The truncated moment problem on plane curves

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Bivariate truncated moment problem (TMP)

Question

Let $k \in \mathbb{N}$ and

$$\beta = \beta^{(k)} = (\beta_{i,j})_{i,j \in \mathbb{Z}_+, i+j \le k}$$

a bivariate sequence of real numbers of degree k.

$K \subseteq \mathbb{R}^2$ is a closed subset.

The bivariate truncated moment problem on K (K–TMP): characterize the existence of a positive Borel measure μ on $\mathbb{R}^2$ with support in K, such that

$$\beta_{i,j} = \int_K x^i y^j d\mu(x)$$

for $i, j \in \mathbb{Z}_+$, $i + j \le k$.

μ is called a K–representing measure (K–RM) of β.

Bivariate moment matrix

The moment matrix $M(k)$ associated to β with the rows and columns indexed by $X^iY^j,\ i+j \le k,$ in degree-lexicographic order

$$1, X, Y, X^2, XY, Y^2, \ldots, X^k, X^{k-1}Y, \ldots, Y^k$$

is defined by where

$$M(k) := \begin{array}{c|ccccccc|}
 & 1 & X & Y & \cdots & X^{i_2}Y^{j_2} & \cdots & Y^k \\
\hline
1 & \beta_{0,0} & \beta_{1,0} & \beta_{0,1} & \cdots & \beta_{i_2,j_2} & \cdots & \beta_{0,k} \\
X & \beta_{1,0} & \beta_{2,0} & \beta_{1,1} & \cdots & \beta_{i_2+1,j_2} & \cdots & \beta_{1,k} \\
Y & \beta_{0,1} & \beta_{1,1} & \beta_{0,2} & \cdots & \beta_{i_2,j_2+1} & \cdots & \beta_{0,k+1} \\
\vdots & \vdots & & & \ddots & & & \vdots \\
X^{i_1}Y^{j_1} & \beta_{i_1,j_1} & \beta_{i_1+1,j_1} & \beta_{i_1,j_1+1} & \cdots & \beta_{i_1+i_2,j_1+j_2} & \cdots & \beta_{i_1,j_1+k} \\
\vdots & \vdots & & & & \vdots & \ddots & \vdots \\
Y^k & \beta_{0,k} & \beta_{1,k} & \beta_{0,k+1} & \cdots & \beta_{i_2,j_2+k} & \cdots & \beta_{0,2k} \\
\hline
\end{array}$$

Known results on the bivariate TMP

1. **Quadratic TMP, i.e. $\beta = \beta^{(2)}$:** Completely solved. Curto & Fialkow, '96
2. **Cubic TMP, i.e. $\beta = \beta^{(3)}$:** Completely solved. Kimsey, '14, Curto & Yoo, '18
3. **Quartic TMP, i.e. $\beta = \beta^{(4)}$:** Completely solved.
   $M(2)$ singular: Curto & Fialkow, '02
   $M(2)$ nonsingular: Fialkow & Nie, '10, Curto & Yoo, '16
4. **Quintic TMP, i.e. $\beta = \beta^{(5)}$:** Completely solved. El Azhar, Harrat, Idrissi, Zerouali, '19
5. **Sextic TMP, i.e. $\beta = \beta^{(6)}$:** Partially solved.
   - Extremal case - $\text{rank}\, M(3) = \text{card}\, \mathcal{V}$ Curto & Fialkow & Möller, '05
   - On variety $y = x^3$ Fialkow, '11
   - $\text{rank}\, M(3) \in \{7, 8\}$ Curto, Yoo, '14, '15
   - On special cases of reducible varieties Yoo, '17
   - $M(3)$ invertible Fialkow, '17, Fialkow & Blekherman, '20
6. **TMP on quadratic curves:** Completely solved. Curto & Fialkow, '02, '04, '05, '14
7. **TMP on cubic curves, i.e. $\beta = \beta^{(2k)}$:** Cases solved.
   - Infinite variety: $y = x^3$ Fialkow, '11
   - Finite variety: $z^3 = itz + u\bar{z}$, $t, u \in \mathbb{R}$ Curto, Yoo '14, '15
8. **Bounds on the number of atoms:** Riener & Schweighofer, '18, di Dio & Schmüdgen, '18

K–TMP for K being a curve $p(x, y) = 0$

Explicit solution

This is the solution in terms of numerical conditions on β and is most desired.

Solution based on feasibility of a LMI

If an explicit solution does not exist, then we are satisfied with a LMI based solution with bounded sizes of LMIs.

Solution based on the size of PSD extensions

Existence of such solution gives bounds on the degrees in the sum of squares certificates of positivity of polynomials on the curve or disproves the existence of these bounds.

Special case of feasibility of a LMI based solution.

K–TMP for K being a curve $p(x, y) = 0$

Explicit solution

Irreducible curves:

1. $\deg p = 1$ (Curto-Fialkow,'96)
2. $\deg p = 2$ (Curto-Fialkow,'02-'14)
3. $p = y - x^3$ (Fialkow,'11)
4. $p = y^2 - x^3$ (Z.,'21)
5. $p = xy^2 - 1$ (Z.,'22)
6. 'symmetric' Weierstrass case $p = y^2 - x^3 - ax - b$ (with Bhardwaj,'23-)

Reducible curves:

1. $\deg p = 2$ (Curto-Fialkow,'05,'15)
2. $p = y(1 + yq(x, y))$ (Z.,'22)
3. $p = y(x + yq(x, y))$
4. $p = y(x^2 + yq(x, y))$

Solution based on the size m of PSD extensions of $M(k)$:

$$\left.\begin{array}{l} 1.\ y = q(x) \\ 2.\ y^j x^i = 1, \text{ irred.} \end{array}\right\} \quad m = \left\lceil \frac{\deg p - 1}{2} \right\rceil$$

(Z.,'23)

3. $yq(x) = 1$: $m = O(k \deg(q))$ (Fialkow,11')

✗ $y^j = x^i$, irred., $i, j > 1$

Solution based on feasibility of a LMI:

1. $x = r(t), y = q(t)$, $r, q \in \mathbb{R}[t]$, $\gcd(\deg r, \deg q) = 1$
2. $x = t^i, y = \left(\frac{1}{t}\right)^j$, $\gcd(i, j) = 1$
3. $p = y(1 + ax^2 + yq(x, y))$

Column relations of a moment matrix

To every polynomial $p := \sum_{i,j} a_{i,j} x^i y^j \in \mathbb{R}[x,y]_k$, we associate the vector

$$p(X,Y) = \sum_{i,j} a_{i,j} X^i Y^j = a_{0,0} \cdot \begin{array}{c} 1 \\ \begin{pmatrix} \beta_{0,0} \\ \beta_{1,0} \\ \beta_{0,1} \\ \vdots \\ \beta_{0,k} \end{pmatrix} \end{array} + a_{1,0} \cdot \begin{array}{c} X \\ \begin{pmatrix} \beta_{1,0} \\ \beta_{2,0} \\ \beta_{1,1} \\ \vdots \\ \beta_{1,k} \end{pmatrix} \end{array} + \cdots + a_{0,k} \cdot \begin{array}{c} Y^k \\ \begin{pmatrix} \beta_{0,k} \\ \beta_{1,k} \\ \beta_{0,k+1} \\ \vdots \\ \beta_{0,2k} \end{pmatrix} \end{array}$$

from the column space of the matrix $M(k)$.

If $p(X,Y) = \mathbf{0}$, then p is a column relation.

Necessary conditions for the existence of a RM

The matrix $M(k)$ is recursively generated (RG) if for $p, q, pq \in \mathbb{R}[x, y]_k$

$$p(X, Y) = \mathbf{0} \quad \Rightarrow \quad (pq)(X, Y) = \mathbf{0}.$$

The matrix $M(k)$ is p–pure, if there are no other column relations expect those coming from p by RG.

Proposition (Curto and Fialkow, 96')

If $\beta^{(2k)}$ has a representing measure μ, then

$M(k)$ is positive semidefinite (PSD) and RG.

Bivariate p–pure TMPs with concrete solutions

p irreducible

p**–pure** . . . only relations are those coming from p by RG

NC . . . numerical conditions,   **FE** . . . flat extension

$\#$**atoms** $=$ rank $M(k) +$ **i**

proved by FE technique   proved by univariate reduction technique

deg p	p	Solution	FE exists	i
1	y	PSD	✓	0
2	$x^2 + y^2 - 1$	PSD	✓	0
	$y - x^2$	PSD	✓	0
	$xy - 1$	PSD	✓	0
3	$y - x^3$	NC	✓	0
	$y^2 - x^3$	NC	✓	0
	$xy^2 - 1$	NC	✓	0
	‘symmetric’ $y^2 - x^3 - ax - b$	NC	X	≤ 3

p–pure TMP for $p(x, y) = y^2 - x^3 - ax - b$

$k \geq 3$, $\beta := \{\beta_{ij}\}_{i,j\in\mathbb{Z}_+, i+j\leq 2k}$, analysis of the existence of a flat extension

$$M(k+1) = \begin{pmatrix} M(k) & B(k+1) \\ (B(k+1))^T & C(k+1) \end{pmatrix}$$

of $M(k)$ following Fialkow's $p(x, y) = y - x^3$ approach:

1. The block $B(k+1)$ restricted to rows of degree k is of the form :

$$\begin{array}{c} \\ X^k \\ X^{k-1}Y \\ \vdots \\ \vdots \\ X^2Y^{k-2} \\ XY^{k-1} \\ Y^k \end{array} \begin{array}{c} \begin{array}{ccccccc} X^{k+1} & X^kY & \cdots & \cdots & X^2Y^{k-1} & XY^k & Y^{k+1} \end{array} \\ \begin{pmatrix} \beta_{2k+1,0} & \beta_{2k,1} & \cdots & \cdots & \beta_{k+2,k-1} & \beta_{k+1,k} & \beta_{k,k+1} \\ \beta_{2k,1} & \beta_{2k-1,2} & \iddots & \iddots & \beta_{k+1,k} & \beta_{k,k+1} & \beta_{k-1,k+2} \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \beta_{k+3,k-2} & \beta_{k+2,k-1} & \iddots & \iddots & \iddots & \iddots & \theta \\ \beta_{k+2,k-1} & \beta_{k+1,k} & \iddots & \iddots & \iddots & \theta & \phi \\ \beta_{k+1,k} & \beta_{k,k+1} & \cdots & \cdots & \theta & \phi & \psi \end{pmatrix} \end{array},$$

where

$$\beta_{i,2k+1-i} = \beta_{i-3,2k+3-i} - a\beta_{i-2,2k+1-i} - b\beta_{i-3,2k+1-i} \quad \text{for } 3 \leq i \leq 2k+1$$

and θ, ϕ, ψ are arbitrary.

2. $C(k+1) := (B(k+1))^T M(k)^\dagger B(k+1)$ has a moment structure iff:

$$C_{k,k} = C_{k+1,k-1},$$

$$C_{k+1,k} = C_{k+2,k-1},$$

$$C_{k+1,k+1} = C_{k+2,k}$$

2. $C(k+1) := (B(k+1))^T M(k)^\dagger B(k+1)$ has a moment structure iff:

$$C_{k,k} = C_{k+1,k-1},$$

$$\phi = f_2\theta^2 + f_1\theta + f_0$$

$$C_{k+1,k} = C_{k+2,k-1},$$

$$C_{k+1,k+1} = C_{k+2,k}$$

2. $C(k+1) := (B(k+1))^T M(k)^\dagger B(k+1)$ has a moment structure iff:

$$\begin{aligned} C_{k,k} &= C_{k+1,k-1}, \\ \phi &= f_2\theta^2 + f_1\theta + f_0 \\ C_{k+1,k} &= C_{k+2,k-1}, \\ \psi &= j_{11}\phi\theta + j_{10}\phi + j_{02}\theta^2 + j_{01}\theta + j_{00} \\ C_{k+1,k+1} &= C_{k+2,k} \end{aligned}$$

2. $C(k+1) := (B(k+1))^T M(k)^\dagger B(k+1)$ has a moment structure iff:

$$\begin{aligned}
C_{k,k} &= C_{k+1,k-1}, \\
\phi &= f_2\theta^2 + f_1\theta + f_0 \\
C_{k+1,k} &= C_{k+2,k-1}, \\
\psi &= j_{11}\phi\theta + j_{10}\phi + j_{02}\theta^2 + j_{01}\theta + j_{00} \\
C_{k+1,k+1} &= C_{k+2,k}
\end{aligned}$$

$$k_{101}\psi\theta + k_{100}\psi + k_{011}\phi\theta + k_{010}\phi + k_{002}\theta^2 + k_{001}\theta + k_{000} =$$

$$\ell_{20}\phi^2 + \ell_{11}\phi\theta + \ell_{10}\phi + \ell_{02}\theta^2 + \ell_{01}\theta + \ell_{00}$$

2. $C(k+1) := (B(k+1))^T M(k)^\dagger B(k+1)$ has a moment structure iff:

$$\begin{aligned} C_{k,k} &= C_{k+1,k-1}, \\ \phi &= f_2\theta^2 + f_1\theta + f_0 \\ C_{k+1,k} &= C_{k+2,k-1}, \\ \psi &= j_{11}\phi\theta + j_{10}\phi + j_{02}\theta^2 + j_{01}\theta + j_{00} \\ C_{k+1,k+1} &= C_{k+2,k} \\ k_{101}\psi\theta + k_{100}\psi + k_{011}\phi\theta + k_{010}\phi + k_{002}\theta^2 + k_{001}\theta + k_{000} &= \\ \ell_{20}\phi^2 + \ell_{11}\phi\theta + \ell_{10}\phi + \ell_{02}\theta^2 + \ell_{01}\theta + \ell_{00} & \end{aligned}$$

3. A short computation shows that the last equation is of the form

$$\alpha_2\theta^2 + \alpha_1\theta + \alpha_0 = 0$$

and a flat extension $M(k+1)$ exists iff it has a real root θ.

p–pure TMP for $p(x,y) = y^2 - x^3 - ax - b$

Example (A measure exists, but there is no flat extension.)

Generating $M(3)$ with 10 atoms $(x_i, y_i), (x_i, -y_i)$ where

$$x_i = \frac{1}{i}, \quad y_i = \sqrt{x_i^3 - \frac{524287}{262144}x_i + 1}, \quad i = 1, \ldots, 5,$$

$M(3)$ is of rank 9 having a column relation

$$p(X,Y) = Y^2 - X^3 + \frac{524287}{262144}X - 1 = 0.$$

A flat extension $M(4)$ does not exist, since in

$$\alpha_2\theta^2 + \alpha_1\theta + \alpha_0 = 0$$

α_2, α_0 are rationals of the same sign, $\alpha_1 = 0$ and hence a real solution θ does not exist.

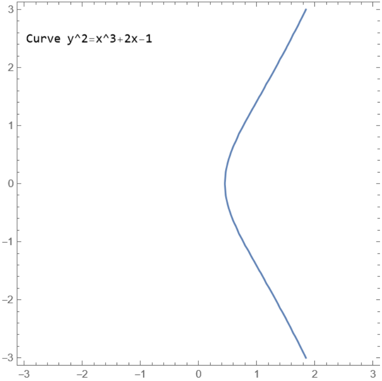
Symmetric p–pure TMP for $p(x,y)=y^2-x^3-ax-b$

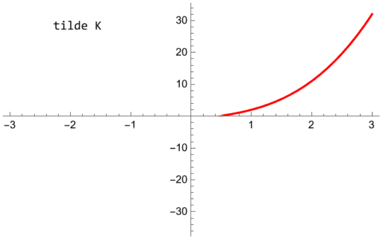
$k \geq 3$, $\beta := \{\beta_{ij}\}_{i,j\in\mathbb{Z}_+, i+j\leq 2k}$, symmetric: $\beta_{ij}=0$ if j is odd.

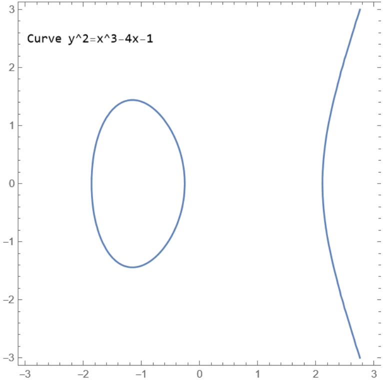
1. Substitution $Z = Y^2$:

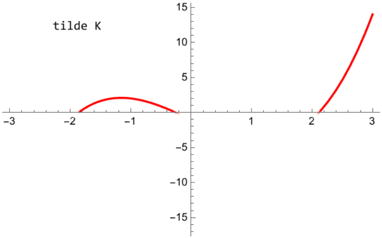
$$\widetilde{\beta} := \widetilde{\beta}_{ij} = \beta_{i,2j} \quad \text{for } i,j \in \mathbb{Z}_+ \text{ with } i+2j \leq 2k.$$

Define $\widetilde{p}(x,z) := z - x^3 - ax - b$ and $\widetilde{K} := \mathcal{Z}(\widetilde{p}) \cap (\mathbb{R}\times\mathbb{R}^+)$:









2. β has a $\mathcal{Z}(p)$–RM. $\Leftrightarrow$ $\widetilde{\beta}$ has a $\widetilde{K}$–RM.

($\Leftarrow$): If $(x_1, z_1), \dots, (x_m, z_m)$ are atoms in the measure for $\widetilde{\beta}$ with densities $\rho_1, \dots, \rho_m$, then

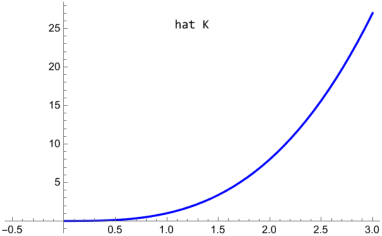
$$(x_1, \sqrt{z_1}), (x_1, -\sqrt{z_1}), \dots, (x_m, \sqrt{z_m}), (x_m, -\sqrt{z_m})$$

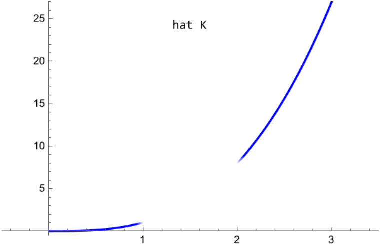
are atoms in the measure for β with densities $\frac{\rho_1}{2}, \frac{\rho_1}{2}, \dots, \frac{\rho_m}{2}, \frac{\rho_m}{2}$.

3. Affine linear transformation $\varphi(x, z) = (x, z - ax - b - x_0)$:

$$\widehat{\beta} := \varphi(\widetilde{\beta})$$

Define $\widehat{p}(x, z) := z - x^3$ and $\widehat{K} := \mathcal{Z}(\widehat{p}) \cap (\mathbb{R}^+ \times \mathbb{R}^+)$:





4. $\widetilde{\beta}$ has a $\widetilde{K}$–RM. $\Leftrightarrow$ $\widehat{\beta}$ has a $\widehat{K}$–RM.

5. The corresponding univariate sequence:

$$\gamma_{i+3j} = \widehat{\beta}_{ij}, \qquad \text{for } i, j \in \mathbb{Z}_+ \ \text{ with } i + 2j \leq 2k.$$

Note that $\gamma := (\gamma_t)_{t \leq 3k}$ is a degree $3k$ sequence.

6. $\widehat{\beta}$ has a $\widehat{K}$–RM. $\quad\Leftrightarrow\quad$ γ has a $\mathrm{pr}_x(\widehat{K})$–RM.

$\mathrm{pr}_x \ldots$ projection to x-coordinate.

Note that $\mathrm{pr}_x(\widehat{K})$ is of the form $[0, \infty)$ or $[0, c] \cup [d, \infty)$.

7.1 $\mathrm{pr}_x(\widetilde{K}) = [0, \infty)$: Use the solution to the Stieltjes TMP (Curto, Fialkow, 91').

7.2 $\mathrm{pr}_x(\widetilde{K}) = [0, c] \cup [d, \infty)$:

- By the truncated Riesz-Haviland theorem (Curto, Fialkow, 08'), the functional $L_\gamma : \mathbb{R}[x]_{\leq 3k} \to \mathbb{R}$, defined by

$$L_\gamma(p) := \sum_{0 \leq i \leq 3k} a_i \gamma_i, \qquad \text{where} \quad p = \sum_{0 \leq i \leq 3k} a_i x^i,$$

must have a $\mathrm{pr}_x(\widetilde{K})$–positive extension $L_{\gamma^{(3k+2)}}$ if k is even and $L_{\gamma^{(3k+1)}}$ if k is odd.

7.2 $\mathrm{pr}_x(\widetilde{K}) = [0, c] \cup [d, \infty)$:

- By the Positivstellensatz on $\mathrm{pr}_x(\widetilde{K})$ (Kuhlmann, Marshall, Schwartz, 05'), $f|_{\mathrm{pr}_x(\widetilde{K})} \geq 0$ if and only if
$$f = \sigma_0 + \sigma_1 x + \sigma_2 (x-c)(x-d) + \sigma_3 x(x-c)(x-d),$$
where $\sigma_0, \sigma_1, \sigma_2, \sigma_3 \in \sum \mathbb{R}[x]^2$ and
$$\deg \sigma_0, \deg(\sigma_1 x), \deg(\sigma_2 (x-c)(x-d)), \deg(\sigma_3 x(x-c)(x-d)) \leq \deg f.$$
- Finally, the solution in case of $\mathrm{pr}_x(\widetilde{K}) = [0, c] \cup [d, \infty)$ can be concretely characterized in terms of the localizing Hankel matrices at $1, x, (x-c)(x-d), x(x-c)(x-d)$:

$$\begin{pmatrix} \gamma_0 & \cdots & \gamma_m \\ \vdots & & \vdots \\ \gamma_m & \cdots & \gamma_{2m} \end{pmatrix}, \quad \begin{pmatrix} \gamma_1 & \cdots & \gamma_{m+1} \\ \vdots & & \vdots \\ \gamma_{m+1} & \cdots & \gamma_{2m+1} \end{pmatrix},$$

$$\begin{pmatrix} cd\gamma_0 - (c+d)\gamma_1 + \gamma_2 & \cdots & cd\gamma_{m-1} - (c+d)\gamma_m + \gamma_{m+1} \\ \vdots & & \vdots \\ cd\gamma_{m-1} - (c+d)\gamma_m + \gamma_{m+1} & \cdots & cd\gamma_{2m-2} - (c+d)\gamma_{2m-1} + \gamma_{2m} \end{pmatrix},$$

$$\begin{pmatrix} cd\gamma_1 - (c+d)\gamma_2 + \gamma_3 & \cdots & cd\gamma_m - (c+d)\gamma_{m+1} + \gamma_{m+2} \\ \vdots & & \vdots \\ cd\gamma_m - (c+d)\gamma_{m+1} + \gamma_{m+2} & \cdots & cd\gamma_{2m-1} - (c+d)\gamma_{2m} + \gamma_{2m+1} \end{pmatrix}.$$

Bivariate *p*–pure TMPs with concrete solutions

p reducible

p–**pure** . . . only relations are those coming from *p* by RG

NC . . . numerical conditions, **FE** . . . flat extension

$\#$**atoms** $=\operatorname{rank} M(k)+$ **i**

proved by FE technique proved by univariate reduction technique

deg p	p	Solution	FE exists	i
2	xy	PSD	✗	1
	y^2-1	PSD	✓	0
3	$y(x+yq(x,y))$	PSD		≤ 2
	$y(1+yq(x,y))$	NC	✓	0
	$y(x^2+yq(x,y))$	NC		≤ 3

Solving the TMP on reducible rational curves

Basic idea

1. Study decompositions

$$\beta = \beta^{(1)} + \beta^{(2)},$$

where

$\beta^{(1)}$: a moment sequence on one irreducible component of $\mathcal{C}$,

$\beta^{(2)}$: a moment sequence on the complement of $\mathcal{C}$.

2. Apply the solution to the TMP on each summand $\beta^{(i)}$, $i = 1, 2$.

Let $\mathcal{B}$ be the basis for the column space and $\vec{X} = (1, X, \ldots, X^k)$. Then

$$(M(k))|_{\mathcal{B}} = \begin{array}{c|cc|} & \vec{X} & \mathcal{B}\backslash\vec{X} \\ (\vec{X})^T & A_{11} & A_{12} \\ (\mathcal{B}\backslash\vec{X})^T & (A_{12})^T & A_{22} \end{array} = M_1(k) + M_2(k),$$

Due to the relation $Ys(X, Y) = 0$ in $M(k)$:

$$M_1(k) = \begin{array}{c|cc|} & \vec{X} & \mathcal{B}\backslash\vec{X} \\ (\vec{X})^T & * & 0 \\ (\mathcal{B}\backslash\vec{X})^T & 0 & 0 \end{array}, \qquad M_2(k) = \begin{array}{c|cc|} & \vec{X} & \mathcal{B}\backslash\vec{X} \\ (\vec{X})^T & * & A_{12} \\ (\mathcal{B}\backslash\vec{X})^T & (A_{12})^T & A_{22} \end{array}.$$

There are only two paramaters in $*$, $*$:

1. $s(x, y) = x + yq(x, y)$: $\beta^{(2)}_{0,0}, \beta^{(2)}_{2k,0}$. Easy to analyze.
2. $s(x, y) = 1 + q(x, y)$: $\beta^{(2)}_{2k-1,0}, \beta^{(2)}_{2k,0}$. A bit more demanding.
3. $s(x, y) = x^2 + q(x, y)$: $\beta^{(2)}_{0,0}, \beta^{(2)}_{1,0}$. Involved analysis.
4. $s(x, y) = 1 + ax^2 + q(x, y)$: $\beta^{(2)}_{0,0}, \beta^{(2)}_{1,0}$ all over $*$. Intractable to analyze.

Property $(S_{k,m})$

Solution to the TMP based on the size of PSD extensions

$\mathcal{Z}(p) = \{(x, y) \in \mathbb{R}^2 \colon p(x, y) = 0\}$

$\mathcal{Z}(p)$ has property $(S_{k,m})$ if the following are equivalent:

1. $\beta^{(2k)}$ has a $\mathcal{Z}(p)$–RM.
2. $M(k)$ satisfies $p(X, Y) = 0$ and admits a PSD extension $M(k+m)$.

$\mathcal{Z}(p)$ has property $(A_{k,m})$ if every $f \in \mathbb{R}[x, y]_{\leq 2k+2}$ with $f|_{\mathcal{Z}(p)} > 0$ is of the form

$$f = \sum_i f_i^2 + p \sum_j g_j^2 - p \sum_\ell h_\ell^2,$$

where $f_i^2, pg_j^2, ph_\ell^2 \in \mathbb{R}[x, y]_{\leq 2m}$.

Theorem (Curto and Fialkow, 08')

$$(A_{k,k+m}) \Rightarrow (S_{k,m}) \quad \textit{and} \quad (S_{k,m}) \Rightarrow (A_{k-1,k+m}).$$

Bivariate TMP on $p(x, y) = 0$ with $\deg p \geq 4$

proved through property $(A_{k,m(k)})$ (Fialkow,11')

proved by univariate reduction technique

LMI . . . feasibility problem of a linear matrix inequality

$\deg p$	p	$(S_{k,m})$	m	Solution	# atoms
$\ell \geq 4$	$y - q(x)$	✓	$O(k\ell)$ $\lceil \frac{\ell-1}{2} \rceil$	LMI	$k\ell$
	$y^j x^{\ell-j} - 1$, irred.	✓	$\lceil \frac{\ell-1}{2} \rceil$	LMI	$k\ell$
	$y^j - x^\ell$, $j > 1$, irred.	✗	✗	LMI	$k\ell$

Hankel matrix

Let $k \in \mathbb{N}$. For

$$\gamma = (\gamma_0, \ldots, \gamma_{2k}) \in \mathbb{R}^{2k+1}$$

we write

$$A_\gamma = \begin{array}{cc} & \begin{array}{ccccc} 1 & T & T^2 & \cdots & T^k \end{array} \\ \begin{array}{c} 1 \\ T \\ T^2 \\ \vdots \\ T^k \end{array} & \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \iddots & \iddots & \gamma_{k+1} \\ \gamma_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \gamma_{2k-1} \\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k-1} & \gamma_{2k} \end{pmatrix} \end{array}$$

$p(x,y)=y^{\ell_2}x^{\ell_1}-1$, $\gcd(\ell_1,\ell_2)=1$, has $(S_{k,k+\lceil\frac{\ell_1+\ell_2}{2}\rceil})$

1. Parametrization: $x=t^{\ell_2}$, $y=t^{-\ell_1}$.
2. The univariate sequence: $\beta_{ij} \;\leftrightarrow\; \gamma_{i\ell_2-j\ell_1}$.

$$\gamma := (\gamma_{-2k\ell_1},\ldots,\gamma_{2k\ell_2}) \text{ has some gaps.}$$

3. $\beta^{(2k)}$ has a $\mathcal{Z}(p)$–RM $\quad\Leftrightarrow\quad$ γ has a $(\mathbb{R}\setminus\{0\})$–RM.
4. Solution of the strong $(\mathbb{R}\setminus\{0\})$-TMP (Z,22'), i.e., TFAE:
   - γ has a $(\mathbb{R}\setminus\{0\})$–RM.
   - γ can be extended to the sequence

$$\widetilde{\gamma} := (\gamma_{-2k\ell_1-2},\ldots,\gamma_{2k\ell_2+2}) \quad \text{without gaps and} \quad A_{\widetilde{\gamma}} \text{ is PSD.}$$

5. $M(k+\lceil\frac{\ell_1+\ell_2}{2}\rceil)$ PSD $\quad\Rightarrow\quad$ $A_{\widetilde{\gamma}}$ PSD.

LMI based solution for $p(x, y) = y^{\ell_2} x^{\ell_1} - 1$

Theorem

TFAE:

1. $\beta^{(2k)}$ *has a* $\mathcal{Z}(p)$*–RM.*
2. $\beta_{i+\ell_1, j+\ell_2} = \beta_{i,j}$ *for every* $i, j \in \mathbb{Z}_+$*, such that* $i + j \leq 2k - \ell_1 - \ell_2$ *and there exist missing values* γ_i *in the sequence*

   $$\widetilde{\gamma} = (\gamma_{-2k\ell_1-2}, \gamma_{-2k\ell_1-1}, \cdots, \gamma_{2k\ell_2+1}, \gamma_{2k\ell_2+2})$$

   generated by

   $$\gamma_{i\ell_2 - j\ell_1} = \beta_{i,j}$$

   such that

   $$A_{\widetilde{\gamma}} \succeq 0.$$

Example: $p(x, y) = yx^2 - 1$

The matrix $A_{\tilde{\gamma}}$ is equal to

$$\left(\begin{array}{c|cccccc|c}
\boldsymbol{\gamma_{-4k-2}} & \boldsymbol{\gamma_{-4k-1}} & \gamma_{-4k} & \boldsymbol{\gamma_{-4k+1}} & \gamma_{-4k+2} & \cdots & \gamma_k & \gamma_{k+1} \\
\hline
\boldsymbol{\gamma_{-4k-1}} & \gamma_{-4k} & \boldsymbol{\gamma_{-4k+1}} & \gamma_{-4k+2} & \iddots & & \gamma_{k+1} & \gamma_{k+2} \\
\gamma_{-4k} & \boldsymbol{\gamma_{-4k+1}} & \gamma_{-4k+2} & \iddots & & & \gamma_{k+2} & \vdots \\
\boldsymbol{\gamma_{-4k+1}} & \beta_{0,2k-1} & \iddots & & & & \vdots & \vdots \\
\gamma_{-4k+2} & \iddots & & & & & \vdots & \gamma_{2k} \\
\vdots & & & & & & \gamma_{2k} & \boldsymbol{\gamma_{2k+1}} \\
\hline
\gamma_{k+1} & \cdots & \cdots & \cdots & \gamma_{2k-1} & \gamma_{2k} & \boldsymbol{\gamma_{2k+1}} & \boldsymbol{\gamma_{2k+2}}
\end{array}\right)$$

We need to complete the **bold** moments such that $A_{\tilde{\gamma}}$ is PSD.

$p(x,y)=y^{\ell_2}-x^{\ell_1}$, $\ell_2>\ell_1>1$, irreducible does not have property $(S_{k,m})$ for every m

1. Parametrization: $x=t^{\ell_2}$, $y=t^{\ell_1}$.
2. The univariate sequence: $\beta_{ij} \;\leftrightarrow\; \gamma_{i\ell_2+j\ell_1}$.

   $$\gamma := \gamma_0,\ldots,\gamma_{2k\ell_2} \text{ has some gaps.}$$
3. $\beta^{(2k)}$ has a $\mathcal{Z}(p)$–RM $\quad\Leftrightarrow\quad$ γ has a $\mathbb{R}$–RM.
4. Solution of the $\mathbb{R}$-TMP: γ has a $\mathbb{R}$–RM $\;\Leftrightarrow\;$ γ can be extended to the sequence

   $$\gamma^{(2k\ell_2+2)}=(\gamma_0,\ldots,\gamma_{2k\ell_2+2}) \quad \text{without gaps and} \quad A_{\gamma^{(2k\ell_2+2)}} \text{ is PSD.}$$
5. One can construct a sequence γ such that A_γ is not even partially PSD, but it can be extended with $\gamma_{2k\ell_2+1},\gamma_{2k\ell_2+2},\ldots$ to a matrix such that the submatrices corresponding to matrices $M(k+m)$ are PSD.

- Columns of $M(\ell)$ correspond to columns

$$\mathcal{T}_\ell = \{T^s \colon s = a\ell_1 + b\ell_2,\ a, b = 0, \ldots, \ell\} = \{1, T^{s_1}, T^{s_2}, \ldots, T^{s_{r_\ell}}\}$$

  of the univariate Hankel matrix $A_{\gamma^{(2\ell\ell_2)}}$.

- Then

$$A_{\gamma^{(2\ell\ell_2)}} = \begin{array}{c} \\ 1 \\ \vdots \\ T^{s_1} \\ \vdots \\ T^{s_2} \\ \vdots \\ T^{s_{r_\ell}} \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & \cdots & T^{s_1} & \cdots & T^{s_2} & \cdots & T^{s_{r_\ell}} \end{array} \\ \left(\begin{array}{ccccccc} \gamma_0 & & \gamma_{s_1} & & \gamma_{s_2} & & \gamma_{s_{r_\ell}} \\ & & & & & & \\ \gamma_{s_1} & & \gamma_{2s_1} & & \gamma_{s_1+s_2} & & \gamma_{s_1+s_{r_\ell}} \\ & & & & & & \\ \gamma_{s_2} & & \gamma_{s_1+s_2} & & \gamma_{2s_2} & & \gamma_{s_2+s_{r_\ell}} \\ & & & & & & \\ \gamma_{s_{r_\ell}} & & \gamma_{s_{r_\ell}+s_1} & & \gamma_{s_{r_\ell}+s_2} & & \gamma_{2s_{r_\ell}} \end{array} \right) \end{array}$$

  The specified part of $A_{\gamma^{(2\ell\ell_2)}}$ corresponds to $M(\ell)|_{\text{rows/columns in the basis}}$.

If $M(k)|_{\text{basis}}$ is PD, then A is PD and it has infinitely many PD extensions:

$$\begin{array}{c|cccccccc}
 & 1 & \cdots & T^{s_1} & \cdots & T^{s_2} & \cdots & T^{s_{r_\ell}} & T^{s_{r_\ell}+1} \\
\hline
1 & \gamma_0 & & \gamma_{s_1} & & \gamma_{s_2} & & \gamma_{s_{r_\ell}} & \gamma_{s_{r_\ell}+1} \\
\vdots & & & & & & & & \\
T^{s_1} & \gamma_{s_1} & & \gamma_{2s_1} & & \gamma_{s_1+s_2} & & \gamma_{s_1+s_{r_\ell}} & \gamma_{s_1+s_{r_\ell}+1} \\
\vdots & & & & & & & & \\
T^{s_2} & \gamma_{s_2} & & \gamma_{s_1+s_2} & & \gamma_{2s_2} & & \gamma_{s_2+s_{r_\ell}} & \gamma_{s_2+s_{r_\ell}+1} \\
\vdots & & & & & & & & \\
T^{s_{r_\ell}} & \gamma_{s_{r_\ell}} & & \gamma_{s_{r_\ell}+s_1} & & \gamma_{s_{r_\ell}+s_2} & & \gamma_{2s_{r_\ell}} & \gamma_{2s_{r_\ell}+1} \\
T^{s_{r_\ell}+1} & \gamma_{s_{r_\ell}+1} & & \gamma_{s_{r_\ell}+1+s_1} & & \gamma_{s_{r_\ell}+1+s_2} & & \gamma_{2s_{r_\ell}+1} & \gamma_{2s_{r_\ell}+2}
\end{array}$$

- $\gamma_{2s_{r_\ell}+1}$ is chosen arbitrarily, while $\gamma_{2s_{r_\ell}+2}$ must be such that the Schur complement is positive.
- One can continue in this way to determine $T^{s_{r_\ell}+2}, T^{s_{r_\ell}+3}, \ldots$. On the side of β one gets a sequence of extensions $\beta^{(2k)}, \beta^{(2k+2)}, \beta^{(2k+4)}, \ldots$ such that $M(k+1), M(k+2), \ldots$ are PSD.
- So one gets a full sequence $\beta^{(\infty)}$ with $M(\infty)$ PSD.

- γ can be chosen such that it does not have a measure, even though $(A)|_{\mathcal{T}_k}$ is PD. Consequently, we will get β with infinitely many extensions but without a measure.

Case 1: One of ℓ_1, ℓ_2 is even. Say $\ell_1 = 2\ell_1'$. Then

$$\begin{array}{c|cccccccc|}
 & 1 & \dots & T^{\ell_1'} & \dots & T^{\ell_1} & \dots & T^{\ell_1'(\ell_2-1)} & \dots \\
\hline
1 & \gamma_0 & & & & \gamma_{\ell_1} & & & \\
\vdots & & & & & & & & \\
T^{\ell_1'} & & & \gamma_{\ell_1} & & & & \gamma_{\ell_1'\ell_2} & \\
\vdots & & & & & & & & \\
T^{\ell_1} & \gamma_{\ell_1} & & & & & & & \\
\vdots & & & & & & & & \\
T^{\ell_1'(\ell_2-1)} & & & \gamma_{\ell_1'\ell_2} & & & & \gamma_{\ell_1(\ell_2-1)} & \\
\vdots & & & & & & & &
\end{array}.$$

1. Generate any sequence $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{2s_{r_k}})$ such that A_γ is PD.
2. You decrease γ_{ℓ_1} such that the submatrix $(A_\gamma)|_{\{T^{\ell_1'}, T^{\ell_1'(\ell_2-1)}\}}$ is not PSD.
3. Since γ_{ℓ_1} occurs in $(A_\gamma)|_{\mathcal{T}_k}$ only twice at non-diagonal places, you can increase γ_0 such that $(A_\gamma)|_{\mathcal{T}_k}$ is PD.

Nonnegative but not sos polynomial on $\mathcal{Z}(p)$

Let $(v_1, v_2) \in \mathbb{R}^2$ be the eigenvector of the negative eigenvalue of

$$\begin{pmatrix} \gamma_{\ell_1} & \gamma_{\ell_1' \ell_2} \\ \gamma_{\ell_1' \ell_2} & \gamma_{\ell_1(\ell_2-1)} \end{pmatrix}.$$

Then

$$\left(v_1 t^{\ell_1'} + v_2 t^{\ell_1'(\ell_2-1)}\right)^2 = v_1^2 y + 2 v_1 v_2 x^{\ell_1'} + v_2^2 y^{\ell_2 - 1}$$

is nonnegative on $\mathcal{Z}(p)$, but not sos.

Case 2: Both ℓ_1, ℓ_2 are odd. Then

$$\begin{array}{c|cccccccc}
 & 1 & \cdots & T^{\frac{\ell_1+\ell_2}{2}} & \cdots & T^{\ell_1+\ell_2} & \cdots & T^{\frac{\ell_2(\ell_1-1)}{2}} & \cdots \\
\hline
1 & \gamma_0 & & & & \gamma_{\ell_1+\ell_2} & & & \\
\vdots & & & & & & & & \\
T^{\frac{\ell_1+\ell_2}{2}} & & & \gamma_{\ell_1+\ell_2} & & & & \frac{\ell_2(\ell_1-1)}{2} & \\
\vdots & & & & & & & & \\
T^{\ell_1+\ell_2} & \gamma_{\ell_1+\ell_2} & & & & & & & \\
\vdots & & & & & & & & \\
T^{\frac{\ell_2(\ell_1-1)}{2}} & & & \frac{\ell_2(\ell_1-1)}{2} & & & & \ell_2(\ell_1-1) & \\
\vdots & & & & & & & &
\end{array}$$

1. Generate any sequence $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{2s_{r_k}})$ such that A_γ is PD.
2. You decrease $\gamma_{\ell_1+\ell_2}$ such that the submatrix $A|_{\left\{T^{\ell_1+\ell_2}, T^{\frac{\ell_2(\ell_1-1)}{2}}\right\}}$ is not PSD.
3. Since $\gamma_{\ell_1+\ell_2}$ occurs in $(A_\gamma)|_{\mathcal{T}_k}$ only twice at non-diagonal places, you can increase γ_0 such that $(A_\gamma)|_{\mathcal{T}_k}$ is PD.

Nonnegative but not sos polynomial on $\mathcal{Z}(p)$

Let $(v_1, v_2) \in \mathbb{R}^2$ be the eigenvector of the negative eigenvalue of

$$\begin{pmatrix} \gamma_{\ell_1+\ell_2} & \gamma_{\frac{\ell_2(\ell_1-1)}{2}} \\ \gamma_{\frac{\ell_2(\ell_1-1)}{2}} & \gamma_{\ell_2(\ell_1-1)} \end{pmatrix}.$$

Then

$$\left(v_1 t^{\frac{\ell_1+\ell_2}{2}} + v_2 t^{\frac{\ell_2(\ell_1-1)}{2}}\right)^2 = v_1^2 xy + 2v_1 v_2 y^{\frac{1+\ell_2}{2}} + v_2^2 x^{\ell_1-1}$$

is nonnegative on $\mathcal{Z}(p)$, but not sos.

Thank you for your attention!