Solving bivariate truncated moment problems by reduction to the univariate setting

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Bivariate truncated moment problem

- $k \in \mathbb{N}$ and
$$\beta = \beta^{(k)} = (\beta_{i,j})_{i,j \in \mathbb{Z}_+, i+j \leq k}$$
a bivariate sequence of real numbers of degree k.
- K a closed subset of $\mathbb{R}^2$
- The **bivariate truncated moment problem on K (K–TMP)**: characterize the existence of a positive Borel measure μ on $\mathbb{R}^2$ with support in K, such that
$$\beta_{i,j} = \int_K x^i y^j d\mu(x) \quad \text{for} \quad i, j \in \mathbb{Z}_+, i + j \leq 2k. \tag{1}$$
- μ satisfying (1) is a K–representing measure (K–RM) of β.

Bivariate moment matrix

The moment matrix $M(k)$ associated to β with the rows and columns indexed by X^iY^j, $i+j \le k$, in degree-lexicographic order

$$1, X, Y, X^2, XY, Y^2, \ldots, X^k, X^{k-1}Y, \ldots, Y^k$$

is defined by

$$M(k) = (\beta_{i+j})_{i,j=0}^k = \begin{bmatrix} M[0,0](\beta) & M[0,1](\beta) & \cdots & M[0,k](\beta) \\ M[1,0](\beta) & M[1,1](\beta) & \cdots & M[1,k](\beta) \\ \vdots & \vdots & \ddots & \vdots \\ M[k,0](\beta) & M[k,1](\beta) & \cdots & M[k,k](\beta) \end{bmatrix},$$

where

$$M[i,j](\beta) := \begin{array}{c c} & \begin{array}{ccccc} X^j & X^{j-1}Y & X^{j-2}Y^2 & \cdots & Y^j \end{array} \\ \begin{array}{c} X^i \\ X^{i-1}Y \\ X^{i-2}Y^2 \\ \vdots \\ Y^i \end{array} & \begin{bmatrix} \beta_{i+j,0} & \beta_{i+j-1,1} & \beta_{i+j-2,2} & \cdots & \beta_{i,j} \\ \beta_{i+j-1,1} & \beta_{i+j-2,2} & \beta_{i+j-3,3} & \cdots & \beta_{i-1,j+1} \\ \beta_{i+j-2,2} & \beta_{i+j-3,3} & \beta_{i+j-4,4} & \cdots & \beta_{i-2,j+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{j,i} & \beta_{j-1,i+1} & \beta_{j-2,i+2} & \cdots & \beta_{0,i+j} \end{bmatrix} \end{array}$$

are Hankel matrices.

Necessary conditions

- To every polynomial $p := \sum_{i,j\in\mathbb{Z}_+, i+j\le k} a_i x^i y^j \in \mathbb{R}[x,y]_k$, we associate the vector

$$p(X,Y) = \sum_{i,j\in\mathbb{Z}_+, i+j\le k} a_i X^i Y^j$$

from the column space of the matrix $M(k)$.

- The matrix $M(k)$ is recursively generated (RG) if for $p, q, pq \in \mathbb{R}[x,y]_k$ and $p(X,Y) = \mathbf{0}$, also $(pq)(X,Y) = \mathbf{0}$.

- The matrix $M(k)$ satisfies the variety condition (VC) if

$$\operatorname{rank} M(k) \le \operatorname{card} \mathcal{V},$$

where $\mathcal{V} := \bigcap_{\substack{g\in\mathbb{R}[x,y]_{\le k}, \\ g(X,Y)=\mathbf{0} \text{ in } M(k)}} \left\{(x,y)\in\mathbb{R}^2 : g(x,y) = 0\right\}$.

Proposition

If $\beta^{(2k)}$ has a representing measure μ, then

$M(k)$ is PSD, RG and satisfies VC.

Known results on the bivariate TMP

Theorem

1. ***Quadratic TMP, i.e.*** $\beta = \beta^{(2)}$: *Completely solved.* *Curto & Fialkow, '96*
2. ***Cubic TMP, i.e.*** $\beta = \beta^{(3)}$: *Completely solved.* *Kimsey, '14, Curto & Yoo, '18*
3. ***Quartic TMP, i.e.*** $\beta = \beta^{(4)}$: *Completely solved.*
   $M(2)$ *singular:* *Curto & Fialkow, '02*
   $M(2)$ *nonsingular:* *Fialkow & Nie, '10, Curto & Yoo, '16*
4. ***Quintic TMP, i.e.*** $\beta = \beta^{(5)}$: *Completely solved.* *El Azhar, Harrat, Idrissi, Zerouali, '19*
5. ***Sextic TMP, i.e.*** $\beta = \beta^{(6)}$: *Partially solved.*
   - *Extremal case* - rank $M(3)$ = card $\mathcal{V}$
     Curto & Fialkow & Möller, '05
   - *On variety* $y = x^3$ *Fialkow, '11*
   - rank $M(3) \in \{7, 8\}$ *Curto, Yoo, '14, '15*
   - *On special cases of reducible varieties* *Yoo, '17*
   - $M(3)$ *invertible* *Fialkow, '17, Fialkow & Blekherman, '20*

Known results on the bivariate TMP

Theorem

6. ***TMP on quadratic curves:*** *Completely solved.* *Curto & Fialkow, '02, '04, '05, '14*

   Let $p(x, y)$ be a quadratic polynomial. $\beta^{(2k)}$ has a representing measure supported on $\{(x, y) \in \mathbb{R}^2 : p(x, y) = 0\}$ iff $M(k)$ is PSD, RG, satisfies VC and $p(X, Y) = 0$ is a column relation.

7. ***TMP on cubic curves, i.e.*** $\beta = \beta^{(2k)}$***:*** *Cases solved.*
   - *Infinite variety: $y = x^3$* *Fialkow, '11*
   - *Finite variety: $z^3 = itz + u\bar{z}$, $t, u \in \mathbb{R}$* *Curto, Yoo '14, '15*

- The main technique is the application of the Flat extension theorem by extending $M(k)$ to $M(k+1)$ of the same rank or in some case to a psd $M(k+2)$ such that $M(k+2)$ is a FE of $M(k+1)$.
- This analysis is very demanding in some cases (e.g., on $xy = 1$, $y = x^3$, $y^2 = 1$).

A new approach to the TMP - the reduction to the univariate case

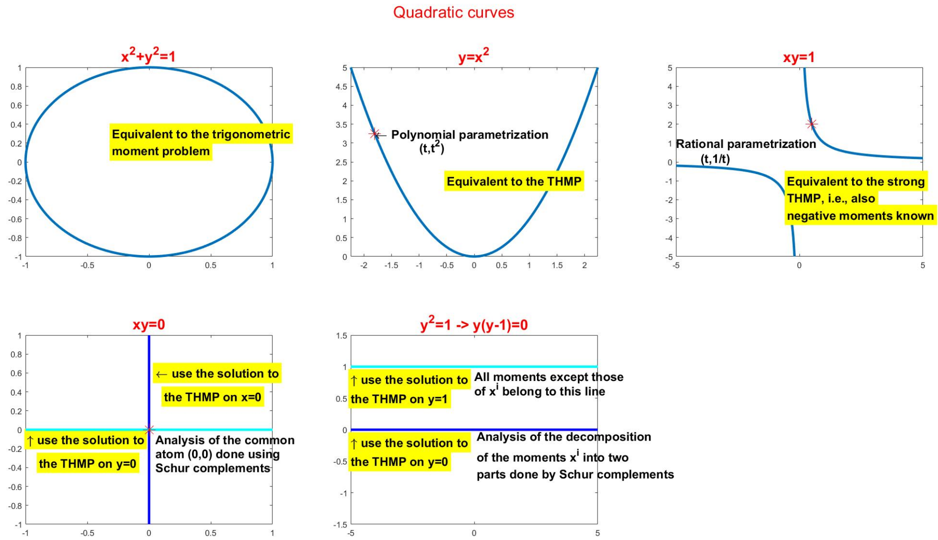
Basic ideas:

1. For irreducible curves:
   - Get rid of one variable by expressing it with the other using the RG relations.
   - Use the solution to the corresponding univariate full (strong) TMP or the (strong) TMP with gaps.
2. For reducible curves:
   - Apply the appropriate affine linear transformation to transform the variety into the form, where it is possible to analyze the decompositions of the moments into the parts corresponding to different lower degree varieties.
   - Use the solution of the (strong) TMP on each variety of the decomposition.

Scope of this approach:

1. Solution to the TMP on quadratic curves, some new cases of cubic curves and special cases of higher degree curves.
2. Possible to restrict further the support within the curve.
3. Solving new cases of the univariate (strong) TMP with gaps solves the bivariate TMP on a new curve.

The TMP on quadratic curves - a new approach



Solution to the TMP on $xy = 0$

Let $\beta^{(2k)}$ be a sequence with $M(k)$ satisfying the column relation $XY = 0$.

We write $\vec{X} = (X, \dots, X^k)$ and $\vec{Y} = (Y, \dots, Y^k)$.

Since $M(k)$ must be rg, we have that

$$M(k) = \begin{array}{c c} & \begin{array}{ccc} 1 & \vec{X} & \vec{Y} \end{array} \\ \begin{array}{c} 1 \\ \vec{X}^T \\ \vec{Y}^T \end{array} & \begin{bmatrix} 1 & a^t & b^t \\ a & A & 0 \\ b & 0 & B \end{bmatrix} \end{array} \bigoplus \mathbf{0},$$

with A, B, a, b equal to

$$\begin{bmatrix} \beta_{2,0} & \cdots & \beta_{k+1,0} \\ \vdots & \iddots & \vdots \\ \beta_{k+1,0} & \cdots & \beta_{2k,0} \end{bmatrix}, \begin{bmatrix} \beta_{0,2} & \cdots & \beta_{0,k+1} \\ \vdots & \iddots & \vdots \\ \beta_{0,k+1} & \cdots & \beta_{0,2k} \end{bmatrix}, \begin{bmatrix} \beta_{1,0} \\ \vdots \\ \beta_{k,0} \end{bmatrix}, \begin{bmatrix} \beta_{0,1} \\ \vdots \\ \beta_{0,k} \end{bmatrix}.$$

Let

$$M_1(\alpha) := \begin{bmatrix} \alpha & a^t \\ a & A \end{bmatrix} \quad \text{and} \quad M_2(\gamma) := \begin{bmatrix} \gamma & b^t \\ b & B \end{bmatrix}.$$

Using Schur complements and the THMP it is easy to analyse when there is $a \in [0, 1]$ such that $M_1(a)$ and $M_2(1 - a)$ admit $\mathbb{R}$–representing measures.

Theorem (Solution to the TMP on $xy = 0$)

The sequence β admits a RM on $\{(x, y) : xy = 0\}$ if and only if $M(k)$ is PSD, $\beta_{i+1,j+1} = \beta_{i,j}$ for every $0 \le i, j \le 2k$ such that $i + j \le 2k - 2$, and one of the following conditions holds:

1. $\operatorname{rank} M(k) > \operatorname{rank}(A \oplus B)$ *and*
   1. $\operatorname{rank} A = k$ *or* $\operatorname{rank} A = \operatorname{rank} A|_{\{X,\ldots,X^{k-1}\}}$,
   2. $\operatorname{rank} B = k$ *or* $\operatorname{rank} B = \operatorname{rank} B|_{\{Y,\ldots,Y^{k-1}\}}$,
2. $\operatorname{rank} M(k) = \operatorname{rank}(A \oplus B)$ *and*
   1. $\operatorname{rank} M_1(a^t A^\dagger a) = \operatorname{rank} \left(M_1(a^t A^\dagger a)\right)|_{\{X,\ldots,X^{k-1}\}}$,
   2. $\operatorname{rank} M_2(b^t B^\dagger b) = \operatorname{rank} \left(M_2(b^t B^\dagger b)\right)|_{\{Y,\ldots,Y^{k-1}\}}$.

If the RM exists, then there is a $\left(\operatorname{rank} M(k)\right)$–atomic RM unless (1) holds and neither $M_1(a^t A^\dagger a)$ nor $M_2(b^t B^\dagger B)$ admit a RM, in which case there is a $\left(\operatorname{rank} M(k) + 1\right)$–atomic RM.

Let $\beta^{(2k)}$ be a sequence with $M(k)$ satisfying the column relation $Y^2 = Y$.

We write $\vec{X}^{(i)} = (1, X, \ldots, X^{k-i})$ and $Y^j\vec{X}^{(i)} = (Y^j, \ldots, Y^jX^{k-i})$.

Let

$$M = M(k)|_{\{\vec{X}^{(0)} \cup Y\vec{X}^{(1)}\}} = \begin{array}{c} \\ (\vec{X}^{(0)})^T \\ X^k \\ (Y\vec{X}^{(1)})^T \end{array} \begin{array}{c} \begin{array}{ccc} \vec{X}^{(0)} & X^k & Y\vec{X}^{(1)} \end{array} \\ \begin{pmatrix} A_1 & a & B_1 \\ a^T & \beta_{2k,0} & b^T \\ B_1 & b & B_1 \end{pmatrix} \end{array},$$

$$N = M(k)|_{\{\vec{X}^{(1)} \cup Y\vec{X}^{(1)}\}} = \begin{array}{c} \\ (\vec{X}^{(1)})^T \\ (Y\vec{X}^{(1)})^T \end{array} \begin{array}{c} \begin{array}{cc} \vec{X}^{(1)} & Y\vec{X}^{(1)} \end{array} \\ \begin{pmatrix} A_1 & B_1 \\ B_1 & B_1 \end{pmatrix} \end{array},$$

Let

$$M_1(\alpha) = \begin{array}{c} \\ (\vec{X}^{(1)})^T \\ X^k \end{array} \begin{array}{cc} \vec{X}^{(1)} & X^k \\ \multicolumn{2}{c}{\begin{pmatrix} B_1 & b \\ b^T & \alpha \end{pmatrix}} \end{array}, \quad M_2(\alpha) = \begin{array}{c} \\ (\vec{X}^{(1)})^T \\ X^k \end{array} \begin{array}{cc} \vec{X}^{(1)} & X^k \\ \multicolumn{2}{c}{\begin{pmatrix} A_1 - B_1 & a - b \\ (a-b)^T & \beta_{2k,0} - \alpha \end{pmatrix}} \end{array}.$$

Using Schur complements and the THMP it is easy to analyse when there is a such that $M_1(a)$ and $M_2(a)$ admit $\mathbb{R}$–representing measures.

Theorem (Solution to the TMP on $y^2 = y$)

The sequence β admits a RM on $\{(x, y) : y^2 = y\}$ if and only if $M(k)$ is PSD, $\beta_{i,j+2} = \beta_{i,j+1}$ hold for every $i, j \in \mathbb{Z}_+$ with $i + j \leq 2k - 2$ and one of the following conditions holds:

1. *B_1 is invertible.*
2. *$A_1 - B_1$ is invertible.*
3. $\operatorname{rank} M = \operatorname{rank} N$.

If the RM exists, then there always exists a $(\operatorname{rank} M(k))$*–atomic RM.*

Solution to the TMP on $xy = 1$

Let $\beta^{(2k)}$ be a sequence with $M(k)$ satisfying the column relation $XY = 1$.

We write $\vec{X} = (1, X, \ldots, X^k)$ and $\vec{Y} = (Y, \ldots, Y^k)$.

Theorem (Solution to the TMP on $xy = 1$)

The sequence β admits a RM on $\{(x, y) : xy = 1\}$ if and only if $M(k)$ is PSD, $\beta_{i+1,j+1} = \beta_{i,j}$ hold for every $i, j \in \mathbb{Z}_+$ with $i + j \le 2k - 2$, and one of the following statements hold:

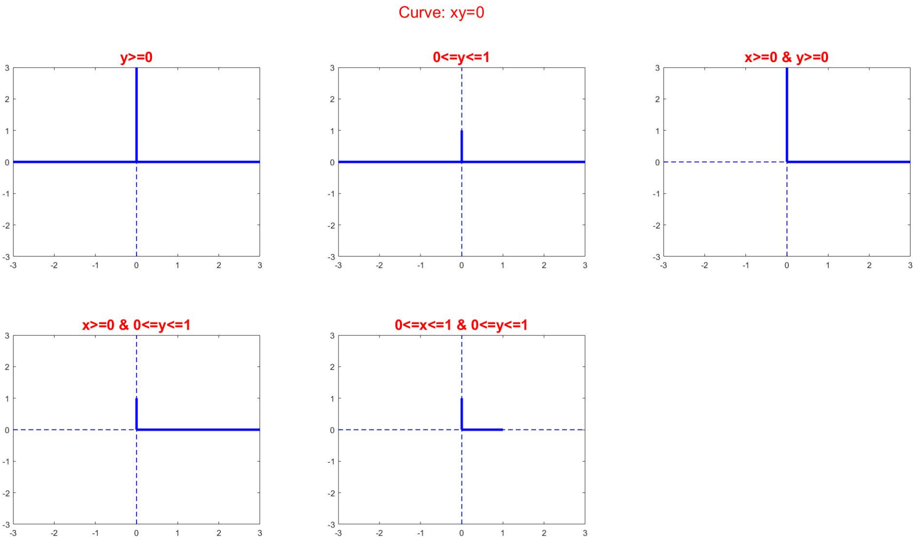
1. $M(k)|_{\{\vec{X}\cup\vec{Y}\}}$ *is invertible.*
2. $\mathrm{rank}(M(k)_{\{\vec{X}\cup\vec{Y}\}}) = \mathrm{rank}(M(k)_{\{\vec{X}\cup\vec{Y}\setminus\{X^k\}\}}) = \mathrm{rank}(M(k)_{\{\vec{X}\cup\vec{Y}\setminus\{Y^k\}\}})$.

If the RM exists, then there always exists a $(\mathrm{rank}\, M(k))$*–atomic RM.*

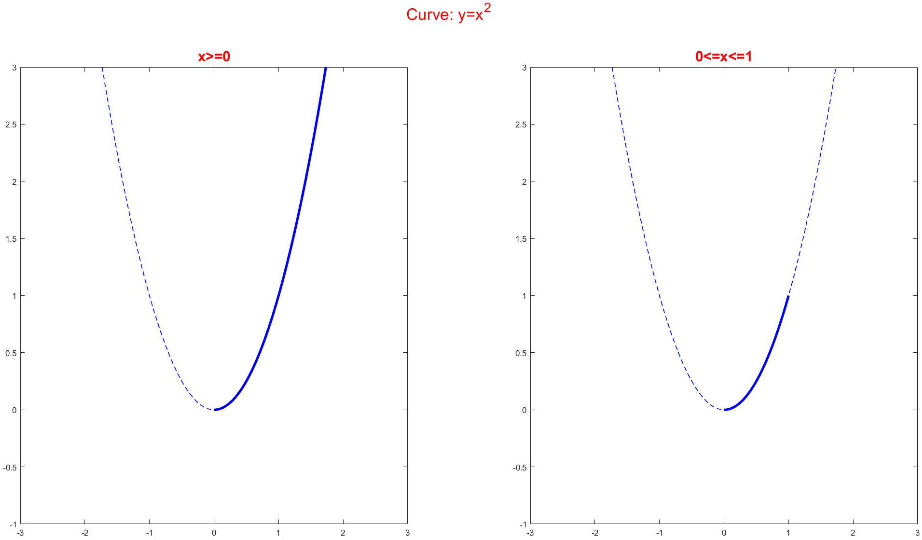
The new approach for the TMP on quadratic curves

- the variety condition from the solutions removed and replaced by rank conditions of certain matrices
- only 'pure' RG conditions need to be checked (those coming from the given quadratic relation), while others follow from rank conditions
- the solutions without variety and 'non–pure' RG conditions suitable to use when solving TMPs on higher degree reducible curves
- for $xy = 1$ also the existence of a minimal measure established (i.e., of rank $M(k)$)
- Replacing the use of the (strong) THMP with the use of the solutions to the (strong) Stieltjes or (strong) Hausdorff TMP one obtains the solution of the TMP on various subsets of the curve

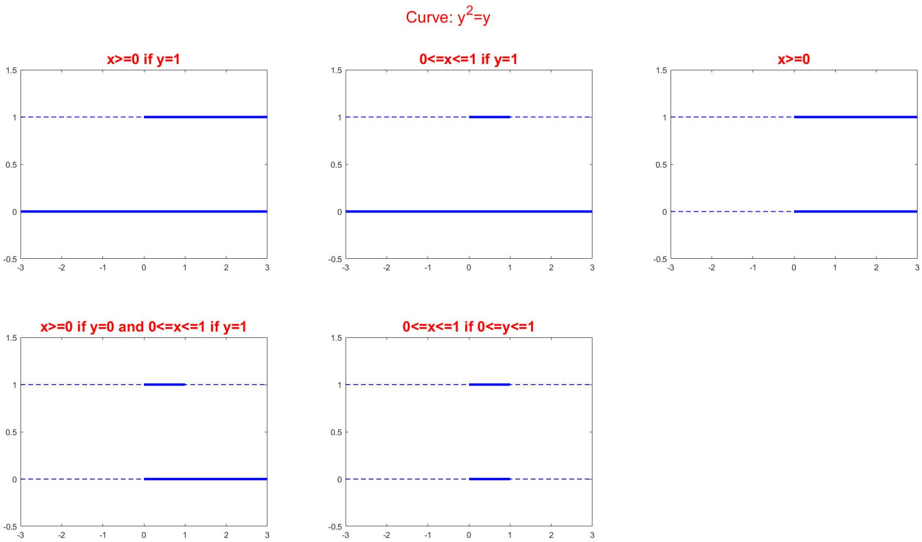
Solutions to the TMP on degenerate hyperbola $xy = 0$



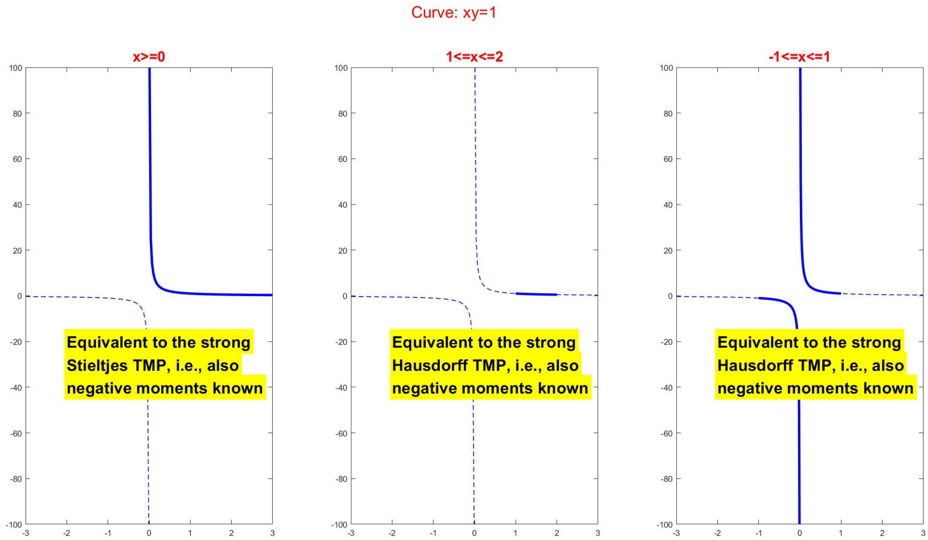
Solutions to the TMP on parabola $y = x^2$



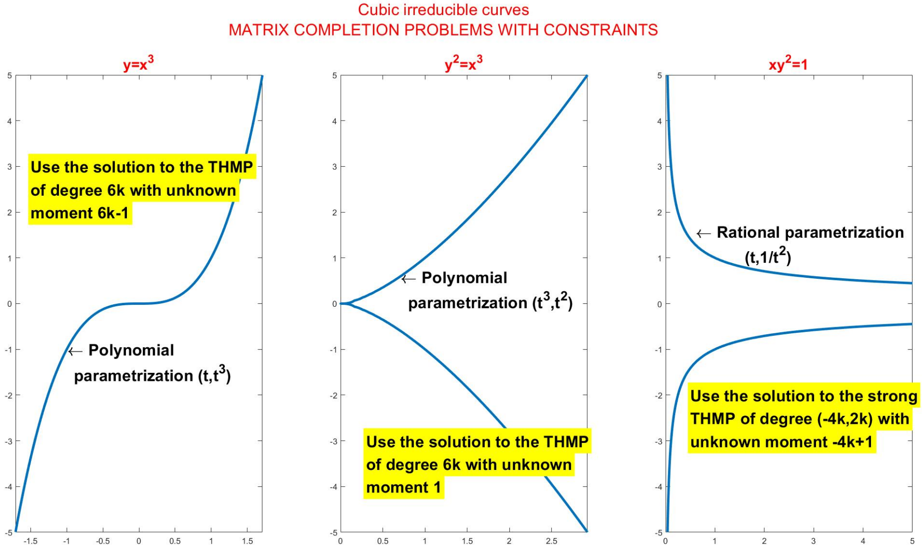
Solutions to the TMP on parallel lines $y^2 = y$



Solutions to the TMP on hyperbola $xy = 1$



The TMP on irreducible cubic curves - a new approach



Solution to the TMP on $y = x^3$

Let $\beta^{(2k)}$ be a sequence with $M(k)$ satisfying the column relation $Y = X^3$.

Every atom must be of the form (t, t^3) for some $t \in \mathbb{R}$. So $\beta_{i,j}$ corresponds to the moment of z^{i+3j}.

As i, j run over $0, 1, \ldots, 2k$ such that $i + j \leq 2k$, the sum $i + 3j$ runs over the set

$$\{0, 1, \ldots, 6k - 2, 6k\}.$$

The problem is equivalent to the **THMP with a gap** γ_{6k-1}, i.e., does there exist $x \in \mathbb{R}$ such that

$$(\gamma_0, \gamma_1, \ldots, \gamma_{6k-2}, x, \gamma_{6k})$$

admits a measure on $\mathbb{R}$. This is a PSD matrix completion problem with constraints.

Matrix completion result

Proposition

Let

$$A(?) := \begin{bmatrix} A_1 & a & b \\ a^T & \alpha & ? \\ b^T & ? & \beta \end{bmatrix} = \begin{bmatrix} A_1 & a & * \\ a^T & \alpha & * \\ * & * & * \end{bmatrix} = \begin{bmatrix} A_1 & * & b \\ * & * & * \\ b^T & * & \beta \end{bmatrix}$$

be a $n \times n$ matrix, where A_1 is a symmetric matrix, $a, b \in \mathbb{R}^{n-2}$ are vectors, $\alpha, \beta \in \mathbb{R}$ real numbers and x is a variable. Let A_2 and A_3 be the colored submatrices of $A(x)$ and

$$x_{\pm} := b^T A_1^{\dagger} a \pm \sqrt{(A_2/A_1)(A_3/A_1)} \in \mathbb{R},$$

where $A_2/A_1 = \alpha - a^T A^{\dagger} a$ and $A_3/A_1 = \beta - b^T B^{\dagger} b$. Then:

1. $A(x_0)$ is PSD if and only if A_2, A_3 are PSD and $x_0 \in [x_-, x_+]$.
2. $$\operatorname{rank} A(x_0) = \max \left\{ \operatorname{rank} A_2, \operatorname{rank} A_3 \right\} + \begin{cases} 0, & \text{for } x_0 \in \{x_-, x_+\}, \\ 1, & \text{for } x_0 \in (x_-, x_+). \end{cases}$$

Notation - Hankel matrix

Let $k \in \mathbb{N}$. For $\gamma = (\gamma_0, \ldots, \gamma_{2k}) \in \mathbb{R}^{2k+1}$ we define the corresponding Hankel matrix as

$$A_\gamma := \left[\gamma_{i+j}\right]_{i,j=0}^{k} = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \iddots & \iddots & \gamma_{k+1} \\ \gamma_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \gamma_{2k-1} \\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k-1} & \gamma_{2k} \end{pmatrix}.$$

We use

$$A_\gamma(m)$$

to denote the restriction of A to the first m rows and columns.

THMP of degree $2k$ with a gap γ_{2k-1}

Theorem

Let $k > 1$ *and* $\gamma(x) := (\gamma_0, \gamma_1, \ldots, \gamma_{2k-2}, x, \gamma_{2k}),$

be a sequence, where x *is a variable,* $\gamma^{(1)} = (\gamma_0, \gamma_1, \ldots, \gamma_{2k-2}),$ $\gamma^{(2)} = (\gamma_0, \gamma_1, \ldots, \gamma_{2k-4})$ *with the moment matrix*

$$A_{\gamma(x)} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A_{\gamma^{(1)}}} & \begin{matrix} v \\ x \end{matrix} \\ \hline v^T & x & \gamma_{2k} \end{array}\right] = \left[\begin{array}{cc|c} A_{\gamma^{(2)}} & u & v \\ u^T & \gamma_{2k-2} & x \\ \hline v^T & x & \gamma_{2k} \end{array}\right],$$

where $v = (\gamma_k, \ldots, \gamma_{2k-2})$ *and* $u = (\gamma_{k-1}, \ldots, \gamma_{2k-3})$*. TFAE:*

1. *There exists* $x_0 \in \mathbb{R}$ *and a RM for* $\gamma(x_0)$*.*
2. $A_{\gamma^{(1)}}$ *and* $\begin{bmatrix} A_{\gamma^{(2)}} & v \\ v^T & \gamma_{2k} \end{bmatrix}$ *are PSD and one of the following conditions is true:*
   a) $A_{\gamma^{(1)}}$ *is PD.*
   b) $\operatorname{rank} A_{\gamma^{(2)}} = \operatorname{rank} A_{\gamma^{(1)}} = \operatorname{rank} \begin{bmatrix} A_{\gamma^{(2)}} & v \\ v^T & \gamma_{2k} \end{bmatrix}.$

Solution to the TMP on $y = x^3$

Let $\beta^{(2k)}$ be a sequence with $M(k)$ satisfying the column relation $Y = X^3$.

We write

$$\mathcal{B} := (1, X, X^2, Y, YX, YX^2, \ldots, Y^{k-1}, Y^{k-1}X, Y^k).$$

Theorem (Solution to the TMP on $Y = X^3$)

The sequence β admits a RM on $\{(x, y) : y = x^3\}$ if and only if $M(k)$ is PSD, $\beta_{i+3,j} = \beta_{i,j+1}$ hold for every $i, j \in \mathbb{Z}_+$ with $i + j \leq 2k - 3$, and denoting

$$\gamma(\mathbf{x}) = (\beta_{0,0}, \beta_{1,0}, \ldots, \beta_{2k,0}, \beta_{2k-1,1}, \beta_{2k-2,2}, \ldots, \beta_{1,2k-1}, \mathbf{x}, \beta_{0,2k})$$

one of the following statements hold:

1. $A_{\gamma(x)}(3k)$ *is PD.*
2. $A_{\gamma(x)}(3k)$ *is PSD and* rank $A_{\gamma(x)}(3k-1)$ = rank $A_{\gamma(x)}(3k)$ = rank $M(k)|_{\{\mathcal{B}\}}$.

If the RM exists, then there is a (rank $M(k)$)*–atomic RM unless* (1) *holds and* rank $M(k) = 3k - 1$, *in which case there is a* $(3k)$*–atomic RM.*

Solution to the TMP on $y^2 = x^3$

Let $\beta^{(2k)}$ be a sequence with $M(k)$ satisfying the column relation $Y^2 = X^3$.

Every atom must be of the form (t^2, t^3) for some $t \in \mathbb{R}$. So $\beta_{i,j}$ corresponds to the moment of z^{2i+3j}, where z is a variable.

As i, j run over $0, 1, \ldots, 2k$ such that $i + j \leq 2k$, the sum $2i + 3j$ runs over the set

$$\{0, 2, 3, \ldots, 6k\}.$$

The problem is equivalent to the **THMP with a gap** γ_1, i.e., does there exist $x \in \mathbb{R}$ such that

$$(\gamma_0, x, \gamma_2, \ldots, \gamma_{6k-1}, \gamma_{6k})$$

admits a measure on $\mathbb{R}$. This is a PSD matrix completion problem with constraints.

THMP of degree $2k$ with a gap γ_1

Theorem

Let $k > 1$ and

$$\gamma(x) := (\gamma_0, x, \gamma_2, \ldots, \gamma_{2k}),$$

be a sequence, where x is a variable, $\gamma^{(1)} = (\gamma_2, \ldots, \gamma_{2k})$, $\gamma^{(2)} = (\gamma_4, \ldots, \gamma_{2k})$ with the moment matrix

$$A_{\gamma(x)} := \left[\begin{array}{c|cc} \gamma_0 & x & u^T \\ \hline x & \multicolumn{2}{c}{\multirow{2}{*}{$A_{\gamma^{(1)}}$}} \\ u & & \end{array}\right] = \left[\begin{array}{c|cc} \gamma_0 & x & u^T \\ \hline x & \gamma_2 & w^T \\ u & w & A_{\gamma^{(2)}} \end{array}\right]$$

where $u^T = (\gamma_2, \ldots, \gamma_k)$ and $w^T = (\gamma_3, \ldots, \gamma_{k+1})$. TFAE:

1. *There exists $x_0 \in \mathbb{R}$ and a RM for $\gamma(x_0)$.*
2. $A_{\gamma^{(1)}}$ *and* $\begin{bmatrix} \gamma_0 & u^T \\ u & A_{\gamma^{(2)}} \end{bmatrix}$ *are PSD and one of the following conditions is true:*

   a) $A_{\gamma^{(1)}}$ *and* $\begin{bmatrix} \gamma_0 & u^T \\ u & A_{\gamma^{(2)}} \end{bmatrix}(k)$ *are PD.*

   b) $\operatorname{rank} A_{\gamma^{(1)}}(k) = \operatorname{rank} A_{\gamma^{(1)}} = \operatorname{rank} A_{\gamma^{(2)}}$.

Solution to the TMP on $y^2 = x^3$

Let $\beta^{(2k)}$ be a sequence with $M(k)$ satisfying the column relation $Y^2 = X^3$.

We write

$$\mathcal{B} := (1, X, Y, X^2, XY, Y^2, X^2Y, \ldots, XY^{k-1}, Y^k).$$

Theorem (Solution to the TMP on $Y^2 = X^3$)

The sequence β admits a RM on $\{(x,y)\colon y^2 = x^3\}$ if and only if $M(k)$ is PSD, $\beta_{i+3,j} = \beta_{i,j+2}$ hold for every $i, j \in \mathbb{Z}_+$ with $i + j \leq 2k - 3$, and denoting

$$\gamma(x) = (\beta_{0,0}, x, \underbrace{\beta_{1,0}, \beta_{0,1}, \overbrace{\beta_{2,0}, \ldots, \beta_{1,2k-1}, \beta_{0,2k}}^{\gamma^{(2)}}}_{\gamma^{(1)}})$$

one of the following conditions is true:

a) $A_{\gamma^{(1)}}$ *and* $M(k)|_{\{\mathcal{B} \setminus Y^k\}}$ *are PD.*

b) $\operatorname{rank} A_{\gamma^{(1)}}(k) = \operatorname{rank} A_{\gamma^{(1)}} = \operatorname{rank} A_{\gamma^{(2)}}$.

If the RM exists, then there is a $\big(\operatorname{rank} M(k)\big)$*–atomic RM unless* (1) *holds and* $\operatorname{rank} M(k) = 3k - 1$*, in which case there is a* $(3k)$*–atomic RM.*

Strong THMP of degree $(-2k_1, 2k_2)$ with a gap γ_{-2k_1+1}

Theorem

Let $k > 1$ and $\gamma(x) := (\gamma_{-2k_1}, x, \gamma_{-2k_1+2}, \ldots, \gamma_{2k_2})$,
be a sequence, where x is a variable, $\gamma^{(1)} = (\gamma_{-2k_1+2}, \ldots, \gamma_{-2k_2})$, $\gamma^{(2)} = (\gamma_{-2k_1+4}, \ldots, \gamma_{2k_2})$ *with the moment matrix*

$$A_{\gamma(x)} := \left[\begin{array}{c|cc} \gamma_{-2k_1} & x & u^T \\ \hline x & \multicolumn{2}{c}{\multirow{2}{*}{$A_{\gamma^{(1)}}$}} \\ u & & \end{array}\right] = \left[\begin{array}{c|cc} \gamma_{-2k_1} & x & u^T \\ \hline x & \gamma_2 & w^T \\ u & w & A_{\gamma^{(2)}} \end{array}\right]$$

where $u^T = (\gamma_{-2k_1+2}, \ldots, \gamma_{-k_1+k_2+1})$ *and* $w^T = (\gamma_{-2k_1+2}, \ldots, \gamma_{-k_1+k_2})$. *TFAE:*

1. *There exists $x_0 \in \mathbb{R}$ and a RM for $\gamma(x_0)$.*
2. $A_{\gamma^{(1)}}$ *and* $\begin{bmatrix} \gamma_{-2k_1} & u^T \\ u & A_{\gamma^{(2)}} \end{bmatrix}$ *are PSD and one of the following conditions is true:*

   a) $A_{\gamma^{(1)}}$ *and* $\begin{bmatrix} \gamma_{-2k_1} & u^T \\ u & A_{\gamma^{(2)}} \end{bmatrix}(k_1 + k_2 - 1)$ *are PD.*

   b) $\operatorname{rank} A_{\gamma^{(1)}}(k_1 + k_2 - 1) = \operatorname{rank} A_{\gamma^{(1)}} = \operatorname{rank} A_{\gamma^{(2)}} = \operatorname{rank} \begin{bmatrix} \gamma_{-2k_1} & u^T \\ u & A_{\gamma^{(2)}} \end{bmatrix}$.

Solution to the TMP on $xy^2 = 1$

Let $\beta^{(2k)}$ be a sequence with $M(k)$ satisfying the column relation $XY^2 = 1$.

We write

$$\mathcal{B} := (X^k, X^{k-1}, X^{k-1}Y, \ldots, 1, Y, \ldots, Y^k).$$

Theorem (Solution to the TMP on $XY^2 = 1$)

The sequence β admits a RM on $\{(x, y) : xy^2 = 1\}$ if and only if $M(k)$ is PSD, $\beta_{i+1,j+2} = \beta_{i,j}$ hold for every $i, j \in \mathbb{Z}_+$ with $i + j \leq 2k - 3$, and denoting

$$\gamma(x) = (\beta_{2k,0}, x, \underbrace{\beta_{2k-1,0}, \beta_{2k-1,1}, \overbrace{\beta_{2k-2,0}, \ldots, \beta_{0,0}, \beta_{0,1}, \ldots, \beta_{0,2k}}^{\gamma^{(2)}}}_{\gamma^{(1)}})$$

one of the following conditions is true:

a) $A_{\gamma^{(1)}}$ *and* $M(k)|_{\{\mathcal{B}\setminus Y^k\}}$ *are PD.*

b) $\operatorname{rank} A_{\gamma^{(1)}}(3k-1) = \operatorname{rank} A_{\gamma^{(1)}} = \operatorname{rank} A_{\gamma^{(2)}} = \operatorname{rank} M(k)$.

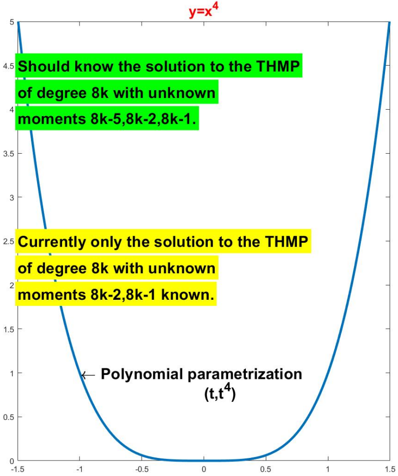
If the RM exists, then there is a $\big(\operatorname{rank} M(k)\big)$–atomic RM unless (13) *holds and* $\operatorname{rank} M(k) = 3k - 1$*, in which case there is a* $(3k)$*–atomic RM.*

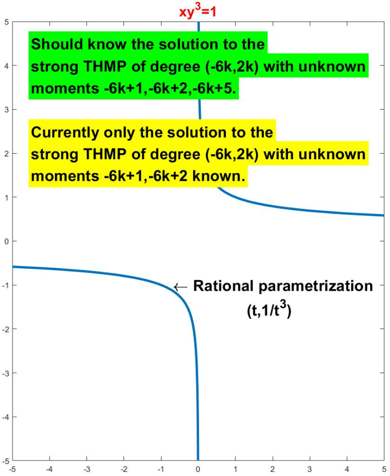
Advantages of the new approach

- Replacing the use of THMP with the use of the solutions to the truncated Stieltjes or Hausdorff MP one obtains the solution of the TMP on various subsets of the curve
- There is a potential to extend this approach to higher degree curves with polynomial or rational parametrization, but one needs to solve the the corresponding univariate TMP with gaps.

The TMP on higher degree curves - a new approach

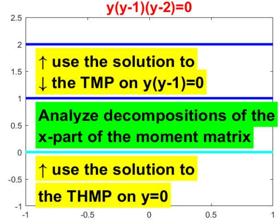
Higher degree irreducible curves
MATRIX COMPLETION PROBLEMS WITH CONSTRAINTS

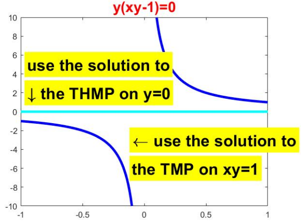


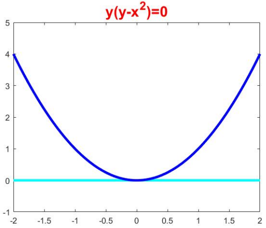


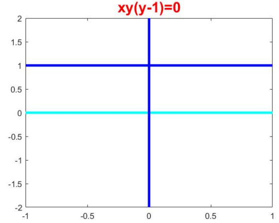
The TMP on reducible cubic curves - a new approach

Cubic reducible curves
MATRIX COMPLETION PROBLEMS WITH CONSTRAINTS









Solution to the TMP on $y(y-\alpha_1)(y-\alpha_2)$

Let $\beta^{(2k)}$ be a sequence with $M(k)$ satisfying the column relation $Y(Y-\alpha_1)(Y-\alpha_2)=\mathbf{0}$.

Then β has a RM on $\{y=0\}\cup\{y=\alpha_1\}\cup\{y=\alpha_2\}$ if and only if it can be decomposed as

$$\beta=\widetilde{\beta}+\widehat{\beta},$$

where $\widetilde{\beta}$ has a RM on $\{y=0\}$ and $\widehat{\beta}$ has a RM on the union $\{y=\alpha_1\}\cup\{y=\alpha_2\}$.

It turns out that all the moments of $\widetilde{\beta},\widehat{\beta}$ are uniquely determined except

$$\widetilde{\beta}_{2k-1,0},\widetilde{\beta}_{2k,0},\widehat{\beta}_{2k-1,0},\widehat{\beta}_{2k-1,0},$$

which satisfy the relations $\beta_{2k-j,0}=\widetilde{\beta}_{2k-j,0}+\widehat{\beta}_{2k-j,0}$, $j=0,1$.

Using Schur complements it is possible to characterize precisely when $\widetilde{\beta}_{2k-1,0},\widetilde{\beta}_{2k,0}$ exist such that $M^{(\widetilde{\beta})}(k)$ and $M^{(\widehat{\beta})}(k)$ have RM on $\mathbb{R}$ and $(y-\alpha_1)(y-\alpha_2)=0$, respectively.

Solution to the TMP on $y(y-1)(y-\alpha)=0$

Let $\beta^{(2k)}$ be a sequence with $M(k)$ satisfying the column relation $Y(Y-1)(Y-\alpha)=0$.

We write

$$\vec{X}^{(i)} := (1, X, \ldots, X^{k-i}) \qquad \text{and} \qquad Y^j\vec{X}^{(i)} := (Y^j, Y^jX, \ldots, Y^jX^{k-i}).$$

Let

$$M = \begin{array}{c c} & \begin{array}{ccc} \vec{X}^{(0)} & Y\vec{X}^{(1)} & Y^2\vec{X}^{(2)} \end{array} \\ \begin{array}{c} (\vec{X}^{(0)})^T \\ (Y\vec{X}^{(1)})^T \\ (Y^2\vec{X}^{(2)})^T \end{array} & \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ (A_{01})^T & A_{11} & A_{12} \\ (A_{02})^T & (A_{12})^T & A_{22} \end{pmatrix} \end{array} = \begin{pmatrix} A_{00} & B \\ B^T & C \end{pmatrix},$$

$$N = \begin{array}{c c} & \begin{array}{cccc} \vec{X}^{(1)} & Y\vec{X}^{(1)} & Y^2\vec{X}^{(2)} & Y^2X^{k-1} \end{array} \\ \begin{array}{c} (\vec{X}^{(1)})^T \\ (Y\vec{X}^{(1)})^T \\ (Y^2\vec{X}^{(2)})^T \\ Y^2X^{k-1} \end{array} & \begin{pmatrix} \widehat{A}_{00} & \widehat{A}_{01} & \widehat{A}_{02} & c \\ (\widehat{A}_{01})^T & \widetilde{A}_{11} & \widetilde{A}_{12} & b \\ (\widehat{A}_{02})^T & (\widetilde{A}_{12})^T & \widetilde{A}_{22} & a \\ c^T & b^T & a^T & \beta_{2k-2,4} \end{pmatrix} \end{array},$$

Solution to the TMP on $y(y-1)(y-\alpha)=0$

$$
\begin{aligned}
\widetilde{B}_{00} &= \alpha^{-1}\cdot\Big((1+\alpha)\cdot\widehat{A}_{01}-\begin{pmatrix}\widehat{A}_{02} & c\end{pmatrix}\Big)\in\mathbb{R}^{(k-1)\times(k-1)},\\
h &= \alpha^{-1}\cdot\big(\alpha\cdot\widehat{a}_{01}-\widetilde{a}_{02}\big)\in\mathbb{R}^{k-1},\\
\widetilde{A}_{01} &= \begin{pmatrix}\widehat{A}_{01}\\ (\widetilde{a}_{01})^T\end{pmatrix}=\begin{pmatrix}\widehat{A}_{01} & \\ (\widehat{a}_{01})^T & \widetilde{\beta}_{2k-1,1}\end{pmatrix},\quad \widetilde{A}_{02}=\begin{pmatrix}\widehat{A}_{02}\\ (\widetilde{a}_{02})^T\end{pmatrix},
\end{aligned}
$$

Solution to the TMP on $y(y-1)(y-\alpha)=0$

Theorem (Solution to the TMP on $Y(Y-1)(Y-\alpha)=0$)

The sequence β admits a RM on $\{(x,y)\colon y(y-1)(y-\alpha)=0\}$ if and only if $M(k)$ is PSD, $\beta_{i,j+3}=(1+\alpha)\beta_{i,j+2}$ hold for every $i,j\in\mathbb{Z}_+$ with $i+j\leq 2k-3$, N is PSD and one of the following conditions is true:

1. $\widehat{A}_{01}-\widetilde{B}_{00}$ *or* $\alpha\widetilde{B}_{00}-\widehat{A}_{01}$ *is invertible.*
2. *Denoting by $v\in\mathbb{R}^{k-2}$ any vector such that $(\alpha\widetilde{B}_{00}-\widehat{A}_{01})\begin{pmatrix} v^T & 1\end{pmatrix}^T=0$, defining real numbers*

$$t'=\frac{(\widehat{a}_{01})^Tv-\alpha h^Tv+\widetilde{\beta}_{2k-1,1}}{\alpha},\quad u'=\begin{pmatrix} h^T\\ t'\\ (\widetilde{a}_{01})^T\end{pmatrix}^T\begin{pmatrix} \widetilde{B}_{00} & \widehat{A}_{01}\\ (\widehat{A}_{01})^T & \widetilde{A}_{11}\end{pmatrix}^{\dagger}\begin{pmatrix} h\\ t'\\ \widetilde{a}_{01}\end{pmatrix}$$

and a Hankel matrix

$$A_\gamma:=A_{00}-\left(\begin{array}{c|c} \widetilde{B}_{00} & \begin{matrix} h\\ t'\end{matrix}\\ \hline \begin{matrix} h^T & t'\end{matrix} & u'\end{array}\right)\in S_{k+1},$$

where $\gamma\in\mathbb{R}^{2k+1}$, it holds that A_γ is invertible or $\operatorname{rank}A_\gamma=\operatorname{rank}A_\gamma(k)$.

If the RM exists, then there is a $\big(\operatorname{rank}M(k)\big)$*–atomic RM if* $\operatorname{rank}N\leq\operatorname{rank}M(k)$ *and* $\big(\operatorname{rank}M(k)+1\big)$*–atomic otherwise.*

Thank you for your attention!