Operator Positivstellensätze for noncommutative polynomials positive on matrix convex sets

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Main results

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1. **Operator linear Positivstellensatz:** the characterization of the inclusion of free Hilbert spectrahedra.
2. **Matrix linear Gleichstellensatz:** the characterization of the equality of free spectrahedra.
3. **Operator convex Positivstellensatz:** the characterization of the inclusion of a free Hilbert spectrahedron in the free positivity domain of a matrix polynomial.

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Context: Operator version of the results of Helton, Klep and McCullough.

Notation

$\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, \mathcal{K}, \mathcal{G}$ ... separable real Hilbert space
$B(\mathcal{H})$ ... an algebra of bounded linear operators on $\mathcal{H}$
$\mathbb{S}_{\mathcal{H}}$ ... a vector space of self-adjoint operators on $\mathcal{H}$
$I_{\mathcal{H}}$ ... the identity operator on $\mathcal{H}$
$\mathbb{S}_n$ ... real symmetric $n \times n$ matrices

Linear pencils and LOI sets

For $A_0, A_1, \dots, A_g \in \mathbb{S}_{\mathscr{H}}$, the expression

$$L(x) = A_0 + \sum_{j=1}^{g} A_j x_j$$

is a **linear operator pencil (LOP)**.

1. If $\dim(\mathscr{H}) < \infty$, then $L(x)$ is a **linear matrix pencil (LMP)**.
2. If $A_0 = I_{\mathscr{H}}$, then L is **monic**.

Linear pencils and LOI sets

For a tuple $X = (X_1, \dots, X_g) \in \mathbb{S}_n^g$, the **evaluation** $L(X)$ is defined as

$$L(X) = A_0 \otimes I_n + \sum_{j=1}^{g} A_j \otimes X_j,$$

where $\otimes$ stands for a tensor product of vector spaces.

Linear pencils and LOI sets

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where $\otimes$ stands for a tensor product of vector spaces.
We call the set

$$D_L(1) = \{x \in \mathbb{R}^g \colon L(x) \succeq 0\}$$

a **Hilbert spectrahedron** or a **LOI domain** and the set

$$D_L = (D_L(n))_n \quad \text{where} \quad D_L(n) = \{X \in \mathbb{S}_n^g \colon L(X) \succeq 0\},$$

a **free Hilbert spectrahedron** or a **free LOI set**.

Inclusion and equality of free Hilbert spectrahedra

Given L_1 and L_2 monic linear operator pencils

$$L_1(x) := I_{\mathscr{H}_1} + \sum_{j=1}^{g} A_j x_j, \quad L_2(x) := I_{\mathscr{H}_2} + \sum_{j=1}^{g} B_j x_j,$$

where $A_j \in \mathbb{S}_{\mathscr{H}_1}$ and $B_j \in \mathbb{S}_{\mathscr{H}_2}$, we are interested in the algebraic characterization of the inclusion and equality of the free LOI sets:

1. When does $D_{L_1} \subseteq D_{L_2}$ hold?
2. When does $D_{L_1} = D_{L_2}$ hold?

Operator linear Positivstellensatz

Theorem (Z.; Davidson, Dor-On, Shalit, Solel)

For LOPs $L_1 \in \mathbb{S}_{\mathscr{H}_1}\langle x\rangle$, $L_2 \in \mathbb{S}_{\mathscr{H}_2}\langle x\rangle$ the inclusion $D_{L_1} \subseteq D_{L_2}$ is true if and only if there exist:

1. *a separable real Hilbert space $\mathscr{K}$,*
2. *a contraction $V : \mathscr{H}_2 \to \mathscr{K}$,*
3. *a positive semidefinite operator $S \in B(\mathscr{H}_2)$ and*
4. *a $*$-homomorphism $\pi : B(\mathscr{H}_1) \to B(\mathscr{K})$ such that*

$$L_2 = S + V^* \pi(L_1) V.$$

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$$L_2 = S + V^*\pi(L_1)V.$$

Moreover, if $D_{L_1}(1)$ is bounded, then V can be chosen to be isometric and π a unital $*$-homomorphism.

Monicity necessary

Example

Let

$$L(y) = \begin{bmatrix} 1 & y \\ y & 0 \end{bmatrix}, \quad \ell(y) = y,$$

be a non-monic LMP and a polynomial, respectively. Then

$$\cup_n \{0_n\} = D_L \subseteq D_\ell = \cup_n \{X \in \mathbb{S}_n : X \succeq 0\},$$

but the conclusion of LPsatz is not true.

Polar duals and operator convex hulls

The **operator free polar dual** $\mathcal{K}^{\mathscr{K},\circ}$ of a free set $\mathcal{K} \subseteq \mathbb{S}^g$ in $\mathscr{K}$ is

$$\mathcal{K}^{\mathscr{K},\circ} = \left\{ A \in \mathbb{S}^g_{\mathscr{K}} : L_A(X) = I_{\mathscr{K}} \otimes I + \sum_{j=1}^{g} A_j \otimes X_j \succeq 0 \text{ for all } X \in \mathcal{K} \right\}.$$

The **operator Hilbert convex hull** $\text{oper-conv}_{\mathscr{K}}\{A\}$ of $A := (A_1, \dots, A_g) \in \mathbb{S}^g_{\mathscr{H}}$ in $\mathscr{K}$ is the set

$$\text{oper-conv}_{\mathscr{K}}\{A\} := \bigcup_{(\mathscr{G},\pi,V)\in\Pi} (V^*\pi(A_1)V, \dots, V^*\pi(A_g)V),$$

where Π is the set of all triples $(\mathscr{G}, \pi, V)$ of a separable real Hilbert space $\mathscr{G}$, a contraction $V : \mathscr{K} \to \mathscr{G}$ and a unital $*$-homomorphism $\pi : B(\mathscr{H}) \to B(\mathscr{G})$.

Polar duals and operator convex hulls

Corollary

Suppose

$$L := I_{\mathscr{H}} + \sum_{j=1}^{g} A_j x_j \in \mathbb{S}_{\mathscr{H}}\langle x \rangle$$

is a monic LOP. Then

$$(D_L)^{\mathscr{K},\circ} = \text{oper-conv}_{\mathscr{K}}\{(A_1, \ldots, A_g)\}.$$

Equality of free Hilbert spectrahedra - definitions

Minimality of a pencil: Let H be a closed subspace of $\mathscr{H}$ such that

$$A_j H \subseteq H \quad \text{for } j = 0, \ldots, g.$$

Then L is unitarily equivalent to

$$\begin{pmatrix} L|_H & 0 \\ 0 & L|_{H^\perp} \end{pmatrix} := \begin{pmatrix} I_H + \sum_{j=1}^g (A_j)|_H x_j, & 0 \\ 0 & I_{H^\perp} + \sum_{j=1}^g (A_j)|_{H^\perp} x_j \end{pmatrix}.$$

If there is no proper closed subspace of $\mathscr{H}$ such that $D_L = D_{L|_H}$, then L is σ-**minimal** pencil.

Equality of free spectrahedra - solution

Theorem (Linear Gleichstellensatz)

Let L_1, L_2 be monic σ-minimal LMIs. Then $D_{L_1} = D_{L_2}$ if and only there is a unitary matrix U such that

$$L_2 = U^* L_1 U.$$

For LMIs with bounded D_{L_1} LG was proved for LMIs by Helton, Klep, McCullough in 2010, while for LOIs with compact operator coefficients and bounded D_{L_1} by Davidson, Dor-On, Shalit, Solel in 2016.

Nonexistence of σ-minimal operator subpencil

Example

Let

$$L(x) = I_{\ell^2} + \operatorname{diag}\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}} x$$

be a diagonal linear operator pencil with coefficients from $B(\ell^2(\mathbb{N}))$. Then

$$D_L(m) = \{X \in \mathbb{S}_m \colon X \succeq -I_{\ell^2}\}$$

and there does not exist a σ-minimal whole subpencil of L.

Counterexample to the operator Linear Gleichstellensatz

Example

Let $S_1, S_2 \in B(\ell^2(\mathbb{N}))$ be defined by

$$e_i \mapsto e_{2i-1} \quad \text{and} \quad e_i \mapsto e_{2i} \quad \text{for } i \in \mathbb{N}$$

respectively. Cuntz C^*-algebra $C^*(S_1, S_2)$ has a unique $*$-isomorphism θ such that

$$\theta(S_1) = S_2, \quad \theta(S_2) = S_1.$$

Let

$$A_1 := S_1 + S_1^*, \quad A_2 := S_2 + S_2^*,$$
$$A_3 := i(S_1 - S_1^*), \quad A_4 := i(S_2 - S_2^*).$$

Counterexample to the operator Linear Gleichstellensatz

Example

The LOPs

$$\begin{aligned} L_1(x) &= I_{\ell^2} + A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4, \\ L_2(x) &= I_{\ell^2} + A_2x_1 + A_1x_2 + A_4x_3 + A_3x_4 \end{aligned}$$

are σ-minimal pencils with $D_{L_1} = D_{L_2}$, but there is no unitary operator $U : \ell^2 \to \ell^2$ such that

$$L_2 = U^* L_1 U \quad \text{or} \quad L_2 = U^* \overline{L_1} U.$$

Noncommutative (nc) polynomials

$\langle x \rangle \ldots$ free monoid generated by $x = (x_1, \ldots, x_g)$
$\mathbb{R}\langle x \rangle \ldots$ the associative $\mathbb{R}$-algebra freely generated by x
$f \in \mathbb{R}\langle x \rangle \ldots$ noncommutative (nc) polynomial
$\deg(f) \ldots$ the length of the longest word in f

Involution * fixes $\mathbb{R} \cup \{\emptyset\}$, reverses the order of words, and acts linearly on polynomials.

Polynomials invariant under this involution are **symmetric**.

Noncommutative (nc) polynomials

Operator-valued nc polynomials are the elements of the form

$$P = \sum_{w \in \langle x \rangle} A_w \otimes w \in B(\mathscr{H}_1, \mathscr{H}_2) \otimes \mathbb{R}\langle x \rangle ,$$

where the sum is finite.

The involution * extends to $B(\mathscr{H}) \otimes \mathbb{R}\langle x \rangle$ by

$$P^* = \sum_{w \in \langle x \rangle} A_w^* \otimes w^* \in B(\mathscr{H}_2, \mathscr{H}_1) \otimes \mathbb{R}\langle x \rangle .$$

If $P = P^*$, then we say P is **symmetric**.

Polynomial evaluations

If $P \in B(\mathscr{H}) \otimes \mathbb{R}\langle x \rangle$ and $X \in M_n^g$, then the **evaluation**

$$P(X) \in B(\mathscr{H}) \otimes M_n$$

is defined by replacing x_i by X_i and sending the empty word to the identity operator on $\mathscr{K}$.

$P = P^*$ determines the **free Hilbert semialgebraic set** by

$$D_P = (D_P(n))_n \quad \text{where} \quad D_P(n) = \{X \in \mathbb{S}_n^g \colon P(X) \succeq 0\}.$$

Positivstellensatz problem

Suppose $L \in \mathbb{S}_{\mathscr{H}}\langle x\rangle$ is a monic linear operator pencil (LOP) and

$$P = P^* \in B(\mathscr{K}) \otimes \mathbb{R}\langle x\rangle$$

a symmetric operator-valued nc polynomial such that

$$D_L \subseteq D_P.$$

The problem is to find an algebraic expression for the polynomial P in terms of the polynomial L.

Operator convex multivariate Positivstellensatz

Theorem (Operator convex multivariate Positivstellensatz)

Let $L \in \mathbb{S}_{\mathscr{H}}\langle x\rangle$ be a monic LOP and $P = P^* \in \mathbb{R}^{\nu\times\nu}\langle x\rangle$ a matrix-valued nc polynomial. Then $D_L \subseteq D_P$ is true if and only if there exist:

1. *a separable real Hilbert space $\mathscr{K}$,*
2. *a $*$-homomorphism $\pi : B(\mathscr{H}) \to B(\mathscr{K})$,*
3. *matrix polynomials $R_j \in \mathbb{R}^{\nu\times\nu}\langle x\rangle$ and*
4. *operator polynomials $Q_k \in B(\mathbb{R}^\nu, \mathscr{K}) \otimes \mathbb{R}\langle x\rangle$*

all of degree at most $\frac{\deg(P)+2}{2}$ such that

$$P = \sum_j R_j^* R_j + \sum_k Q_k^* \pi(L) Q_k.$$

Operator convex univariate Positivstellensatz

Theorem (Operator convex univariate Positivstellensatz)

Let $L = I_{\mathscr{H}} + A_1 y \in \mathbb{S}_{\mathscr{H}}\langle y \rangle$ be a univariate monic LOP and $P = P^* \in B(\mathscr{K}) \otimes \mathbb{R}\langle x \rangle$ an operator-valued nc polynomial. Then $D_L \subseteq D_P$ is true if and only if there exist:

1. *a separable real Hilbert space $\mathscr{G}$,*
2. *a $*$-homomorphism $\pi : B(\mathscr{H}) \to B(\mathscr{G})$ and*
3. *operator polynomials $R_j \in B(\mathscr{K}) \otimes \mathbb{R}\langle x \rangle$ and $Q_k \in B(\mathscr{K}, \mathscr{G}) \otimes \mathbb{R}\langle x \rangle$*

all of degree at most $\frac{\deg(P)+2}{2}$ such that

$$P = \sum_j R_j^* R_j + \sum_k Q_k^* \pi(L) Q_k.$$

Monicity necessary

Example

Let

$$L(y) = \operatorname{diag}\left(-\frac{1}{n} + \frac{y}{n^2}\right)_{n\in\mathbb{N}}$$

be a diagonal LOP and $\ell(y) = -1$ a constant polynomial. Then

$$\emptyset = D_L = D_\ell$$

but the conclusion of CPsatz is not true.

Thank you for your attention!