Positive polynomials and the truncated moment problem on plane cubic curves

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Bivariate truncated moment problem (TMP)

Question

Let $k \in \mathbb{N}$ and

$$L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$$

a linear functional.

$C \subseteq \mathbb{R}^2$ is a plane cubic.

The bivariate truncated moment problem on C (C–TMP): characterize the existence of a positive Borel measure μ on $\mathbb{R}^2$ with support in C, such that

$$L(f) = \int_C f \, d\mu$$

for $i, j \in \mathbb{Z}_+, i + j \leq k$.

If μ exists, it is called a C–representing measure (C–RM) of L and L is called a C–moment functional.

Classification of plane cubics

Up to invertible affine change of coordinates

Irreducible cases:

$$(I)\ y = p(x), \quad (II)\ xy = p(x), \quad (III)\ y^2 = p(x),$$
$$(IV)\ xy^2 + ay = p(x),$$

where $p(x) = bx^3 + cx^2 + dx + e$.

Reducible cases:

$$(i)\ y(a+y)(b+y),\ a, b \in \mathbb{R}_*, a \neq b,$$
$$(ii)\ y(ay + x^2 + y^2),\ a \in \mathbb{R}_*, \quad (iii)\ y(x - y^2),$$
$$(iv)\ y(1 - xy), \quad (v)\ y(x + y + axy),\ a \in \mathbb{R}_*,$$
$$(vi)\ y(ay + x^2 - y^2), a \in \mathbb{R}, \quad (vii)\ yx(y+1),$$
$$(viii)\ y(1 + ay + bx^2 + cy^2),\ a, b, c \in \mathbb{R}, b \neq 0.$$

TMP: $p(x,y) = y^2 - x^3 - ax - b$

$\mathcal{M}(k)$	the moment matrix representing $L : \mathbb{R}[x,y]_{\leq 2k} \to \mathbb{R}$
I	the ideal in $\mathbb{R}[x,y]$ generated by p
I_j	$I \cap \mathbb{R}[x,y]_{\leq j}$
$\ker L$	the kernel of the bilinear form $\overline{L} : \mathbb{R}[x,y]_{\leq k} \times \mathbb{R}[x,y]_{\leq k} \to \mathbb{R}$ induced by L

Theorem (Bhardwaj, Z, 23+)

Assume $\mathcal{M}(k) \succeq 0$ and $\ker L = I_k$. The following statements are equivalent:

1. *L has a $(\operatorname{rank} \mathcal{M}(k))$–atomic $\mathcal{Z}(p)$–representing measure.*
2. *Certain quadratic polynomial $\mathcal{Q}(\theta)$ has a real root.*

Using a recent result by Baldi, Blekherman and Sinn this gives a concrete algebraic condition for the existence of a measure in case of projectively smooth curves with one connected component.

Where does $\mathcal{Q}(\theta)$ come from?

Analysis of the existence of a flat extension of $\mathcal{M}(k)$

$$\mathcal{M}(k+1) = \begin{pmatrix} \mathcal{M}(k) & B(k+1) \\ (B(k+1))^T & C(k+1) \end{pmatrix}$$

following Fialkow's $p(x, y) = y - x^3$ approach ($\beta_{i,j} = L(x^i y^j)$):

1. The block $B(k+1)$ restricted to rows of degree k is of the form :

$$\begin{array}{c} \\ X^k \\ X^{k-1}Y \\ \vdots \\ \vdots \\ X^2Y^{k-2} \\ XY^{k-1} \\ Y^k \end{array} \begin{array}{c} \begin{array}{ccccccc} X^{k+1} & X^kY & \cdots & \cdots & X^2Y^{k-1} & XY^k & Y^{k+1} \end{array} \\ \begin{pmatrix} \beta_{2k+1,0} & \beta_{2k,1} & \cdots & \cdots & \beta_{k+2,k-1} & \beta_{k+1,k} & \beta_{k,k+1} \\ \beta_{2k,1} & \beta_{2k-1,2} & \iddots & \iddots & \beta_{k+1,k} & \beta_{k,k+1} & \beta_{k-1,k+2} \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \beta_{k+3,k-2} & \beta_{k+2,k-1} & \iddots & \iddots & \iddots & \iddots & \theta \\ \beta_{k+2,k-1} & \beta_{k+1,k} & \iddots & \iddots & \iddots & \theta & \phi \\ \beta_{k+1,k} & \beta_{k,k+1} & \cdots & \cdots & \theta & \phi & \psi \end{pmatrix} \end{array},$$

where

$$\beta_{i,2k+1-i} = \beta_{i-3,2k+3-i} - a\beta_{i-2,2k+1-i} - b\beta_{i-3,2k+1-i} \quad \text{for } 3 \le i \le 2k+1$$

and θ, ϕ, ψ are arbitrary.

2. $C(k+1) := (B(k+1))^T \mathcal{M}(k)^\dagger B(k+1)$ has a moment structure iff:

$$C_{k,k} = C_{k+1,k-1},$$
$$\phi = f_2\theta^2 + f_1\theta + f_0$$
$$C_{k+1,k} = C_{k+2,k-1},$$
$$\psi = j_{11}\phi\theta + j_{10}\phi + j_{02}\theta^2 + j_{01}\theta + j_{00}$$
$$C_{k+1,k+1} = C_{k+2,k},$$
$$k_{101}\psi\theta + k_{100}\psi + k_{011}\phi\theta + k_{010}\phi + k_{002}\theta^2 + k_{001}\theta + k_{000} =$$
$$\ell_{20}\phi^2 + \ell_{11}\phi\theta + \ell_{10}\phi + \ell_{02}\theta^2 + \ell_{01}\theta + \ell_{00}$$

3. A short computation shows that the last equation is of the form

$$\alpha_2\theta^2 + \alpha_1\theta + \alpha_0 = 0$$

and a flat extension $\mathcal{M}(k+1)$ exists iff it has a real root θ.

Some definitions

$C = \mathcal{Z}(P)$ a plane cubic, $I = \langle P \rangle \subseteq \mathbb{R}[x, y]$ an ideal generated by P,

$L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ a linear functional

$\mathbb{R}[C] = \mathbb{R}[x, y]/I$	a coordinate ring of C
$\mathbb{R}[C]_{\leq m}$	an image of $\mathbb{R}[x, y]_{\leq m}$ under the restriction map $f \mapsto f\|_C$
$Q(\mathbb{R}[C])$	a quotient ring of $\mathbb{R}[C]$
$L_C : \mathbb{R}[C]_{\leq 2k} \to \mathbb{R}$	an induced functional
$\ker L_C$	the kernel of the bil. form $\overline{L}_C : \mathbb{R}[C]_{\leq k} \times \mathbb{R}[C]_{\leq k} \to \mathbb{R}$ induced by L_C
$\mathrm{Pos}_{2k}(C)$	a set of all $p \in \mathbb{R}[C]_{\leq 2k}$ with $p(x) \geq 0$ for $x \in C$
V	a finite–dimensional vector space in $Q(\mathbb{R}[C])$
f	an element of $\mathbb{R}[C]$
V_f	a vector space generated by $\{fgh \colon g, h \in V\}$

Assume that $V_f \subseteq \mathbb{R}[C]_{\leq k}$. Then the functional

$$L_{C,(V,f)} : V_f \to \mathbb{R}, \quad L_{C,(V,f)}(g) := L_C(g)$$

if well–defined and called a **V_f–localizing functional of** L_C.

General solution to the TMP

Assume $V_f \subseteq \mathbb{R}[C]_{\leq k}$.

L_C is **strictly positive** if $L_C(p) > 0$ for every $0 \neq p \in \mathrm{Pos}_{2k}(C)$.

L_C is **strictly square positive** if $L_C(g^2) > 0$ for every $0 \neq g \in \mathbb{R}[C]_{\leq k}$.

L_C is V_f**–locally strictly square positive** if $L_C(fg^2) > 0$ for every $g \in V$.

Since every strictly positive functional L_C is a C–moment functional, the following is a concrete solution to the nonsingular C–TMP for irreducible C or reducible C without non-real intersection points.

Theorem (Kummer, Z., 24+)

Let $L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ be a linear functional with $\ker L = I_{2k}$. There are $f \in \mathbb{R}[C]$ and a finite–dimensional vector space V in $Q(\mathbb{R}[C])$ with $V_f \subseteq \mathbb{R}[C]_{\leq k}$ such that the following are equivalent:

1. *L_C is strictly positive.*
2. *L_C is strictly square positive and V_f–locally strictly square positive.*

General solution to the TMP

Assume that C is reducible with non-real intersection points defined by

$$P(x, y) = y(1 + x^2 + ay + by^2).$$

Theorem (Kummer, Z., 24+)

Let $L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ be a linear functional. Assume that $1 + x^2 + ay + by^2 = 0$ lies in the upper half-plane. Then the following are equivalent:

1. *L_C is strictly positive.*
2. *L_C is strictly square positive, $(\mathbb{R}[C]_{\leq k-1})_y$–locally strictly square positive and $(\mathbb{R}[C]_{\leq k-1})_{1+x^2+ay+y^2}$–locally strictly square positive*

Specifying V and f for irreducible cases

$C = \mathcal{Z}(P)$, $\mathcal{B}_k$ is a basis for $\mathbb{R}[C]_{\leq k}$, $\mathcal{B}_V$ is a basis for V,

$$\Phi_1(p(x,y)) := p(t^2, t^3 - t),$$
$$\Phi_2(p(x,y)) := p(t^2+1, t^3+t).$$

P	$\mathcal{B}_k$	$\mathcal{B}_V$	f
$y^2 - x(x-a)(x-b)$, $a, b \in \mathbb{R}$, $0 < a < b$	$\{1, x, y, \dots, x^2y^{i-2}, xy^{i-1}, y^i, \dots x^2y^{k-2}, xy^{k-1}, y^k\}$	$\mathcal{B}_k \setminus \{y^k\} \cup \{\frac{y}{x}\}$	x
$y^2 - x(x^2 + c)$, $c \in (0, \infty)$	$\{1, x, y, \dots, x^2y^{i-2}, xy^{i-1}, y^i, \dots x^2y^{k-2}, xy^{k-1}, y^k\}$	$\mathcal{B}_k \setminus \{y^k\} \cup \{\frac{y}{x}\}$	x
$y^2 - x^3$	$\{1, x, y, \dots, x^2y^{i-2}, xy^{i-1}, y^i, \dots x^2y^{k-2}, xy^{k-1}, y^k\}$	$\mathcal{B}_k \setminus \{1\} \cup \{\frac{y}{x}\}$	1
$y^2 - x(x-1)^2$	$\Phi_1^{-1}(\{1, t^2-1, t^3-t, \dots, t^{k-1} - t^{k-3}, t^k - t^{k-2}\})$	$\mathcal{B}_k \setminus \{1\} \cup \{\frac{y}{x-1}\}$	1
$y^2 - x^2(x-1)$	$\Phi_2^{-1}(\{1, t^2+1, t^3+t, \dots, t^{k-1} + t^{k-3}, t^k + t^{k-2}\})$	$\mathcal{B}_k \setminus \{1\} \cup \{\frac{y}{x}\}$	1
$yx - c(x)$, c of degree 3, $c(0) \neq 0$	$\{1, x, y, \dots, x^2y^{i-2}, xy^{i-1}, y^i, \dots x^2y^{k-2}, xy^{k-1}, y^k\}$	$\mathcal{B}_k \setminus \{y^k\} \cup \{y^k - 2[x^{2k}]\}$	1
$xy^2 + ax - by - c$, $a, b, c \in \mathbb{R}$, $c \neq 0$ or $ab \neq 0$	$\{x^k, x^{k-1}, x^{k-1}y, \dots, x, xy, 1, y, \dots, y^k\}$	$\mathcal{B}_k \setminus \{x^k\} \cup \{x^ky\}$	1

Specifying V and f for reducible cases

$C = \mathcal{Z}(P)$, $\mathcal{B}_k$ is a basis for $\mathbb{R}[C]_{\leq k}$, $\mathcal{B}_V$ is a basis for V, f is always 1

P	$\mathcal{B}_k$	$\mathcal{B}_V$
$y(ay + x^2 + y^2)$, $a \in \mathbb{R} \setminus \{0\}$	$\{1, x, y, \ldots, x^j, x^{j-1}y, x^{j-2}y^2, \ldots x^k, x^{k-1}y, x^{k-2}y^2\}$	$\mathcal{B}_k \setminus \{1\} \cup \{\frac{ay+x^2+y^2}{x}\}$
$y(1 + ay - x^2 - y^2)$, $a \in \mathbb{R}$	$\{1, x-1, x^2-1, \ldots, x^{k-2}(x^2-1), y, yx, \ldots, yx^{k-1}, y^2, \ldots, y^2x^{k-2}\}$	$\mathcal{B}_k \setminus \{1\} \cup \{1 - 2\frac{1+ay-x^2-y^2}{1-x^2}\}$
$y(x - y^2)$	$\{1, x, \ldots, x^k, y, y^2, yx, y^2x, \ldots, yx^j, y^2x^j, \ldots, y^2x^{k-2}, yx^{k-1}\}$	$\mathcal{B}_k \setminus \{x^k\} \cup \{x^k - 2y^2x^{k-1}\}$
$y(1 + y - x^2)$	$1, x-1, x^2-1, \ldots, x^{k-2}(x^2-1), y, yx, y^2, y^2x \ldots, y^{k-1}x, y^k\}$	$\mathcal{B}_k \setminus \{x^k\} \cup \{1 - x - 2\frac{1+y-x^2}{1+x}\}$
$y(x - y)(x + y)$	$\{1, x, y, x^2, xy, y^2, \ldots x^k, x^{k-1}y, y^k\}$	$\mathcal{B}_k \setminus \{1\} \cup \{\frac{x^2-y^2}{x}]\}$
$yx(y + 1)$	$\{1, x, y, x^2, xy, y^2, \ldots x^k, x^{k-1}y, y^k\}$	$\mathcal{B}_k \setminus \{x^k\} \cup \{x^k + 2yx^k\}$

Main method in proofs

$C = \mathcal{Z}(P)$, $P = \prod_{i=1}^{r} P_i$ with P_i irreducible

Theorem (Baldi, Blekherman, Sinn, 24+ & Kummer, 24+)

Assume that C is irreducible or reducible without non-real intersection points. If the restriction of $Q \in \mathbb{R}[x, y]_{\leq 2k}$ to C generates an extreme ray of $\mathrm{Pos}_{2k}(C)$, then, for every $i = 1, \ldots, r$, either Q is divisible by P_i or the common zero set of P_i and Q in $\mathbb{C}^2$ consists of real points only.

In the reducible case with non-real intersections points, Q as above has precisely a pair of non-real intersections points as zeroes except real points in case it is divisible by some P_i.

V and f appearing in the tables above also appear in the following Positivstellensatz.

Theorem

There are $f \in \mathbb{R}[C]$ and a finite–dimensional vector space V in $Q(\mathbb{R}[C])$ with $V_f \subseteq \mathbb{R}[C]_{\leq k}$ such that the following are equivalent:

1. $p \in \mathrm{Pos}_{2k}(C)$.
2. *There exist finitely many $g_i \in \mathbb{R}[C]_{\leq k}$ and $h_j \in V$ such that* $p = \sum_i g_i^2 + f \sum_j h_j^2$.

TMP for $y^2 - x(x-a)(x-b) = 0$, $a, b \in \mathbb{R}, 0 < a < b$

A C–degree function $\deg_C$:

$$\deg_C(x^i y^j) = 2i + 3j \quad \text{including negative } i, j.$$

A basis $\mathcal{B}_k$ for $\mathbb{R}[C]_{\leq k}$ and V for V:

$\mathcal{B}_k$	1	x	y	$\dots$	x^2y^{i-2}	xy^{i-1}	y^i	$\dots$	x^2y^{k-2}	xy^{k-1}	y^k
$\deg_C$	0	2	3	$\dots$	$3i-2$	$3i-1$	$3i-2$	$\dots$	$3k-2$	$3k-1$	$3k/1$
$\mathcal{B}_v$	1	x	y	$\dots$	x^2y^{i-2}	xy^{i-1}	y^i	$\dots$	x^2y^{k-2}	xy^{k-1}	$\frac{y}{x}$

Theorem

Let $p \in \mathrm{Pos}_{2k}(C)$. Then there exist finitely many $g_i \in \mathbb{R}[C]_{\leq k}$ and $h_j \in V$ such that $p = \sum_i g_i^2 + x \sum_j h_j^2$.

Sketch of the proof:

- Let $u \in \mathrm{Pos}_{2k}(C)$ be an extreme ray. and $u^h(x, y, z) = z^{2k} u(\frac{x}{z}, \frac{y}{z})$ a homogenization of u.
- Then u^h has only real zeroes P_i, $i = 1, \dots, 3k$, of the form $P_i = [x_i : y_i : 1]$, $x_i, y_i \in \mathbb{R}$ or $P_i = [0 : 1 : 0]$, each of multiplicity 2.
- Known fact: $P := P_1 \oplus \dots \oplus P_{3k}$ is a 2–torsion point in the group law of C.
- If P is the point at infinity $O := [0 : 1 : 0]$, then $u^h = (u_1^h)^2$ for some $u_1^h \in \mathbb{R}[x, y, z]_{\leq k}$ and $u = u_1^2$ is a square of $u_1(x, y) = u_1^h(x, y, 1) \in \mathbb{R}[C]_{\leq k}$.
- Otherwise $P = [0 : 0 : 1]$ and $xzu^h = (u_2^h)^2$ for some $u_2^h \in \mathbb{R}[x, y, z]_{\leq k+1}$. Then $u = x(\frac{u_2}{x})^2$, where $u_2 = u_2^h(x, y, 1)$. Considering $\deg_C$ of both sides, u_2 cannot contain 1, y^{k+1} or xy^k.

TMP for $y^2 - x(x-a)(x-b) = 0$, $a, b \in \mathbb{R}, 0 < a < b$

Example: $2k = 6$, $\beta_{ij} = L(x^i y^j)$

L_C strict square potivity and V_x–local strict square positivity are equivalent to positive definiteness of the following matrices:

$$
\begin{array}{c} \\ 1 \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \\ X^2Y \\ XY^2 \\ Y^3 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 1 & X & Y & X^2 & XY & Y^2 & X^2Y & XY^2 & Y^3 \end{array} \\
\begin{bmatrix}
\beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} \\
\beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} \\
\beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} \\
\beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{23} \\
\beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{14} \\
\beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} & \beta_{23} & \beta_{14} & \beta_{05} \\
\beta_{21} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{23} & \beta_{42} & \beta_{33} & \beta_{24} \\
\beta_{12} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{14} & \beta_{33} & \beta_{24} & \beta_{15} \\
\beta_{03} & \beta_{13} & \beta_{04} & \beta_{23} & \beta_{14} & \beta_{05} & \beta_{24} & \beta_{15} & \beta_{06}
\end{bmatrix}
\end{array},
$$

$$
\begin{array}{c} \\ 1 \\ Y/X \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \\ X^2Y \\ XY^2 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} X & Y & X^2 & XY & X^3 & X^2Y & XY^2 & X^3Y & X^2Y^2 \end{array} \\
\begin{bmatrix}
\beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} \\
\beta_{01} & L((x-a)(x-b)) & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} \\
\beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} \\
\beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} \\
\beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{50} & \beta_{41} & \beta_{32} & \beta_{51} & \beta_{42} \\
\beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{23} & \beta_{42} & \beta_{33} \\
\beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{14} & \beta_{33} & \beta_{24} \\
\beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{51} & \beta_{42} & \beta_{33} & \beta_{52} & \beta_{43} \\
\beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{42} & \beta_{33} & \beta_{24} & \beta_{43} & \beta_{34}
\end{bmatrix}
\end{array}.
$$

TMP for nodal cubic $y^2 - x(x-1)^2 = 0$

Parametrization of C:

$$x(t) = t^2,\ y(t) = t^3 - t, \quad t \in \mathbb{R}.$$

Let

$$\mathrm{Nodal} := \{s \in \mathbb{R}[t] \colon s(1) = s(-1)\}, \quad \mathrm{Nodal}_{\leq i} := \{s \in \mathrm{Nodal} \colon \deg s \leq i\}.$$

The map

$$\Phi : \mathbb{R}[C] \to \mathrm{Nodal}, \quad \Phi(p(x,y)) = p(t^2, t^3 - t)$$

is a ring isomorphism. The vector subspace $\mathbb{R}[C]_{\leq i}$ is in one-to-one correspondence with the set $\mathrm{Nodal}_{\leq 3i}$ under Φ.

Let

$$\mathrm{Pos}(\mathrm{Nodal}_{\leq i}) := \{f \in \mathrm{Nodal}_{\leq i} \colon f(t) \geq 0 \text{ for every } t \in \mathbb{R}\},$$
$$\widetilde{\mathrm{Nodal}}_{\leq i} := \{s \in \mathbb{R}[t]_{\leq i} \colon s(1) = -s(-1)\}.$$

Theorem

Let $p \in \mathrm{Pos}(\mathrm{Nodal}_{\leq 6k})$. Then there exist finitely many $g_i \in \mathrm{Nodal}_{\leq 3k}$ and $h_j \in \widetilde{\mathrm{Nodal}}_{\leq 3k}$ such that $p = \sum_i g_i^2 + \sum_j h_j^2$.

TMP for nodal cubic $y^2 - x(x-1)^2 = 0$

The basis for $\mathrm{Nodal}_{\leq i}$ is the following:

$$\mathcal{B}_{\mathrm{Nodal}_{\leq i}} := \{1, t^2 - 1, t^3 - t, t^4 - t^2, \ldots, t^{i-1} - t^{i-3}, t^i - t^{i-2}\}.$$

The basis for $\widetilde{\mathrm{Nodal}}_{\leq i}$ is the following:

$$\mathcal{B}_{\widetilde{\mathrm{Nodal}}_{\leq i}} := \{t, t^2 - 1, t^3 - t, t^4 - t^2, \ldots, t^{i-1} - t^{i-3}, t^i - t^{i-2}\}.$$

We have that

$$\frac{y}{x-1}$$

maps to t under Φ. So this is a replacement for 1 in the basis for V.

This approach also gives an idea for constructive solution to the TMP working also in singular cases

Using correspondence Φ above the C–TMP for L is equivalent to the $\mathbb{R}$–TMP for

$$L_{\mathrm{Nodal}_{\leq 6k}} : \mathrm{Nodal}_{\leq 6k} \to \mathbb{R}, \quad L_{\mathrm{Nodal}_{\leq 6k}}(p) = L_C(\Phi^{-1}(p)).$$

Using the basis $\mathcal{B}_{\mathrm{Nodal}_{\leq 3k}} \cup \mathcal{B}_{\widetilde{\mathrm{Nodal}}_{\leq 3k}}$ the moment matrix of $L_{\mathrm{Nodal}_{\leq 6k}}$ is

$$\begin{array}{c} \\ 1 \\ T \\ T^2-1 \\ T^3-T \\ \vdots \\ T^{3k}-T^{3k-2} \end{array}
\begin{array}{c} \begin{array}{cccccc} 1 & T & T^2-1 & T^3-T & \cdots & T^{3k}-T^{3k-2} \end{array} \\
\begin{bmatrix}
\mathcal{L}(1) & {\color{red}?} & \mathcal{L}(t^2-1) & \mathcal{L}(t^3-t) & \cdots & \mathcal{L}(t^{3k}-t^{3k-2}) \\
{\color{red}?} & \mathcal{L}(t^2) & \mathcal{L}(t^3-t) & \mathcal{L}(t^4-t^2) & \cdots & \mathcal{L}(t^{3k+1}-t^{3k-1}) \\
\mathcal{L}(t^2-1) & \mathcal{L}(t^3-t) & \mathcal{L}((t^2-1)^2) & \mathcal{L}(t(t^2-1)^2) & \cdots & \mathcal{L}(t^{3k-2}(t^2-1)^2) \\
\mathcal{L}(t^3-t) & \mathcal{L}(t(t^3-t)) & \mathcal{L}(t(t^2-1)^2) & \mathcal{L}((t^3-t)^2) & \cdots & \mathcal{L}(t^{3k-1}(t^2-1)^2) \\
\vdots & & & & \ddots & \vdots \\
\mathcal{L}(t^{3k}-t^{3k-2}) & \cdots & & & \cdots & \mathcal{L}((t^{3k}-t^{3k-2})^2)
\end{bmatrix} \end{array}.$$

From here it is easy to characterize when $L_{\mathrm{Nodal}_{\leq 6k}}$ is a $\mathbb{R}$–moment functional and construct a measure after completing the only ? position in the matrix above.

However, it is not clear whether one needs rank $\overline{L}_{\mathrm{Nodal}_{\leq 6k}}$ or rank $\overline{L}_{\mathrm{Nodal}_{\leq 6k}} + 1$ atoms in a minimal measure.

TMP for nodal cubic $y^2 - x(x-1)^2 = 0$

$\Phi : \mathbb{R}[C]_{\leq 2k} \to \text{Nodal}_{\leq 6k}, \Phi(p(x,y)) = p(t^2, t^3 - t),$

$V = \text{span}\{\Phi^{-1}(\mathcal{B}_{\text{Nodal}_{\leq 3k}})\}$

L_C is **singular** if $\ker L_C \neq \{0\}$.

L_C is $(V,1)$**–locally singular** if $\ker L_{C,(V,1)} \neq \{0\}$.

Theorem

Let $L : \mathbb{R}[x,y]_{\leq 2k}$ be a linear functional such that $I_{\leq 2k} \subseteq \ker L$ and L_C is singular or $(V,1)$–locally singular. Then the following are equivalent:

1. *L is a C–moment functional.*
2. *L_C is square positive and $(V,1)$–locally square positive and one of the following holds:*
   - 2.1 $\text{rank}\, \overline{L}_C = \text{rank}(\overline{L}_C)|_{(\Phi^{-1}(\mathcal{B}_{\text{Nodal}_{\leq 3k-1}}))}$.
   - 2.2 $\text{rank}\, \overline{L}_{C,(V,1)} = \text{rank}(\overline{L}_{C,(V,1)})|_{(\Phi^{-1}(\mathcal{B}_{\widetilde{\text{Nodal}}_{\leq 3k-1}}))}$.

TMP for $y(ay + x^2 + y^2) = 0$

A line C_1 an a circle C_2 with one double intersection point

Parametrization of C:

$$C_1 : \{(s,0), s \in \mathbb{R}\}; \quad C_2 : \Big\{(-\frac{a}{2}\frac{t^2-1}{t^2+1}, -\frac{a}{2}\frac{(t+1)^2}{t^2+1}\Big\}, \quad t \in \mathbb{R}.$$

Let

$$\mathrm{Circ} = \{(f(s), g(t)) \in \mathbb{R}[s] \times \mathbb{R}\Big[\frac{1}{t^2+1}, \frac{t}{t^2+1}\Big] : f(0) = g(-1),\ f'(0) = \frac{2g'(-1)}{a}\},$$

$$\mathrm{Circ}_{\leq i} = \{(f(s), g(t)) \in \mathbb{R}[s]_{\leq i} \times \mathbb{R}\Big[\frac{1}{t^2+1}, \frac{t}{t^2+1}\Big]_{\leq i, \leq i} : f(0) = g(-1),\ f'(0) = \frac{2g'(-1)}{a}\}.$$

The map

$$\Phi : \mathbb{R}[C] \to \mathrm{Circ}_1,\ \Phi(p(x,y)) = \left(p(s,0), p\Big(-\frac{a}{2}\frac{t^2-1}{t^2+1}, -\frac{a}{2}\frac{(t+1)^2}{t^2+1}\Big)\right)$$

is a ring isomorphism. The vector subspace $\mathbb{R}[C]_{\leq i}$ is in one-to-one correspondence with the set $\mathrm{Circ}_{\leq 3i}$ under Φ.

Let

$$\mathrm{Pos}(\mathrm{Circ}_{\leq i}) := \{(f(s), g(t)) \in (\mathrm{Circ}_1)_{\leq i} : f(s) \geq 0,\ g(t) \geq 0 \text{ for every } (s,t) \in \mathbb{R}^2\},$$

$$\widetilde{\mathrm{Circ}}_{\leq i} := \{(f(s), g(t)) \in \mathbb{R}[s]_{\leq i} \times \mathbb{R}\Big[\frac{1}{t^2+1}, \frac{t}{t^2+1}\Big]_{\leq i, \leq i} : f(0) = g(-1) = 0\}.$$

TMP for $y(ay + x^2 + y^2) = 0$

Theorem

Let $(p_1, p_2) \in \mathrm{Pos}(\mathrm{Circ}_{\leq 2k})$. *Then there exist finitely many* $(g_{1;i}, g_{2,i}) \in \mathrm{Circ}_{\leq k}$ *and* $(h_{1;j}, h_{2;j}) \in \widetilde{\mathrm{Circ}}_{\leq k}$ *such that*

$$(p_1, p_2) = \sum_i (g_{1;i}^2, g_{2;i}^2) + \sum_j (h_{1;j}^2, h_{2;j}^2).$$

The basis for $\mathrm{Circ}_{\leq i}$ is the following:

$$\mathcal{B}_{\mathrm{Circ}_{\leq i}} := \Phi(\{1, x, y, x^2, xy, y^2, \ldots, x^j, x^{j-1}y, x^{j-2}y^2, \ldots x^i, x^{i-1}y, x^{i-2}y^2\})$$

The basis for $\widetilde{\mathrm{Circ}}_{\leq i}$ is the following:

$$\mathcal{B}_{\widetilde{\mathrm{Circ}}_{\leq i}} := \mathcal{B}_{\mathrm{Circ}_{\leq i}} \setminus \{(1,1)\} \cup \{(s,0)\}$$

We have that

$$\frac{ay + x^2 + y^2}{x}$$

maps to $(s, 0)$ under Φ. So this is a replacement for 1 in the basis for V.

TMP for $y(ay + x^2 + y^2) = 0$

Constructive approach   joint work with Yoo

$\beta = \{\beta_{i,j}\}_{i,j\in\mathbb{Z}_+,\ 0\le i+j\le 2k}$ has a C–RM on if and only if it can be decomposed as

$$\beta = \beta^{(\ell)} + \beta^{(c)},$$

where

$\beta^{(\ell)} = \{\beta^{(\ell)}_{i,j}\}_{i,j\in\mathbb{Z}_+,\ 0\le i+j\le 2k}$ admits a $\mathbb{R}$–rm,

$\beta^{(c)} = \{\beta^{(c)}_{i,j}\}_{i,j\in\mathbb{Z}_+,\ 0\le i+j\le 2k}$ admits a $\mathcal{Z}(ay + x^2 + y^2)$–rm.

It turns out that all the moments of $\beta^{(\ell)}, \beta^{(c)}$ are uniquely determined except

$$\beta^{(\ell)}_{0,0}, \beta^{(\ell)}_{1,0}, \beta^{(c)}_{0,0}, \beta^{(c)}_{1,0},$$

which satisfy the relations

$$\beta_{j,0} = \beta^{(\ell)}_{j,0} + \beta^{(c)}_{j,0}, \quad j = 0, 1.$$

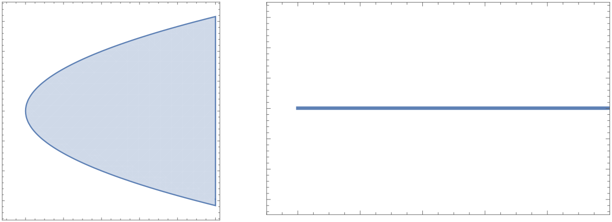
TMP for $y(ay + x^2 + y^2) = 0$

Moment matrices of $\beta^{(\ell)}$ and $\beta^{(c)}$ are of the form

$$\begin{array}{c} \\ 1 \\ X \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} 1 & X & \cdots \end{array} \\ \begin{bmatrix} \beta_{0,0} - \mathbf{t} & \beta_{1,0} - \mathbf{u} & * \\ \beta_{1,0} - \mathbf{u} & * & * \\ * & * & * \end{bmatrix} \end{array}, \quad \begin{array}{c} \\ 1 \\ X \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} 1 & X & \cdots \end{array} \\ \begin{bmatrix} \mathbf{t} & \mathbf{u} & * \\ \mathbf{u} & * & * \\ * & * & * \end{bmatrix} \end{array},$$

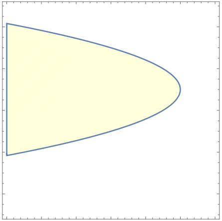
where $\mathbf{t}$ and $\mathbf{u}$ are parameters.

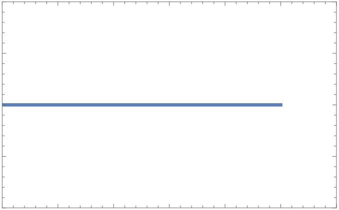
$(\mathbf{t}, \mathbf{u})$–region $\mathcal{R}_1$ of positive semidefiniteness of $\beta^{(\ell)}$ has one of the following forms:



TMP for $y(ay + x^2 + y^2) = 0$

$(\mathbf{t}, \mathbf{u})$–region $\mathcal{R}_2$ of positive semidefiniteness of $\beta^{(c)}$ has one of the following forms:





To solve the TMP we need to construct the existence of a point

$$(t_0, u_0) \in \mathcal{R}_1 \cap \mathcal{R}_2$$

such that both completions satisfy the conditions in the solutions to the corresponding TMP. To obtain measures with the least number of atoms, t_0, u_0 belonging to the boundaries $\partial\mathcal{R}_1$ and $\partial\mathcal{R}_2$ are desired.

TMP for $y(ay + x^2 + y^2) = 0$

Main result: A technically involved analysis using computations with Schur complements gives:

1. Concrete numerical conditions for the existence of the measure.
2. Number of atoms needed in a minimal measure.
3. In the nonsingular case with a measure, i.e., $\operatorname{rank} \mathcal{M}(k) = 3k$, a minimal measure has $3k$–atoms (which is the Caratheodory number in this case).

Thank you for your attention!