The tracial moment problem on quadratic varieties

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joint work with

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Notation

- $\langle X, Y \rangle$ ... the **free monoid** generated by the noncommuting letters X, Y, i.e., **words** in X, Y.
- $\mathbb{R}\langle X, Y \rangle$ ... the free algebra of polynomials in X, Y (**noncommutative (nc) polynomials**)
  involution $p \mapsto p^*$ fixes $\mathbb{R} \cup \{X, Y\}$ and reverses words

Example

$(3XY^2 - YX)^* = 3Y^2X - XY.$

- A word v is **cyclically equivalent** to w ($v \overset{\text{cyc}}{\sim} w$) iff

$$\exists u_1, u_2 : \qquad v = u_1 u_2, \quad w = u_2 u_1.$$

Bivariate truncated tracial sequence

Bivariate truncated tracial sequence (BTTS) of order n is a sequence of real numbers,

$$\beta \equiv \beta^{(2n)} = (\beta_w)_{|w| \le 2n},$$

indexed by words w in X, Y of length at most $2n$ such that

1. $\beta_v = \beta_w$ whenever $v \overset{\text{cyc}}{\sim} w$,
2. $\beta_w = \beta_{w^*}$ for all $|w| \le 2n$,

Example

For $t \in \mathbb{N}$ and a pair (A, B) of symmetric real $t \times t$ matrices,

$$\beta_w = \operatorname{tr}(w(A, B))$$

is a BTTS of order n.

Bivariate truncated tracial moment problem

Question

Which BTTS's are convex combinations (finite) of BTTS's as in the example above?

Remark

- The size of the matrices is not bounded.
- Studying probability measures on pairs of symmetric matrices is not a more general problem
  Burgdorf, Cafuta, Klep, Povh, 2013.

For future reference we call the pairs of matrices in the convex combination **atoms**, and their weights **densities**.

Truncated moment matrix $\mathcal{M}_n$

Index rows and columns of $\mathcal{M}_n$ by words in $\mathbb{R}\langle X, Y\rangle_{\leq n}$ in the degree-lexicographic order.

The entry in a row w_1 and a column w_2 of $\mathcal{M}_n$ is $\beta_{w_1^* w_2}$:

$$\mathcal{M}_n = \begin{array}{c} \\ \mathbb{1} \\ \mathbb{X} \\ \vdots \\ w_1 \\ \vdots \\ \mathbb{Y}^n \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbb{1} & \mathbb{X} & \cdots & w_2 & \cdots & \mathbb{Y}^n \end{array} \\ \begin{pmatrix} \beta_1 & \beta_X & \cdots & \beta_{w_2} & \cdots & \beta_{Y^n} \\ \beta_X & \beta_{X^2} & \cdots & \beta_{Xw_2} & \cdots & \beta_{XY^n} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \beta_{w_1} & \beta_{w_1^* X} & \cdots & \beta_{w_1^* w_2} & \cdots & \beta_{w_1^* Y^n} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \beta_{Y^n} & \beta_{XY^n} & \cdots & \beta_{Y^n w_2} & \cdots & \beta_{Y^{2n}} \end{pmatrix} \end{array}.$$

$n = 2$: 7×7 moment matrix $\mathcal{M}_2$

$$
\begin{array}{c c}
 & \begin{array}{ccccccc} \mathbb{1} & \mathbb{X} & \mathbb{Y} & \mathbb{X}^2 & \mathbb{X}\mathbb{Y} & \mathbb{Y}\mathbb{X} & \mathbb{Y}^2 \end{array} \\
\begin{array}{c} \mathbb{1} \\ \mathbb{X} \\ \mathbb{Y} \\ \mathbb{X}^2 \\ \mathbb{X}\mathbb{Y} \\ \mathbb{Y}\mathbb{X} \\ \mathbb{Y}^2 \end{array} &
\begin{pmatrix}
\beta_1 & \beta_X & \beta_Y & \beta_{X^2} & \beta_{XY} & \beta_{XY} & \beta_{Y^2} \\
\beta_X & \beta_{X^2} & \beta_{XY} & \beta_{X^3} & \beta_{X^2Y} & \beta_{X^2Y} & \beta_{XY^2} \\
\beta_Y & \beta_{XY} & \beta_{Y^2} & \beta_{X^2Y} & \beta_{XY^2} & \beta_{XY^2} & \beta_{Y^3} \\
\beta_{X^2} & \beta_{X^3} & \beta_{X^2Y} & \beta_{X^4} & \beta_{X^3Y} & \beta_{X^3Y} & \beta_{X^2Y^2} \\
\beta_{XY} & \beta_{X^2Y} & \beta_{XY^2} & \beta_{X^3Y} & \color{red}{\beta_{X^2Y^2}} & \color{red}{\beta_{XYXY}} & \beta_{XY^3} \\
\beta_{XY} & \beta_{X^2Y} & \beta_{XY^2} & \beta_{X^3Y} & \color{red}{\beta_{XYXY}} & \color{red}{\beta_{X^2Y^2}} & \beta_{XY^3} \\
\beta_{Y^2} & \beta_{XY^2} & \beta_{Y^3} & \beta_{X^2Y^2} & \beta_{XY^3} & \beta_{XY^3} & \beta_{Y^4}
\end{pmatrix}
\end{array}
$$

If $\beta_{X^2Y^2} = \beta_{XYXY}$, then the BQTMP reduces to the classical, commutative bivariate quartic moment problem.

Bivariate TMP - known results

Commutative TMP:

- *Curto and Fialkow (1996-2014):*

  a complete solution in case of a singular $\mathcal{M}_2$ using rank-preserving extension of $\mathcal{M}_n$ to $\mathcal{M}_{n+1}$.

- *Fialkow, Nie (2010) and Curto, Yoo (2016):*

  a solution of the quartic TMP with a non-singular $\mathcal{M}_2$.

Tracial TMP:

- *Burgdorf, Klep (2010, 2012):*

  A solution of the tracial quartic TMP with a non-singular $\mathcal{M}_2$.

Our motivation:

- Study the tracial TMP with a singular $\mathcal{M}_2$.

Our results

$\mathcal{M}_2$ is *singular* in $\mathcal{M}_n$ and $\beta_{X^2Y^2} \neq \beta_{XYXY}$.

1. Already for $n = 2$ the existence of a rank-preserving extension of $\mathcal{M}_2$ to $\mathcal{M}_3$ is mostly not a necessary condition for the existence of a measure.

2. If $\text{rank}(\mathcal{M}_2) \leq 3$, then β does not admit a measure.

   Easy observation: $\mathbb{1}, \mathbb{X}, \mathbb{Y}, \mathbb{XY}$ *must be linearly independent.*

3. For $\text{rank}(\mathcal{M}_2) \in \{4, 5\}$, we characterize when a measure exists, construct the minimal measure and describe its uniqueness.

   - *rank 4: 1 atom of size 2*
   - *rank 5, using ALT: one relation* $\mathbb{XY} + \mathbb{YX} = \mathbf{0}$ *and the second*

   $$\mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1} \quad \textit{or} \quad \mathbb{Y}^2 - \mathbb{X}^2 = \mathbb{1} \quad \textit{or} \quad \mathbb{Y}^2 = \mathbb{1} \quad \textit{or} \quad \mathbb{Y}^2 = \mathbb{X}^2.$$

   - *rank 5: atoms of size 2 suffice, in the quartic case 1 atom of size 2 and at most 3 atoms of size 1.*

Our results

4 If $n = 2$ and rank$(\mathcal{M}_2) = 6$, then the existence of a measure is almost always equivalent to the feasibilty of certain linear matrix inequalities and atoms of size 2 suffice.

- *using ALT: the column relation is one of*

$$\mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X} = \mathbf{0} \text{ or } \mathbb{X}^2 + \mathbb{Y}^2 = \mathbb{1} \text{ or } \mathbb{Y}^2 - \mathbb{X}^2 = \mathbb{1} \text{ or } \mathbb{Y}^2 = \mathbb{1}.$$

- *first 3 cases: 1 atom of size 2 and at most 6 of size 1*
- $\mathbb{Y}^2 = \mathbb{1}$: *size 3 atoms not needed - a brute force argument*

5 For $\mathcal{M}_3$ with $\mathbb{Y}^2 = 1$ atoms of size 2 not sufficient, on the contrary to the other three rank 6 relations.

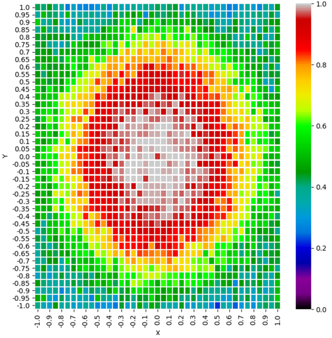
6 Simplification of the proof of the cm TMP satisfying $\mathbb{X}\mathbb{Y} = 0$.

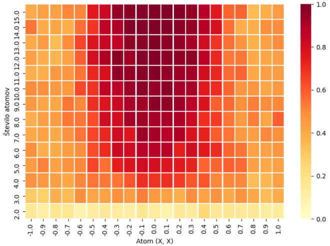
- *reduction to one variable and the use of the solution of the truncated Hamburger MP*
- *original proof by Curto and Fialkow uses bivariate flat extension theorem*

Our results

7 Some statistical analysis for the non-singular case of $\mathcal{M}_2$ (joint with Nace Gorenc)

- *by Burgdorf and Klep result, at most 15 size 2 atoms needed*
- *conjecture: 1 atom of size 2 and up to 6 atoms of size 1 suffice*
- *Curto and Yoo constructive proof for non-singular $\mathcal{M}_2$: subtract $\mathcal{M}_2^{(x,y)}$ from $\mathcal{M}_2$ for some (x, y) to rank 5 matrix and extend flatly*





Thank you for your attention!