There are many more positive maps than completely positive maps

joint work with I. Klep, S. McCullough and K. Šivic

LAW 2017

Main results

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A main tool is the real algebraic geometry techniques developed by Blekherman to study the gap between positive polynomials and sums of squares.

Notation

$\mathbb{F}$ . . . the field $\{\mathbb{R}$ or $\mathbb{C}\}$
$M_n(\mathbb{F})$ . . . $n \times n$ matrices over $\mathbb{F}$ equipped with (conjugate) transposition as the involution $*$
$\mathbb{S}_n$ . . . real symmetric matrices
$A \succeq 0$ . . . the matrix A is positive semidefinite

Positive and completely positive maps

For $n, m \in \mathbb{N}$, a linear map $\Phi : M_n(\mathbb{F}) \to M_m(\mathbb{F})$ is:

1. $*$-**linear** if $\Phi(A^*) = \Phi(A)^*$ for every $A \in M_n(\mathbb{F})$.
2. **positive** if $\Phi(A) \succeq 0$ for every $A \succeq 0$.
3. **completely positive (cp)** if for all $k \in \mathbb{N}$ the ampliations

   $$I_k \otimes \Phi : M_k(\mathbb{F}) \otimes M_n(\mathbb{F}) \to M_k(\mathbb{F}) \otimes M_m(\mathbb{F}), \quad M \otimes A \mapsto M \otimes \Phi(A)$$

   are positive.

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Theorem (Arveson, 2009)

Let $n, m \geq 2$. Then the probability p that a positive map $\varphi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is cp satisfies $0 < p < 1$.

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Can we find more precise bounds for p in Arveson's theorem?

Question

How to construct positive map Φ which are not cp?

Main results

Theorem

For integers $n, m \geq 2$, the probability $p^{\mathbb{F}}_{n,m}$ that a random positive map $\Phi : M_n(\mathbb{F}) \to M_m(\mathbb{F})$ is completely positive, is bounded by

$$p^{\mathbb{F}}_{n,m} < \left(\left(2^{28-\dim_{\mathbb{R}} \mathbb{F}}\right)^{\frac{1}{2}} \cdot 3^{-\frac{5}{2}} \cdot 5^2 \cdot 10^{\frac{2}{9}} \cdot \frac{1}{\sqrt{\min(n,m) - \frac{1}{2}}} \right)^{D_{\mathcal{M}_{\mathcal{C}_{\mathbb{F}}}}},$$

where $D_{\mathcal{M}_{\mathcal{C}_{\mathbb{F}}}} = \begin{cases} n^2m^2 - 1, & \text{if } \mathbb{F} = \mathbb{C}, \\ \frac{nm(nm+1)}{2}, & \text{if } \mathbb{F} = \mathbb{R}. \end{cases}$

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If $\min(n,m) \geq \left(2^{28-\dim_{\mathbb{R}} \mathbb{F}}\right) \cdot 3^{-5} \cdot 5^4 \cdot 10^{\frac{4}{9}}$, *then*

$$\lim_{\max(n,m)\to\infty} p^{\mathbb{F}}_{n,m} = 0.$$

Main results

Theorem

For integers $n, m \geq 3$ the probability $p_{n,m}$ that a positive map $\Phi : \mathbb{S}_n \to \mathbb{S}_m$ is completely positive, is bounded by

$$\left(\frac{3\sqrt{3}}{2^{10} \cdot 7^2 \cdot \sqrt{\min(n,m)}}\right)^{D_{\mathcal{M}}} < p_{n,m} < \left(\frac{2^{12} \cdot 5^2 \cdot 6^{\frac{1}{2}} \cdot 10^{\frac{2}{9}}}{3^3 \cdot \sqrt{\min(n,m)+1}}\right)^{D_{\mathcal{M}}},$$

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where $D_{\mathcal{M}} = \binom{n+1}{2}\binom{m+1}{2} - 1.$

If $\min(n,m) \geq \frac{2^{25} \cdot 5^4 \cdot 10^{\frac{4}{9}}}{3^5}$, *then*

$$\lim_{\max(n,m)\to\infty} p_{n,m} = 0.$$

Positive maps and biforms

$\mathcal{L}(\mathbb{S}_n, \mathbb{S}_m) \ldots$ the vector space of all linear maps from $\mathbb{S}_n$ to $\mathbb{S}_m$
$\mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2} \ldots$ biforms in $\mathrm{x} := (x_1, \ldots, x_n)$ and $\mathrm{y} := (y_1, \ldots, y_m)$ of bidegree (2,2)

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$$\Gamma : \mathcal{L}(\mathbb{S}_n, \mathbb{S}_m) \to \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2}, \quad \Phi \mapsto p_\Phi(\mathrm{x}, \mathrm{y}) := \mathrm{y}^* \Phi(\mathrm{x}\mathrm{x}^*)\mathrm{y}.$$

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Proposition

Let $\Phi : \mathbb{S}_n \to \mathbb{S}_m$ be a linear map. Then

1. *Φ is positive iff p_Φ is nonnegative;*
2. *Φ is completely positive iff p_Φ is a sum of squares.*

Corollary

Estimating the probability that a positive map $\Phi : \mathbb{S}_n \to \mathbb{S}_m$ is cp, is equivalent to estimating the probability that a positive polynomial $p \in \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2}$ is a sum of squares (sos) of polynomials, i.e., $p = \sum_i q_i^2$ for some $q_i \in \mathbb{R}[\mathrm{x}, \mathrm{y}]_{1,1}$.

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Now one can employ powerful techniques, based on harmonic analysis and classical convexity, developed by Barvinok and Blekherman, to obtain bounds on the probability.

Constructing positive maps that are not cp

The Blekherman-Smith-Velasco algorithm (2013) produces positive forms of degree 2 that are not sos on nondegenerate totally-real subvariety $X \subseteq \mathbb{P}^n$ such that $\deg(X) > 1 + \mathrm{codim}(X)$.

The Segre variety $X := \sigma_{n,m}(\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}) \subseteq \mathbb{P}^{nm-1}$ where

$$\begin{aligned}\sigma_{n,m}([x_1 : \ldots : x_n], [y_1 : \ldots : y_m]) = \\ = [x_1 y_1 : x_1 y_2 : \ldots : x_1 y_m : \ldots : x_n y_m],\end{aligned}$$

is an example of such subvariety of degree $\binom{n+m-2}{n-1}$, dimension $n+m-2$ and codimension $(n-1)(m-1)$.

Constructing positive maps that are not cp

X is the zero locus of the ideal $I_{n,m} \subseteq \mathbb{R}[z_{11}, z_{12}, \ldots, z_{1m}, \ldots, z_{nm}]$ generated by all 2×2 minors of the matrix $(z_{ij})_{i,j}$.

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$$\sigma^{\#}_{n,m} : \mathbb{C}[\mathrm{z}]/I_{n,m} \to \mathbb{C}[\mathrm{x}, \mathrm{y}], \quad \sigma^{\#}_{n,m}(z_{ij} + I_{n,m}) = x_i y_j$$

for $1 \le i \le n, 1 \le j \le m$.

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for $1 \leq i \leq n, 1 \leq j \leq m$. Moreover,

$$\sigma^{\#}_{n,m}(\mathbb{R}[\mathrm{z}]_2/I_{n,m}) = \mathbb{R}[\mathrm{x}, \mathrm{y}]_{2,2}.$$

Algorithm

Let $d := n + m - 2 = \dim(X), e := (n-1)(m-1) = \text{codim}(X)$.

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    1. Choose $e+1$ random points $x^{(i)} \in \mathbb{R}^n$ and $y^{(i)} \in \mathbb{R}^m$ and calculate their Kronecker tensor products $z^{(i)} = x^{(i)} \otimes y^{(i)} \in \mathbb{R}^{nm}$.

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   2. Choose d random vectors $v_1, \ldots v_d \in \mathbb{R}^{nm}$ from the kernel of the matrix
   $$\begin{pmatrix} z^{(1)} & \ldots & z^{(e+1)} \end{pmatrix}^* .$$
   The corresponding linear forms $h_1, \ldots, h_d$ are
   $$h_j(\mathrm{z}) = v_j^* \cdot \mathrm{z} \in \mathbb{R}[\mathrm{z}] \quad \text{for } j = 1, \ldots, d.$$

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   3. Choose a random vector v_0 from the kernel of the matrix
   $$\begin{pmatrix} z^{(1)} & \ldots & z^{(e)} \end{pmatrix}^* .$$
   (Note that we have omitted $z^{(e+1)}$.) The corresponding linear form h_0 is
   $$h_0(\mathrm{z}) = v_0^* \cdot \mathrm{z} \in \mathbb{R}[\mathrm{z}].$$

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   1. Let $g_1(\mathrm{z}), \ldots, g_{\binom{n}{2}\binom{m}{2}}(\mathrm{z})$ be the generators of the ideal $I_{n,m}$, i.e., 2×2 minors of the matrix $(z_{ij})_{i,j}$. For each $i = 1, \ldots, e$ compute a basis $\{w_1^{(i)}, \ldots, w_{d+1}^{(i)}\} \subseteq \mathbb{R}^{nm}$ of the kernel of the matrix
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   $$\begin{pmatrix} \nabla g_1(z^{(i)}) & \cdots & \nabla g_{\binom{n}{2}\binom{m}{2}}(z^{(i)}) \end{pmatrix}^* .$$
   2. Let e_i denote the i-th standard basis vector of the corresponding vector space. Choose a random vector $v \in \mathbb{R}^{n^2 m^2}$ from the intersection of the kernels of the matrices
   $$\begin{pmatrix} z^{(i)} \otimes w_1^{(i)} & \cdots & z^{(i)} \otimes w_{d+1}^{(i)} \end{pmatrix}^* \quad \text{for } i = 1, \dots, e$$
   with the kernels of the matrices
   $$\left(\mathrm{e}_i \otimes \mathrm{e}_j - \mathrm{e}_j \otimes \mathrm{e}_i\right)^* \quad \text{for } 1 \le i < j \le nm.$$

Algorithm

3. Construction of a quadratic form in $\mathbb{R}[z]/I_{n,m}$ that is positive but not a sum of squares.

   Calculate the greatest $\delta_0 > 0$ such that $\delta_0 f + \sum_{i=0}^{d} h_i^2$ is nonnegative on $V_{\mathbb{R}}(I_{n,m})$. Then for every $0 < \delta < \delta_0$ the quadratic form

   $$(\delta f + \sum_{i=0}^{d} h_i^2)(z)$$

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An example

$$\begin{aligned} p_\Phi(x,y) = {} & 104x_1^2y_1^2+283x_1^2y_2^2+18x_1^2y_3^2-310x_1^2y_1y_2+18x_1^2y_1y_3+ \\ & +4x_1^2y_2y_3+310x_1x_2y_1^2-18x_1x_3y_1^2-16x_1x_2y_2^2+52x_1x_3y_2^2+4x_1x_2y_3^2- \\ & -26x_1x_3y_3^2-610x_1x_2y_1y_2-44x_1x_3y_1y_2+36x_1x_2y_1y_3-200x_1x_3y_1y_3- \\ & -44x_1x_2y_2y_3+322x_1x_3y_2y_3+285x_2^2y_1^2+16x_3^2y_1^2+4x_2x_3y_1^2 \\ & +63x_2^2y_2^2+9x_3^2y_2^2+20x_2x_3y_2^2+7x_2^2y_3^2+125x_3^2y_3^2-20x_2x_3y_3^2+16x_2^2y_1y_2+ \\ & +4x_3^2y_1y_2-60x_2x_3y_1y_2+52x_2^2y_1y_3+26x_3^2y_1y_3-330x_2x_3y_1y_3- \\ & -20x_2^2y_2y_3+20x_3^2y_2y_3-100x_2x_3y_2y_3. \end{aligned}$$

Thank you for your attention!