A gap between positive even quartics and sums of squares ones

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MoPAT-24 –
Moments and Polynomials:
Applications and Theory,
March 11th, 2024
Konstanz

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Outline

1. Preliminaries

   Problem:

   - copositive vs completely positive matrices (size comparison, examples)

   Converting to polynomials:

   - pos vs sos even quartic forms

2. Discussion on volume estimation

3. Proofs

   real algebraic geometry, asymptotic convex analysis, harmonic analysis

4. Algorithm and Examples

   free probability inspired, implementation: semidefinite programming

1. Preliminaries

Copositive and completely positive matrices

Definitions

$\mathbb{S}_n \ldots$ real symmetric $n \times n$ matrices

A matrix

$$A = (a_{ij})_{i,j} \in \mathbb{S}_n$$

is:

- positive semidefinite (PSD) if $v^T A v \geq 0$ for every $v \in \mathbb{R}^n$.

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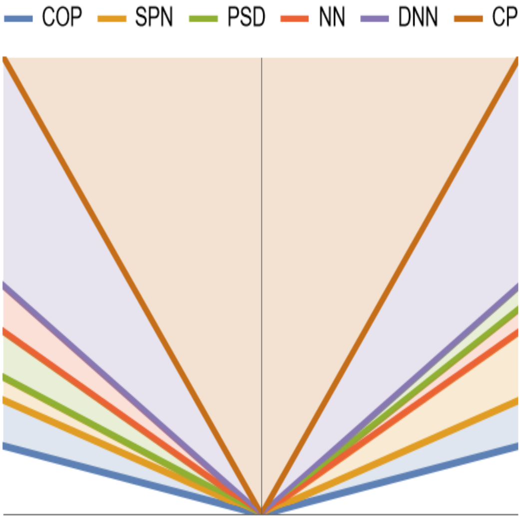
- positive semidefinite (PSD) if $v^T A v \geq 0$ for every $v \in \mathbb{R}^n$.
- nonnegative (NN) if $a_{ij} \geq 0$ for every i, j.
- SPN if $A = P + N$ for some P PSD and N NN.
- doubly nonnegative (DNN) if $A = P \cap N$ for some P PSD and N NN.
- completely positive (CP) if $A = BB^T$ for some $B \in \mathbb{R}^{n \times k}_{\geq 0}$.

Copositive and completely positive matrices

Mental picture



Copositive vs completely positive matrices

Problems and a small sample of existing literature

Problem 1: Establish asymptotically exact quantitative bounds on the fraction of COP matrices that are CP.

Problem 2: Derive algorithm to produce COP matrices that are not CP.

Copositive vs completely positive matrices

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Problem 1: Establish asymptotically exact quantitative bounds on the fraction of COP matrices that are CP.

Problem 2: Derive algorithm to produce COP matrices that are not CP.

- Maxfield, Minc (1962), Hall, Newman (1963): $\mathrm{COP}_n = \mathrm{SPN}_n$ holds only for $n \leq 4$.
- Parrilo (2000): $\mathrm{int}(\mathrm{COP}_n) \subseteq \bigcup_r K_n^{(r)}$, where $(\mathrm{x}^2 = (x_1^2, \ldots, x_n^2))$

$$K_n^{(r)} := \{A \in \mathbb{S}_n : (\sum_{i=1}^{n} x_i^2)^r \cdot (\mathrm{x}^2)^T A \mathrm{x}^2 \text{ is a sum of squares of forms}\}.$$

- Dickinson, Dür, Gijben, Hildebrand (2013): $\mathrm{COP}_5 \neq K_5^{(r)}$ for any $r \in \mathbb{N}$.
- Laurent, Schweighofer, Vargas (2022, 23+): $\mathrm{COP}_5 = \bigcup_r K_5^{(r)}$ and $\mathrm{COP}_6 \neq \bigcup_r K_6^{(r)}$.

Copositive matrices meet RAG

$\mathbb{R}[\mathrm{x}^2]_{4,e}$ ... forms in $\mathrm{x}^2 = (x_1^2, \ldots, x_n^2)$ of degree 4, i.e., *quartic even forms*.

There is a natural bijection

$$\Gamma : \mathbb{S}_n \to \mathbb{R}[\mathrm{x}]_{4,e}, \quad A \mapsto q_A(\mathrm{x}) := (\mathrm{x}^2)^T A \mathrm{x}^2 = \sum_{i,j=1}^{n} a_{ij} x_i^2 x_j^2.$$

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Proposition

Let $A \in \mathbb{S}_n$ be a matrix. Then:

1. A is COP iff q_A is nonnegative. (q_A ... *POS*)
2. A is PSD iff q_A is of the form $\sum_i \left(\sum_j f_{ij} x_j^2\right)^2$. ($q_A$ ... *lin-SOS*)
3. A is NN iff q_A has nonnegative coefficients. (q_A ... *NN*)
4. A is SPN iff q_A is of the form $\sum_i \left(\sum_j f_{ij} x_i x_j\right)^2$ (Parrilo, 00') (q_A ... *SOS*)
5. A is DNN iff q_A is ℓ-SOS and NN. (q_A ... *DNN*)
6. A is CP iff q_A is of the form $\sum_i \left(\sum_j f_{ij} x_j^2\right)^2$ with $f_{ij} \geq 0$. (q_A ... *CP*)

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Corollary. The gaps between COP/PSD/NN/SPN/DNN/CP matrices correspond to the gaps between POS/ℓ-SOS/NN/SOS/DNN/CP even quartics.

Gap between positive and sos polynomials

$$\mathbb{R}[\mathrm{x}]_{2k} \quad \ldots \quad \text{forms in } \mathrm{x} = (x_1, \ldots, x_n) \text{ of degree } 2k$$

Theorem (Blekherman, 2006)

For $n \geq 3$ and fixed k the probability p_n that a positive polynomial $f \in \mathbb{R}[\mathrm{x}]_{2k}$ is sum of squares, satisfies

$$\left(C_1 \cdot \frac{1}{n^{(k-1)/2}}\right)^{\dim \mathbb{R}[x]_{2k} - 1} \leq p_n \leq \left(C_2 \cdot \frac{1}{n^{(k-1)/2}}\right)^{\dim \mathbb{R}[x]_{2k} - 1},$$

where C_1, C_2 are absolute constants.

In particular, for $2k = 4$,

$$p_n \in \Theta\left(\left(\frac{1}{\sqrt{n}}\right)^{\dim \mathbb{R}[x]_4 - 1}\right).$$

Solutions to Problems 1 and 2

Theorem: For $n > 4$ the probability p_n that a positive even quartic $f \in \mathbb{R}[\mathrm{x}^2]_{4,e}$ is sum of squares, satisfies

$$\left(2^{-8} \cdot 3^{-2}\right)^{\dim \mathbb{R}[\mathrm{x}^2]_{4,e-1}} \leq p_n.$$

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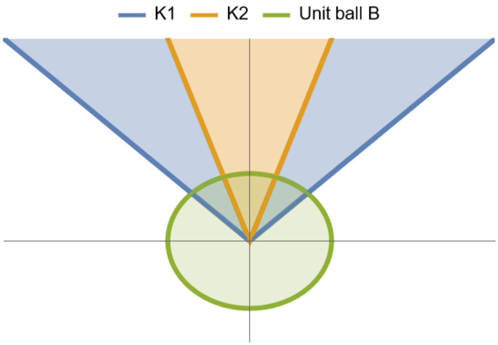
Problem 2

Free probability inspired construction of $\mathrm{DNN}_n \setminus \mathrm{CP}_n$, $n \geq 5$, matrices. Dually, we obtain matrices from $\mathrm{COP}_n \setminus \mathrm{SPN}_n$, or equivalently pos but not sot even quartics.

2. Discussion on volume estimates

Cones in question

Intersect with a unit ball in some metric



- Goal: Compare the sizes of $K_1 \cap B$ and $K_2 \cap B$.
- Beware 1: The choice of the measure influences the results.
- Beware 2: The ambient vector space V must be an inner product space for the pushforward of the Lebesgue measure to be independent of the isomorphism $\phi : V \to \mathbb{R}^{\dim V}$.
- Beware 3: The choice of the inner product and the metric for the ball B influence the results.

Volume radius

Proper measure of the asymptotic sizes of a sequence of compact sets

The volume radius $\mathrm{vrad}(C)$ of a compact set $C \subseteq \mathbb{R}^n$, equipped with an inner product $\langle \cdot, \cdot \rangle$ and a measure μ, is

$$\mathrm{vrad}(C) = \left(\frac{\mathrm{Vol}(C)}{\mathrm{Vol}(B)} \right)^{1/n},$$

where B is the unit ball in $\langle \cdot, \cdot \rangle$.

- Since we are concerned with the asymptotic behavior as n goes to infinity, we need to eliminate the dimension effect when dilating K by some factor c.
- A dilation multiplies the volume of C by c^n, but a more appropriate effect would be multiplication by c.

Gap between positive and sos polynomials asymptotically not visible in the ball of the ℓ^1 norm

- $\mathbb{R}[\mathrm{x}]_{2k}$ is equipped with the natural L^2 inner product

$$\langle f, g\rangle = \int_{S^{n-1}} fg \,\mathrm{d}\sigma,$$

where and σ is the rotation invariant probability measures on the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

- Let $\|\cdot\|_1$ the ℓ^1 norm on the vector of coefficients, i.e.,

$$\|\sum_\alpha a_\alpha \mathrm{x}^\alpha\|_1 = \sum_\alpha |a_\alpha|.$$

- E.g., for $k = 2$, due to the equality (and Rogers-Shepard inequality)

$$x_i x_j x_k x_\ell = \frac{1}{2}(x_i x_j + x_k x_\ell)^2 - \frac{1}{2}x_i^2 x_j^2 - \frac{1}{2}x_k^2 x_\ell^2,$$

the volume radii of positive and sos polynomials in the unit ball B_1 of $\|\cdot\|_1$ are bounded by absolute constants.

Blekherman's result on the gap between positive and sos polynomials refers to the unit ball in the L^2 norm

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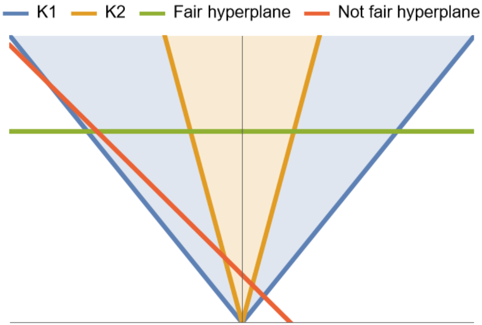
- Let B_2 be the unit ball in the L^2 norm.
- Direct volume estimates for the sections $\mathrm{POS}_{2k} \cap B_2$ and $\mathrm{SOS}_{2k} \cap B_2$ are difficult to obtain.
- Instead, it is natural to compare POS_{2k} and SOS_{2k} when intersected with some affine hyperplane.

Choice of the affine hyperplane for comparison of the cones



1. In case the cones share a unique line of symmetry, it is natural to take the hyperplane whose normal is this line of symmetry.
2. Under the action $O \cdot f(\mathrm{x}) := f(O^{-1}\mathrm{x})$ for $O \in O(n)$, POS_{2k} and SOS_{2k} are invariant, while $\alpha(x_1^2 + \ldots + x_n^2)^2$, $\alpha \in \mathbb{R}$, are the only fixed points.
3. So the hyperplane with the normal $(x_1^2 + \ldots + x_n^2)^2$ is the 'fairest' choice.

A general procedure to obtain the volume estimates

Inputs:

- A convex cone K in a finite-dimensional inner product space V.
- A norm $\|\cdot\|$ w.r.t. which the size of K is to be estimated.

Output: Quantitative bounds on the size of K.

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3. Choose a fair affine hyperplane $\mathcal{H}$: ...such that $K' = K \cap \mathcal{H}$ is bounded.
4. Translate $\mathcal{H}$ to a hyperplane $\mathcal{M}$.
5. Equip $\mathcal{M}$ with a pushforward measure of the Lebesgue measure and estimate $\mathrm{vrad}(K \cap \mathcal{H})$ in $\mathcal{M}$.

3. Proofs

Procedure applied to our problem

1. $\mathbb{R}[\mathrm{x}]_{4,e}$ is equipped with the natural L^2 inner product

$$\langle f, g \rangle = \int_{S^{n-1}} fg \, \mathrm{d}\sigma,$$

   where σ is the rotation invariant probability measures on the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

2. $\mathcal{H}$ is the affine hyperplane of forms from $\mathbb{R}[\mathrm{x}]_{4,e}$ of average 1 on S^{n-1}:

$$\mathcal{H} = \left\{ f \in \mathbb{R}[\mathrm{x}]_{4,e} \colon \int_{S^{n1}} f \, \mathrm{d}\sigma = 1 \right\}.$$

3. $z := \left(\sum_{i=1}^n x_i^2\right)^2$ and thus

$$\mathcal{M} = \mathcal{H} - z = \left\{ f \in \mathbb{R}[\mathrm{x}]_{4,e} \colon \int_{S^{n-1}} f \, \mathrm{d}\sigma = 0 \right\}.$$

4. Let μ be the pushforward of the Lebesgue measure on $\mathbb{R}^{\dim \mathcal{M}}$ to $\mathcal{M}$.

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Observation 2: $\widetilde{(\mathrm{NN})^*_d} = \widetilde{\mathrm{NN}}$

$\Rightarrow$ By a version of the reverse Blaschke-Santaló inequality $\boxed{\frac{1}{\sqrt{2n}} \leq \mathrm{vrad}(\widetilde{\mathrm{NN}})}$.

Here $(\cdot)^*_d$ stands for the dual in the differential inner product, i.e., for

$$f(\mathrm{x}) = \sum_{1\leq i,j,k,\ell\leq n} a_{ijk\ell} x_i x_j x_k x_\ell \in \mathbb{R}[\mathrm{x}]_4$$

and $g \in \mathbb{R}[\mathrm{x}]_4$ we have

$$\langle f, g\rangle_d = \sum_{1\leq i,j,k,\ell\leq n} a_{ijk\ell} \frac{\partial^4 g}{\partial x_i \partial x_j \partial x_k \partial x_\ell}.$$

Procedure applied to our problem

Let

$$\mathrm{LF} := \left\{ \mathrm{pr}(f) \in \mathbb{R}[\mathrm{x}]_{4,e} : f = \sum_i f_i^4 \quad \text{for some } f_i \in \mathbb{R}[\mathrm{x}]_1 \right\}$$

and $\mathrm{pr} : \mathbb{R}[\mathrm{x}]_4 \to \mathbb{R}[\mathrm{x}]_{4,e}$ is the projection defined by:

$$\mathrm{pr}\Big(\sum_{1 \le i \le j \le k \le \ell \le n} a_{ijk\ell} x_i x_j x_k x_\ell \Big) = \sum_{1 \le i \le j \le n} a_{iijj} x_i^2 x_j^2 . \tag{1}$$

Observation 3: $\widetilde{(\mathrm{LF})}_d^* = \widetilde{\mathrm{POS}}$ and $\widetilde{\mathrm{LF}}$ is 'central enough'

for the Blaschke-Santaló inequality to apply.

$\Rightarrow$ By a version of the Blaschke-Santaló inequality

$$\boxed{\mathrm{vrad}(\widetilde{\mathrm{LF}})\, \mathrm{vrad}(\widetilde{\mathrm{POS}}) \le \frac{9}{n^2}.}$$

4. Algorithms and Examples

DNN matrices that are not CP of size $n \geq 5$

Algorithm

1. The setting:

   $L^2[0,1]\ldots$ an ambient space,

   $\mathcal{B} := \{1\} \cup \{\sqrt{2}\cos(2k\pi) \colon k \in \mathbb{N}\} \cup \{\sqrt{2}\sin(2k\pi) \colon k \in \mathbb{N}\}\ldots$ a basis,

   $M_f : L^2[0,1] \to L^2[0,1],\ M_f(g) = fg\ldots$ the multiplication operator.

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2. The idea: Find a closed infinite dimensional subspace $\mathcal{H}$ and $f \in \mathcal{H}$ such that

   $$\boxed{M_f^{\mathcal{H}} := P_{\mathcal{H}} M_f P_{\mathcal{H}}}$$

   has all finite principal submatrices DNN but not CP, where $P_{\mathcal{H}} : L^2[0,1] \to \mathcal{H}$ is the orthogonal projection onto $\mathcal{H}$.

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3. Choice of $\mathcal{H}$ and $f \in \mathcal{H}$:

$$\mathcal{H} \subseteq L^2[0,1] \dots \quad \text{a closed subspace spanned by } \cos(2k\pi), k \in \mathbb{N}_0,$$

$$\boxed{f = 1 + 2\sum_{k=1}^{m} a_k \cos(2k\pi)}, \quad m \in \mathbb{N},$$

DNN matrices that are not CP of size $n \geq 5$

4. Certificates:
   4.1 NN: $a_1 \geq 0, \ldots, a_m \geq 0$.
   4.2 PSD: $f = \sum_i h_i^2$.
   4.3 Not CP:

$\mathcal{H}_n \ldots$ a subspace spanned by $1, \cos(2\pi), \ldots, \cos(2(n-1)\pi)$,

$P_n : \mathcal{H} \to \mathcal{H}_n \ldots$ the orthogonal projection onto $\mathcal{H}_n$,

$A^{(n)} := P_n M_f^{\mathcal{H}} P_n$,

$$H = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix} \in \mathrm{COP} \setminus \mathrm{SPN},$$

(Horn matrix; Hall, Newman, 1963)

We demand

$$\langle A^{(5)}, H \rangle < 0,$$

with $\langle \cdot, \cdot \rangle$ the usual Frobenius inner product on symmetric matrices.

DNN matrices that are not CP of size $n \geq 5$

1. NN is certified by the following equation:

$$\int_0^1 \cos(2j\pi x)\cos(2k\pi x)\cos(2\ell\pi x)dx = \begin{cases} \frac{1}{2}, & \text{if } j = \ell, k = 0, \\ \frac{1}{4}, & \text{if } k \neq 0 \text{ and } j \in \{\ell + k, \ell - k\}, \\ 0, & \text{otherwise.} \end{cases}$$

   In particular,

$$A^{(5)} = \begin{pmatrix} 1 & \sqrt{2}a_1 & \sqrt{2}a_2 & \sqrt{2}a_3 & \sqrt{2}a_4 \\ \sqrt{2}a_1 & a_2 + 1 & a_1 + a_3 & a_2 + a_4 & a_3 + a_5 \\ \sqrt{2}a_2 & a_1 + a_3 & a_4 + 1 & a_1 + a_5 & a_2 + a_6 \\ \sqrt{2}a_3 & a_2 + a_4 & a_1 + a_5 & 1 + a_6 & a_1 \\ \sqrt{2}a_4 & a_3 + a_5 & a_2 + a_6 & a_1 & 1 \end{pmatrix}.$$

2. PSD is certified by

$$M_f^{\mathcal{H}} = \sum_i \left(M_{h_i}^{\mathcal{H}}\right)^2 = \sum_i M_{h_i}^{\mathcal{H}} \left(M_{h_i}^{\mathcal{H}}\right)^*.$$

3. Not CP is certified by

$$\text{COP}^* = \text{CP} \quad \text{(in the Frobenius inner product).}$$

DNN matrices that are not CP of size $n \geq 5$

The feasibility semidefinite program (SDP) implements the algorithm above:

$$\mathrm{tr}(A^{(5)}H) = -\frac{1}{20},$$

$$f = v^{\mathsf{T}} B v \quad \text{with} \quad B \succeq 0 \text{ of size } 4 \times 4,$$

$$a_i \geq 0, \quad i = 1, \ldots, 6,$$

where

$$v^T = \begin{pmatrix} 1 & \cos(2\pi x) & \cos(4\pi x) & \cos(6\pi x) \end{pmatrix}.$$

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Implementation and an example

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$$v^T = \begin{pmatrix} 1 & \cos(2\pi x) & \cos(4\pi x) & \cos(6\pi x) \end{pmatrix}.$$

Solving this SDP, we get

$$A^{(5)} = \begin{pmatrix} 1 & \frac{16\sqrt{2}}{27} & \frac{\sqrt{2}}{123} & \frac{1}{147\sqrt{2}} & \frac{5\sqrt{2}}{21} \\ \frac{16\sqrt{2}}{27} & \frac{124}{123} & \frac{1577}{2646} & \frac{212}{861} & \frac{1205}{8526} \\ \frac{\sqrt{2}}{123} & \frac{1577}{2646} & \frac{26}{21} & \frac{572}{783} & \frac{1777340\sqrt{2}-2413803}{3254580} \\ \frac{1}{147\sqrt{2}} & \frac{212}{861} & \frac{572}{783} & \frac{1777340\sqrt{2}+814317}{3254580} & \frac{16}{27} \\ \frac{5\sqrt{2}}{21} & \frac{1205}{8526} & \frac{1777340\sqrt{2}-2413803}{3254580} & \frac{16}{27} & 1 \end{pmatrix}.$$

COP matrices that are not SPN of size $n \geq 5$

Algorithm and an example

Let $A^{(n)}$ be a DNN not CP matrix. To obtain a matrix $C \in \text{COP} \setminus \text{SPN}$ of size $n \times n$ we demand

$$\langle A^{(n)}, C \rangle < 0, \tag{2}$$

$$\Big(\sum_{i=1}^{n} x_i^2\Big)^k \big((\mathrm{x}^2)^T C \mathrm{x}^2\big) \quad \text{is SOS for some } k \in \mathbb{N}. \tag{3}$$

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we obtain (with $\langle A^{(5)}, C \rangle = -\frac{1}{10}$ and $k = 1$)

$$C = \begin{pmatrix} 17 & -\frac{91}{5} & \frac{33}{2} & \frac{38}{3} & -\frac{36}{5} \\ -\frac{91}{5} & \frac{59}{3} & -\frac{53}{4} & 8 & \frac{33}{4} \\ \frac{33}{2} & -\frac{53}{4} & \frac{39}{4} & -\frac{13}{2} & 8 \\ \frac{38}{3} & 8 & -\frac{13}{2} & \frac{16}{3} & -\frac{13}{3} \\ -\frac{36}{5} & \frac{33}{4} & 8 & -\frac{13}{3} & \frac{1373628701}{353935575} \end{pmatrix}.$$

Thank you for your attention!