The bivariate truncated moment problem on quadratic and some higher degree curves

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Classical truncated moment problem

- Let $\beta = \beta^{(2k)} = (\beta_i)_{i\in\mathbb{Z}_+^d, |i|\leq 2k}$ be a d-dimensional multisequence of real numbers of degree $2k$.

Example

For $d = 2$ and $k = 2$, β is a 15-parametric sequence

$$\beta = (\beta_{0,0}, \beta_{1,0}, \beta_{0,1}, \beta_{2,0}, \beta_{1,1}, \beta_{0,2}, \beta_{3,0}, \beta_{2,1}, \beta_{1,2}, \beta_{0,3}, \beta_{4,0}, \beta_{3,1}, \beta_{2,2}, \beta_{1,3}, \beta_{0,4}).$$

- The **truncated moment problem (TMP)**: characterize the existence of a positive Borel measure μ on $\mathbb{R}^d$ with support in the closed set K, such that

$$\beta_i = \int_K \underline{x}^i d\mu(\underline{x}) \quad \text{for} \quad i \in \mathbb{Z}_+^d, |i| \leq 2k,$$

where $\underline{x}^i := x_1^{i_1} \cdots x_d^{i_d}$.

Theorem (Richter, 1957; Bayer,Teichmann, 2006)

It suffices to study finitely atomic measures in the TMP.

Tracial truncated moment problem

- Let $\beta \equiv \beta^{(2k)} = (\beta_w)_{|w|\leq 2k}$ be a d-dimensional multisequence indexed by words w in noncommuting letters $X_1, X_2, \ldots, X_d$ of length at most $2k$ such that

$$\beta_{v_1 v_2} = \beta_{v_2 v_1} \quad \text{and} \quad \beta_w = \beta_{w^*},$$

  for every words v_1, v_2, w and w^* is the reverse of w.

- The **tracial truncated moment problem (TTMP)**: characterize the existence of a positive Borel measure μ on the set of tuples of real symmetric matrices $S_n(\mathbb{R})^d$ of some size n, such that

$$\beta_w = \int_{S_n(\mathbb{R})^d} \mathrm{Tr}(w(\underline{A}))d\mu(\underline{A}) \quad \text{for every word } w, |w| \leq 2k,$$

  where Tr denotes the normalized trace of a matrix.

Theorem (Burgdorf, Cafuta, Klep, Povh, 2013)

It suffices to study finitely atomic measures in the TTMP.

Tracial truncated moment problem

Example

For $d = 2$ and $k = 2$, β is a 16-parametric sequence

$$\beta = \big(\beta_1, \beta_X, \beta_Y, \beta_{X^2}, \beta_{XY} = \beta_{YX}, \beta_{Y^2}, \beta_{X^3}, \beta_{X^2Y} = \beta_{XYX} = \beta_{YX^2},$$
$$\beta_{XY^2} = \beta_{YXY} = \beta_{Y^2X}, \beta_{Y^3}, \beta_{X^4}, \beta_{X^3Y} = \beta_{X^2YX} = \beta_{XYX^2} = \beta_{YX^3},$$
$$\beta_{X^2Y^2} = \beta_{XY^2X} = \beta_{Y^2X^2} = \beta_{YX^2Y}, \beta_{XYXY} = \beta_{YXYX},$$
$$\beta_{XY^3} = \beta_{YXY^2} = \beta_{Y^2XY} = \beta_{Y^3X}, \beta_{Y^4}\big),$$

Remark

If $\beta_{X^2Y^2} = \beta_{XYXY}$, then every atom (X, Y) in the measure must satisfy $XY = YX$, and the problem becomes a classical moment problem.

Classical truncated moment matrix

The moment matrix (mm) $M(k)$ associated with a commutative sequence β with the rows and columns indexed by monomials X^i, $|i| \leq k$, in degree-lexicographic order, is defined by

$$M(k) = (\beta_{i+j})_{i,j\in\mathbb{Z}_+^d, |i|,|j|\leq k}.$$

Example

$$d = 1, k = 4: \ M(4) = \begin{array}{c} \\ 1 \\ X \\ X^2 \\ X^3 \\ X^4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & X & X^2 & X^3 & X^4 \end{array} \\ \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 \\ \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 \end{pmatrix} \end{array}.$$

$$d = k = 2: \ M(2) = \begin{array}{c} \\ 1 \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & X & Y & X^2 & XY & Y^2 \end{array} \\ \begin{pmatrix} \beta_{0,0} & \beta_{1,0} & \beta_{0,1} & \beta_{2,0} & \beta_{1,1} & \beta_{0,2} \\ \beta_{1,0} & \beta_{2,0} & \beta_{1,1} & \beta_{3,0} & \beta_{2,1} & \beta_{1,2} \\ \beta_{0,1} & \beta_{1,1} & \beta_{0,2} & \beta_{2,1} & \beta_{1,2} & \beta_{0,3} \\ \beta_{2,0} & \beta_{3,0} & \beta_{2,1} & \beta_{4,0} & \beta_{3,1} & \beta_{2,2} \\ \beta_{1,1} & \beta_{2,1} & \beta_{1,2} & \beta_{3,1} & \beta_{2,2} & \beta_{1,3} \\ \beta_{0,2} & \beta_{1,2} & \beta_{0,3} & \beta_{2,2} & \beta_{1,3} & \beta_{0,4} \end{pmatrix} \end{array}.$$

Tracial truncated moment matrix

The tracial moment matrix $M_{\mathrm{tr}}(k)$ associated with a tracial sequence β with the rows and columns indexed by words w in nc letters $X_1, \ldots, X_d$, $|w| \leq k$, in degree-lexicographic order, is defined by

$$M_{\mathrm{tr}}(k) = (\beta_{w_1^* w_2})_{w_1, w_2}.$$

Example

For $d = k = 2$ we have:

$$M_{\mathrm{tr}}(2) = \begin{array}{c} \\ 1 \\ X \\ Y \\ X^2 \\ XY \\ YX \\ Y^2 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & X & Y & X^2 & XY & YX & Y^2 \end{array} \\ \begin{pmatrix} \beta_1 & \beta_X & \beta_Y & \beta_{X^2} & \beta_{XY} & \beta_{XY} & \beta_{Y^2} \\ \beta_X & \beta_{X^2} & \beta_{XY} & \beta_{X^3} & \beta_{X^2Y} & \beta_{X^2Y} & \beta_{XY^2} \\ \beta_Y & \beta_{XY} & \beta_{Y^2} & \beta_{X^2Y} & \beta_{XY^2} & \beta_{XY^2} & \beta_{Y^3} \\ \beta_{X^2} & \beta_{X^3} & \beta_{X^2Y} & \beta_{X^4} & \beta_{X^3Y} & \beta_{X^3Y} & \beta_{X^2Y^2} \\ \beta_{XY} & \beta_{X^2Y} & \beta_{XY^2} & \beta_{X^3Y} & \color{red}{\beta_{X^2Y^2}} & \color{red}{\beta_{XYXY}} & \beta_{XY^3} \\ \beta_{XY} & \beta_{X^2Y} & \beta_{XY^2} & \beta_{X^3Y} & \color{red}{\beta_{XYXY}} & \color{red}{\beta_{X^2Y^2}} & \beta_{XY^3} \\ \beta_{Y^2} & \beta_{XY^2} & \beta_{Y^3} & \beta_{X^2Y^2} & \beta_{XY^3} & \beta_{XY^3} & \beta_{Y^4} \end{pmatrix} \end{array}.$$

Properties of the classical moment matrix

- To every polynomial $p := \sum_{i \in \mathbb{Z}_+^d, |i| \leq k} a_i \underline{x}^i \in \mathbb{R}[\underline{x}]_k$, we associate the vector
$$p(\underline{X}) = \sum_{i \in \mathbb{Z}_+^d, |i| \leq k} a_i \underline{X}^i$$
from the column space $\mathcal{C}(M(k))$ of the matrix $M(k)$.
- $M(k)$ is **recursively generated (rg)** if:
$$p, q, pq \in \mathbb{R}[\underline{x}]_k \quad \text{and} \quad p(\underline{X}) = \mathbf{0}, \quad \text{then} \quad (pq)(\underline{X}) = \mathbf{0}.$$
- $M(k)$ satisfies the **variety condition** if
$$\text{rank}\, M(k) \leq \text{card}\left(\bigcap_{\substack{g \in \mathbb{R}[\underline{x}]_{\leq k}, \\ g(\underline{X}) = \mathbf{0} \text{ in } M(k)}} \left\{ \underline{x} \in \mathbb{R}^d : g(\underline{x}) = 0 \right\} \right).$$

Proposition

Assume that β has a representing measure μ. Then:

- $M(k)$ is psd, rg and satisfies the variety condition.
- The support $\text{supp}\, \mu$ is a subset of $\mathcal{Z}_p := \{\underline{x} \in \mathbb{R}^d : p(\underline{x}) = 0\}$ if and only if $p(\underline{X}) = \mathbf{0}$.

Properties of the tracial moment matrix

- To every nc polynomial $p := \sum_{|w| \leq k} a_w w$, we associate the vector

$$p(\underline{X}) = \sum_{|w| \leq k} a_i w(\underline{X})$$

from the column space $\mathcal{C}(M_{\mathrm{tr}}(k))$ of the matrix $M_{\mathrm{tr}}(k)$.

- The matrix $M_{\mathrm{tr}}(k)$ is **recursively generated (rg)** if:

$$p, q, pq \in \mathbb{R}\langle \underline{X} \rangle_k \quad \text{and} \quad p(\underline{X}) = \mathbf{0}, \quad \text{then} \quad (pq)(\underline{X}) = \mathbf{0}.$$

Proposition

Assume that β has a representing measure μ. Then:

- $M_{\mathrm{tr}}(k)$ is psd and rg.
- The support $\mathrm{supp}\, \mu$ is a subset of $\mathcal{Z}_p^{\mathrm{nc}} := \bigcup_{n=1}^{\infty} \left\{ \underline{X} \in S_n(\mathbb{R})^d : p(\underline{X}) = 0 \right\}$ if and only if $p(\underline{X}) = \mathbf{0}$.

Truncated Hamburger moment problem (THMP)

Theorem (Curto & Fialkow, 1991)

For $k \in \mathbb{N}$ and $\beta = (\beta_0, \ldots, \beta_{2k})$ with $\beta_0 > 0$, the following statements are equivalent:

1. *There exists a rm for β supported on $K = \mathbb{R}$.*
2. *There exists a $(\text{rank}\, M(k))$-atomic rm for β supported on $K = \mathbb{R}$.*
3. *One of the following holds:*
   - $M(k) \succ 0$.
   - $M(k) \succeq 0$ *and* $\text{rank}\, M(k) = \text{rank}\, M(k-1)$.

Remark

The tracial THMP in one variable coincides with the classical THMP.

Bivariate TMP for quadratic varieties

Theorem (Curto & Fialkow, 1996-2015)

Let

$$\beta = \beta^{(2k)} = (\beta_{i,j})_{i,j\in\mathbb{Z}_+, i+j\leq 2k}$$

be a bisequence of real numbers of degree $2k$ such that the moment matrix satisfies

$$p(X, Y) = \mathbf{0},$$

where p is a quadratic polynomial.

After applying an affine linear transformation p can be assumed to be one of the polynomials xy, $xy - 1$, $y^2 - y$, $x^2 + y^2 - 1$, $y - x^2$.

Then:

1. *There exists a rm for β.*
2. *$M(k)$ is psd, rg and satisfies the variety condition.*

Flat extension theorem (FET)

The proof of the previous theorem is based on the following theorem.

Theorem (Curto, Fialkow, 1998)

Let $M(k)$ be a moment matrix, which has a psd extensions $M(k+d)$ and $M(k+d+1)$ for some $d \in \mathbb{N}$ such that

$$\operatorname{rank} M(k+d) = \operatorname{rank} M(k+d+1).$$

Then β has a $(\operatorname{rank} M(k+d))$-atomic rm.

The tracial version of this theorem is the following.

Theorem (Burgdorf, Klep, 2012)

Let $M_{\mathrm{tr}}(k)$ be a tracial moment matrix, which has a psd extensions $M_{\mathrm{tr}}(k+d)$ and $M_{\mathrm{tr}}(k+d+1)$ for some $d \in \mathbb{N}$ such that

$$\operatorname{rank} M_{\mathrm{tr}}(k+d) = \operatorname{rank} M_{\mathrm{tr}}(k+d+1).$$

Then β has a rm with atoms of size at most $\operatorname{rank} M_{\mathrm{tr}}(k+d)$.

Bivariate TTMP for quadratic varieties

- **Possible column relations:**

  after applying an appropriate affine linear transformation.

$$XY + YX = \mathbf{0} \quad \text{or} \quad X^2 + Y^2 = 1 \quad \text{or} \quad Y^2 - X^2 = 1 \quad \text{or} \quad Y^2 = 1.$$

- **Analysis of flat extensions:**
  - Flat extension $M_{\mathrm{tr}}(k+1)$ of a psd, rg $M_{\mathrm{tr}}(k)$ mostly does not exist.
  - Analyzing further extensions $M_{\mathrm{tr}}(k+2), M_{\mathrm{tr}}(k+3), \ldots$ is too demanding due to too many parameters.

- **Another approach:**
  - First bound the size and the form of possible nc atoms.
  - Decompose

    $$M_{\mathrm{tr}}(2) = M_{\mathrm{cm}}(2) + M_{\mathrm{nc}}(2),$$

    where $M_{\mathrm{cm}}(2)$ comes from *some* size 1 atoms and $M_{\mathrm{nc}}(2)$ comes from all irreducible atoms of size more than 1 and *some* size 1 atoms, for which you know admit a measure.

Tracial TMP for $M_{\mathrm{tr}}(2)$ with relation $X^2 + Y^2 = 1$

1. The bound on the size of the atoms is 2. Moreover, irreducible size 2 atoms are of the form

$$X = \begin{pmatrix} \gamma & \alpha \\ \alpha & -\gamma \end{pmatrix}, \quad Y = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix} \qquad \text{where } \alpha, \gamma, \mu \in \mathbb{R}.$$

Here we used that X^2 and Y commute and use that $X^2 + Y^2 = I$.

2. It follows that

$$M_{\mathrm{cm}}(2) = \begin{pmatrix} ? & \beta_X & \beta_Y & ? & ? & ? \\ \beta_X & ? & ? & \beta_{X^3} & \beta_{X^2Y} & \beta_{X^2Y} \\ \beta_Y & ? & ? & \beta_{X^2Y} & \beta_X - \beta_{X^3} & \beta_X - \beta_{X^3} \\ ? & \beta_{X^3} & \beta_{X^2Y} & ? & ? & ? \\ ? & \beta_{X^2Y} & \beta_X - \beta_{X^3} & ? & ? & ? \\ ? & \beta_{X^2Y} & \beta_X - \beta_{X^3} & ? & ? & ? \end{pmatrix},$$

$$M_{\mathrm{nc}}(2) = \begin{pmatrix} ? & 0 & 0 & ? & ? & ? \\ 0 & ? & ? & 0 & 0 & 0 \\ 0 & ? & ? & 0 & 0 & 0 \\ ? & 0 & 0 & ? & ? & ? \\ ? & 0 & 0 & ? & ? & ? \\ ? & 0 & 0 & ? & ? & ? \end{pmatrix}.$$

Tracial TMP for $M_{\mathrm{tr}}(2)$ with relation $X^2 + Y^2 = 1$

$$L(a,b,c,d,e) := \begin{pmatrix} a & \beta_X & \beta_Y & b & c & c \\ \beta_X & b & c & \beta_{X^3} & \beta_{X^2Y} & \beta_{X^2Y} \\ \beta_Y & c & a-b & \beta_{X^2Y} & \beta_X - \beta_{X^3} & \beta_X - \beta_{X^3} \\ b & \beta_{X^3} & \beta_{X^2Y} & d & e & e \\ c & \beta_{X^2Y} & \beta_X - \beta_{X^3} & e & b-d & b-d \\ c & \beta_{X^2Y} & \beta_X - \beta_{X^3} & e & b-d & b-d \end{pmatrix}.$$

Theorem (Bhardway, Z., 2018)

β admits a measure if and only if there exist $a, b, c, d, e \in \mathbb{R}$ such that

- $L(a,b,c,d,e) \succeq 0, \quad M_{\mathrm{tr}}(2) - L(a,b,c,d,e) \succeq 0,$
- $(M_{\mathrm{tr}}(2) - L(a,b,c,d,e))_{\{1,X,Y,XY\}} \succ 0,$
- *$L(a,b,c,d,e)$ is rg and satisfies the variety condition.*

Remark

Using this theorem examples where $M_{\mathrm{tr}}(2)$ being psd and rg does not imply the existence of a measure can be obtained.

Tracial TMP for $M_{\mathrm{tr}}(k)$ with two quadratic relations

- **Possible column relations:**

  after applying an appropriate affine linear transformation.

  - $XY + YX = \mathbf{0}$
  - $X^2 + Y^2 = 1$ or $Y^2 - X^2 = 1$ or $Y^2 = 1$ or $Y^2 = X^2$.

- **Analysis of flat extensions:** still too demanding

- **Another approach:**
  - The bound on the size of the atoms is 2 and irreducible size 2 atoms are of the form

$$X = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}, \qquad \text{where } \alpha, \mu \in \mathbb{R}.$$

  - It suffices to study the restriction due to column relations

$$M(k)|_{\{\vec{X}, Y\vec{X}'\}} = \begin{array}{c} \\ \vec{X} \\ Y\vec{X}' \end{array} \begin{array}{c} \begin{array}{cc} \vec{X} & Y\vec{X}' \end{array} \\ \begin{pmatrix} A & B \\ B & C \end{pmatrix} \end{array},$$

where $\vec{X} := (1, X, \ldots, X^k)$, $Y\vec{X}' := (Y, YX, \ldots, YX^{k-1})$.

Tracial TMP for $M_{\mathrm{tr}}(k)$ with two quadratic relations

Since there are only 4 possible size 1 atoms $((\pm 1,0),(0,\pm 1))$, the *best* candidate for $M_{\mathrm{cm}}(k)$ is

$$M_{\mathrm{cm}}(k) = |\beta_X|\cdot M(k)^{(\mathrm{sign}(\beta_X)1,0)} + |\beta_Y|\cdot M(k)^{(0,\mathrm{sign}(\beta_Y)1)},$$

where $M(k)^{(x,y)}$ stands for the mm generated by $(x,y)\in\mathbb{R}^2$, and

$$M_{\mathrm{nc}}(k)|_{\{\vec{X},Y\vec{X}'\}} = \begin{array}{c} \\ \vec{X} \\ Y\vec{X}' \end{array}\begin{array}{c} \begin{array}{cc} \vec{X} & Y\vec{X}' \end{array} \\ \begin{pmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & C_1 \end{pmatrix} \end{array}.$$

Solving the TMP $M_{\mathrm{nc}}(k)|_{\{\vec{X},Y\vec{X}'\}}$ is in fact the classical TMP on $\mathbb{R}$ or $[-1,1]$. If the atoms $x_1,\ldots,x_m$ represent A_1, then $M_{\mathrm{nc}}(k)|_{\{\vec{X},Y\vec{X}'\}}$ is represented by:

- if $X^2+Y^2=1$: $\left(\begin{pmatrix} 0 & x_i \\ x_i & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{1-x_i^2} & 0 \\ 0 & -\sqrt{1-x_i^2} \end{pmatrix}\right)$,
- Similarly for the other three cases.

Theorem (Bhardwaj, Z., 2021)

$M_{\mathrm{tr}}(k)$ admits a nc measure $\Leftrightarrow$ $M_{\mathrm{nc}}(k)$ is psd and rg.

Application of the techniques to the classical TMP

Question

The bivariate tracial TMP with two quadratic column relations can be reduced to the use of the univariate classical TMP.

1. Is the same true for the bivariate classical TMP with one quadratic column relation?
   Given by $p(x, y) = 0$ where $p(x, y)$ is one of xy, $xy - 1$, $y^2 - y$, $x^2 + y^2 - 1$, $y - x^2$.
2. If the answer to (1) is yes, can this technique by applied to cubic/higher degree column relations?

The answer to both questions above is yes.

1. - $p(x, y) \in \{xy, y^2 - y, y - x^2\} \ldots$ reduction to the TMP for $\mathbb{R}$.
   - $p(x, y) = xy - 1 \ldots$ reduction to the TMP for $\mathbb{R} \setminus \{0\}$, where negative moment are also known.
   - $p(x, y) = x^2 + y^2 - 1 \ldots$ reduction to the trigonometric TMP.

Application of the techniques to the classical TMP

2.

$p(x,y)$	reduces to the TMP of degree	with gaps at degrees
$y-x^3$	$6k$ for $K=\mathbb{R}$	$6k-1$
y^2-x^3	$6k$ for $K=\mathbb{R}$	1
x^2y-1	$(-4k,2k)$ for $K=\mathbb{R}\setminus\{0\}$	$-4k+1$
$y-x^4$	$8k$ for $K=\mathbb{R}$	$8k-5, 8k-2, 8k-1$
y^3-x^4	$8k$ for $K=\mathbb{R}$	$1,2,5$
x^3y-1	$(-6k,2k)$ for $K=\mathbb{R}\setminus\{0\}$	$-6k+1, -6k+2, -6k+5$

All problems above are psd matrix completion problems with one additional constraint in case the completion is only singular:

- for $K=\mathbb{R}$: the last column is in the span of the others.
- for $K=\mathbb{R}\setminus\{0\}$: the last and first column must be in the span of the others.

Also $p(x,y)=y(y-\alpha_1)(y-\alpha_2)$ reduces to the TMP for $\mathbb{R}$ by decomposing

$$M(k)=M_1(k)+M_2(k)+M_3(k),$$

where each $M_i(k)$ corresponds to one line.

Thank you for your attention!