The singular bivariate quartic tracial moment problem

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Notation

- $\langle X, Y \rangle$ ... the **free monoid** generated by the noncommuting letters X, Y, i.e., **words** in X, Y.
- $\mathbb{R}\langle X, Y \rangle$ ... the free algebra of polynomials in X, Y (**noncommutative (nc) polynomials**), endowed with the involution $p \mapsto p^*$ fixing $\mathbb{R} \cup \{X, Y\}$ and reversing the order of letters in each word.

Example

$(XY^2 - YX)^* = Y^2X - XY.$

- The **degree** $|p|$ of $p \in \mathbb{R}\langle X, Y \rangle$ is the length of the longest word in p. We write $\mathbb{R}\langle X, Y \rangle_{\leq n}$ for the set of all polynomials of degree at most n.
- A word v is **cyclically equivalent** to w ($v \overset{\text{cyc}}{\sim} w$) iff v is a cyclic permutation of w, i.e., there exist words u_1, u_2 such that $v = u_1u_2$, $w = u_2u_1$.

Bivariate truncated tracial sequence

Bivariate truncated tracial sequence (BTTS) of order n is a sequence of real numbers,

$$\beta \equiv \beta^{(2n)} = (\beta_w)_{|w| \le 2n},$$

indexed by words w in X, Y of length at most $2n$ such that

1. $\beta_v = \beta_w$ whenever $v \overset{\text{cyc}}{\sim} w$,
2. $\beta_w = \beta_{w^*}$ for all $|w| \le 2n$,

Example

For $t \in \mathbb{N}$ and $(A, B) \in (\mathbb{SR}^{t \times t})^2$ (where $\mathbb{SR}^{t \times t}$ denotes symmetric real $t \times t$ matrices), the sequence

$$\beta_w = \mathrm{tr}(w(A, B)) \quad \text{where} \quad |w| \le 2n$$

is a BTTS of order n.

Bivariate truncated tracial moment problem

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We call β a **bivariate truncated tracial moment sequence (BTTMS) of order** n if there exist $N \in \mathbb{N}$, $t_i \in \mathbb{N}$, $\lambda_i \in \mathbb{R}_{>0}$ with $\sum_{i=1}^{N} \lambda_i = 1$ and pairs of $t_i \times t_i$ real symmetric matrices (A_i, B_i), such that

$$\beta_w = \sum_{i=1}^{N} \lambda_i \cdot \frac{1}{t_i} \mathrm{tr}(w(A_i, B_i)), \quad \text{for all } |w| \leq 2n.$$

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Remark

Restricting t_i's to 1 we get the classical truncated moment problem studied extensively by Curto and Fialkow.

Bivariate truncated tracial moment problem

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- The measure is of **type** $(m_1, m_2, \ldots, m_r)$ if it consists of exactly $m_i \in \mathbb{N} \cup \{0\}$ atoms of size i and $m_r \neq 0$.
- A measure for β of type $(m_1^{(1)}, m_2^{(1)}, \ldots, m_{r_1}^{(1)})$ is **minimal**, if there does not exist another measure for β of type $(m_1^{(2)}$, $m_2^{(2)}$,..., $m_{r_2}^{(2)})$ such that

$$(\underbrace{0, \ldots, 0}_{r_1 - r_2}, m_{r_2}^{(2)}, m_{r_2-1}^{(2)}, \ldots, m_1^{(2)}) \prec_{\text{lex}} (m_{r_1}^{(1)}, m_{r_1-1}^{(1)}, \ldots, m_1^{(1)}).$$

Remark

1. Replacing an atom $(A, B) \in (\mathbb{SR}^{t \times t})^2$ with any atom
$$(UAU^t, UBU^t) \in (\mathbb{SR}^{t \times t})^2$$
where $U \in \mathbb{R}^{t \times t}$ is an orthogonal matrix, generates the same BTTS.

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2. By the tracial version of Bayer-Teichmann theorem, studying finite atomic measures is equivalent to studying probability measures on $(\mathbb{SR}^{t \times t})^2$ such that

$$\beta_w = \int_{(\mathbb{SR}^{t \times t})^2} \mathrm{tr}(w(A, B))\, \mathrm{d}\mu(A, B).$$

Bivariate quartic tracial moment problem (BQTMP)

For $n = 2$ the sequence $\beta^{(4)}$ has 16 parameters:

3 of degree 1: $\beta_1, \beta_X, \beta_Y$

3 of degree 2: $\beta_{X^2}, \beta_{XY} = \beta_{YX}, \beta_{Y^2}$

4 of degree 3: $\beta_{X^3}, \beta_{X^2Y} = \beta_{XYX} = \beta_{YX^2}, \beta_{XY^2} = \beta_{YXY} = \beta_{Y^2X}, \beta_{Y^3},$

6 of degree 4: $\beta_{X^4}, \beta_{X^3Y} = \beta_{X^2YX} = \beta_{XYX^2} = \beta_{YX^3},$

$\beta_{X^2Y^2} = \beta_{XY^2X} = \beta_{Y^2X^2} = \beta_{YX^2Y},$

$\beta_{XYXY} = \beta_{YXYX},$

$\beta_{XY^3} = \beta_{YXY^2} = \beta_{Y^2XY} = \beta_{Y^3X}, \beta_{Y^4}.$

Truncated moment matrix $\mathcal{M}_n$

Index rows and columns of $\mathcal{M}_n$ by words in $\mathbb{R}\langle X, Y\rangle_{\leq n}$ in the degree-lexicographic order.

The entry in a row w_1 and a column w_2 of $\mathcal{M}_n$ is $\beta_{w_1^* w_2}$:

$$\mathcal{M}_n = \begin{array}{c} \\ \mathbb{1} \\ \mathbb{X} \\ \vdots \\ w_1 \\ \vdots \\ \mathbb{Y}^n \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbb{1} & \mathbb{X} & \cdots & w_2 & \cdots & \mathbb{Y}^n \end{array} \\ \begin{pmatrix} \beta_1 & \beta_X & \cdots & \beta_{w_2} & \cdots & \beta_{Y^n} \\ \beta_X & \beta_{X^2} & \cdots & \beta_{Xw_2} & \cdots & \beta_{XY^n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{w_1} & \beta_{w_1^* X} & \cdots & \beta_{w_1^* w_2} & \cdots & \beta_{w_1^* Y^n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{Y^n} & \beta_{XY^n} & \cdots & \beta_{Y^n w_2} & \cdots & \beta_{Y^{2n}} \end{pmatrix} \end{array}.$$

$n = 2$: 7×7 moment matrix $\mathcal{M}_2$

$$\begin{array}{c|ccccccc}
 & \mathbb{1} & \mathbb{X} & \mathbb{Y} & \mathbb{X}^2 & \mathbb{X}\mathbb{Y} & \mathbb{Y}\mathbb{X} & \mathbb{Y}^2 \\
\hline
\mathbb{1} & \beta_1 & \beta_X & \beta_Y & \beta_{X^2} & \beta_{XY} & \beta_{XY} & \beta_{Y^2} \\
\mathbb{X} & \beta_X & \beta_{X^2} & \beta_{XY} & \beta_{X^3} & \beta_{X^2Y} & \beta_{X^2Y} & \beta_{XY^2} \\
\mathbb{Y} & \beta_Y & \beta_{XY} & \beta_{Y^2} & \beta_{X^2Y} & \beta_{XY^2} & \beta_{XY^2} & \beta_{Y^3} \\
\mathbb{X}^2 & \beta_{X^2} & \beta_{X^3} & \beta_{X^2Y} & \beta_{X^4} & \beta_{X^3Y} & \beta_{X^3Y} & \beta_{X^2Y^2} \\
\mathbb{X}\mathbb{Y} & \beta_{XY} & \beta_{X^2Y} & \beta_{XY^2} & \beta_{X^3Y} & \color{red}{\beta_{X^2Y^2}} & \color{red}{\beta_{XYXY}} & \beta_{XY^3} \\
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\mathbb{Y}^2 & \beta_{Y^2} & \beta_{XY^2} & \beta_{Y^3} & \beta_{X^2Y^2} & \beta_{XY^3} & \beta_{XY^3} & \beta_{Y^4}
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\end{array}
$$

If $\beta_{X^2Y^2} = \beta_{XYXY}$, then the BQTMP reduces to the classical bivariate quartic moment problem.

Bivariate quartic moment problem - results

- Curto, Fialkow (1996-2014): a complete solution of the classical *singular* case, i.e., $\mathcal{M}_2$ is *non-invertible*.

  Main tool: a rank-preserving extension of $\mathcal{M}_2$ to $\mathcal{M}_3$.

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  Proof: not constructive (duality with trace polynomials) but 15 atoms of size 2 are sufficient.

Our motivation

- **Motivation:** *Solve a singular tracial moment problem for* $\mathcal{M}_2$.

  ?? **Main tool:** *a rank-preserving extension of* $\mathcal{M}_2$ *to* $\mathcal{M}_3$??

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Our results

We assume that $\mathcal{M}_n$, $n \geq 2$, is such that $\mathcal{M}_2$ is *non-invertible* and $\beta_{X^2Y^2} \neq \beta_{XYXY}$.

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3. For $\text{rank}(\mathcal{M}_2) \in \{4, 5\}$, we can characterize exactly when a measure exists, what is the type of a minimal measure and describe its uniqueness.

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3. For $\text{rank}(\mathcal{M}_2) \in \{4, 5\}$, we can characterize exactly when a measure exists, what is the type of a minimal measure and describe its uniqueness.
4. If $\text{rank}(\mathcal{M}_2) = 6$, then the existence of a measure is almost always equivalent to the feasibilty of certain linear matrix inequalities and atoms of size 2 suffice.

Main techical tools

Assume that $\beta^{(2n)}$ admits a measure consisting of atoms

$$(X_1, Y_1) \in (\mathbb{SR}^{t_1 \times t_1})^2, \ldots, (X_N, Y_N) \in (\mathbb{SR}^{t_N \times t_N})^2.$$

Then:

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2. **Support of a measure:** For $p \in \mathbb{R}\langle X, Y\rangle_{\leq n}$

$$\underbrace{p(X_1, Y_1) = \ldots = p(X_N, Y_N) = 0}_{\text{usual evaluations}} \quad \text{iff} \quad \underbrace{p(\mathbb{X}, \mathbb{Y}) = \mathbf{0} \text{ in } \mathcal{M}_n.}_{\text{replacing words by columns of } \mathcal{M}_n}$$

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3. **Recursive generation**: For $p, q \in \mathbb{R}\langle X, Y\rangle_{\leq n}$ such that $pq \in \mathbb{R}\langle X, Y\rangle_{\leq n}$

$$p(\mathbb{X}, \mathbb{Y}) = \mathbf{0} \text{ in } \mathcal{M}_n \quad \Rightarrow \quad pq(\mathbb{X}, \mathbb{Y}) = \mathbf{0} \text{ in } \mathcal{M}_n.$$

Main techical tools

4. **Affine linear transformations**: For $a, b, c, d, e, f \in \mathbb{R}$ with $bf - ce \neq 0$ we define

$$\phi(x, y) = (\phi_1(x, y), \phi_2(x, y)) := (a + bx + cy, d + ex + fy).$$

Let $\widetilde{\beta}^{(2n)}$ be the sequence obtained by the rule

$$\widetilde{\beta}_w = \sum_{w'} a_{w'} \beta'_{w'},$$

where $w(\phi_1(X, Y), \phi_2(X, Y)) = \sum_{w'} a_{w'} w'$.

Solving MP for $\mathcal{M}_n$ is equivalent to solving MP for $\widetilde{\mathcal{M}}_n$.

Main techical tools

Example

For

$$\phi(x,y) = (\phi_1(x,y), \phi_2(x,y)) := (1 + x + y, x - y)$$

we get

$$\widetilde{\beta}_{XY} = \beta_X - \beta_Y + \beta_{X^2} - \beta_X\beta_Y + \beta_Y\beta_X - \beta_{Y^2}$$

since

$$XY \mapsto (1 + X + Y)(X - Y) = X - Y + X^2 - XY + YX - Y^2.$$

Curto & Fialkow result explicitly

Theorem (Curto, Fialkow)

Suppose $\beta \equiv \beta^{(4)}$ is a commutative sequence with the associated moment matrix $\mathcal{M}_2$. Let

$$\mathcal{V} := \bigcap_{\substack{g \in \mathbb{R}[x,y]_{\leq 2} \\ g(\mathbb{X},\mathbb{Y})=\mathbf{0}}} \mathcal{V}(g)$$

be the variety associated to $\mathcal{M}_2$ and $p \in \mathbb{R}[x,y]$ a polynomial of degree 2. TFAE:

1. *β admits a measure supported in $\mathcal{V}(p)$.*
2. *$\mathcal{M}(2)$ is positive semidefinite, recursively generated, satisfies $\operatorname{rank}(M(2)) \leq \operatorname{card} \mathcal{V}$ and has a column dependency relation $p(\mathbb{X}, \mathbb{Y}) = 0$.*

Rank-preserving extension of $\mathcal{M}_2$ to $\mathcal{M}_3$

$$\mathcal{M}_3 = \begin{pmatrix} \mathcal{M}_2 & B_3 \\ B_3^t & C_3 \end{pmatrix} \text{ where } B_3 \in \mathbb{R}^{7\times 8} \text{ and } C_3 \in \mathbb{R}^{8\times 8}:$$

$$\begin{array}{c} \\ \mathbb{1} \\ \mathbb{X} \\ \mathbb{Y} \\ \mathbb{X}^2 \\ \mathbb{XY} \\ \mathbb{YX} \\ \mathbb{Y}^2 \end{array}
\begin{array}{c} \begin{array}{cccccccc} \mathbb{X}^3 & \mathbb{X}^2\mathbb{Y} & \mathbb{XYX} & \mathbb{XY}^2 & \mathbb{YX}^2 & \mathbb{YXY} & \mathbb{Y}^2\mathbb{X} & \mathbb{Y}^3 \end{array} \\
\begin{pmatrix}
\beta_{X^3} & \beta_{X^2Y} & \beta_{X^2Y} & \beta_{XY^2} & \beta_{X^2Y} & \beta_{XY^2} & \beta_{XY^2} & \beta_{Y^3} \\
\beta_{X^4} & \beta_{X^3Y} & \beta_{X^3Y} & \beta_{X^2Y^2} & \beta_{X^3Y} & \beta_{XYXY} & \beta_{X^2Y^2} & \beta_{XY^3} \\
\beta_{X^3Y} & \beta_{X^2Y^2} & \beta_{XYXY} & \beta_{XY^3} & \beta_{X^2Y^2} & \beta_{XY^3} & \beta_{XY^3} & \beta_{Y^4} \\
\beta_{X^5} & \beta_{X^4Y} & \beta_{X^4Y} & \beta_{X^3Y^2} & \beta_{X^4Y} & \beta_{X^2YXY} & \beta_{X^3Y^2} & \beta_{X^2Y^3} \\
\beta_{X^4Y} & \beta_{X^3Y^2} & \beta_{X^2YXY} & \beta_{X^2Y^3} & \beta_{X^2YXY} & \beta_{XY^2XY} & \beta_{XY^2XY} & \beta_{XY^4} \\
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\end{pmatrix} \end{array},$$

$$\begin{array}{c} \\ \mathbb{X}^3 \\ \mathbb{X}^2\mathbb{Y} \\ \mathbb{XYX} \\ \mathbb{XY}^2 \\ \mathbb{YX}^2 \\ \mathbb{YXY} \\ \mathbb{Y}^2\mathbb{X} \\ \mathbb{Y}^3 \end{array}
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\begin{pmatrix}
\beta_{X^6} & \beta_{X^5Y} & \beta_{X^5Y} & \beta_{X^4Y^2} & \beta_{X^5Y} & \beta_{X^3YXY} & \beta_{X^4Y^2} & \beta_{X^3Y^3} \\
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\beta_{X^5Y} & \beta_{X^2YX^2Y} & \beta_{X^3YXY} & \beta_{X^2Y^2XY} & \beta_{X^4Y^2} & \beta_{X^2Y^2XY} & \beta_{X^3Y^3} & \beta_{X^2Y^4} \\
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\end{pmatrix} \end{array}.$$

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- B_3 must satisfy

$$
\begin{aligned}
\beta_{X^2Y^3} = \beta_{XY^2XY} = \beta_{X^2Y} - q, \qquad & \beta_{Y^5} = \beta_Y - 2\beta_{X^2Y} + q, \\
\beta_{X^3Y^2} = \beta_{X^2YXY} = \beta_{X^3} - p, \qquad & \beta_{X^5} = p, \\
\beta_{XY^4} = \beta_X - 2\beta_{X^3} + p, \qquad & \beta_{X^4Y} = q,
\end{aligned}
$$

where $p, q \in \mathbb{R}$ are parameters.

Rank-preserving extension of $\mathcal{M}_2$ to $\mathcal{M}_3$

$\mathcal{M}_2$ of rank 6 satisfying the relation $\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1}$.

- B_3 must satisfy

$$\beta_{X^2Y^3} = \beta_{XY^2XY} = \beta_{X^2Y} - q, \qquad \beta_{Y^5} = \beta_Y - 2\beta_{X^2Y} + q,$$
$$\beta_{X^3Y^2} = \beta_{X^2YXY} = \beta_{X^3} - p, \qquad \beta_{X^5} = p,$$
$$\beta_{XY^4} = \beta_X - 2\beta_{X^3} + p, \qquad \beta_{X^4Y} = q,$$

  where $p, q \in \mathbb{R}$ are parameters.

- Define

$$M_1 := \{\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{X}^2, \mathbb{X}\mathbb{Y}, \mathbb{Y}\mathbb{X}\}$$
$$M_2 := \{\mathbb{X}^3, \mathbb{X}^2\mathbb{Y}, \mathbb{X}\mathbb{Y}\mathbb{X}, \mathbb{X}\mathbb{Y}^2, \mathbb{Y}\mathbb{X}^2, \mathbb{Y}\mathbb{X}\mathbb{Y}, \mathbb{Y}^2\mathbb{X}, \mathbb{Y}^3\}.$$

  and calculate 6×10 matrix

$$W = (\mathcal{M}_2|_{M_1})^{-1} B_3|_{M_1,M_2}.$$

Rank-preserving extension of $\mathcal{M}_2$ to $\mathcal{M}_3$

- Then the only candidate for C_3 is equal to

$$C_3 := W^t \mathcal{M}_2|_{M_1} W$$

and $\mathcal{M}_3$ has a moment structure if and only if

$$C_{47} = C_{66}, \qquad C_{16} = C_{23}, \qquad C_{28} = C_{44},$$
$$C_{25} = C_{33}, \qquad C_{48} = C_{68}, \qquad C_{26} = C_{27}.$$
$$C_{12} = C_{13}, \qquad C_{14} = C_{22},$$

Rank 6: $\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1}$, example

For $\beta_{X^4} \in (\frac{1}{4}, \frac{1}{2})$, the following matrices are psd moment matrices of rank 6 satisfying the relation $\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1}$,

$$\mathcal{M}_2(\beta_{X^4}) = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \beta_{X^4} & 0 & 0 & \frac{1}{2} - \beta_{X^4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} - \beta_{X^4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} - \beta_{X^4} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} - \beta_{X^4} & 0 & 0 & \beta_{X^4} \end{pmatrix}.$$

Rank 6: $\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1}$, example

For $\beta_{X^4} \in (\frac{1}{4}, \frac{1}{2})$, the following matrices are psd moment matrices of rank 6 satisfying the relation $\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1}$,

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None of them admit a rank-preserving extension to $\mathcal{M}_3$, but it turns out that they all admit a measure of type $(4, 1)$.

Rank 6: $\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1}$, example

For $\beta_{X^4} \in (\frac{1}{4}, \frac{1}{2})$, the following matrices are psd moment matrices of rank 6 satisfying the relation $\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1}$,

$$\mathcal{M}_2(\beta_{X^4}) = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \beta_{X^4} & 0 & 0 & \frac{1}{2} - \beta_{X^4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} - \beta_{X^4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} - \beta_{X^4} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} - \beta_{X^4} & 0 & 0 & \beta_{X^4} \end{pmatrix}.$$

None of them admit a rank-preserving extension to $\mathcal{M}_3$, but it turns out that they all admit a measure of type $(4, 1)$.

However, the relation $\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1}$ does not imply there always exists a measure.

$\mathcal{M}_2$ of rank at most 3

Proposition

Suppose $n \geq 2$ and $\beta^{(2n)}$ is a sequence such that $\beta_{X^2Y^2} \neq \beta_{XYXY}$ and admits a measure. Then the columns

$$\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{XY}$$

of $\mathcal{M}_n$ are linearly independent.

Proposition

Suppose $n \geq 2$ and $\beta^{(2n)}$ is a sequence such that $\beta_{X^2Y^2} \neq \beta_{XYXY}$ and admits a measure. Then the columns

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of $\mathcal{M}_n$ are linearly independent.

Proof.

$$\mathbf{0} = a \cdot \mathbb{1} + b \cdot \mathbb{X} + c \cdot \mathbb{Y} + d \cdot \mathbb{XY}$$

where $a, b, c, d \in \mathbb{R}$.

- If $d \neq 0$, then $\beta_{X^2Y^2} = \beta_{XYXY}$. $\rightarrow\leftarrow$
- If $d = 0$, the recursive generation implies that

$$\mathbf{0} = a \cdot \mathbb{X} + b \cdot \mathbb{X}^2 + c \cdot \mathbb{XY} = a \cdot \mathbb{Y} + b \cdot \mathbb{XY} + c \cdot \mathbb{Y}^2.$$

  If $b \neq 0$ or $c \neq 0$, it follows that $\beta_{X^2Y^2} = \beta_{XYXY}$. $\rightarrow\leftarrow$ Hence $b = c = 0$. Finally $\mathbf{0} = a \cdot \mathbb{1}$ implies that $a = 0$.

Theorem

Assume that $\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{X}\mathbb{Y}$ *are linearly independent and write*

$$\begin{aligned} \mathbb{X}^2 &= a_1 \cdot \mathbb{1} + b_1 \cdot \mathbb{X} + c_1 \cdot \mathbb{Y} + d_1 \cdot \mathbb{X}\mathbb{Y}, \\ \mathbb{Y}\mathbb{X} &= a_2 \cdot \mathbb{1} + b_2 \cdot \mathbb{X} + c_2 \cdot \mathbb{Y} + d_2 \cdot \mathbb{X}\mathbb{Y}, \\ \mathbb{Y}^2 &= a_3 \cdot \mathbb{1} + b_3 \cdot \mathbb{X} + c_3 \cdot \mathbb{Y} + d_3 \cdot \mathbb{X}\mathbb{Y} \end{aligned}$$

where $a_j, b_j, c_j, d_j \in \mathbb{R}$ *for* $j = 1, 2, 3$.

Theorem

Assume that $\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{X}\mathbb{Y}$ *are linearly independent and write*

$$\mathbb{X}^2 = a_1 \cdot \mathbb{1} + b_1 \cdot \mathbb{X} + c_1 \cdot \mathbb{Y} + d_1 \cdot \mathbb{X}\mathbb{Y},$$
$$\mathbb{Y}\mathbb{X} = a_2 \cdot \mathbb{1} + b_2 \cdot \mathbb{X} + c_2 \cdot \mathbb{Y} + d_2 \cdot \mathbb{X}\mathbb{Y},$$
$$\mathbb{Y}^2 = a_3 \cdot \mathbb{1} + b_3 \cdot \mathbb{X} + c_3 \cdot \mathbb{Y} + d_3 \cdot \mathbb{X}\mathbb{Y}$$

where $a_j, b_j, c_j, d_j \in \mathbb{R}$ *for* $j = 1, 2, 3$*. Then*

1. $d_1 = d_3 = 0$, $d_2 = -1$.

Theorem

Assume that $\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{X}\mathbb{Y}$ *are linearly independent and write*

$$\begin{aligned} \mathbb{X}^2 &= a_1 \cdot \mathbb{1} + b_1 \cdot \mathbb{X} + c_1 \cdot \mathbb{Y}, \\ \mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X} &= a_2 \cdot \mathbb{1} + b_2 \cdot \mathbb{X} + c_2 \cdot \mathbb{Y}, \\ \mathbb{Y}^2 &= a_3 \cdot \mathbb{1} + b_3 \cdot \mathbb{X} + c_3 \cdot \mathbb{Y} \end{aligned}$$

where $a_j, b_j, c_j \in \mathbb{R}$ *for* $j = 1, 2, 3$. *Then*

2. β *admits a measure iff* $\mathcal{M}_n$ *is recursively generated,* $\mathcal{M}_2$ *is psd and*

$$c_1 = b_3 = 0, \quad b_2 = c_3, \quad c_2 = b_1. \tag{1}$$

Theorem

Assume that $\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{XY}$ are linearly independent and write

$$\mathbb{X}^2 = a_1 \cdot \mathbb{1} + b_1 \cdot \mathbb{X} + c_1 \cdot \mathbb{Y},$$
$$\mathbb{XY} + \mathbb{YX} = a_2 \cdot \mathbb{1} + b_2 \cdot \mathbb{X} + c_2 \cdot \mathbb{Y},$$
$$\mathbb{Y}^2 = a_3 \cdot \mathbb{1} + b_3 \cdot \mathbb{X} + c_3 \cdot \mathbb{Y}$$

where $a_j, b_j, c_j \in \mathbb{R}$ for $j = 1, 2, 3$. Then

2. *β admits a measure iff $\mathcal{M}_n$ is recursively generated, $\mathcal{M}_2$ is psd and*

$$c_1 = b_3 = 0, \quad b_2 = c_3, \quad c_2 = b_1. \tag{1}$$

Moreover, if $n > 2$ then the equations (1) follow from $\mathcal{M}_n$ being recursively generated.

Theorem

Assume that $\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{XY}$ are linearly independent, $\mathcal{M}_2$ is psd and there are $a_1, a_2, a_3, b_1, b_2 \in \mathbb{R}$ such that

$$\mathbb{X}^2 = a_1 \cdot \mathbb{1} + b_1 \cdot \mathbb{X},$$

$$\mathbb{XY} + \mathbb{YX} = a_2 \cdot \mathbb{1} + b_2 \cdot \mathbb{X} + b_1 \cdot \mathbb{Y},$$

$$\mathbb{Y}^2 = a_3 \cdot \mathbb{1} + b_2 \cdot \mathbb{Y}.$$

3. *The minimal measure is of type* $(0, 1)$ *with a* ***unique*** *(up to orthogonal equivalence) atom* $(X, Y) \in (\mathbb{SR}^{2\times 2})^2$ *given by*

$$\left(\begin{pmatrix} \sqrt{a_1 + \frac{b_1^2}{4}} + \frac{b_1}{2} & 0 \\ 0 & -\sqrt{a_1 + \frac{b_1^2}{4}} + \frac{b_1}{2} \end{pmatrix}, \ c \cdot \begin{pmatrix} a + b_2 & \sqrt{4 - a^2} \\ \sqrt{4 - a^2} & -a + b_2 \end{pmatrix} \right),$$

where $a = \frac{4a_2 + 2b_1 b_2}{\sqrt{(4a_1 + b_1^2)(4a_3 + b_2^2)}}$, $c = \frac{1}{2}\sqrt{a_3 + \frac{b_2^2}{4}}$.

$\mathcal{M}_2$ of rank 5 or 6 - basic reduction 1

Proposition (Basic column relations)

Suppose $\beta \equiv \beta^{(2n)}$ generates $\mathcal{M}_n$ with $\mathcal{M}_2$ of rank 5 or 6. If β admits a measure, then we may assume (by applying an affine linear transformation on β) that:

1. If rank($\mathcal{M}_2$) = 5, then $\mathcal{M}_n$ satisfies

$$\boxed{\mathbb{XY} + \mathbb{YX} = \mathbf{0}}$$

   and one of

$$\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1} \quad \text{or} \quad \mathbb{Y}^2 - \mathbb{X}^2 = \mathbb{1} \quad \text{or} \quad \mathbb{Y}^2 = \mathbb{1} \quad \text{or} \quad \mathbb{Y}^2 = \mathbb{X}^2.$$

2. If rank($\mathcal{M}_2$) = 6, then $\mathcal{M}_n$ satisfies one of

$$\mathbb{XY}+\mathbb{YX} = \mathbf{0} \quad \text{or} \quad \mathbb{X}^2+\mathbb{Y}^2 = \mathbb{1} \quad \text{or} \quad \mathbb{Y}^2-\mathbb{X}^2 = \mathbb{1} \quad \text{or} \quad \mathbb{Y}^2 = \mathbb{1}.$$

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1. If rank($\mathcal{M}_2$) = 5, then $\mathcal{M}_n$ satisfies

$$\boxed{\mathbb{XY} + \mathbb{YX} = \mathbf{0}} \quad \Rightarrow \quad \text{many 0's in } \mathcal{M}_2$$

   and one of

$$\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1} \quad \text{or} \quad \mathbb{Y}^2 - \mathbb{X}^2 = \mathbb{1} \quad \text{or} \quad \mathbb{Y}^2 = \mathbb{1} \quad \text{or} \quad \mathbb{Y}^2 = \mathbb{X}^2.$$

2. If rank($\mathcal{M}_2$) = 6, then $\mathcal{M}_n$ satisfies one of

$$\mathbb{XY}+\mathbb{YX} = \mathbf{0} \quad \text{or} \quad \mathbb{X}^2+\mathbb{Y}^2 = \mathbb{1} \quad \text{or} \quad \mathbb{Y}^2-\mathbb{X}^2 = \mathbb{1} \quad \text{or} \quad \mathbb{Y}^2 = \mathbb{1}.$$

Basic reduction 1: idea of the proof

Case 1: The set $\left\{\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{X}^2, \mathbb{X}\mathbb{Y}\right\}$ is the basis for $\mathcal{C}_{\mathcal{M}_2}$.

- $\exists a_j, b_j, c_j, d_j, e_j \in \mathbb{R}$ for $j = 1, 2$ such that

$$\mathbb{Y}\mathbb{X} = a_1\mathbb{1} + b_1\mathbb{X} + c_1\mathbb{Y} + d_1\mathbb{X}^2 + e_1\mathbb{X}\mathbb{Y},$$
$$\mathbb{Y}^2 = a_2\mathbb{1} + b_2\mathbb{X} + c_2\mathbb{Y} + d_2\mathbb{X}^2 + e_2\mathbb{X}\mathbb{Y}.$$

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Case 1: The set $\left\{\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{X}^2, \mathbb{X}\mathbb{Y}\right\}$ **is the basis for** $\mathcal{C}_{\mathcal{M}_2}$**.**

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$$\mathbb{Y}^2 = a_2\mathbb{1} + b_2\mathbb{X} + c_2\mathbb{Y} + d_2\mathbb{X}^2 + e_2\mathbb{X}\mathbb{Y}.$$

- Comparing rows $\mathbb{X}\mathbb{Y}$ and $\mathbb{Y}\mathbb{X}$: $e_1 = -1$ and $e_2 = 0$.

Basic reduction 1: idea of the proof

Case 1: The set $\left\{\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{X}^2, \mathbb{X}\mathbb{Y}\right\}$ is the basis for $\mathcal{C}_{\mathcal{M}_2}$.

- $\exists a_j, b_j, c_j, d_j \in \mathbb{R}$ for $j = 1, 2$ such that

$$\begin{aligned} \mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X} &= a_1\mathbb{1} + b_1\mathbb{X} + c_1\mathbb{Y} + d_1\mathbb{X}^2, \\ \mathbb{Y}^2 &= a_2\mathbb{1} + b_2\mathbb{X} + c_2\mathbb{Y} + d_2\mathbb{X}^2. \end{aligned}$$

Basic reduction 1: idea of the proof

Case 1: The set $\left\{\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{X}^2, \mathbb{X}\mathbb{Y}\right\}$ is the basis for $\mathcal{C}_{\mathcal{M}_2}$.

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- Focus on $\mathbb{Y}^2$:
  - **Case 1.1:** $d_2 < 0$:

Basic reduction 1: idea of the proof

Case 1: The set $\left\{\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{X}^2, \mathbb{X}\mathbb{Y}\right\}$ **is the basis for** $\mathcal{C}_{\mathcal{M}_2}$**.**

- $\exists a_j, b_j, c_j, d_j \in \mathbb{R}$ for $j = 1, 2$ such that

$$\begin{aligned} \mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X} &= a_1\mathbb{1} + b_1\mathbb{X} + c_1\mathbb{Y} + d_1\mathbb{X}^2, \\ \mathbb{Y}^2 &= a_2\mathbb{1} + b_2\mathbb{X} + c_2\mathbb{Y} + d_2\mathbb{X}^2. \end{aligned}$$

- Focus on $\mathbb{Y}^2$:
  - **Case 1.1:** $d_2 < 0$:

$$\underbrace{(\mathbb{Y}^2 - \frac{c_2}{2})^2}_{\phi_2(X,Y)} = -(\underbrace{\sqrt{|d_2|}\mathbb{X} - \frac{b_2}{2\sqrt{|d_2|}}}_{\phi_1(X,Y)})^2 + (\underbrace{a_2 + \frac{c_2^2}{4} + \frac{b_2^2}{4d_2}}_{=:C>0})\mathbb{1}.$$

$$\phi(X, Y) = (\frac{1}{\sqrt{C}}\phi_1(X, Y), \frac{1}{\sqrt{C}}\phi_2(X, Y))$$

Basic reduction 1: idea of the proof

Case 1: The set $\left\{\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{X}^2, \mathbb{X}\mathbb{Y}\right\}$ is the basis for $\mathcal{C}_{\mathcal{M}_2}$.

- $\exists a_j, b_j, c_j, d_j \in \mathbb{R}$ for $j = 1, 2$ such that

$$\mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X} = a_1\mathbb{1} + b_1\mathbb{X} + c_1\mathbb{Y} + d_1\mathbb{X}^2,$$
$$\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1}.$$

- Focus on $\mathbb{Y}^2$:
  - **Case 1.1:** $d_2 < 0$:

$$\underbrace{(\mathbb{Y}^2 - \frac{c_2}{2})^2}_{\phi_2(X,Y)} = -\underbrace{(\sqrt{|d_2|}\mathbb{X} - \frac{b_2}{2\sqrt{|d_2|}})^2}_{\phi_1(X,Y)} + \underbrace{(a_2 + \frac{c_2^2}{4} + \frac{b_2^2}{4d_2})}_{=:C>0}\mathbb{1}.$$

$$\phi(X, Y) = \left(\frac{1}{\sqrt{C}}\phi_1(X, Y), \frac{1}{\sqrt{C}}\phi_2(X, Y)\right)$$

Basic reduction 1: idea of the proof

Case 1: The set $\left\{\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{X}^2, \mathbb{X}\mathbb{Y}\right\}$ is the basis for $\mathcal{C}_{\mathcal{M}_2}$.

- $\exists a_1, b_1, c_1, d_1 \in \mathbb{R}$ such that

$$\mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X} = a_1\mathbb{1} + b_1\mathbb{X} + c_1\mathbb{Y} + d_1\mathbb{X}^2,$$
$$\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1}.$$

Basic reduction 1: idea of the proof

Case 1: The set $\left\{\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{X}^2, \mathbb{X}\mathbb{Y}\right\}$ is the basis for $\mathcal{C}_{\mathcal{M}_2}$.

- $\exists a_1, b_1, c_1, d_1 \in \mathbb{R}$ such that

$$\begin{aligned} \mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X} &= a_1\mathbb{1} + b_1\mathbb{X} + c_1\mathbb{Y} + d_1\mathbb{X}^2, \\ \mathbb{X}^2 + \mathbb{Y}^2 &= \mathbb{1}. \end{aligned}$$

- RG relations:

$$\begin{aligned} \mathbb{X}^2\mathbb{Y} + \mathbb{X}\mathbb{Y}\mathbb{X} &= a_1\mathbb{X} + b_1\mathbb{X}^2 + c_1\mathbb{X}\mathbb{Y} + d_1\mathbb{X}^3, \\ \mathbb{Y}\mathbb{X}\mathbb{Y} + \mathbb{Y}^2\mathbb{X} &= a_1\mathbb{Y} + b_1\mathbb{Y}\mathbb{X} + c_1\mathbb{Y}^2 + d_1\mathbb{Y}\mathbb{X}^2, \\ \mathbb{X}^3 + \mathbb{Y}^2\mathbb{X} &= \mathbb{X}, \quad \mathbb{Y}\mathbb{X}^2 + \mathbb{Y}^3 = \mathbb{Y} \\ \mathbb{X}^2\mathbb{Y} + \mathbb{Y}^3 &= \mathbb{Y}, \end{aligned}$$

Case 1: The set $\left\{\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{X}^2, \mathbb{X}\mathbb{Y}\right\}$ is the basis for $\mathcal{C}_{\mathcal{M}_2}$.

- $\exists a_1, b_1, c_1, d_1 \in \mathbb{R}$ such that

$$\begin{aligned} \mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X} &= a_1 \mathbb{1} + b_1 \mathbb{X} + c_1 \mathbb{Y} + d_1 \mathbb{X}^2, \\ \mathbb{X}^2 + \mathbb{Y}^2 &= \mathbb{1}. \end{aligned}$$

- RG relations:

$$\begin{aligned} \mathbb{X}^2\mathbb{Y} + \mathbb{X}\mathbb{Y}\mathbb{X} &= a_1 \mathbb{X} + b_1 \mathbb{X}^2 + {\color{blue}0}\mathbb{X}\mathbb{Y} + d_1 \mathbb{X}^3, \\ \mathbb{Y}\mathbb{X}\mathbb{Y} + \mathbb{Y}^2\mathbb{X} &= a_1 \mathbb{Y} + {\color{red}0}\mathbb{Y}\mathbb{X} + c_1 \mathbb{Y}^2 + d_1 \mathbb{Y}\mathbb{X}^2, \\ \mathbb{X}^3 + \mathbb{Y}^2\mathbb{X} &= \mathbb{X}, \quad \mathbb{Y}\mathbb{X}^2 + \mathbb{Y}^3 = \mathbb{Y}, \\ \mathbb{X}^2\mathbb{Y} + \mathbb{Y}^3 &= \mathbb{Y}, \end{aligned}$$

Basic reduction 1: idea of the proof

Case 1: The set $\{\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{X}^2, \mathbb{X}\mathbb{Y}\}$ is the basis for $\mathcal{C}_{\mathcal{M}_2}$.

- $\exists a_1, d_1 \in \mathbb{R}$ such that

$$\mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X} = a_1\mathbb{1} + d_1\mathbb{X}^2,$$
$$\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1}.$$

Basic reduction 1: idea of the proof

Case 1: The set $\{\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{X}^2, \mathbb{X}\mathbb{Y}\}$ is the basis for $\mathcal{C}_{\mathcal{M}_2}$.

- $\exists a_1, d_1 \in \mathbb{R}$ such that

$$\mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X} = a_1\mathbb{1} + d_1\mathbb{X}^2,$$
$$\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1}.$$

- Continue the analysis and we end up with:

$$\mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X} = \mathbf{0},$$
$$\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1},$$

or

$$\mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X} = \mathbf{0},$$
$$\mathbb{Y}^2 = \mathbb{1},$$

Basic reduction 2

Proposition (Form of the atoms)

Suppose $\beta \equiv \beta^{(2n)}$ generates $\mathcal{M}_n$ satisfying one of:

$$\mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X} = \mathbf{0} \quad \text{or} \quad \mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1} \quad \text{or} \quad \mathbb{Y}^2 - \mathbb{X}^2 = \mathbb{1}.$$

If β admits a measure, then:

(1) There exists a measure with atoms of the following two forms:

- $(x_i, y_i) \in \mathbb{R}^2$.
- $(X_i, Y_i) \in (\mathbb{SR}^{2\times 2})^2$ such that

$$X_i = \begin{pmatrix} \gamma_i & b_i \\ b_i & -\gamma_i \end{pmatrix} \quad \text{and} \quad Y_i = \begin{pmatrix} \mu_i & \mathbf{0} \\ \mathbf{0} & -\mu_i \end{pmatrix}$$

where $\gamma_i \geq 0$, $\mu_i \neq 0$ and $b_i \in \mathbb{R}$.

Basic reduction 2

Proposition (Form of the atoms)

Suppose $\beta \equiv \beta^{(2n)}$ generates $\mathcal{M}_n$ satisfying one of:

$$\mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X} = \mathbf{0} \quad \text{or} \quad \mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1} \quad \text{or} \quad \mathbb{Y}^2 - \mathbb{X}^2 = \mathbb{1}.$$

If β admits a measure, then:

(2) In the measure from (1) all the moments of the form $\beta_{X^{2i}Y^{2j-1}}$ and $\beta_{X^{2i-1}Y^{2j}}$ come from atoms of size 1.

Basic reduction 2: idea of the proof

Let $(X, Y) \in \mathbb{SR}^{t\times t}$ be the atom of a measure.

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2. $\mathbf{XY} + \mathbf{YX}$ **diagonal** :

$$X = \begin{pmatrix} D_1 & B \\ B^t & D_2 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} \mu I_{n_1} & \mathbf{0} \\ \mathbf{0} & -\mu I_{n_2} \end{pmatrix},$$

where $\mu > 0$, $n_1, n_2 \in \mathbb{N}$, $D_1 \in \mathbb{R}^{n_1\times n_1}$ and $D_2 \in \mathbb{R}^{n_2\times n_2}$ are diagonal matrices and $B \in \mathbb{R}^{n_1\times n_2}$.

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3. Using the relation we may assume that $n_1 = n_2$, $D_1 = -D_2 = \gamma I_{n_1}$ for some $\gamma \geq 0$.
4. By a further reduction $n_1 = 1$.

$\mathcal{M}_n$ with relations $\mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X} = \mathbf{0}$ and $\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1}$.

If $\mathcal{M}_n$ is recursively generated, then its column space is spanned by the columns

$$\mathbb{1}, \mathbb{X}, \mathbb{X}^2, \ldots, \mathbb{X}^n, \mathbb{Y}, \mathbb{X}\mathbb{Y}, \ldots, \mathbb{X}^{n-1}\mathbb{Y}.$$

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In this basis the moment matrix has the form

$$\widetilde{\mathcal{M}_n}=\begin{pmatrix}\mathcal{M}_n^X & B_n\\ B_n & \mathcal{M}_n^Y\end{pmatrix}$$

where $\mathcal{M}_n^X$, $\mathcal{M}_n^Y$ and B_n are equal to

$$\begin{array}{c} \\ \mathbb{1}\\ \mathbb{X}\\ \mathbb{X}^2\\ \vdots\\ \mathbb{X}^{2k}\\ \vdots\\ \mathbb{X}^n\end{array}\begin{array}{c}\begin{array}{ccccccc}\mathbb{1} & \mathbb{X} & \mathbb{X}^2 & \cdots & \mathbb{X}^{2k} & \cdots & \mathbb{X}^n\end{array}\\ \left(\begin{array}{ccccccc}\beta_1 & \beta_X & \beta_{X^2} & \cdots & \beta_{X^{2k}} & \cdots & \\ \beta_X & \beta_{X^2} & \beta_X & \cdots & \beta_X & \cdots & \\ \beta_{X^2} & \beta_X & \beta_{X^4} & \cdots & \beta_{X^{2k+2}} & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ \beta_{X^{2k}} & \beta_X & \beta_{X^{2k+2}} & \cdots & \beta_{X^{4k}} & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \\ & & & & & & \beta_{X^{2n}}\end{array}\right)\end{array},$$

$$
\begin{array}{c|cccccc}
 & \mathbb{Y} & \mathbb{XY} & \cdots & \mathbb{X}^{2k-1}\mathbb{Y} & \cdots & \mathbb{X}^{n-1}\mathbb{Y} \\
\hline
\mathbb{Y} & \beta_1 - \beta_{X^2} & 0 & \cdots & 0 & \cdots & \\
\mathbb{XY} & 0 & \beta_{X^2} - \beta_{X^4} & \cdots & \beta_{X^{2k}} - \beta_{X^{2k+2}} & \cdots & \\
\vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \\
\mathbb{X}^{2k-1}\mathbb{Y} & 0 & \beta_{X^{2k}} - \beta_{X^{2k+2}} & \cdots & \beta_{X^{4k-2}} - \beta_{X^{4k}} & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \\
\mathbb{X}^{n-1}\mathbb{Y} & & & & & &
\end{array},
$$

$$
\begin{array}{c|ccccc}
 & \mathbb{Y} & \mathbb{XY} & \mathbb{X}^2\mathbb{Y} & \cdots & \mathbb{X}^{n-1}\mathbb{Y} \\
\hline
\mathbb{1} & \beta_Y & 0 & 0 & \cdots & 0 \\
\mathbb{X} & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & & & & \\
\mathbb{X}^n & 0 & 0 & 0 & \cdots & 0
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$$

By the form of the atoms we know that the blue moments must come from the atoms of size 1.

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Hence $\widetilde{\mathcal{M}}_n$ admits a measure if and only if

$$\widehat{\mathcal{M}}_n := \widetilde{\mathcal{M}}_n - |\beta_X|\widetilde{\mathcal{M}}_n^{(\text{sign}(\beta_X)1,0)} - |\beta_Y|\widetilde{\mathcal{M}}_n^{(0,\text{sign}(\beta_Y)1)}$$

admits a measure where $\widetilde{\mathcal{M}}_n^{(x,y)}$ is the moment matrix generated by the atom $(x, y) \in \mathbb{R}^2$.

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$\widehat{\mathcal{M}}_n$ is of the form

$$\widehat{\mathcal{M}_n} = \begin{pmatrix} \widehat{\mathcal{M}}_n^X & 0 \\ 0 & \widehat{\mathcal{M}}_n^Y \end{pmatrix},$$

$\mathcal{M}_n$ with relations $\mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X} = \mathbf{0}$ and $\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1}$.

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$$
\begin{array}{c}
\\ \mathbb{1} \\ \mathbb{X} \\ \mathbb{X}^2 \\ \vdots \\ \mathbb{X}^{2k} \\ \vdots \\ \mathbb{X}^n
\end{array}
\begin{array}{c}
\begin{array}{ccccccc}
\mathbb{1} & \mathbb{X} & \mathbb{X}^2 & \cdots & \mathbb{X}^{2k} & \cdots & \mathbb{X}^n
\end{array} \\
\left(\begin{array}{ccccccc}
\beta_1 - |\beta_X| - |\beta_Y| & 0 & \beta_{X^2} - |\beta_X| & \cdots & \beta_{X^{2k}} - |\beta_X| & \cdots & \\
0 & \beta_{X^2} - |\beta_X| & 0 & \cdots & 0 & \cdots & \\
\beta_{X^2} - |\beta_X| & 0 & \beta_{X^4} - |\beta_X| & \cdots & \beta_{X^{2k+2}} - |\beta_X| & \cdots & \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\
\beta_{X^{2k}} - |\beta_X| & 0 & \beta_{X^{2k+2}} - |\beta_X| & \cdots & \beta_{X^{4k}} - |\beta_X| & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \\
 & & & & & & \beta_{X^{2n}} - |\beta_X|
\end{array}\right)
\end{array}
$$

$$
\begin{array}{c}
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\end{array} \\
\left(\begin{array}{cccccc}
\beta_1 - \beta_{X^2} - |\beta_Y| & 0 & \cdots & 0 & \cdots & \\
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\vdots & \vdots & \ddots & \vdots & \cdots & \\
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\end{array}\right),
\end{array}
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$\mathcal{M}_n$ with relations $\mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X} = \mathbf{0}$ and $\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1}$.

By the solution of the truncated Hamburger moment problem (Curto & Fialkow, 1991), $\widehat{\mathcal{M}}_n^X$ admits a measure iff $\widehat{\mathcal{M}}_n^X$ is psd and recursively generated.

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Moreover, $\widehat{\mathcal{M}}_n^X$ admits a minimal measure with exactly m atoms (say $x_1, \ldots, x_m$) iff $\widehat{\mathcal{M}}_n^X$ is of rank m.

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If also $\widehat{\mathcal{M}}_n^Y$ is psd, then the atoms which represent $\widehat{\mathcal{M}}_n$ are

$$\left(\begin{pmatrix} 0 & x_i \\ x_i & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{1-x_i^2} & 0 \\ 0 & -\sqrt{1-x_i^2} \end{pmatrix} \right) \quad i = 1, \ldots, m$$

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Moreover, it can be shown that the minimal measures are of one of the types

$$(1, m-2) \quad \text{or} \quad (2, m-2) \quad \text{or} \quad (3, m-2).$$

$\mathcal{M}_2$ with relations $\mathbb{X}\mathbb{Y}+\mathbb{Y}\mathbb{X}=\mathbf{0}$ and $\mathbb{X}^2+\mathbb{Y}^2=\mathbb{1}$.

Theorem

For $\beta=\beta^{(4)}$ we have:

1. *$\mathcal{M}_2$ is positive semidefinite if and only if*

$$|\beta_X|<\beta_{X^2}<1,\quad |\beta_Y|<(1-\beta_{X^2}),\quad c<\beta_{X^4}<\beta_{X^2},$$

where $c:=\frac{-\beta_{X^2}^3+\beta_{X^2}^4-\beta_X^2+\beta_Y^2\beta_X^2+3\beta_{X^2}\beta_X^2-2\beta_{X^2}^2\beta_X^2}{-\beta_{X^2}+\beta_Y^2\beta_{X^2}+\beta_{X^2}^2+\beta_X^2-\beta_{X^2}\beta_X^2}.$

2. *β admits a measure if and only if*

$$|\beta_Y|<1-|\beta_X|,\ |\beta_X|<\beta_{X^2}<1-|\beta_Y|,\ d\leq\beta_{X^4}<\beta_{X^2},$$

where $d=\frac{-\beta_{X^2}^2-|\beta_X|+2\beta_{X^2}|\beta_X|+|\beta_Y\beta_X|}{-1+|\beta_Y|+|\beta_X|}.$

3. *Around 70.5% of β-s with psd $\mathcal{M}_2$ admit a measure. (We integrate w.r.t. the Lebesgue measure.)*

Theorem

4. *The minimal measure is unique (up to orthogonal equivalence) and of type:*
   - $(1,1)$ *if and only if* $\beta_X\beta_Y = 0$ *and* $\beta_{X^4} = c$.

   There are two minimal measures (up to orthogonal equivalence) of type:
   - $(2,1)$ *if and only if* $\beta_X = \beta_Y = 0$ *or* $(\beta_X\beta_Y \neq 0$ *and* $\beta_{X^4} = c)$.
   - $(3,1)$ *if and only if* $\beta_X\beta_Y \neq 0$ *and* $\beta_{X^4} \neq c$.

$\mathcal{M}_2$ (without $\mathbb{Y}^2$ row/column) is of the form

$$\begin{pmatrix}
\beta_1 & \beta_X & \beta_Y & \beta_{X^2} & \beta_{XY} & \beta_{XY} \\
\beta_X & \beta_{X^2} & \beta_{XY} & \beta_{X^3} & \beta_{X^2Y} & \beta_{X^2Y} \\
\beta_Y & \beta_{XY} & \beta_1 - \beta_{X^2} & \beta_{X^2Y} & \beta_X - \beta_{X^3} & \beta_X - \beta_{X^3} \\
\beta_{X^2} & \beta_{X^3} & \beta_{X^2Y} & \beta_{X^4} & \beta_{X^3Y} & \beta_{X^3Y} \\
\beta_{XY} & \beta_{X^2Y} & \beta_X - \beta_{X^3} & \beta_{X^3Y} & \beta_{X^2} - \beta_{X^4} & \beta_{XYXY} \\
\beta_{XY} & \beta_{X^2Y} & \beta_X - \beta_{X^3} & \beta_{X^3Y} & \beta_{XYXY} & \beta_{X^2} - \beta_{X^4}
\end{pmatrix}.$$

Rank 6: relation $\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1}$

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By the form of the atoms we know that the blue moments must come from the atoms of size 1.

Rank 6: relation $\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1}$

We define the linear matrix polynomial $L(a, b, c, d, e)$ by

$$\begin{pmatrix} a & \beta_X & \beta_Y & b & c & c \\ \beta_X & b & c & \beta_{X^3} & \beta_{X^2Y} & \beta_{X^2Y} \\ \beta_Y & c & a-b & \beta_{X^2Y} & \beta_X - \beta_{X^3} & \beta_X - \beta_{X^3} \\ b & \beta_{X^3} & \beta_{X^2Y} & d & e & e \\ c & \beta_{X^2Y} & \beta_X - \beta_{X^3} & e & b-d & b-d \\ c & \beta_{X^2Y} & \beta_X - \beta_{X^3} & e & b-d & b-d \end{pmatrix}.$$

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Theorem

1. $\beta^{(6)}$ *admits a measure if and only if there exist* $a, b, c, d, e \in \mathbb{R}$ *such that*
   - $L(a, b, c, d, e) \succeq 0, \quad \mathcal{M}_2 - L(a, b, c, d, e) \succeq 0,$
   - $(\mathcal{M}_2 - L(a, b, c, d, e))_{\{\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{X}\mathbb{Y}\}} \succ 0,$
   - *L is recursively generated and*

$$\operatorname{rank}(L(a, b, c, d, e)) \leq \operatorname{card} \mathcal{V}_L..$$

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     $$\operatorname{rank}(L(a,b,c,d,e)) \leq \operatorname{card} \mathcal{V}_L..$$
2. *If* $\beta_X = \beta_Y = \beta_{X^3} = \beta_{X^2Y} = 0$, *then the measure always exists and is of type (4,1).*

Open questions

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2. Constructive solution for the non-singular $\mathcal{M}_2$?
(Since for tracial $\mathcal{M}_2$ of rank 6 being psd and rg is not sufficient for the existence of a measure, Curto-Yoo's constructive solution for the nonsingular commutative $\mathcal{M}_2$ does not extend to the tracial case.)

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3. Analysis of $\mathcal{M}_3$.

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 (Since for tracial $\mathcal{M}_2$ of rank 6 being psd and rg is not sufficient for the existence of a measure, Curto-Yoo's constructive solution for the nonsingular commutative $\mathcal{M}_2$ does not extend to the tracial case.)
3. Analysis of $\mathcal{M}_3$.
 (There are examples of $\mathcal{M}_3$ generated by 1 atom of size 3 with empty commutative variety and without a representing measure with atoms of size at most 2.)

Thank you for your attention!