Truncated Moment Problems On Algebraic Curves

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joint work with

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Outline

1. Preliminaries
   - Types of solutions to the truncated moment problem
   - Moment matrices and necessary conditions for existence
2. Flat Extension Theorem
   - Applications to lines and conics in the plane
   - Application to the quartic case in the plane
   - Application to the cubic curve $y = x^3$
3. Truncated Riesz–Haviland Theorem
   - Applications to planar cubic curves
4. Core Variety Theorem
   - Trichotomy in the sextic case in $\mathbb{R}^2$
   - Trichotomy for irreducible planar cubics
   - Quadrichotomy for reducible planar cubics
5. The curve $y = x^4$

1. Preliminaries

- Types of solutions to the truncated moment problem
- Moment matrices and necessary conditions for existence

The Classical Truncated K-Moment Problem (K-TMP)

Let $m \in \mathbb{N}$ and $\boldsymbol{i} := (i_1, \dots, i_n) \in \mathbb{Z}_+^n$ with $|\boldsymbol{i}| := i_1 + \cdots + i_n$.

$$\beta \equiv \beta^{(m)} = \{\beta_{\boldsymbol{i}} : \ \boldsymbol{i} \in \mathbb{Z}_+^n, \ |\boldsymbol{i}| \le m\}$$

an n–dimensional real multisequence of degree m.

$K \subseteq \mathbb{R}^n$ is a closed subset.

The classical truncated K–moment problem (K–TMP): characterize the existence of a positive Borel measure μ on $\mathbb{R}^n$ with $\operatorname{supp} \mu \subseteq K$, such that

$$\beta_{\boldsymbol{i}} = \int_K \boldsymbol{x}^{\boldsymbol{i}} \, d\mu(\boldsymbol{x}) \qquad (|\boldsymbol{i}| \le m)$$

where $\boldsymbol{x} := (x_1, \dots, x_n)$ and $\boldsymbol{x}^{\boldsymbol{i}} := x_1^{i_1} \cdots x_n^{i_n}$.

μ is called a K–representing measure of β.

Types of Solutions to the K–TMP

By degree of explicitness

Solutions to the K–TMP can be classified by how explicitly they characterize the existence of a K–representing measure.

1. **Abstract solutions** *(existence criteria)*
   Typically partially algorithmic. Foundational for the theory, but often hard to use in concrete numerical instances.

2. **Concrete solutions** *(checkable conditions)*
   Give necessary and sufficient conditions that are directly verifiable in explicit examples. Highly useful in applications, but often difficult to obtain.

3. **Constructive solutions** *(measure produced)*
   Concrete criteria that additionally produce a representing measure. Strongest form of solvability in applications.

Milestone Abstract Solutions to the K–TMP

Three foundational existence criteria

1. **Flat Extension Theorem** (Curto & Fialkow, 1996)

   Solvability $\iff$ existence of a *rank-preserving PSD extension* of the moment matrix.

   Often constructive on curves (all conics; some cubics).

2. **Truncated Riesz–Haviland Theorem** (Curto & Fialkow, 2008)

   Solvability $\iff$ existence of an extension whose Riesz functional is *K–positive* (nonnegative on all polynomials ≥ 0 on K).

   Relies on explicit nonnegativity certificates.

3. **Core Variety Theorem** (Blekherman & Fialkow, 2020)

   Solvability $\iff$ *non-emptiness of a kernel-determined variety* (the core variety).

   Provides a geometric and coordinate-free viewpoint.

Riesz Functional and its Positivity

Let $\mathbb{R}[\boldsymbol{x}] := \mathbb{R}[x_1, \ldots, x_n]$ and, for $m \in \mathbb{N}$, set

$$\mathbb{R}[\boldsymbol{x}]_{\leq m} := \{\, p \in \mathbb{R}[\boldsymbol{x}] : \deg p \leq m \,\}.$$

Given a truncated moment sequence $\beta \equiv \beta^{(m)} = \{\beta_{\boldsymbol{i}}\}_{|\boldsymbol{i}| \leq m}$, define the Riesz functional

$$L_\beta : \mathbb{R}[\boldsymbol{x}]_{\leq m} \to \mathbb{R}, \qquad p(\boldsymbol{x}) = \sum_{|\boldsymbol{i}| \leq m} a_{\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}} \longmapsto L_\beta(p) := \sum_{|\boldsymbol{i}| \leq m} a_{\boldsymbol{i}}\, \beta_{\boldsymbol{i}}.$$

Proposition. If β has a K–representing measure μ, then L_β is K–positive, i.e.,

$$p \in \mathbb{R}[\boldsymbol{x}]_{\leq m}, \;\; p|_K \geq 0 \quad \Longrightarrow \quad L_\beta(p) \geq 0.$$

Proof.

Assume $\beta_\alpha = \int_K \boldsymbol{x}^{\boldsymbol{i}}\, d\mu(\boldsymbol{x})$ for all $|\boldsymbol{i}| \leq m$. Then for any $p = \sum a_{\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}} \in \mathbb{R}[\boldsymbol{x}]_{\leq m}$,

$$L_\beta(p) = \sum a_{\boldsymbol{i}} \beta_{\boldsymbol{i}} = \int_K \Big(\sum a_{\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}} \Big)\, d\mu = \int_K p\, d\mu \;\geq\; 0,$$

since $p \geq 0$ on K and μ is a positive measure. □

Moment Matrix

The moment matrix $M_k \equiv M_k(\beta)$ of $\beta \equiv \beta^{(2k)} = \{\beta_{\boldsymbol{i}}\}_{|\boldsymbol{i}|\leq 2k}$ is the symmetric matrix whose rows and columns are indexed by the monomials

$$\boldsymbol{X}^{\boldsymbol{i}} := X_1^{i_1} \cdots X_n^{i_n}, \qquad |\boldsymbol{i}| \leq k,$$

listed in degree–lexicographic order, and whose entries are

$$(M_k)_{\boldsymbol{i},\boldsymbol{j}} := L_\beta(\boldsymbol{x}^{\boldsymbol{i}+\boldsymbol{j}}) = \beta_{\boldsymbol{i}+\boldsymbol{j}}, \qquad |\boldsymbol{i}|, |\boldsymbol{j}| \leq k.$$

Example ($n = 2$). In degree–lexicographic order the monomials are

$$1,\ X,\ Y,\ X^2,\ XY,\ Y^2,\ \ldots,\ X^k,\ X^{k-1}Y,\ \ldots,\ Y^k.$$

$$M_k = \begin{array}{c} \\ 1 \\ X \\ Y \\ \vdots \\ X^{i_1}Y^{j_1} \\ \vdots \\ Y^k \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & X & Y & \cdots & X^{i_2}Y^{j_2} & \cdots & Y^k \end{array} \\ \left[\begin{array}{ccccccc} \beta_{0,0} & \beta_{1,0} & \beta_{0,1} & \cdots & \beta_{i_2,j_2} & \cdots & \beta_{0,k} \\ \beta_{1,0} & \beta_{2,0} & \beta_{1,1} & \cdots & \beta_{i_2+1,j_2} & \cdots & \beta_{1,k} \\ \beta_{0,1} & \beta_{1,1} & \beta_{0,2} & \cdots & \beta_{i_2,j_2+1} & \cdots & \beta_{0,k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \beta_{i_1,j_1} & \beta_{i_1+1,j_1} & \beta_{i_1,j_1+1} & \cdots & \beta_{i_1+i_2,\,j_1+j_2} & \cdots & \beta_{i_1,\,j_1+k} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \beta_{0,k} & \beta_{1,k} & \beta_{0,k+1} & \cdots & \beta_{i_2,\,j_2+k} & \cdots & \beta_{0,2k} \end{array}\right] \end{array}.$$

Functional Calculus on the Moment Matrix

Let $\beta \equiv \beta^{(2k)}$. For $p, q \in \mathbb{R}[\boldsymbol{x}]_{\leq k}$, denote by $\widehat{p}, \widehat{q}$ their coefficient vectors with respect to the ordered monomial basis of $\mathbb{R}[\boldsymbol{x}]_{\leq k}$. Then

$$\langle M_d \, \widehat{p}, \widehat{q} \rangle \; := \; L_\beta(pq).$$

The evaluation of $p(\boldsymbol{x}) = \sum_{|\boldsymbol{i}| \leq k} a_i x^{\boldsymbol{i}} \in \mathbb{R}[\boldsymbol{x}]_{\leq k}$ on M_d is equal to

$$p(\boldsymbol{X}) \equiv \sum_{|\boldsymbol{i}| \leq k} a_{\boldsymbol{i}} \, \boldsymbol{X}^{\boldsymbol{i}} := M_d \, \widehat{p}.$$

Example. For $n = 2$ and $p := \sum_{i,j} a_{i,j} x^i y^j \in \mathbb{R}[x, y]_{\leq k}$:

$$p(X,Y) = \sum_{i,j} a_{i,j} X^i Y^j = a_{0,0} \cdot \overset{1}{\begin{pmatrix} \beta_{0,0} \\ \beta_{1,0} \\ \beta_{0,1} \\ \vdots \\ \beta_{0,k} \end{pmatrix}} + a_{1,0} \cdot \overset{X}{\begin{pmatrix} \beta_{1,0} \\ \beta_{2,0} \\ \beta_{1,1} \\ \vdots \\ \beta_{1,k} \end{pmatrix}} + \cdots + a_{0,k} \cdot \overset{Y^k}{\begin{pmatrix} \beta_{0,k} \\ \beta_{1,k} \\ \beta_{0,k+1} \\ \vdots \\ \beta_{0,2k} \end{pmatrix}}$$

Necessary Conditions for a Representing Measure

Assume $\beta^{(2k)}$ admits a representing measure μ on K. Then:

1. Positivity. The moment matrix $M_k(\beta)$ is positive semidefinite:

$$\langle M_k \widehat{p}, \widehat{p} \rangle \;=\; L_\beta(p^2) \;=\; \int p^2 \, d\mu \;\geq\; 0, \qquad \forall\, p \in \mathbb{R}[\boldsymbol{x}]_{\leq k}.$$

2. Support-size bound. Any representing measure must satisfy

$$\operatorname{card}(\operatorname{supp} \mu) \;\geq\; \operatorname{rank} M_k.$$

3. Variety condition. Define the variety of M_k by

$$V(M_k) := \Big\{ x \in \mathbb{R}^n : \; p(x) = 0 \;\; \forall\, p \in \mathbb{R}[\boldsymbol{x}]_{\leq k} \text{ with } p(\boldsymbol{X}) = 0 \Big\}.$$

   Then

$$\operatorname{supp} \mu \subseteq V(M_k), \qquad \text{and} \qquad \operatorname{card} V(M_k) \;\geq\; \operatorname{rank} M_k.$$

4. Recursive generation. M_k is recursively generated in the sense that

$$p(\boldsymbol{X}) = 0 \;\Longrightarrow\; (pq)(\boldsymbol{X}) = 0 \quad \text{whenever } p, q, pq \in \mathbb{R}[\boldsymbol{x}]_{\leq k},$$

2. Flat Extension Theorem and Applications

- Applications to lines and conics in the plane
- Application to the quartic case in the plane
- Application to the cubic curve $y = x^3$

Flat Extension Theorems

Flat Extension Theorem I (Curto & Fialkow, 1996). The following statements are equivalent:

1. $\beta^{(2k)}$ admits a $(\operatorname{rank} M_k)$–atomic representing measure.
2. M_k is positive semidefinite and there exists an extension M_{k+1} such that

$$\operatorname{rank} M_{k+1} = \operatorname{rank} M_k.$$

In this case, M_{k+1} has a *unique representing measure*, which is $(\operatorname{rank} M_k)$–atomic, and whose support is $V(M_{k+1})$.

Flat Extension Theorem II (Curto & Fialkow, 2005). The following statements are equivalent:

1. $\beta^{(2k)}$ has a representing measure.
2. M_k admits a positive semidefinite extension M_{k+d} (for some d, $0 \leq d \leq 2 \dim \mathbb{R}[\boldsymbol{x}]_{\leq 2k} - d$) which, in turn, admits a flat extension M_{k+d+1}, i.e.,

$$\operatorname{rank} M_{k+d+1} = \operatorname{rank} M_{k+d}.$$

Applications of Flat Extension Theorem I

TMP on conics in the plane

Theorem (Curto & Fialkow 2002, 2004, 2005; Fialkow 2016).

Let $p(x, y) \in \mathbb{R}[x, y]_{\leq 2}$ and define the quadratic curve

$$\mathcal{Z}(p) := \{(x, y) \in \mathbb{R}^2 : \ p(x, y) = 0\}.$$

Fix $k \geq \deg p$. Then the following are equivalent:

1. $\beta^{(2k)}$ admits a $\mathcal{Z}(p)$–representing measure.
2. M_k is positive semidefinite and recursively generated, satisfies the variety condition, and the column relation $p(X, Y) = 0$ holds in M_k.

Main technique:

1. Reduce the conic $\mathcal{Z}(p)$ to one of five canonical forms ($x^2 + y^2 = 1$, $y = x^2$, $xy = 1$, $y^2 = y$, $xy = 0$) using affine linear transformations, then analyze each case via flat extensions.
2. Show that either M_k admits a flat extension M_{k+1}, or it admits a positive extension to M_{k+1} from which a flat extension occurs at the next step (i.e., M_{k+2} is flat).

Applications of Flat Extension Theorem II

Nonsingular quartic binary TMP

Theorem (Fialkow & Nie 2010; Curto & Yoo 2014).

Assume $\beta^{(4)}$ is such that

$$M_2 \text{ is positive definite.}$$

Then $\beta^{(4)}$ admits a representing measure.

Moreover, M_2 admits a flat extension M_3, whence there exists a 6–atomic representing measure.

Main technique:

1. Existence (Fialkow & Nie): uses convex-geometric methods together with a description of positive bivariate quartics.
2. Constructive step (Curto & Yoo): produces a flat extension and hence an explicit finitely atomic measure via the Flat Extension Theorem.

Applications of Flat Extension Theorem III

A cubic planar curve $y = x^3$

We say that M_k is $(y - x^3)$–pure if its column dependence relations are exactly those obtained from the basic relation $Y = X^3$ by linearity and recursiveness.

Theorem (Fialkow, 2011).

Assume M_k is positive semidefinite and $(y - x^3)$–pure. Then the following are equivalent:

1. β admits a representing measure supported on $\{(x, y) : y = x^3\}$;
2. β admits a finitely atomic representing measure;
3. M_k admits a flat extension M_{k+1};
4. $\beta_{1,\,2k-1} > \phi(\beta)$, where $\phi(\beta)$ is an explicit rational expression in the moments.

3. Truncated Riesz–Haviland Theorem

- Applications to planar cubic curves

Truncated Riesz–Haviland Theorem

Riesz–Haviland Theorem (M. Riesz - $n = 1$, 1923; Haviland - $n > 1$, 1935). The following statements are equivalent:

1. $\beta^{(\infty)}$ admits a K–representing measure.
2. The associated Riesz functional L_β is K–positive, i.e.,

$$p \in \mathbb{R}[\boldsymbol{x}], \;\; p|_K \geq 0 \quad \Longrightarrow \quad L_\beta(p) \geq 0.$$

Theorem (Stochel, 2001). $\beta^{(\infty)}$ has a K–representing measure if and only if $\beta^{(m)}$ has a K–representing measure for each $m \geq 1$.

Truncated Riesz–Haviland Theorem (TRHT) (Curto & Fialkow, 2008). The following statements are equivalent:

1. $\beta = \beta^{(2k)}$ or $\beta = \beta^{(2k+1)}$ admits a K–representing measure.
2. β admits an extension $\widetilde{\beta} \equiv \beta^{(2k+2)}$ such that the associated Riesz functional $L_{\widetilde{\beta}}$ is K–positive, i.e.,

$$p \in \mathbb{R}[\boldsymbol{x}]_{\leq 2k+2}, \;\; p|_K \geq 0 \quad \Longrightarrow \quad L_{\widetilde{\beta}}(p) \geq 0.$$

Nonsingular Truncated Riesz–Haviland Theorem

Theorem (di Dio & Schmüdgen, 2018). If L_β is K–strictly positive, i.e.,

$$0 \neq p \in \mathbb{R}[\boldsymbol{x}]_{\leq 2k}, \quad p|_K \geq 0 \quad \Longrightarrow \quad L_{\beta^{(2k)}}(p) > 0,$$

then $\beta^{(2k)}$ admits a K–representing measure μ.

Remark. Checking K–positivity is often difficult, since K–positive polynomials of bounded degree are hard to describe. When Hilbert's SOS theorem applies (e.g., $n = 1$; or $k = 1$; or $n = k = 2$), every $p \geq 0$ with $\deg p \leq 2k$ satisfies $p = \sum_i p_i^2$. Thus, if $M_k(\beta) \succeq 0$ then

$$L_\beta(p) = \sum_i L_\beta(p_i^2) = \sum_i \left\langle M_k(\beta)\, \widehat{p}_i, \widehat{p}_i \right\rangle \geq 0,$$

so L_β is positive.

Irreducible Cubic Normal Forms

13 affine equivalence classes

Proposition. Let $P \in \mathbb{R}[x, y]$ be an irreducible cubic polynomial. Up to an invertible affine linear change of variables, P is equivalent to *exactly one* of the following normal forms:

(Weierstraß)

1. *Smooth Weierstraß type 1:*

   $y^2 - x(x-a)(x-b),\ 0 < a < b,$

2. *Smooth Weierstraß type 2:*

   $y^2 - x(x^2 + c^2),\ c \neq 0.$

(Rational)

3. *Neile's parabola:* $y^2 - x^3.$
4. *Nodal cubic:* $y^2 - x(x-1)^2.$
5. *Isolated point:* $y^2 - x^2(x-1).$
6. *Cubic parabola:* $y - x^3.$
7. *Rational cubic type 1:*

   $xy - x^3 - q_2 x^2 - q_1 x - q_0, \quad q_0 \neq 0.$

8. *Rational cubic type 2:* $xy^2 + ay - dx - e.$

(Newton)

9. *Non-Weierstraß type 1:* $xy^2 + ay - x^2 - dx - e.$
10. *Non-Weierstraß type 2:*
    $xy^2 - x^3 - cx^2 - dx - e,\ e \neq 0.$
11. *Non-Weierstraß type 3:*
    $xy^2 + x^3 - cx^2 - dx - e,\ e \neq 0.$
12. *Non-Weierstraß type 4:*
    $xy^2 + ay - x^3 - cx^2 - dx - e,\ ae \neq 0.$
13. *Non-Weierstraß type 5:*
    $xy^2 + ay + x^3 - cx^2 - dx - e,\ ae \neq 0.$

Reducible Cubic Normal Forms

16 affine equivalence classes

Proposition. Up to an invertible affine linear change of variables, every reducible cubic polynomial P such that $\mathcal{Z}(P) \not\subseteq \mathcal{Z}(Q)$ for any $Q \in \mathbb{R}[x,y]_{\leq 2}$ is equivalent to *exactly one* of the following forms:

1. *Circular type 1:* $P_{14}(x,y) = y(ay + x^2 + y^2)$, $a \in \mathbb{R} \setminus \{0\}$.
2. *Circular type 2:* $P_{15}(x,y) = y(1 + ay + x^2 + y^2)$, $|a| > 2$.
3. *Circular type 3:* $P_{16}(x,y) = y(1 + ay - x^2 - y^2)$, $a \in \mathbb{R}$.
4. *Parabolic type 1:* $P_{17}(x,y) = y(x^2 - y)$.
5. *Parabolic type 2:* $P_{18}(x,y) = y(x - y^2)$.
6. *Parabolic type 3:* $P_{19}(x,y) = y(1 + y + x^2)$.
7. *Parabolic type 4:* $P_{20}(x,y) = y(1 + y - x^2)$.
8. *Hyperbolic type 1:* $P_{21}(x,y) = y(1 - xy)$.
9. *Hyperbolic type 2:* $P_{22}(x,y) = y(x + y + axy)$, $a \in \mathbb{R} \setminus \{0\}$.
10. *Hyperbolic type 3:* $P_{23}(x,y) = y(ay + x^2 - y^2)$, $a \in \mathbb{R} \setminus \{0\}$.
11. *Hyperbolic type 4:* $P_{24}(x,y) = y(1 + ay + x^2 - y^2)$, $|a| \neq 2$.
12. *Hyperbolic type 5:* $P_{25}(x,y) = y(1 + ay - x^2 + y^2)$.
13. *Parallel lines type:* $P_{26}(x,y) = y(a + y)(b + y)$, $a, b \in \mathbb{R} \setminus \{0\}$, $a \neq b$.
14. *Intersecting lines type 1:* $P_{27}(x,y) = y(x - y)(x + y)$.
15. *Intersecting lines type 2:* $P_{28}(x,y) = yx(y + 1)$.
16. *Intersecting lines type 3:* $P_{29}(x,y) = y(1 + x - y)(1 - x - y)$.

Some Essential Definitions

Let $P \in \mathbb{R}[x, y]_{\leq 3}$ be a cubic polynomial, and set

$$C := \mathcal{Z}(P) \subseteq \mathbb{R}^2.$$

Coordinate ring. Define the coordinate ring of C by

$$\mathbb{R}[C] \ := \ \mathbb{R}[x, y]/I, \qquad I := (P).$$

Total ring of fractions. Let

$$Q(\mathbb{R}[C])$$

denote the total ring of fractions of $\mathbb{R}[C]$. If C is irreducible, then $Q(\mathbb{R}[C])$ is the quotient field of $\mathbb{R}[C]$.

Degree truncations. For $m \in \mathbb{N}_0$, let

$$\mathbb{R}[C]_{\leq m}$$

be the image of $\mathbb{R}[x, y]_{\leq m}$ under the canonical map $\mathbb{R}[x, y] \to \mathbb{R}[C]$.

Applications of the TRHT

Cubic planar curves

Theorem I (Kummer & Z., 2025). Let $P \in \mathbb{R}[x,y]_{\leq 3}$ be cubic such that:

(1) P is irreducible and $K := \mathcal{Z}(P)$ has no isolated points; or

(2) $P = P_1 P_2$ is reducible where $P_1, P_2 \in \mathbb{R}[x,y]_{\leq 2}$, and the system $P_1(x,y) = P_2(x,y) = 0$ has no non-real solutions.

Then there exist rational functions $f, f_1 \in Q(\mathbb{R}[C])$ and an increasing sequence $\{V^{(\ell)}\}_\ell$ of vector subspaces from $Q(\mathbb{R}[C])$ with $\dim V^{(\ell)} = 3\ell$ and a spanning description

$$V^{(\ell)} = \mathrm{span}\{f_1,\, f_2, \ldots, f_{3\ell}\}, \qquad f_i \in \mathbb{R}[C]_{\leq \ell} \quad \text{for } i = 2, \ldots, 3\ell,$$

such that the following statements are equivalent for every $k \in \mathbb{N}$:

(i) $p|_C \geq 0$ where $p \in \mathbb{R}[C]_{\leq 2k}$.

(ii) There exist finitely many $g_i \in \mathbb{R}[C]_{\leq k}$, $h_j \in V^{(k)}$ such that

$$p = \sum_i g_i^2 \; + \; f \sum_j h_j^2.$$

Remark. In case (1), if $\mathcal{Z}(P)$ has an isolated point, then $V^{(k)}$ is replaced by $V^{(k-1)}$ in (ii).

$(f, V^{(\ell)})$–Localizing Moment Matrix

$V^{(\ell)} = \mathrm{span}\{f_1, f_2, \ldots, f_{3\ell}\}$

The $(f, V^{(\ell)})$–localizing moment matrix

$$M_{f,V^{(\ell)}} \equiv M_{f,V^{(\ell)}}(\beta)$$

associated to $\beta \equiv \beta^{(2\ell)} = \{\beta_{i,j}\}_{i+j\le 2\ell}$ is the symmetric matrix whose rows and columns are indexed by

$$f_1(X, Y), \; f_2(X, Y), \; \ldots, \; f_{3\ell}(X, Y),$$

and whose entries are given by

$$(M_{f,V^{(\ell)}})_{i,j} := L_\beta\big(f\, f_i f_j\big), \qquad 1 \le i, j \le 3\ell.$$

$(f, V^{(\ell)})$–Localizing Moment Matrix

Example. $P(x,y) = y^2 - x(x^2 + ax + b) = y^2 - xq(x), \quad q(x) := x^2 + ax + b$

$$f = x, \qquad f_1 = \frac{y}{x}, \qquad V^{(3)} = \mathrm{span}\left\{1,\ \frac{y}{x},\ x,\ y,\ x^2,\ xy,\ y^2,\ x^2y,\ xy^2\right\}.$$

$$M_{x,V^{(3)}} = \begin{array}{c} \\ 1 \\ Y/X \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \\ X^2Y \\ XY^2 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & Y/X & X & Y & X^2 & XY & Y^2 & X^2Y & XY^2 \end{array} \\ \begin{bmatrix} \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} \\ \beta_{01} & L_\beta(q(x)) & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} \\ \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} \\ \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} \\ \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{50} & \beta_{41} & \beta_{32} & \beta_{51} & \beta_{42} \\ \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{23} & \beta_{42} & \beta_{33} \\ \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{14} & \beta_{33} & \beta_{24} \\ \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{51} & \beta_{42} & \beta_{33} & \beta_{52} & \beta_{43} \\ \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{42} & \beta_{33} & \beta_{24} & \beta_{43} & \beta_{34} \end{bmatrix} \end{array},$$

$\beta_{52},\ \beta_{43},\ \beta_{34}$ can be expressed in terms of lower-degree moments whenever β has a representing measure supported on $K = \mathcal{Z}(P)$, since

$$L_\beta(x^i y^j P) = 0 \qquad \text{for all } i, j \in \mathbb{N} \cup \{0\}.$$

Applications of the TRHT

Cubic planar curves

M_k is P**–pure** if its column dependence relations are *exactly those forced by* $P(X, Y) = 0$ by linearity and recursive generation.

Theorem (Kummer & Z., 2025). Assume the setting of Theorem I.(1) above. Then $\beta^{(2k)}$ admits a $\mathcal{Z}(P)$–representing measure if and only if one of the following holds:

(i) $M_k \succeq 0$ is P–pure and $M_{f,V^{(k)}} \succ 0$.

(ii) There exists a unique candidate extension $\widetilde{\beta} \equiv \beta^{(2k+2)}$ such that

$$M_{k+1}(\widetilde{\beta}) \succeq 0 \qquad \text{and} \qquad M_{f,V^{(k+1)}}(\widetilde{\beta}) \succeq 0.$$

Remark. In case (ii), at least one of M_k and $M_{f,V^{(k)}}$ fails to have rank $3k$. Moreover, the proof in (ii) is constructive: one explicitly builds $\widetilde{\beta}$ and hence a representing measure.

In contrast, case (i) is existential and not constructive.

Application of the TRHT

Cubic planar curves

Theorem II (Kummer & Z., 2025). Let $P = P_1 P_2 \in \mathbb{R}[x,y]_{\leq 3}$ be a reducible cubic where $P_1, P_2 \in \mathbb{R}[x,y]_{\leq 2}$, and the system

$$P_1(x,y) = P_2(x,y) = 0$$

has two non-real solutions.

Then the following statements are equivalent for every $k \in \mathbb{N}$:

1. $p|_K \geq 0$ for some $p \in \mathbb{R}[C]_{\leq 2k}$.
2. There exist finitely many $f_i \in \mathbb{R}[C]_{\leq k}$, $h_j, g_\ell \in \mathbb{R}[C]_{\leq k}$ such that

$$p = \sum_i f_i^2 + \chi_1 P_1 \sum_j h_j^2 + \chi_2 P_2 \sum_\ell g_\ell^2,$$

   where

$$\chi_1 = \begin{cases} 1, & \text{if } P_1 \text{ is nonnegative on } \mathcal{Z}(P_2), \\ -1, & \text{if } P_1 \text{ is nonpositive on } \mathcal{Z}(P_2), \\ 0, & \text{if } P_1 \text{ changes sign on } \mathcal{Z}(P_2), \end{cases}$$

$$\chi_2 = \begin{cases} 1, & \text{if } P_2 \text{ is nonnegative on } \mathcal{Z}(P_1), \\ -1, & \text{if } P_2 \text{ is nonpositive on } \mathcal{Z}(P_1). \end{cases}$$

Irreducible Cubic Normal Forms

13 affine equivalence classes

Legend: Constructive solutions are available. Constructive solutions are missing.

(Weierstraß)

1. *Smooth Weierstraß type 1:*

   $y^2 - x(x-a)(x-b),\ 0 < a < b,$

2. *Smooth Weierstraß type 2:*

   $y^2 - x(x^2 + c^2),\ c \neq 0.$

(Rational)

3. *Neile's parabola:* $y^2 - x^3$.
4. *Nodal cubic:* $y^2 - x(x-1)^2$.
5. *Isolated point:* $y^2 - x^2(x-1)$.
6. *Cubic parabola:* $y - x^3$.
7. *Rational cubic type 1:*

   $xy - x^3 - q_2x^2 - q_1x - q_0, \quad q_0 \neq 0.$

8. *Rational cubic type 2:* $xy^2 + ay - dx - e$.

(Newton)

9. *Non-Weierstraß type 1:* $xy^2 + ay - x^2 - dx - e$.
10. *Non-Weierstraß type 2:* $xy^2 - x^3 - cx^2 - dx - e,\ e \neq 0.$
11. *Non-Weierstraß type 3:* $xy^2 + x^3 - cx^2 - dx - e,\ e \neq 0.$
12. *Non-Weierstraß type 4:* $xy^2 + ay - x^3 - cx^2 - dx - e,\ ae \neq 0.$
13. *Non-Weierstraß type 5:* $xy^2 + ay + x^3 - cx^2 - dx - e,\ ae \neq 0.$

Smooth Weierstraß type

$p_1(x,y) = y^2 - x(x-a)(x-b),\ 0 < a < b$ or $p_2(x,y) = y^2 - x(x^2+c^2),\ c \neq 0$

Theorem (Bhardwaj & Z., 2026). The following statements are equivalent:

1. $\beta^{(2k)}$ admits a $(\operatorname{rank} M_k)$–atomic $\mathcal{Z}(p)$–representing measure.
2. M_k admits a flat extension M_{k+1}.
3. A certain quadratic polynomial, completely determined by $\beta^{(2k)}$, has a real root.

Remark. Using a recent result (2024+) by Baldi, Blekherman and Sinn on the number of atoms in a minimal measure, for type 2 (i.e., p_2), the conditions (1)–(3) are equivalent to the existence of a $\mathcal{Z}(p)$–representing measure for β.

In type 1 (i.e., p_1) a representing measure may require $(\operatorname{rank} M_k + 1)$ atoms.

Open problem 1. *Find a complete constructive solution in the type 1 case, i.e., characterize when a $(\operatorname{rank} M_k + 1)$–atomic representing measure exists.*

Open problem 2. *Extend the theorem to Newton-type non-rational cubics, i.e., to curves of the form $xy^2 + ay = bx^3 + cx^2 + dx + e$.*

TMP on the nodal cubic (constructive)

$p_4(x,y) = y^2 - x(x-1)^2$

Parametrization of $\mathcal{Z}(p_4)$. $(x(t), y(t)) = (t^2,\ t^3 - t), \qquad t \in \mathbb{R}.$

Define

$$\mathrm{Nodal} := \{\, s \in \mathbb{R}[t] : s(1) = s(-1) \,\}, \qquad \mathrm{Nodal}_{\le i} := \{\, s \in \mathrm{Nodal} : \deg s \le i \,\}.$$

Correspondence. The map

$$\Phi : \mathbb{R}[C] \to \mathrm{Nodal}, \qquad \Phi\big(p(x,y)\big) := p(t^2, t^3 - t),$$

is a ring isomorphism. Moreover, $\mathbb{R}[C]_{\le i}$ corresponds bijectively to $\mathrm{Nodal}_{\le 3i}$.

Define the auxiliary subspace

$$\widetilde{\mathrm{Nodal}}_{\le i} := \{\, s \in \mathbb{R}[t]_{\le i} : s(1) = -s(-1) \,\}.$$

Theorem (Positivstellensatz). Let $p \in \mathrm{Nodal}_{\le 6k}$ satisfy $p(t) \ge 0$ for all $t \in \mathbb{R}$. Then there exist finitely many $g_i \in \mathrm{Nodal}_{\le 3k}$ and $h_j \in \widetilde{\mathrm{Nodal}}_{\le 3k}$ such that

$$p = \sum_i g_i^2 + \sum_j h_j^2.$$

TMP on the nodal cubic (constructive)

$p_4(x, y) = y^2 - x(x-1)^2$

Bases in the univariate model. The basis for $\mathrm{Nodal}_{\leq i}$ is

$$\mathcal{B}_{\mathrm{Nodal}_{\leq i}} := \left\{1,\ t^2 - 1,\ t^3 - t,\ t^4 - t^2,\ \ldots,\ t^{i-1} - t^{i-3},\ t^i - t^{i-2}\right\}.$$

The basis for $\widetilde{\mathrm{Nodal}}_{\leq i}$ is

$$\mathcal{B}_{\widetilde{\mathrm{Nodal}}_{\leq i}} := \left\{t,\ t^2 - 1,\ t^3 - t,\ t^4 - t^2,\ \ldots,\ t^{i-1} - t^{i-3},\ t^i - t^{i-2}\right\}.$$

Key observation. Under $\Phi(p) = p(t^2, t^3 - t)$ we have

$$\Phi\left(\frac{y}{x-1}\right) = t.$$

Thus $\frac{y}{x-1}$ plays the role of the generator t (and hence replaces 1 in the basis for $V^{(k)}$).

TMP on the nodal cubic (constructive)

$p_4(x,y) = y^2 - x(x-1)^2$

Using the correspondence Φ above, the C–TMP for β is equivalent to the $\mathbb{R}$–TMP for

$$\mathcal{L} : \mathrm{Nodal}_{\leq 6k} \to \mathbb{R}, \qquad \mathcal{L}(p) := L_C\big(\Phi^{-1}(p)\big).$$

Using the ordered basis $\mathcal{B}_{\mathrm{Nodal}_{\leq 3k}} \cup \mathcal{B}_{\widetilde{\mathrm{Nodal}}_{\leq 3k}}$, the moment matrix of $L_{\mathrm{Nodal}_{\leq 6k}}$ has the form

$$\begin{array}{c} 1 \\ T \\ T^2-1 \\ T^3-T \\ \vdots \\ T^{3k}-T^{3k-2} \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & T & T^2-1 & T^3-T & \cdots & T^{3k}-T^{3k-2} \end{array} \\ \left[\begin{array}{cccccc} \mathcal{L}(1) & ? & \mathcal{L}(t^2-1) & \mathcal{L}(t^3-t) & \cdots & \mathcal{L}(t^{3k}-t^{3k-2}) \\ ? & \mathcal{L}(t^2) & \mathcal{L}(t^3-t) & \mathcal{L}(t^4-t^2) & \cdots & \mathcal{L}(t^{3k+1}-t^{3k-1}) \\ \mathcal{L}(t^2-1) & \mathcal{L}(t^3-t) & \mathcal{L}((t^2-1)^2) & \mathcal{L}(t(t^2-1)^2) & \cdots & \mathcal{L}(t^{3k-2}(t^2-1)^2) \\ \mathcal{L}(t^3-t) & \mathcal{L}(t(t^3-t)) & \mathcal{L}(t(t^2-1)^2) & \mathcal{L}((t^3-t)^2) & \cdots & \mathcal{L}(t^{3k-1}(t^2-1)^2) \\ \vdots & & & & \ddots & \vdots \\ \mathcal{L}(t^{3k}-t^{3k-2}) & \cdots & & & \cdots & \mathcal{L}((t^{3k}-t^{3k-2})^2) \end{array}\right] \end{array}.$$

From here, one can characterize when $\mathcal{L}$ is an $\mathbb{R}$–moment functional and then construct a measure by completing the single unknown entry $\mathcal{L}(t)$.

Open problem 3. *Does a minimal measure require* $3k$ *or* $3k+1$ *atoms?*

Reducible Cubic Normal Forms

Open problem 4. *Find constructive solutions for missing cases below.*

Legend: Constructive solutions are available. Constructive solutions are missing.

1. *Circular type 1:* $P_{14}(x, y) = y(ay + x^2 + y^2)$, $a \in \mathbb{R} \setminus \{0\}$.
2. *Circular type 2:* $P_{15}(x, y) = y(1 + ay + x^2 + y^2)$, $|a| > 2$.
3. *Circular type 3:* $P_{16}(x, y) = y(1 + ay - x^2 - y^2)$, $a \in \mathbb{R}$.
4. *Parabolic type 1:* $P_{17}(x, y) = y(x^2 - y)$.
5. *Parabolic type 2:* $P_{18}(x, y) = y(x - y^2)$.
6. *Parabolic type 3:* $P_{19}(x, y) = y(1 + y + x^2)$.
7. *Parabolic type 4:* $P_{20}(x, y) = y(1 + y - x^2)$.
8. *Hyperbolic type 1:* $P_{21}(x, y) = y(1 - xy)$.
9. *Hyperbolic type 2:* $P_{22}(x, y) = y(x + y + axy)$, $a \in \mathbb{R} \setminus \{0\}$.
10. *Hyperbolic type 3:* $P_{23}(x, y) = y(ay + x^2 - y^2)$, $a \in \mathbb{R} \setminus \{0\}$.
11. *Hyperbolic type 4:* $P_{24}(x, y) = y(1 + ay + x^2 - y^2)$, $|a| \neq 2$.
12. *Hyperbolic type 5:* $P_{25}(x, y) = y(1 + ay - x^2 + y^2)$.
13. *Parallel lines type:* $P_{26}(x, y) = y(a + y)(b + y)$, $a, b \in \mathbb{R} \setminus \{0\}$, $a \neq b$.
14. *Intersecting lines type 1:* $P_{27}(x, y) = y(x - y)(x + y)$.
15. *Intersecting lines type 2:* $P_{28}(x, y) = yx(y + 1)$.
16. *Intersecting lines type 3:* $P_{29}(x, y) = y(1 + x - y)(1 - x - y)$.

TMP for $y(ay + x^2 + y^2) = 0$ (constructive)

Joint work with Seonguk Yoo

$$\beta = \{\beta_{i,j}\}_{i,j\in\mathbb{Z}_+,\ 0\le i+j\le 2k}.$$

has a C–representing measure if and only if it admits a decomposition

$$\beta \ = \ \beta^{(\ell)} \ + \ \beta^{(c)},$$

where

$\beta^{(\ell)} = \{\beta^{(\ell)}_{i,j}\}_{0\le i+j\le 2k}$ admits a measure on $y = 0$,

$\beta^{(c)} = \{\beta^{(c)}_{i,j}\}_{0\le i+j\le 2k}$ admits a $\mathcal{Z}(ay + x^2 + y^2)$–representing measure.

Moreover, all moments of $\beta^{(\ell)}$ and $\beta^{(c)}$ are uniquely determined except the four scalars

$$\beta^{(\ell)}_{0,0},\ \beta^{(\ell)}_{1,0}, \qquad \beta^{(c)}_{0,0},\ \beta^{(c)}_{1,0},$$

which are constrained by the matching relations

$$\beta_{j,0} = \beta^{(\ell)}_{j,0} + \beta^{(c)}_{j,0}, \qquad j = 0, 1.$$

4. Core Variety Theorem

- Trichotomy in the sextic case in $\mathbb{R}^2$
- Trichotomy for irreducible planar cubics
- Quadrichotomy for reducible planar cubics

Core Variety Theorem

Let $\beta \equiv \beta^{(2k)}$. The core variety $\mathcal{CV}(L_\beta)$ is defined recursively by:

$$\begin{aligned} V_0 &:= V(M_k), \\ V_{i+1} &:= \bigcap \left\{ \mathcal{Z}(f) : \ f \in \ker L_\beta, \ f\big|_{V_i} \geq 0 \right\} \qquad (i \geq 0), \\ \mathcal{CV}(L_\beta)\} &:= \bigcap_{i \geq 0} V_i. \end{aligned}$$

Core Variety Theorem (Blekherman & Fialkow, 2020). The following statements are equivalent:

1. $\beta^{(2k)}$ admits a representing measure.
2. $\mathcal{CV}(L_\beta)$ is nonempty.

In this case, $\mathcal{CV}(L)$ coincides with the union of supports of all finitely atomic representing measures for L_β.

Core Variety for Lines and Conics ($\deg p \leq 2$)

Let $p(x, y) \in \mathbb{R}[x, y]_{\leq 2}$ and assume $M_k(\beta) \succeq 0$.

- ***p*-pure case:** If $M_k(\beta)$ is p-pure, then

$$\mathcal{CV}(L_\beta) = \mathcal{V}(L_\beta) = \mathcal{Z}(p) = \{(x, y) : p(x, y) = 0\}.$$

- **Not *p*-pure, irreducible *p*:** If $M_k(\beta)$ is recursively generated and $p(X, Y) = 0$, then exactly one holds:

$$\text{(i) } |\mathcal{CV}(L_\beta)| = \operatorname{rank} M_n \;\Rightarrow\; \exists!\ \mu \text{ with } \operatorname{supp}\mu = \mathcal{CV}(L_\beta),$$

$$\text{(ii) } \mathcal{CV}(L_\beta) = \varnothing \;\Rightarrow\; \text{no representing measure.}$$

- **Not *p*-pure, reducible $p = p_1 p_2$:** If $M_k(\beta)$ is recursively generated and $p(X, Y) = 0$, then exactly one holds:
  - (i) $\mathcal{CV}(\beta) = \mathcal{Z}(p_1) \cup \{w_1, \ldots, w_\ell\}$, $0 \leq \ell \leq k$, $w_i \in \mathcal{Z}(p_2) \setminus \mathcal{Z}(p_1)$;
  - (ii) $\mathcal{CV}(\beta) = \mathcal{Z}(p_2) \cup \{w_1, \ldots, w_\ell\}$, $0 \leq \ell \leq k$, $w_i \in \mathcal{Z}(p_1) \setminus \mathcal{Z}(p_2)$;
  - (iii) $|\mathcal{CV}(L_\beta)| = \operatorname{rank} M_n \Rightarrow \exists! \mu,\ \operatorname{supp}\mu = \mathcal{CV}(L_\beta)$;
  - (iv) $\mathcal{CV}(L_\beta) = \varnothing \Rightarrow$ no representing measure.

Core Variety Trichotomy

Theorem (Fialkow, 2017). Assume that the moment matrix $M_3(\beta)$ is positive definite.

Then **exactly one** of the following occurs:

(i) **Full support case:**

$$\mathcal{CV}(L_\beta) = \mathbb{R}^2$$

and L_β admits a representing measure.

(ii) **Finite case:**

$$|\mathcal{CV}(L_\beta)| = 10$$

and L_β admits a representing measure with

$$\operatorname{supp} \mu = \mathcal{CV}(L_\beta).$$

(iii) **Empty case:**

$$\mathcal{CV}(L_\beta) = \emptyset$$

and L_β has no representing measure.

Core Variety Trichotomy for Curves $y = x^d$

Theorem (Fialkow, Z., 2025). Let $\beta \equiv \beta^{(2n)}$ and assume

$$M_n(\beta) \succeq 0 \qquad \text{and} \qquad M_n(\beta) \text{ is } (y - x^d)\text{–pure.}$$

Set

$$\Gamma := \mathcal{Z}(y - x^d).$$

Then **exactly one** of the following occurs:

(i) **Full curve case:**

$$\mathcal{CV}(L_\beta) = \Gamma$$

and L_β admits a representing measure.

(ii) **Finite case:** $\mathcal{CV}(L_\beta)$ is finite, and L_β admits a unique representing measure supported on

$$\mathcal{CV}(L_\beta).$$

(iii) **Empty case:**

$$\mathcal{CV}(L_\beta) = \emptyset$$

and L_β has no representing measure.

Core Variety Trichotomy for Cubic Curves

Theorem (Yoo, Z., in preparation).
Let $P(x,y)$ be an *irreducible* cubic and assume that $M_n(\beta) \succeq 0$ is P–pure.

Then **exactly one** of the following occurs:

(i) **Full curve case:**

$$\mathcal{CV}(L_\beta) = \mathcal{Z}(P)$$

and L_β admits a representing measure.

(ii) **Finite case:**

$$|\mathcal{CV}(L_\beta)| = \operatorname{rank} M_n = 3n$$

There is a unique representing measure, supported on $\mathcal{CV}(L_\beta)$.

(iii) **Empty case:**

$$\mathcal{CV}(L_\beta) = \emptyset$$

and L_β has no representing measure.

Open problem 5. *Does finiteness of the core variety imply uniqueness of the representing measure?*

Applications of the Core Variety Theorem

A cubic planar curve $y = x^3$

We say that M_k is $(y - x^3)$–pure if its column dependence relations are exactly those obtained from the basic relation $Y = X^3$ by linearity and recursiveness.

Theorem (Fialkow, Z., 2025).

Assume M_k is positive semidefinite and $(y - x^3)$–pure. Then the following are equivalent:

1. β admits a representing measure supported on $\{(x, y) : y = x^3\}$;
2. β admits a finitely atomic representing measure;
3. $\mathcal{CV}(L_\beta) = \mathcal{Z}(y - x^3)$;
4. $\beta_{1,2k-1} > \phi(\beta)$, where $\phi(\beta)$ is an explicit rational expression in the moments.

5. The curve $y = x^4$

TMP for the first quartic case: $y = x^4$

Reduction to a univariate TMP. Let $\beta^{(2k)} = (\beta_{i,j})_{0 \le i+j \le 2k}$. Then $\beta^{(2k)}$ has a representing measure supported on $\Gamma := \mathcal{Z}(y - x^4)$ if and only if the univariate sequence $\gamma = (\gamma_m)_{m \le 8k}$, defined by

$$\gamma_{i+4j} = \beta_{i,j} \qquad (0 \le i, j \le k),$$

admits an $\mathbb{R}$–representing measure.

Gaps and completion. Not all moments of γ are determined by $\beta^{(2k)}$: there is a gap at

$$\gamma_{8k-1}, \ \gamma_{8k-2}, \ \gamma_{8k-5}.$$

Hence the associated problem becomes a matrix completion problem. The question is whether one can choose the auxiliary moments of a Hankel matrix so that either:

- the completed matrix is positive definite, or
- it is positive semidefinite and singular and the last column does not increase the rank.

TMP for the first quartic case: $y = x^4$

The corresponding Hankel matrix for $n = d = 4$ is as follows. Undetermined moments are denoted by $\mathbf{A}_{27}$, $\mathbf{A}_{30}$, $\mathbf{A}_{31}$.

$$\begin{pmatrix}
\gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} \\
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} & \gamma_{17} \\
\gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} & \gamma_{17} & \gamma_{18} \\
\gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} & \gamma_{17} & \gamma_{18} & \gamma_{19} \\
\gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} & \gamma_{17} & \gamma_{18} & \gamma_{19} & \gamma_{20} \\
\gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} & \gamma_{17} & \gamma_{18} & \gamma_{19} & \gamma_{20} & \gamma_{21} \\
\gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} & \gamma_{17} & \gamma_{18} & \gamma_{19} & \gamma_{20} & \gamma_{21} & \gamma_{22} \\
\gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} & \gamma_{17} & \gamma_{18} & \gamma_{19} & \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{23} \\
\gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} & \gamma_{17} & \gamma_{18} & \gamma_{19} & \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} \\
\gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} & \gamma_{17} & \gamma_{18} & \gamma_{19} & \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & \gamma_{25} \\
\gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} & \gamma_{17} & \gamma_{18} & \gamma_{19} & \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & \gamma_{25} & \gamma_{26} \\
\gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} & \gamma_{17} & \gamma_{18} & \gamma_{19} & \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & \gamma_{25} & \gamma_{26} & \mathbf{A}_{27} \\
\gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} & \gamma_{17} & \gamma_{18} & \gamma_{19} & \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & \gamma_{25} & \gamma_{26} & \mathbf{A}_{27} & \gamma_{28} \\
\gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} & \gamma_{17} & \gamma_{18} & \gamma_{19} & \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & \gamma_{25} & \gamma_{26} & \mathbf{A}_{27} & \gamma_{28} & \gamma_{29} \\
\gamma_{14} & \gamma_{15} & \gamma_{16} & \gamma_{17} & \gamma_{18} & \gamma_{19} & \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & \gamma_{25} & \gamma_{26} & \mathbf{A}_{27} & \gamma_{28} & \gamma_{29} & \mathbf{A}_{30} \\
\gamma_{15} & \gamma_{16} & \gamma_{17} & \gamma_{18} & \gamma_{19} & \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & \gamma_{25} & \gamma_{26} & \mathbf{A}_{27} & \gamma_{28} & \gamma_{29} & \mathbf{A}_{30} & \mathbf{A}_{31} \\
\gamma_{16} & \gamma_{17} & \gamma_{18} & \gamma_{19} & \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & \gamma_{25} & \gamma_{26} & \mathbf{A}_{27} & \gamma_{28} & \gamma_{29} & \mathbf{A}_{30} & \mathbf{A}_{31} & \gamma_{32}
\end{pmatrix}$$

We index the columns of this Hankel matrix by

$$\mathcal{B} := \{1, T, T^2, \ldots, T^{4n-6}, T^{4n-5}, T^{4n-4}, T^{4n-3}, T^{4n-2}, T^{4n-1}, T^{4n}\},$$

where colored columns contain auxiliary moments in the last row.

TMP for the first quartic case: $y = x^4$

Index sets. Let $\mathcal{B}$ be the ambient monomial basis. Set

$$\mathcal{S}_1 := \mathcal{B} \setminus \{T^{4n-3}, T^{4n-2}, T^{4n-1}, T^{4n}\},$$
$$\mathcal{S}_2 := \mathcal{B} \setminus \{T^{4n-2}, T^{4n-1}, T^{4n}\},$$
$$\mathcal{S}_3 := \mathcal{B} \setminus \{T^{4n-3}, T^{4n-1}, T^{4n}\},$$
$$\mathcal{S}_4 := \mathcal{B} \setminus \{T^{4n-1}, T^{4n}\},$$
$$\mathcal{S}_5 := \mathcal{B} \setminus \{T^{4n-2}, T^{4n-1}\}.$$

Set $C_i := C|_{\mathcal{S}_i}$.

Deterministic blocks. C_1, C_2, C_3 are completely determined by β.

Block structure.

$$C_2 = \begin{pmatrix} C_1 & u \\ u^t & \gamma_{8n-6} \end{pmatrix}, \qquad C_3 = \begin{pmatrix} C_1 & v \\ v^t & \gamma_{8n-4} \end{pmatrix},$$

$$C_4[A_{8n-5}] = \begin{pmatrix} C_2 & w \\ w^t & \beta_{8n-4} \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ A_{8n-5} \end{pmatrix},$$

$$C_5[A_{8n-5}] = \begin{pmatrix} C_2 & z \\ z^t & \gamma_{8n} \end{pmatrix}, \quad z^t = \begin{pmatrix} z_1^t & A_{8n-5} & \gamma_{8n-4} & \gamma_{8n-3} \end{pmatrix}.$$

Extremal choices.

$$(A_{8n-5})_\pm = v^t C_1^{-1} u \pm \sqrt{(C_2/C_1)(C_3/C_1)}.$$

Propagation coefficients. In $C_4[(A_{8n-5})_\pm]$ we have the relation

$$T^{4n-2} = \sum_{i=1}^{4n-2} \varphi_i^{(\pm)} T^{i-1}, \qquad \varphi_i^{(\pm)} \in \mathbb{R}.$$

Define

$$(A_{8n-2})_\pm = \sum_{i=1}^{4n-2} \varphi_i^{(\pm)} [T^{i-1}]_{Y^n},$$

$$(A_{8n-1})_\pm = \sum_{i=1}^{4n-2} \varphi_i^{(\pm)} [T^{i}]_{Y^n}.$$

TMP for the first quartic case: $\Gamma := \mathcal{Z}(y - x^4)$

Theorem (Fialkow, Z., 2025). Suppose $M_n(\beta) \succeq 0$ and $M_n(\beta)$ is $(y - x^4)$–pure. Then $\beta \equiv \beta^{(2n)}$ admits a Γ–representing measure if and only if the following hold:

(i) $C_2 \succ 0$.

(ii) $C_3 \succeq 0$.

(iii) One of the following alternatives holds:

(a) The last column of

$$C\left[(A_{8n-5})_-,\ (A_{8n-2})_-,\ (A_{8n-1})_-\right]$$

is linearly dependent on the previous columns.

(b) The last column of

$$C\left[(A_{8n-5})_+,\ (A_{8n-2})_+,\ (A_{8n-1})_+\right]$$

is linearly dependent on the previous columns.

(c) There exists

$$A_{8n-5} \in \left((A_{8n-5})_-, (A_{8n-5})_+\right)$$

such that $\delta \leq \rho$, where

$$\delta := \left([T^{4n-1}]_{\mathcal{S}_4}\right)^t \left(C_4[A_{8n-5}]\right)^{-1} [T^{4n-1}]_{\mathcal{S}_4},$$

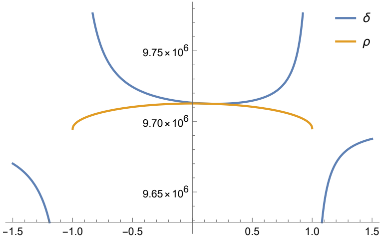
$$\rho := ([T^{4n}]_{\mathcal{S}_2})^t\, C_2{}^{-1}\, [T^{4n-2}]_{\mathcal{S}_2} + \sqrt{\left(C_4[A_{8n-5}]/C_2\right)\left(C_5[A_{8n-5}]/C_2\right)}.$$

Example: Unique measure via $\delta = \rho$

Let $\beta \equiv \beta^{(8)}$ be determined by

$$\begin{aligned}
&\beta_{00}=1,\ \beta_{01}=2,\ \beta_{02}=14,\ \beta_{03}=132, && \beta_{04}=1430,\ \beta_{05}=16796,\ \beta_{06}=208012,\ \beta_{07}=2674440,\\
&\beta_{10}=0,\ \beta_{11}=0,\ \beta_{12}=0,\ \beta_{13}=0, && \beta_{14}=0,\ \beta_{15}=0,\ \beta_{16}=0,\ \beta_{17}\approx 135.39,\\
&\beta_{20}=1,\ \beta_{21}=5,\ \beta_{22}=42,\ \beta_{23}=429, && \beta_{24}=4862,\ \beta_{25}=58786,\ \beta_{26}=742900,\ \beta_{27}=A_{27},\\
&\beta_{30}=0,\ \beta_{31}=0,\ \beta_{32}=0,\ \beta_{33}=0, && \beta_{34}=0,\ \beta_{35}=0,\ \beta_{36}=A_{36},\ \beta_{37}=A_{37},\\
& && \beta_{08}=353576708,
\end{aligned}$$

such that $M_4(\beta) \succeq 0$ and $M_4(\beta)$ is $(y - x^4)$–pure.
Computations reveals:



Conclusion. By Theorem, a representing measure μ exists and is unique. We have

$$\text{card } \mathcal{CV}(L_\beta) = 15 > \text{rank } M_4 = 14,$$

so here card $\mathcal{CV}(L_\beta) >$ rank M_n. This contrasts all previously known examples of finite core variety.

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