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} \\
\beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} \\
\beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} \\
\beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{50} & \beta_{41} & \beta_{32} & \beta_{51} & \beta_{42} \\
\beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{23} & \beta_{42} & \beta_{33} \\
\beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{14} & \beta_{33} & \beta_{24} \\
\beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{51} & \beta_{42} & \beta_{33} & \beta_{52} & \beta_{43} \\
\beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{42} & \beta_{33} & \beta_{24} & \beta_{43} & \beta_{34}
\end{bmatrix}
\end{array}.
$$

TMP for nodal cubic $y^2 - x(x-1)^2 = 0$

Parametrization of C:

$$(x(t), y(t)) = (t^2, t^3 - t), \quad t \in \mathbb{R},$$

Let

$$\text{Nodal} := \{s \in \mathbb{R}[t] \colon s(1) = s(-1)\}, \quad \text{Nodal}_{\leq i} := \{s \in \text{Nodal} \colon \deg s \leq i\}.$$

The map

$$\Phi : \mathbb{R}[C] \to \text{Nodal}, \quad \Phi(p(x,y)) = p(t^2, t^3 - t)$$

is a ring isomorphism. The vector subspace $\mathbb{R}[C]_{\leq i}$ is in one-to-one correspondence with the set $\text{Nodal}_{\leq 3i}$ under Φ.

Let

$$\text{POS}(\text{Nodal}_{\leq i}) := \{f \in \text{Nodal}_{\leq i} \colon f(t) \geq 0 \text{ for every } t \in \mathbb{R}\},$$

$$\widetilde{\text{Nodal}}_{\leq i} := \{s \in \mathbb{R}[t]_{\leq i} \colon s(1) = -s(-1)\}.$$

Theorem

Let $p \in \text{POS}(\text{Nodal}_{\leq 6k})$. Then there exist finitely many $g_i \in \text{Nodal}_{\leq 3k}$ and $h_j \in \widetilde{\text{Nodal}}_{\leq 3k}$ such that $p = \sum_i g_i^2 + \sum_j h_j^2$.

TMP for nodal cubic $y^2 - x(x-1)^2 = 0$

The basis for $\mathrm{Nodal}_{\leq i}$ is the following:

$$\mathcal{B}_{\mathrm{Nodal}_{\leq i}} := \{1, t^2 - 1, t^3 - t, t^4 - t^2, \ldots, t^{i-1} - t^{i-3}, t^i - t^{i-2}\}.$$

The basis for $\widetilde{\mathrm{Nodal}}_{\leq i}$ is the following:

$$\mathcal{B}_{\widetilde{\mathrm{Nodal}}_{\leq i}} := \{t, t^2 - 1, t^3 - t, t^4 - t^2, \ldots, t^{i-1} - t^{i-3}, t^i - t^{i-2}\}.$$

We have that

$$\frac{y}{x-1}$$

maps to t under Φ. So this is a replacement for 1 in the basis for V.

This approach also gives an idea for constructive solution to the TMP working also in singular cases

Using correspondence Φ above the C–TMP for L is equivalent to the $\mathbb{R}$–TMP for

$$L_{\mathrm{Nodal}_{\leq 6k}} : \mathrm{Nodal}_{\leq 6k} \to \mathbb{R}, \quad L_{\mathrm{Nodal}_{\leq 6k}}(p) = L_C(\Phi^{-1}(p)).$$

Using the basis $\mathcal{B}_{\mathrm{Nodal}_{\leq 3k}} \cup \mathcal{B}_{\widetilde{\mathrm{Nodal}}_{\leq 3k}}$ the moment matrix of $L_{\mathrm{Nodal}_{\leq 6k}}$ is

$$\begin{array}{c} \\ 1 \\ T \\ T^2-1 \\ T^3-T \\ \vdots \\ T^{3k}-T^{3k-2} \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & T & T^2-1 & T^3-T & \cdots & T^{3k}-T^{3k-2} \end{array} \\ \left[\begin{array}{cccccc} \mathcal{L}(1) & {\color{red}?} & \mathcal{L}(t^2-1) & \mathcal{L}(t^3-t) & \cdots & \mathcal{L}(t^{3k}-t^{3k-2}) \\ {\color{red}?} & \mathcal{L}(t^2) & \mathcal{L}(t^3-t) & \mathcal{L}(t^4-t^2) & \cdots & \mathcal{L}(t^{3k+1}-t^{3k-1}) \\ \mathcal{L}(t^2-1) & \mathcal{L}(t^3-t) & \mathcal{L}((t^2-1)^2) & \mathcal{L}(t(t^2-1)^2) & \cdots & \mathcal{L}(t^{3k-2}(t^2-1)^2) \\ \mathcal{L}(t^3-t) & \mathcal{L}(t(t^3-t)) & \mathcal{L}(t(t^2-1)^2) & \mathcal{L}((t^3-t)^2) & \cdots & \mathcal{L}(t^{3k-1}(t^2-1)^2) \\ \vdots & & & & \ddots & \vdots \\ \mathcal{L}(t^{3k}-t^{3k-2}) & \cdots & & & \cdots & \mathcal{L}((t^{3k}-t^{3k-2})^2) \end{array}\right] \end{array}.$$

From here it is easy to characterize when $L_{\mathrm{Nodal}_{\leq 6k}}$ is a $\mathbb{R}$–moment functional and construct a measure after completing the only ? position in the matrix above.

However, it is not clear whether one needs $\mathrm{rank}\, \overline{L}_{\mathrm{Nodal}_{\leq 6k}}$ or $\mathrm{rank}\, \overline{L}_{\mathrm{Nodal}_{\leq 6k}} + 1$ atoms in a minimal measure.

TMP for nodal cubic $y^2 - x(x-1)^2 = 0$

$\Phi : \mathbb{R}[C]_{\leq 2k} \to \text{Nodal}_{\leq 6k}, \Phi(p(x,y)) = p(t^2, t^3 - t),$

$V^{(k)} = \text{span}\{\Phi^{-1}(\mathcal{B}_{\text{Nodal}_{\leq 3k}})\}$

L_C is **singular** if $\ker L_C \neq \{0\}$.

L_C is $(V^{(k)}, 1)$**–locally singular** if $\ker L_{C,V^{(k)},1} \neq \{0\}$.

Theorem

Let $L : \mathbb{R}[x,y]_{\leq 2k}$ *be a linear functional such that* $I_{\leq k} \subseteq \ker \overline{L}$ *and (*$\ker \overline{L}_C \neq \{0\}$ *or* $\ker \overline{L}_{C,V^k,1} \neq \{0\}$*). Then the following are equivalent:*

1. *L is a C–moment functional.*
2. *L_C is square positive and $(V^{(k)}, 1)$–locally square positive and one of the following holds:*
   - 2.1 $\text{rank}\, \overline{L}_C = \text{rank}(\overline{L}_C)|_{(\Phi^{-1}(\mathcal{B}_{\text{Nodal}_{\leq 3k-1}}))}$.
   - 2.2 $\text{rank}\, \overline{L}_{C,V^{(k)},1} = \text{rank}(\overline{L}_{C,V^{(k)},1})|_{(\Phi^{-1}(\mathcal{B}_{\widetilde{\text{Nodal}}_{\leq 3k-1}}))}$.

TMP for $y(ay + x^2 + y^2) = 0$

A line C_1 an a circle C_2 with one double intersection point

Parametrization of C:

$$C_1 : \{(s, 0), s \in \mathbb{R}\}; \quad C_2 : \Big\{(-\frac{a}{2}\frac{t^2-1}{t^2+1}, -\frac{a}{2}\frac{(t+1)^2}{t^2+1}\Big\}, \quad t \in \mathbb{R}.$$

Let $D = Q_{\mathrm{i}} + Q_{-\mathrm{i}}$ and

$$\mathrm{Circ} = \{(f(s), g(t)) \in \mathbb{R}[s] \times \mathbb{R}\Big[\frac{1}{t^2+1}, \frac{t}{t^2+1}\Big] : f(0) = g(-1),\ f'(0) = \frac{2g'(-1)}{a}\},$$

$$\mathrm{Circ}_{\leq i} = \{(f(s), g(t)) \in \mathbb{R}[s]_{\leq i} \times \mathcal{L}(iD) \colon f(0) = g(-1),\ f'(0) = \frac{2g'(-1)}{a}\}.$$

The map

$$\Phi : \mathbb{R}[C] \to \mathrm{Circ}_1,\ \Phi(p(x, y)) = \Bigg(p(s, 0), p\Big(-\frac{a}{2}\frac{t^2-1}{t^2+1}, -\frac{a}{2}\frac{(t+1)^2}{t^2+1}\Big)\Bigg)$$

is a ring isomorphism. The vector subspace $\mathbb{R}[C]_{\leq i}$ is in one-to-one correspondence with the set $\mathrm{Circ}_{\leq 3i}$ under Φ.

Let

$$\mathrm{POS}(\mathrm{Circ}_{\leq i}) := \{(f(s), g(t)) \in (\mathrm{Circ}_1)_{\leq i} \colon f(s) \geq 0,\ g(t) \geq 0 \text{ for every } (s, t) \in \mathbb{R}^2\},$$

$$\widetilde{\mathrm{Circ}}_{\leq i} := \{(f(s), g(t)) \in \mathbb{R}[s]_{\leq i} \times \mathcal{L}(iD) \colon f(0) = g(-1) = 0\}.$$

TMP for $y(ay + x^2 + y^2) = 0$

Theorem

Let $(p_1, p_2) \in \mathrm{POS}(\mathrm{Circ}_{\leq 2k})$*. Then there exist finitely many* $(g_{1;i}, g_{2,i}) \in \mathrm{Circ}_{\leq k}$ *and* $(h_{1;j}, h_{2;j}) \in \widetilde{\mathrm{Circ}}_{\leq k}$ *such that*

$$(p_1, p_2) = \sum_i (g_{1;i}^2, g_{2;i}^2) + \sum_j (h_{1;j}^2, h_{2;j}^2).$$

The basis for $\mathrm{Circ}_{\leq i}$ is the following:

$$\mathcal{B}_{\mathrm{Circ}_{\leq i}} := \Phi(\{1, x, y, x^2, xy, y^2, \ldots, x^j, x^{j-1}y, x^{j-2}y^2, \ldots x^i, x^{i-1}y, x^{i-2}y^2\})$$

The basis for $\widetilde{\mathrm{Circ}}_{\leq i}$ is the following:

$$\mathcal{B}_{\widetilde{\mathrm{Circ}}_{\leq i}} := \mathcal{B}_{\mathrm{Circ}_{\leq i}} \setminus \{(1, 1)\} \cup \{(s, 0)\}$$

We have that

$$\frac{ay + x^2 + y^2}{x}$$

maps to $(s, 0)$ under Φ. So this is a replacement for 1 in the basis for V.

Thank you for your attention!