# The truncated moment problem on quadratic, cubic and some higher degree curves

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# Bivariate truncated moment problem

Let $k \in \mathbb{N}$ and

$$\beta = \beta^{(k)} = (\beta_{i,j})_{i,j \in \mathbb{Z}_+, i+j \le k}$$

a bivariate sequence of real numbers of degree $k$.

$K \subseteq \mathbb{R}^2$ is a closed subset.

The **bivariate truncated moment problem on $K$ ($K$–TMP)**: characterize the existence of a positive Borel measure $\mu$ on $\mathbb{R}^2$ with support in $K$, such that

$$\beta_{i,j} = \int_K x^i y^j d\mu(x)$$

for $i, j \in \mathbb{Z}_+$, $i + j \le k$.

$\mu$ is called a $K$–representing measure ($K$–RM) of $\beta$.

# Bivariate moment matrix

The moment matrix $M(k)$ associated to $\beta$ with the rows and columns indexed by $X^iY^j,\ i+j\leq k,$ in degree-lexicographic order

$$1, X, Y, X^2, XY, Y^2, \ldots, X^k, X^{k-1}Y, \ldots, Y^k$$

is defined by

$$M(k)=(\beta_{i+j})_{i,j=0}^{k}=\begin{bmatrix} M[0,0](\beta) & M[0,1](\beta) & \cdots & M[0,k](\beta)\\ M[1,0](\beta) & M[1,1](\beta) & \cdots & M[1,k](\beta)\\ \vdots & \vdots & \ddots & \vdots\\ M[k,0](\beta) & M[k,1](\beta) & \cdots & M[k,k](\beta)\end{bmatrix},$$

where

$$M[i,j](\beta):=\begin{array}{c} \\ X^i\\ X^{i-1}Y\\ X^{i-2}Y^2\\ \vdots\\ Y^i\end{array}\begin{array}{c} \begin{array}{ccccc} X^j & X^{j-1}Y & X^{j-2}Y^2 & \cdots & Y^j\end{array}\\ \begin{bmatrix} \beta_{i+j,0} & \beta_{i+j-1,1} & \beta_{i+j-2,2} & \cdots & \beta_{i,j}\\ \beta_{i+j-1,1} & \beta_{i+j-2,2} & \beta_{i+j-3,3} & \cdots & \beta_{i-1,j+1}\\ \beta_{i+j-2,2} & \beta_{i+j-3,3} & \beta_{i+j-4,4} & \cdots & \beta_{i-2,j+2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \beta_{j,i} & \beta_{j-1,i+1} & \beta_{j-2,i+2} & \cdots & \beta_{0,i+j}\end{bmatrix}\end{array}$$

are Hankel matrices.

# Necessary conditions

- To every polynomial $p := \sum_{i,j} a_{i,j} x^i y^j \in \mathbb{R}[x,y]_k$, we associate the vector

$$p(X,Y) = \sum_{i,j} a_{i,j} X^i Y^j$$

from the column space of the matrix $M(k)$.

- The matrix $M(k)$ is recursively generated (RG) if for $p, q, pq \in \mathbb{R}[x,y]_k$

$$p(X,Y) = \mathbf{0} \quad \Rightarrow \quad (pq)(X,Y) = \mathbf{0}.$$

- The matrix $M(k)$ satisfies the variety condition (VC) if

$$\text{rank}\, M(k) \leq \text{card}\, \mathcal{V},$$

where $\mathcal{V} := \bigcap_{\substack{g \in \mathbb{R}[x,y]_{\leq k},\\ g(X,Y)=\mathbf{0} \text{ in } M(k)}} \underbrace{\left\{(x,y) \in \mathbb{R}^2 : g(x,y) = 0\right\}}_{\mathcal{Z}(p)}$.

## Proposition (Curto and Fialkow, 96')

If $\beta^{(2k)}$ has a representing measure $\mu$, then

$M(k)$ is positive semidefinite (PSD), RG and satisfies VC.

# Solving the TMP by reduction to the univariate case

Basic ideas:

1. For irreducible curve $\mathcal{C}$:
   - Get rid of one variable *(use parametrization of the curve)*.
   - Solve the corresponding univariate TMP.

2. For reducible curve $\mathcal{C}$:
   - Study decompositions $\beta = \beta^{(1)} + \beta^{(2)}$, where $\beta^{(1)}$ is a moment sequence on one irreducible component of $\mathcal{C}$ and $\beta^{(2)}$ on the complement.
   - Apply the solution of the TMP on each summand $\beta^{(i)}$, $i = 1, 2$.

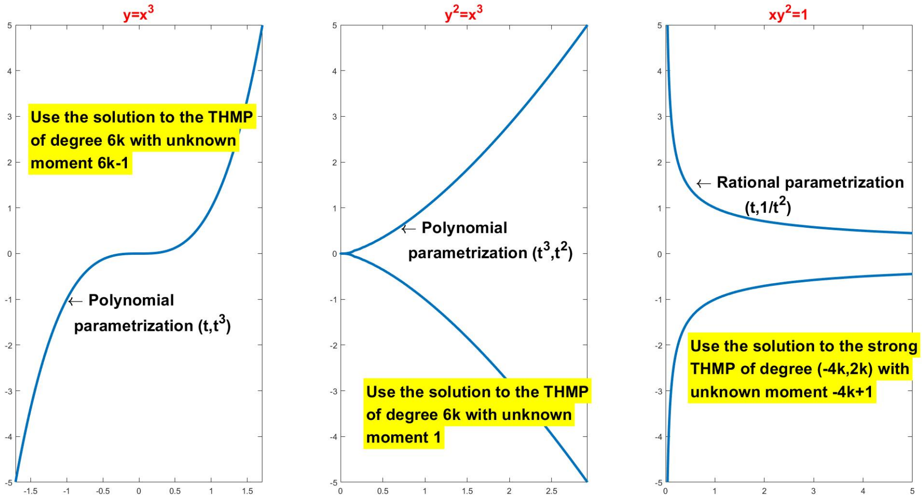
Outcomes of this approach:

1. Concrete solution to the TMP on quadratic (Curto and Fialkow) and some cubic curves.

2. For some higher degree curves two abstract solutions, which are probably most concrete one can hope for, are obtained.

# The univariate reduction solving TMP on some cubics

Cubic irreducible curves
MATRIX COMPLETION PROBLEMS WITH CONSTRAINTS



# The TMP on $y = x^3$ through the flat extension theorem

Let $k \geq 3$, $p(X, Y) = Y - X^3$ and $\beta := \beta^{(2k)} = (\beta_{i,j})_{i,j \in \mathbb{Z}^+, i+j \leq 2k}$.

## Theorem (Fialkow, 11')

*Assume $\beta$ is a p–pure sequence, i.e., p generates all column relations of $M_k$ by RG. TFAE:*

(1) *$\beta$ has a $\mathcal{Z}(p)$–RM.*

(2) *$\beta$ has a (rank $M_k$)–atomic $\mathcal{Z}(p)$–RM.*

(3) *CONCRETE SOLUTION:*
*$M_k$ is PSD and*

$$\boxed{\beta_{1,2k-1} > \psi(\beta)},$$

*where $\psi$ is a rational function in $\beta_{i,j}$.*

(4) *ABSTRACT SOLUTION:*
*$M_k$ admits a PSD, RG extension $M_{k+1}$.*

*Remark:* The solution of the nonpure situation is partly algorithmic.

# The TMP on $y = x^3$ through the univariate reduction

Every atom must be of the form $(t, t^3)$ for some $t \in \mathbb{R}$. So $\beta_{i,j}$ corresponds to the moment of

$$z^{i+3j}.$$

As $i, j$ run over $0, 1, \ldots, 2k$ such that $i + j \leq 2k$, the sum $i + 3j$ runs over the set

$$\{0, 1, \ldots, 6k-2, 6k\}.$$

The problem is equivalent to the

truncated Hamburger MP (THMP) with a gap $\gamma_{6k-1}$,

i.e., does there exist $x \in \mathbb{R}$ such that

$$(\gamma_0, \gamma_1, \ldots, \gamma_{6k-2}, x, \gamma_{6k})$$

admits a measure on $\mathbb{R}$. This is a

PSD matrix completion problem with constraints.

# PSD matrix completion result

## Proposition

Let

$$A(?) := \begin{bmatrix} A_1 & a & b \\ a^T & \alpha & ? \\ b^T & ? & \beta \end{bmatrix} = \begin{bmatrix} A_1 & a & * \\ a^T & \alpha & * \\ * & * & * \end{bmatrix} = \begin{bmatrix} A_1 & * & b \\ * & * & * \\ b^T & * & \beta \end{bmatrix}$$

be a $n \times n$ matrix, where $A_1$ is a symmetric matrix, $a, b \in \mathbb{R}^{n-2}$ are vectors, $\alpha, \beta \in \mathbb{R}$ real numbers and $x$ is a variable. Let $A_2$ and $A_3$ be the colored submatrices of $A(x)$ and

$$x_\pm := b^T A_1^\dagger a \pm \sqrt{(A_2/A_1)(A_3/A_1)} \in \mathbb{R},$$

where $A_2/A_1 = \alpha - a^T A^\dagger a$ and $A_3/A_1 = \beta - b^T B^\dagger b$. Then:

1. $A(x_0)$ is PSD if and only if $A_2$, $A_3$ are PSD and $x_0 \in [x_-, x_+]$.
2. $$\operatorname{rank} A(x_0) = \max\left\{\operatorname{rank} A_2, \operatorname{rank} A_3\right\} + \begin{cases} 0, & \text{for } x_0 \in \{x_-, x_+\}, \\ 1, & \text{for } x_0 \in (x_-, x_+). \end{cases}$$

# Notation - Hankel matrix

Let $k \in \mathbb{N}$. For

$$\gamma = (\gamma_0, \ldots, \gamma_{2k}) \in \mathbb{R}^{2k+1}$$

we define the corresponding Hankel matrix as

$$A_\gamma := \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \iddots & \iddots & \gamma_{k+1} \\ \gamma_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \gamma_{2k-1} \\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k-1} & \gamma_{2k} \end{pmatrix}.$$

# THMP of degree $2k$ with a gap $\gamma_{2k-1}$

## Theorem

*Let $k > 1$ and*

$$\gamma(x) := (\overbrace{\underbrace{\gamma_0, \gamma_1, \ldots, \gamma_{2k-4}}_{\gamma^{(2)}}, \gamma_{2k-3}, \gamma_{2k-2}}^{\gamma^{(1)}}, x, \gamma_{2k}),$$

*be a sequence, where $x$ is a variable, with the moment matrix*

$$A_{\gamma(x)} = \left[\begin{array}{c|c} A_{\gamma^{(1)}} & \begin{array}{c} v \\ x \end{array} \\ \hline \begin{array}{cc} v^T & x \end{array} & \gamma_{2k} \end{array}\right] = \left[\begin{array}{cc|c} A_{\gamma^{(2)}} & u & v \\ u^T & \gamma_{2k-2} & x \\ \hline v^T & x & \gamma_{2k} \end{array}\right],$$

*where $v = (\gamma_k, \ldots, \gamma_{2k-2})$ and $u = (\gamma_{k-1}, \ldots, \gamma_{2k-3})$. TFAE:*

1. *There exists $x_0 \in \mathbb{R}$ and a $\mathbb{R}$–RM for $\gamma(x_0)$.*
2. $A_{\gamma^{(1)}}$ *and* $A := \begin{bmatrix} A_{\gamma^{(2)}} & v \\ v^T & \gamma_{2k} \end{bmatrix}$ *are PSD and one of the following holds:*
   a) $A_{\gamma^{(1)}}$ *is PD.*
   b) $\operatorname{rank} A_{\gamma^{(1)}} = \operatorname{rank} A$.

Let $k \geq 3$, $p(X,Y) = Y - X^3$ and $\beta := \beta^{(2k)}$ a $p(x,y)$–pure sequence. Let

$$\gamma(x) := (\overbrace{\underbrace{\gamma_0, \gamma_1, \ldots, \gamma_{6k-4}}_{\gamma^{(2)}}, \gamma_{6k-3}, \gamma_{6k-2}}^{\gamma^{(1)}}, x, \gamma_{6k}), \quad \text{where } \gamma_{i+3j} = \beta_{i,j}.$$

## Theorem (Fialkow, 11')

*The following statements are equivalent:*

(1) *$\beta$ has a $\mathcal{Z}(p)$–RM.*

(2) *$\beta$ has a (rank $M_k$)–atomic $\mathcal{Z}(p)$–RM.*

(3) *$M_k$ is PSD and*

$$\boxed{\beta_{1,2k-1} > u^T A_{\gamma^{(2)}}^{-1} u}, \quad \textit{where } u = (\gamma_{k-1}, \ldots, \gamma_{2k-3}).$$

(4) *$M_k$ admits a PSD, RG extension $M_{k+1}$.*

# The TMP on $y = x^3$ through the univariate reduction

Let $k \geq 3$, $p(X, Y) = Y - X^3$ and $\beta := \beta^{(2k)}$ a sequence. Let

$$\gamma(x) := (\overbrace{\gamma_0, \gamma_1, \ldots, \gamma_{6k-4}, \gamma_{6k-3}, \gamma_{6k-2}}^{\gamma^{(1)}}, x, \gamma_{6k}), \quad \text{where } \gamma_{i+3j} = \beta_{i,j}.$$

## Theorem

*TFAE:*

(1) *$\beta$ has a $\mathcal{Z}(p)$–RM.*

(2) *$\beta$ has a $(\text{rank } M_k)$– or $(\text{rank } M_k + 1)$–atomic $\mathcal{Z}(p)$–RM.*

(3) *$M_k$ is PSD, $p$–RG ($pq = \mathbf{0}$ if $pq \in \mathbb{R}[X, Y]_{2k}$) and:*

| a) $A_{\gamma^{(1)}}$ is PD. | or | b) $A_{\gamma^{(1)}}$ is PSD and $\text{rank } M_k = \text{rank } A_{\gamma^{(1)}}$. | holds. |
|---|---|---|---|

(4) *$M_k$ admits a PSD, RG extension $M_{k+1}$.*

*Moreover, if the $\mathcal{Z}(p)$–RM for $\beta$ exists:*

- *There is a $(\text{rank } M_k)$–atomic $\mathcal{Z}(p)$–RM unless $\text{rank } M_k = 3k - 1$ and $A_{\gamma^{(1)}}$ is PD.*
- *The $\mathcal{Z}(p)$–RM is unique if $\text{rank } M_k < 3k$. Otherwise two minimal $\mathcal{Z}(p)$–RM exist.*

# The TMP on $yx^2 = 1$ through the univariate reduction

Every atom must be of the form $\left(t, \frac{1}{t^2}\right)$ for some $t \in \mathbb{R}$. So $\beta_{i,j}$ corresponds to the moment of

$$z^{i-2j}.$$

As $i, j$ run over $0, 1, \ldots, 2k$ such that $i + j \leq 2k$, the difference $i - 2j$ runs over the set

$$\{-4k, -4k+2, \ldots, -1, 0, 1, \ldots, 2k\}.$$

The problem is equivalent to the

strong THMP of degree $(-4k, 2k)$ with a gap $\gamma_{-4k+1}$,

i.e., does there exist $x \in \mathbb{R}$ such that

$$(\gamma_{-4k}, x, \gamma_{-4k+2}, \ldots, \gamma_{-1}, \gamma_0, \gamma_1, \ldots, \gamma_{2k})$$

admits a measure on

$$\mathbb{R} \setminus \{0\}.$$

# Strong THMP of degree $(-2k_1, 2k_2)$ with a gap $\gamma_{-2k_1+1}$

## Theorem

*Let $k > 1$ and*

$$\gamma(x) := (\gamma_{-2k_1}, x, \overbrace{\gamma_{-2k_1+2}, \gamma_{-2k_1+3}, \underbrace{\gamma_{-2k_1+4}, \ldots, \gamma_{2k_2}}_{\gamma^{(2)}}}^{\gamma^{(1)}}),$$

*be a sequence, where $x$ is a variable, with the moment matrix*

$$A_{\gamma(x)} := \left[\begin{array}{c|cc} \gamma_{-2k_1} & x & u^T \\ \hline x & \multicolumn{2}{c}{\multirow{2}{*}{$A_{\gamma^{(1)}}$}} \\ u & & \end{array}\right] = \left[\begin{array}{c|cc} \gamma_{-2k_1} & x & u^T \\ \hline x & \gamma_{-2k_1+2} & w^T \\ u & w & A_{\gamma^{(2)}} \end{array}\right]$$

*where $u^T = (\gamma_{-2k_1+2}, \ldots, \gamma_{-k_1+k_2+1})$ and $w^T = (\gamma_{-2k_1+2}, \ldots, \gamma_{-k_1+k_2})$.*
*TFAE:*

1. *There exists $x_0 \in \mathbb{R}$ and a $(\mathbb{R} \setminus \{0\})$–RM for $\gamma(x_0)$.*
2. *$A_{\gamma^{(1)}}$ and $A := \begin{bmatrix} \gamma_{-2k_1} & u^T \\ u & A_{\gamma^{(2)}} \end{bmatrix}$ are PSD and one of the following holds:*
   a) *$A_{\gamma^{(1)}}$ and $A$ without the last row and column are PD.*
   b) rank $A_{\gamma^{(1)}}$ = rank($A_{\gamma^{(1)}}$ *without the last row and column*) = rank $A$.

# The TMP on $YX^2 = 1$

Let $k \geq 3$, $p(x,y) = yx^2 - 1$ and $\beta := \beta^{(2k)}$ a sequence. Let

$$\gamma(x) := (\gamma_{-4k}, x, \overbrace{\gamma_{-4k+2}, \gamma_{-4k+3}, \gamma_{-4k+4}, \ldots, \gamma_{2k}}^{\gamma^{(1)}}), \quad \text{where } \gamma_{i-2j} = \beta_{i,j}.$$

## Theorem

*TFAE:*

(1) *$\beta$ has a $\mathcal{Z}(p)$–representing measure.*

(2) *$\beta$ has a (rank $M_k$)– or (rank $M_k + 1$)–atomic $\mathcal{Z}(p)$–representing measure.*

(3) *$M_k$ is PSD and p–RG, $A_{\gamma^{(1)}}$ is PSD and one of the following holds:*

  a) *$A_{\gamma^{(1)}}$ is PD and $\operatorname{rank}\left(M_k \text{ without column/row } X^k\right) = 3k - 1$.*

  b) *$\operatorname{rank} A_{\gamma^{(1)}} = \operatorname{rank}\left(M_k \text{ without columns/rows } X^k, Y^k\right) = \operatorname{rank} M_k$.*

(4) *$M_k$ admits a PSD, RG extension $M_{k+2}$.*

*Moreover, if the $\mathcal{Z}(p)$–RM for $\beta$ exists:*

- *There is a (rank $M_k$)–atomic $\mathcal{Z}(p)$–RM unless $\operatorname{rank} M_k = 3k - 1$ and $A_{\gamma^{(1)}}$ is PD.*
- *The $\mathcal{Z}(p)$–RM is unique if $\operatorname{rank} M_k < 3k$. Otherwise two minimal $\mathcal{Z}(p)$–RM exist.*

# The TMP on $y^2 = x^3$

Every atom must be of the form $(t^2, t^3)$ for some $t \in \mathbb{R}$. So $\beta_{i,j}$ corresponds to the moment of

$$z^{2(i \bmod 3)+3(j+2\lfloor \frac{i}{3} \rfloor)}.$$

As $i, j$ run over $0, 1, \ldots, 2k$ such that $i + j \leq 2k$, the sum in $z^*$ runs over the set

$$\{0, 2, 3, \ldots, 6k - 1, 6k\}.$$

The problem is equivalent to the

THMP of degree $6k$ with a gap $\gamma_1$,

i.e., does there exist $x \in \mathbb{R}$ such that

$$(\gamma_0, x, \gamma_2, \ldots, \gamma_{6k-1}, \gamma_{6k})$$

admits a measure on

$$\mathbb{R}.$$

# THMP of degree $2k$ with a gap $\gamma_1$

## Theorem

*Let $k > 1$ and*

$$\gamma(x) := (\gamma_0, x, \overbrace{\gamma_2, \gamma_3, \underbrace{\gamma_4, \ldots, \gamma_{2k}}_{\gamma^{(2)}}}^{\gamma^{(1)}}),$$

*be a sequence, where $x$ is a variable, with the moment matrix*

$$A_{\gamma(x)} := \left[\begin{array}{c|cc} \gamma_0 & x & u^T \\ \hline x & & \\ u & & A_{\gamma^{(1)}} \end{array}\right] = \left[\begin{array}{c|cc} \gamma_0 & x & u^T \\ \hline x & \gamma_2 & w^T \\ u & w & A_{\gamma^{(2)}} \end{array}\right]$$

*where $u^T = (\gamma_2, \ldots, \gamma_k)$ and $w^T = (\gamma_3, \ldots, \gamma_{k+1})$. TFAE:*

1. *There exists $x_0 \in \mathbb{R}$ and a $\mathbb{R}$–RM for $\gamma(x_0)$.*
2. *$A_{\gamma^{(1)}}$ and $A := \begin{bmatrix} \gamma_0 & u^T \\ u & A_{\gamma^{(2)}} \end{bmatrix}$ are PSD and one of the following holds:*

   a) *$A_{\gamma^{(1)}}$ and $A$ without the last row and column are PD.*

   b) $\operatorname{rank} A_{\gamma^{(1)}} = \operatorname{rank}(A_{\gamma^{(1)}}$ *without the last row and column*).

# The TMP on $y^2 = x^3$

Let $k \geq 3$, $p(X,Y) = X^3 - Y^2$ and $\beta := \beta^{(2k)}$ a sequence. Let

$$\gamma(x) := (\gamma_0, x, \overbrace{\gamma_2, \gamma_3, \gamma_4, \ldots, \gamma_{6k}}^{\gamma^{(1)}}), \quad \text{where } \gamma_{i-2j} = \beta_{i,j}.$$

## Theorem

*TFAE:*

(1) *$\beta$ has a $\mathcal{Z}(p)$–representing measure.*

(2) *$\beta$ has a $(\operatorname{rank} M_k)$– or $(\operatorname{rank} M_k + 1)$–atomic $\mathcal{Z}(p)$–representing measure.*

(3) *$M_k$ is PSD and p–RG, $A_{\gamma^{(1)}}$ is PSD and one of the following holds:*

a) *$A_{\gamma^{(1)}}$ is PD and $\operatorname{rank}\left(M_k \text{ without column/row } X^k\right) = 3k-1$.*

b) *$\operatorname{rank} A_{\gamma^{(1)}} = \operatorname{rank}\left(M_k \text{ without columns/rows } X^k, Y^k\right)$.*

*Moreover, if the $\mathcal{Z}(p)$–RM for $\beta$ exists:*

- *There is a $(\operatorname{rank} M_k)$–atomic $\mathcal{Z}(p)$–RM unless $\operatorname{rank} M_k = 3k-1$ and $A_{\gamma^{(1)}}$ is PD.*
- *The $\mathcal{Z}(p)$–RM is unique if $\operatorname{rank} M_k < 3k$. Otherwise two minimal $\mathcal{Z}(p)$–RM exist.*

# The TMP on $y^2 = x^3$

Let $k \geq 3$, $p(x, y) = y^2 - x^3$ and $\beta := \beta^{(2k)}$.

## Proposition

The statement

$$\beta \text{ has a } \mathcal{Z}(p)\text{–RM.}$$

is stronger than the statement

$$M_k \text{ admits PSD extensions } M_m \text{ for every } m > k.$$

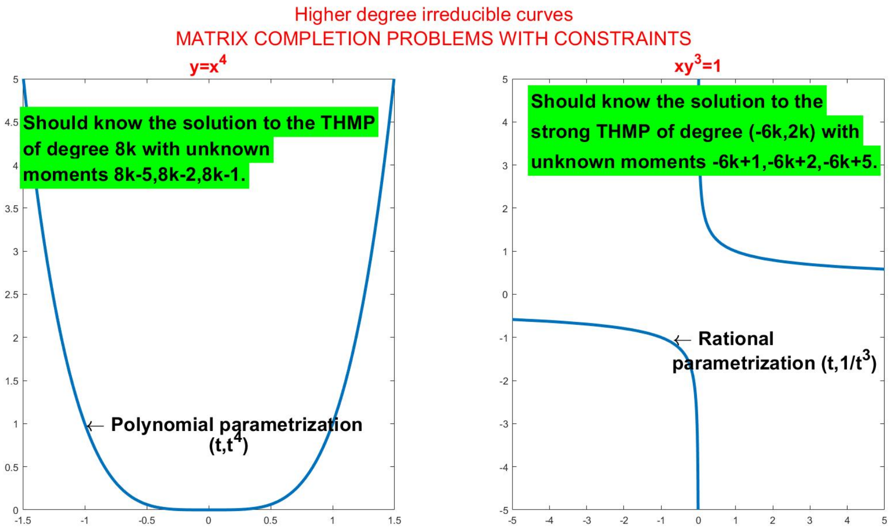
**Idea of the proof.**

- There exists a psd, $p$–RG matrix $M_3$ of rank $3k$ such that $A_{\gamma^{(1)}}$ is not PSD.
- So, $M_3$ does not admit a $\mathcal{Z}(p)$–RM, but one can easily construct PSD extensions $M_m$ for every $m > 3$ in the univariate setting.

## Corollary

*$p$ is not of type A in Stochel's sense.*

# The TMP on higher degree curves - a new approach



# The TMP on $y = x^4$

Every atom must be of the form $(t, t^4)$ for some $t \in \mathbb{R}$. So $\beta_{i,j}$ corresponds to the moment of

$$z^{i+4j}.$$

As $i, j$ run over $0, 1, \ldots, 2k$ such that $i + j \leq 2k$, the sum $i + 4j$ runs over the set

$$\{0, 1, \ldots, 8k-6, 8k-4, 8k-3, 8k\}.$$

The problem is equivalent to the

THMP of degree $8k$ with gaps $\gamma_{8k-5}, \gamma_{8k-2}, \gamma_{8k-1}$,

i.e., do there exist $x_1, x_2, x_3 \in \mathbb{R}$ such that

$$(\gamma_0, \gamma_1, \ldots, \gamma_{8k-6}, x_1, \gamma_{8k-4}, \gamma_{8k-3}, x_2, x_3, \gamma_{8k})$$

admits a measure on

$$\mathbb{R}.$$

# The THMP of degree $8k$ with gaps $\gamma_{8k-5}, \gamma_{8k-2}, \gamma_{8k-1}$

The corresponding Hankel matrix $A_{\gamma(x_1,x_2,x_3)}$ is

$$\begin{pmatrix}
\gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \cdots & & & \cdots & \gamma_k \\
\gamma_1 & \gamma_2 & \gamma_3 & \iddots & & & & & \vdots \\
\gamma_2 & \gamma_3 & \iddots & & & & & & \gamma_{8k-6} \\
\gamma_3 & \iddots & & & & & & \gamma_{8k-6} & x_1 \\
\vdots & & & & & & \gamma_{8k-6} & x_1 & \gamma_{8k-4} \\
 & & & & & \gamma_{8k-6} & x_1 & \gamma_{8k-4} & \gamma_{8k-3} \\
 & & & & \gamma_{8k-6} & x_1 & \gamma_{8k-4} & \gamma_{8k-3} & x_2 \\
\vdots & & & \gamma_{8k-6} & x_1 & \gamma_{8k-4} & \gamma_{8k-3} & x_2 & x_3 \\
\gamma_k & \cdots & \gamma_{8k-6} & x_1 & \gamma_{8k-4} & \gamma_{8k-3} & x_2 & x_3 & \gamma_{8k}
\end{pmatrix}.$$

This is the linear matrix inequality (LMI) feasibility problem with constraints, i.e., the constraint is that in the corank 1 case the last column must be dependent from the others.

# The THMP of degree $8k$ with gaps $\gamma_{8k-5}, \gamma_{8k-2}, \gamma_{8k-1}$

By a simple trick of adding the next row and column the constraint can be removed and this becomes only a LMI feasibility problem, i.e., do there exist $x_1, x_2, x_3$ and $x_4, x_5$ such that

$$\left(\begin{array}{ccccccccc|c}
\gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \cdots & & & \cdots & \gamma_k & \gamma_{k+1} \\
\gamma_1 & \gamma_2 & \gamma_3 & \iddots & & & & & & \vdots \\
\gamma_2 & \gamma_3 & \iddots & & & & & \gamma_{8k-6} & x_1 & \gamma_{8k-4} \\
\gamma_3 & \iddots & & & & & \gamma_{8k-6} & x_1 & \gamma_{8k-4} & \gamma_{8k-3} \\
\vdots & & & & & \gamma_{8k-6} & x_1 & \gamma_{8k-4} & \gamma_{8k-3} & x_2 \\
 & & & & \gamma_{8k-6} & x_1 & \gamma_{8k-4} & \gamma_{8k-3} & x_2 & x_3 \\
\vdots & & & \gamma_{8k-6} & x_1 & \gamma_{8k-4} & \gamma_{8k-3} & x_2 & x_3 & \gamma_{8k} \\
\gamma_k & & \gamma_{8k-6} & x_1 & \gamma_{8k-4} & \gamma_{8k-3} & x_2 & x_3 & \gamma_{8k} & x_4 \\
\hline
\gamma_{k+1} & \cdots & x_1 & \gamma_{8k-4} & \gamma_{8k-3} & x_2 & x_3 & \gamma_{8k} & x_4 & x_5
\end{array}\right)$$

is PSD?

# Algebraic certificate of infeasibility of the LMI

One abstract solution to the TMP on $Y = X^4$

(and all curves of the form $Y = q(X)$ or $YX^\ell = 1$, where $q \in \mathbb{R}[X]$, $\ell \in \mathbb{N}$),

is the following Nonlinear Farkas lemma.

## Theorem (Klep & Schweighofer, 12')

*Let*

$$A(x) := A_0 + A_1 x_1 + \ldots + A_n x_n,$$

*where $A_i$ are real symmetric matrices of size $\alpha$. TFAE:*

1. *$A(x)$ is infeasible.*
2. *$-1 \in M_A^{(2^\ell - 1)}$, where*

$$M_A^{(2^\ell - 1)} = \left\{ \sum_{i=1}^{\ell_1} p_i^2 + \sum_{j=1}^{\ell_2} v_j^T A(x) v_j : p_i \in \mathbb{R}[\underline{x}]_{2^\ell - 1}, v_j \in (\mathbb{R}[\underline{x}]_{2^\ell - 1})^\alpha \right\}$$

*is the $(2^\ell - 1)$–th quadratic module associated to $A(x)$ and $\ell = \min(\alpha, n)$.*

# The TMP on $y = q(x)$

Another abstract solution to the TMP on

all curves of the form $Y = q(X)$, where $q \in \mathbb{R}[X]$,

is the following:

**Theorem (Stochel 92' & Fialkow, 11')**

*TFAE:*

1. $\beta$ *has a* $\mathcal{Z}(p)$*–RM.*
2. $M_k$ *admits a PSD, RG extension* $M_{(2k+1)\deg q-1}$.
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*Remark.* The improvement using the univariate reduction technique in the size of extension is from quadratic in $k, \deg q$ to linear in $k$, $\deg q$.

A similar result holds for curves $yx^{\ell} = 1$, $\ell \in \mathbb{N}$.

Thank you for your attention!