Univariate matrix polynomials positive semidefinite on semialgebraic sets

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Notation

Let $\mathbb{R}[x]$ be the ring of real polynomials. We write $M_n(\mathbb{R}[x])$ for the ring of matrix polynomials equipped with transposition as the involution.

Let

$$\mathbb{S}_n(\mathbb{R}[x]) = \{F \in M_n(\mathbb{R}[x]) \colon F^t = F\}$$

be the set of symmetric matrix polynomials.

Let

$$\sum M_n(\mathbb{R}[x])^2 = \left\{ \sum_{i=1}^{k} A_i^t A_i \colon k \in \mathbb{N}, A_i \in M_n(\mathbb{R}[x]) \right\}$$

be the set of sums of hermitian squares of matrix polynomials.

Problem

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Characterize univariate matrix polynomials, which are positive semidefinite on a union of points and intervals (not necessarily bounded) in $\mathbb{R}$.

Notation

A **closed semialgebraic set** $K_S \subseteq \mathbb{R}$ associated to a finite subset $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ is given by

$$K_S = \{x \in \mathbb{R} \colon g_j(x) \geq 0,\ j = 1, \ldots, s\}.$$

We define the ***n*-th matrix quadratic module** M_S^n by

$$\begin{aligned} M_S^n \quad := \quad & \{\sigma_0 + \sigma_1 g_1 + \ldots + \sigma_s g_s \colon \\ & \sigma_j \in \sum M_n(\mathbb{R}[x])^2 \text{ for } j = 0, \ldots, s\Big\}. \end{aligned}$$

Notation

Let

$$\prod S := \{g_1^{e_1} \cdots g_s^{e_s} : e_j \in \{0,1\},\ j = 1, \ldots, s\}.$$

The *n*-**th matrix preordering** T_S^n is defined as

$$T_S^n = M_{\prod S}^n.$$

Let $\mathrm{Pos}_{\succeq 0}^n(K_S)$ be the set of all $n \times n$ symmetric matrix polynomials, which are positive semidefinite in every point of K_S.

We say a matrix quadratic module M_S^n is **saturated** if

$$M_S^n = \mathrm{Pos}_{\succeq 0}^n(K_S).$$

Notation

Let $K \subseteq \mathbb{R}$ be a closed semialgebraic set. A set

$$S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$$

is the **natural description** of K, if it satisfies the following conditions:

(a) If $a = \min(K)$, then $x - a \in S$.

(b) If $b = \max(K)$, then $b - x \in S$.

(c) If $a, b \in K$, $a < b$ and $c \notin K$ for every $a < c < b$, then

$$(x - a)(x - b) \in S.$$

(d) These are the only elements of S.

Known results - scalar case

Theorem (Kuhlmann, Marshall, 2002)

If S is the natural description of K, then the preordering $T_S^1 = M^1_{\prod S}$ is saturated.

Known results - scalar case

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If S is the natural description of K, then the preordering $T_S^1 = M^1_{\prod S}$ is saturated.

- K not compact: $T^1_{\widetilde{S}}$ is saturated if and only if $\widetilde{S}$ contains each of the polynomials in the natural description of K up to scaling by positive constants.
- K compact: Scheiderer classifed in 2003 exactly when $T^1_{\widetilde{S}}$ is saturated. Moreover, $T^1_{\widetilde{S}} = M^1_{\widetilde{S}}$.

Known results - matrix case

By the results of Gohberg & Krein (1958), Dette & Studden (2002) and Schmüdgen & Savchuk (2012), we have the following:

Theorem

Let K be equal to

$$\mathbb{R} \quad or \quad [0,1] \quad or \quad [0,\infty).$$

Suppose S is the natural description K. Then the n-th quadratic module M_S^n is saturated for every $n \in \mathbb{N}$.

New results

Theorem (Compact Nichtnegativstellensatz for $\mathbb{R}$)

Let K be a compact semialgebraic set in $\mathbb{R}$ with a natural description S. Then the n-th quadratic module M_S^n is saturated for every $n \in \mathbb{N}$.

Sketch of the proof of compact Nsatz

Claim: It suffices to prove that for every $F \in \mathrm{Pos}^n_{\succeq 0}(K_S)$ the ideal

$$I_F := \left\langle h^2 \colon h \in \mathbb{R}[x], h^2 F \in M_S^n \right\rangle$$

is $\mathbb{R}[x]$.

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Indeed, if $I_F = \mathbb{R}[x]$, then by Scheiderer's result there exist $s_1, \ldots, s_k \in \mathrm{Pos}^1_{\succ 0}(K)$ such that

$$s_1 h_1^2 + s_2 h_2^2 + \ldots + s_k h_k^2 = 1,$$

where $I_F = \langle h_1^2, \ldots, h_k^2 \rangle$. Hence,

$$F = \sum_{j=1}^{k} s_j h_j^2 F \in M_S^1 \cdot M_S^n \underbrace{=}_{K \subset \mathbb{R} \text{ compact}} M_S^n,$$

which concludes the proof.

Sketch of the proof of compact Nsatz

To establish the Claim we have to prove that for every $x_0 \in \mathbb{C}$ there exists $h_{x_0} \in \mathbb{R}[x]$ such that

$$h_{x_0}(x_0) \neq 0 \quad \text{and} \quad h_{x_0}^2 F \in M_S^n.$$

Sketch of the proof of compact Nsatz

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The proof of this statement is by induction on the size n of matrix polynomials. For $n = 1$ this is true by the scalar case. We write

$$F(x) = \begin{cases} \underbrace{(x - x_0)^m}_{p(x)} \cdot G(x), & \text{if } x_0 \in \mathbb{R} \\ \underbrace{((x - x_0)(x - \overline{x_0}))^m}_{p(x)} \cdot G(x), & \text{if } x_0 \notin \mathbb{R} \end{cases},$$

where $m \in \mathbb{N}_0$, $G(x) \in M_n(\mathbb{R}[x])$ and $G(x_0) \neq 0$.

Sketch of the proof of Compact Nsatz

Writing $G = \begin{pmatrix} a & \beta \\ \beta^t & C \end{pmatrix}$, where

$$a \in \mathbb{R}[x], \quad \beta \in M_{1,n-1}(\mathbb{R}[x]), \quad C \in M_{n-1}(\mathbb{R}[x]),$$

we may assume $a(x_0) \neq 0$. Then

$$a^4 \cdot G = \begin{pmatrix} a & 0 \\ \beta^t & aI_{n-1} \end{pmatrix} \begin{pmatrix} a^3 & 0 \\ 0 & a(aC - \beta^t \beta) \end{pmatrix} \begin{pmatrix} a & \beta \\ 0 & aI_{n-1} \end{pmatrix},$$

$$\begin{pmatrix} a^3 & 0 \\ 0 & a(aC - \beta^t \beta) \end{pmatrix} = \begin{pmatrix} a & 0 \\ -\beta^t & aI_{n-1} \end{pmatrix} \cdot G \cdot \begin{pmatrix} a & -\beta \\ 0 & aI_{n-1} \end{pmatrix}.$$

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we may assume $a(x_0) \neq 0$. Then

$$a^4 \cdot F = \begin{pmatrix} a & 0 \\ \beta^t & aI_{n-1} \end{pmatrix} \begin{pmatrix} pa^3 & 0 \\ 0 & pa(aC - \beta^t\beta) \end{pmatrix} \begin{pmatrix} a & \beta \\ 0 & aI_{n-1} \end{pmatrix},$$

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Sketch of the proof of compact Nsatz

By induction hypothesis there is $\widetilde{h}_{x_0} \in \mathbb{R}[x]$ such that

$$\widetilde{h}_{x_0}(x_0) \neq 0, \quad \widetilde{h}_{x_0}^2 pa(aC - \beta^t\beta) \in M_S^{n-1}.$$

We also have

$$\widetilde{h}_{x_0}^2 pa^3 \in M_S^1.$$

Therefore

$$a^2\widetilde{h}_{x_0}(x_0) \neq 0 \quad \text{and} \quad (a^2\widetilde{h}_{x_0})^2 F \in M_S^n.$$

Counterexample for the non-compact case

Example

The matrix polynomial

$$F(x) := \begin{bmatrix} x+2 & \sqrt{6} \\ \sqrt{6} & x^2-2x+3 \end{bmatrix}$$

is positive semidefinite on $K := [-1,0] \cup [1,\infty)$, but

$$F \notin T_S^2 = M^2_{\prod S},$$

where S is the natural description of K.

Theorem

Let an unbounded closed semialgebraic set $K \subseteq \mathbb{R}$ satisfy either of the following:

1. *K contains at least two intervals with at least one of them bounded.*
2. *K is a union of an unbounded interval and m isolated points with $m \geq 2$.*
3. *K is a union of two unbounded intervals and m isolated points with $m \geq 2$.*

If $S \subseteq \mathbb{R}[x]$ is a finite set with $K_S = K$, then the 2-nd matrix preordering T_S^2 is not saturated.

Non-compact Nichtnegativstellensatz

Theorem (Non-compact Nichtnegativstellensatz)

Suppose K is an unbounded closed semialgebraic set in $\mathbb{R}$ and S a natural description of K. Then $F \in Pos^n_{\succeq 0}(K)$ if and only if there exists $k \in \mathbb{N}_0$ such that

$$(1 + x^2)^k F \in M^n_S.$$

Thank you for your attention!