Operator Positivstellensätze for noncommutative polynomials positive on matrix convex sets

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Free sets and matrix convex sets

We use $\mathbb{S}_n$ to denote real symmetric $n \times n$ matrices and $\mathbb{S}^g$ for the sequence $(\mathbb{S}_n^g)_n$. A **subset** Γ of $\mathbb{S}^g$ is a sequence $\Gamma = (\Gamma(n))_n$, where $\Gamma(n) \subseteq \mathbb{S}_n^g$ for each n. The subset Γ is **closed with respect to direct sums** if $A = (A_1, \dots, A_g) \in \Gamma(n)$ and $B = (B_1, \dots, B_g) \in \Gamma(m)$ implies

$$A \oplus B = \left(\begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix}, \dots, \begin{bmatrix} A_g & 0 \\ 0 & B_g \end{bmatrix} \right) \in \Gamma(n+m).$$

It is closed with respect to **simultaneous unitary conjugation** if for each n, each $A \in \Gamma(n)$ and each $n \times n$ unitary matrix U,

$$U^* A U = (U^* A_1 U, \dots, U^* A_g U) \in \Gamma(n).$$

Free sets and matrix convex sets

The set Γ is a **free set** if it is closed with respect to direct sums and simultaneous unitary conjugation. If in addition it is closed with respect to **simultaneous isometric conjugation**, i.e., if for each $m \leq n$, each $A = (A_1, \dots, A_g) \in \Gamma(n)$, and each isometry $V : \mathbb{R}^m \to \mathbb{R}^n$,

$$V^*AV = (V^*A_1V, \dots, V^*A_gV) \in \Gamma(m),$$

then Γ is **matrix convex**.

Linear pencils and LOI sets

Let $\mathscr{H}$ be a separable real Hilbert space and $I_{\mathscr{H}}$ the identity operator on $\mathscr{H}$. For self-adjoint operators $A_0, A_1, \dots, A_g \in \mathbb{S}_{\mathscr{H}}$, the expression

$$L(x) = A_0 + \sum_{j=1}^{g} A_j x_j$$

is a **linear operator pencil**. If $\mathscr{H}$ is finite-dimensional, then $L(x)$ is a **linear matrix pencil**. If $A_0 = I_{\mathscr{H}}$, then L is **monic**. To every tuple $A = (A_1, \dots, A_g) \in \mathbb{S}_{\mathscr{H}}^g$ we associate a monic linear pencil L_A by

$$L_A(x) := I_{\mathscr{H}} + \Lambda_A(x), \quad \text{where} \quad \Lambda_A(x) := \sum_{j=1}^{g} A_j x_j.$$

Linear pencils and LOI sets

Given a tuple of self-adjoint matrices $X = (X_1, \dots, X_g) \in \mathbb{S}_n^g$, the **evaluation** $L(X)$ is defined as

$$L(X) = A_0 \otimes I_n + \sum_{j=1}^{g} A_j \otimes X_j.$$

We call the set

$$D_L(1) = \{x \in \mathbb{R}^g : L(x) \succeq 0\}$$

a **Hilbert spectrahedron** or a **LOI domain**, the set

$$D_L = (D_L(n))_n \quad \text{where} \quad D_L(n) = \{X \in \mathbb{S}_n^g : L(X) \succeq 0\},$$

a **free Hilbert spectrahedron** or a **free LOI set**.

Matrix convex sets and LOI sets

Remark

1. *The set* $D_L(1) \subseteq \mathbb{R}^g$ *is a closed convex set and by the classical Hahn-Banach theorem every convex closed subset of* $\mathbb{R}^g$ *is of this form.*
2. *By Effros-Winkler Hahn-Banach theorem every closed matrix convex set which contains 0 is a matrix solution set of a LOI.*

Inclusion of free Hilbert spectrahedra - problem

Let $\mathscr{H}_1, \mathscr{H}_2$ be separable real Hilbert spaces. Given L_1 and L_2 monic linear operator pencils

$$L_1(x) \; := \; I_{\mathscr{H}_1} + \sum_{j=1}^{g} A_j x_j, \quad L_2(x) := I_{\mathscr{H}_2} + \sum_{j=1}^{g} B_j x_j,$$

where $A_j \in \mathbb{S}_{\mathscr{H}_1}$ and $B_j \in \mathbb{S}_{\mathscr{H}_2}$ for $j = 1, \ldots, g$, we are interested in the algebraic characterization of the inclusion of the free LOI sets

$$D_{L_1} \subseteq D_{L_2}.$$

Inclusion of free Hilbert spectrahedra - solution

Theorem (Operator linear Positivstellensatz)

Let L_j, $j = 1,2$, be monic linear operator pencils with coefficients from $B(\mathscr{H}_j)$, $j = 1,2$. Then $D_{L_1} \subseteq D_{L_2}$ if and only if there exist a separable real Hilbert space $\mathscr{K}$, a contraction $V : \mathscr{H}_2 \to \mathscr{K}$, a positive semidefinite operator $S \in B(\mathscr{H}_2)$ and a $*$-homomorphism $\pi : B(\mathscr{H}_1) \to B(\mathscr{K})$ such that

$$L_2 = S + V^* \pi(L_1) V.$$

Moreover, if $D_{L_1}(1)$ is bounded, then V can be chosen to be isometric and π a unital $*$-homomorphism, i.e.,

$$L_2 = V^* \pi(L_1) V.$$

Linear Positivstellensatz - sketch of the proof

Proof.

We define the operator systems:

$$\begin{aligned}\widetilde{\mathcal{S}_1} &:= \operatorname{span}\{I_{\mathscr{H}_1} \oplus 1, A_1 \oplus 0, \dots, A_g \oplus 0\} \subseteq B(\mathscr{H}_1 \oplus \mathbb{R}),\\ \mathcal{S}_2 &:= \operatorname{span}\{I_{\mathscr{H}_2}, B_1, \dots, B_g\} \subseteq B(\mathscr{H}_2),\end{aligned}$$

and the unital $*$-linear map

$$\tilde{\tau} : \widetilde{\mathcal{S}_1} \to \mathcal{S}_2, \quad A_j \oplus 1 \mapsto B_j.$$

Linear Positivstellensatz - sketch of the proof

Proof.

For $n \in \mathbb{N}$, $\tilde{\tau}$ induces the map

$$\tilde{\tau}_n : \mathcal{S}_1^{n \times n} \to \mathcal{S}_2^{n \times n},$$

$$\tilde{\tau}_n\left(\begin{bmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn} \end{bmatrix}\right) = \begin{bmatrix} \tilde{\tau}(T_{11}) & \cdots & \tilde{\tau}(T_{1n}) \\ \vdots & \ddots & \vdots \\ \tilde{\tau}(T_{n1}) & \cdots & \tilde{\tau}(T_{nn}) \end{bmatrix}.$$

We say that $\tilde{\tau}$ is n-**positive** if $\tilde{\tau}_n$ is a positive map. If $\tilde{\tau}$ is n-positive for every $n \in \mathbb{N}$, then $\tilde{\tau}$ is **completely positive (cp)**.

Linear Positivstellensatz - sketch of the proof

Proof.

Lemma

The map $\tilde{\tau}$ is cp if and only if $D_{L_1} \subseteq D_{L_2}$.

Linear Positivstellensatz - sketch of the proof

Proof.

Lemma

The map $\tilde{\tau}$ is cp if and only if $D_{L_1} \subseteq D_{L_2}$.

By the real version of Arveson's extension theorem there exists a cp extension $\tilde{\tau} : B(\mathscr{H}_1) \to B(\mathscr{H}_2)$ for $\tilde{\tau} : \widetilde{\mathcal{S}_1} \to \mathcal{S}_2$.

Linear Positivstellensatz - sketch of the proof

Proof.

Lemma

The map $\tilde{\tau}$ is cp if and only if $D_{L_1} \subseteq D_{L_2}$.

By the real version of Arveson's extension theorem there exists a cp extension $\tilde{\tau} : B(\mathscr{H}_1) \to B(\mathscr{H}_2)$ for $\tilde{\tau} : \widetilde{\mathcal{S}_1} \to \mathcal{S}_2$.

By the Stinespring theorem, there exist a separable real Hilbert space $\mathscr{K}$, a $*$-homomorphism π and an isometry $V : \mathscr{H}_2 \to \mathscr{K}$ such that

$$\tilde{\tau}(C) = V^* \pi(C) V$$

for all $C \in B(\mathscr{H}_1 \oplus \mathbb{R})$. □

Monicity necessary

Example

Let $L(y) = \begin{bmatrix} 1 & y \\ y & 0 \end{bmatrix}$ be a linear matrix polynomial with a spectrahedron $D_L = \{0\}$. The polynomial

$$\ell(y) = y$$

is non-negative on $D_L(1)$, but there do not exist a Hilbert space $\mathscr{K}$, a unital $*$-homomorphism $\pi : B(\mathbb{R}^2) \to B(\mathscr{K})$, polynomials $r_j \in \mathbb{R}\langle y \rangle$ and operator polynomials $q_k \in B(\mathbb{R}, \mathscr{K})\langle y \rangle$ such that

$$y = \sum_j r_j^2 + \sum_k q_k^* \pi(L) q_k.$$

Equality of free Hilbert spectrahedra - problem

Let $\mathscr{H}_1, \mathscr{H}_2$ be separable real Hilbert spaces. Given L_1 and L_2 monic linear operator pencils

$$L_1(x) := I_{\mathscr{H}_1} + \sum_{j=1}^{g} A_j x_j, \quad L_2(x) := I_{\mathscr{H}_2} + \sum_{j=1}^{g} B_j x_j,$$

where $A_j \in \mathbb{S}_{\mathscr{H}_1}$ and $B_j \in \mathbb{S}_{\mathscr{H}_2}$ for $j = 1, \ldots, g$, we are interested in the algebraic characterization of the inclusion of the free LOI sets

$$D_{L_1} = D_{L_2}.$$

Equality of free Hilbert spectrahedra - definitions

Let $A_0, A_1, \ldots, A_g \in \mathbb{S}_{\mathscr{H}}$ be self-adjoint operators and $L(x) = A_0 + \sum_{j=1}^{g} A_j x_j$ a linear pencil. Let $H \subseteq \mathscr{H}$ be a closed subspace of $\mathscr{H}$ which is **invariant** under each A_j, i.e., $A_j H \subseteq H$. Hence, with respect to the decomposition $\mathscr{H} = H \oplus H^{\perp}$, L can be written as the direct sum,

$$L = \tilde{L} \oplus \tilde{L}^{\perp} = \begin{bmatrix} \tilde{L} & 0 \\ 0 & \tilde{L}^{\perp} \end{bmatrix}, \quad \text{where} \quad \tilde{L} = I_H + \sum_{j=1}^{g} \tilde{A}_j x_j,$$

and $\tilde{A}_j$ is the restriction of A_j to H. We say that $\tilde{L}$ is a **subpencil** of L. If H is a proper closed subspace of $\mathscr{H}$, then $\tilde{L}$ is a **proper subpencil** of L. If $D_L = D_{\tilde{L}}$, then $\tilde{L}$ is a **whole subpencil** of L. If L has no proper whole subpencil, then L is σ-**minimal**.

Equality of free spectrahedra - solution

Theorem (Linear Gleichstellensatz)

Let $L_j = I_{d_j} + \sum_{k=1}^{g} A_{k,j} x_j$, $j = 1, 2$, $d_j \in \mathbb{N}$, $A_{k,j} \in \mathbb{S}_{d_j}$, be monic linear matrix pencils. Then $D_{L_1} = D_{L_2}$ if and only if every whole subpencils $\tilde{L}_1$ and $\tilde{L}_2$ of L_1 and L_2 respectively which are σ-minimal, are unitarily equivalent. That is, there is a unitary matrix U such that

$$\tilde{L}_2 = U^* \tilde{L}_1 U.$$

Nonexistence of σ-minimal operator subpencil

Example

Let

$$L(x) = I_{\ell^2} + \mathrm{diag}(\frac{n}{n+1})_{n \in \mathbb{N}}\, x$$

be a diagonal linear operator pencil with coefficients from $B(\ell^2(\mathbb{N}))$. Then

$$D_L(m) = \{X \in \mathbb{S}_m : X \succeq -I_{\ell^2}\}$$

and there does not exist a σ-minimal whole subpencil of L.

Counterexample to operator Linear Gleichstellensatz

Example

Let S_1 and S_2 be bounded operators on $\ell^2(\mathbb{N})$ defined by $e_i \mapsto e_{2i-1}$ for $i \in \mathbb{N}$ and $e_i \mapsto e_{2i}$ for $i \in \mathbb{N}$ respectively. Let $C^*(S_1, S_2)$ be the C^*-algebra generated by S_1, S_2 (Cuntz C^*-algebra). There is a unique $*$-isomorphism

$$\theta : C^*(S_1, S_2) \to C^*(S_1, S_2), \quad \theta(S_1) = S_2, \quad \theta(S_2) = S_1.$$

The linear operator pencils

$$\begin{aligned} L_1(x) &= I_{\ell^2} + A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4, \\ L_2(x) &= I_{\ell^2} + A_2x_1 + A_1x_2 + A_4x_3 + A_3x_4, \end{aligned}$$

where

Counterexample to operator Linear Gleichstellensatz

Example

$$
\begin{aligned}
A_1 &:= S_1 + S_1^* \in B(\ell^2), \quad A_2 := S_2 + S_2^* \in B(\ell^2), \\
A_3 &:= i(S_1 - S_1^*) \in B(\ell^2), \quad A_4 := i(S_2 - S_2^*) \in B(\ell^2),
\end{aligned}
$$

are σ-minimal pencils with $D_{L_1} = D_{L_2}$, but there is no unitary operator $U : \ell^2 \to \ell^2$ such that

$$L_2 = U^* L_1 U \quad \text{or} \quad L_2 = U^* \overline{L_1} U.$$

Noncommutative (nc) polynomials

We write $\langle x \rangle$ for the monoid freely generated by $x = (x_1, \dots, x_g)$, i.e., $\langle x \rangle$ consists of **words** in the g noncommuting letters $x_1, \dots, x_g$. Let $\mathbb{R}\langle x \rangle$ denote the associative $\mathbb{R}$-algebra freely generated by x, i.e., the elements of $\mathbb{R}\langle x \rangle$ are polynomials in the noncommuting variables x with coefficients in $\mathbb{R}$. The elements are called **noncommutative (nc) polynomials**. Endow $\mathbb{R}\langle x \rangle$ with the natural **involution** * which fixes $\mathbb{R} \cup \{\emptyset\}$ pointwise, reverses the order of words, and acts linearly on polynomials. Polynomials invariant under this involution are **symmetric**. The length of the longest word in a nc polynomial $f \in \mathbb{R}\langle x \rangle$ is denoted by $\deg(f)$.

Noncommutative (nc) polynomials

Fix separable Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$. **Operator-valued nc polynomials** are the elements of $B(\mathscr{H}_1, \mathscr{H}_2) \otimes \mathbb{R}\langle x \rangle$. We write

$$P = \sum_{w \in \langle x \rangle} A_w \otimes w \in B(\mathscr{H}_1, \mathscr{H}_2) \otimes \mathbb{R}\langle x \rangle$$

for an element $P \in B(\mathscr{H}_1, \mathscr{H}_2) \otimes \mathbb{R}\langle x \rangle$, where the sum is finite. The involution * extends to operator-valued polynomials by

$$P^* = \sum_{w \in \langle x \rangle} A_w^* \otimes w^* \in B(\mathscr{H}_2, \mathscr{H}_1) \otimes \mathbb{R}\langle x \rangle \,.$$

If $\mathscr{H}_1 = \mathscr{H}_2$ and $P = P^*$, then we say P is **symmetric**.

Polynomial evaluations

If $P \in B(\mathscr{H}_1, \mathscr{H}_2) \otimes \mathbb{R}\langle x \rangle$ is a nc operator-valued polynomial and $X \in B(\mathscr{K})^g$, where $\mathscr{K}$ is a separable Hilbert space, then

$$P(X) \in B(\mathscr{H}_1, \mathscr{H}_2) \otimes B(\mathscr{K})$$

is defined in the natural way by replacing x_i by X_i and sending the empty word to the identity operator on $\mathscr{K}$.

A symmetric operator-valued nc polynomial P determines the **free Hilbert semialgebraic set** by

$$D_P = (D_P(n))_n \quad \text{where} \quad D_P(n) = \{X \in \mathbb{S}_n^g : P(X) \succeq 0\}.$$

Positivstellensatz problem

Let $\mathscr{H}$ and $\mathscr{K}$ be separable real Hilbert spaces. Suppose $L \in \mathbb{S}_{\mathscr{H}}\langle x \rangle$ is a monic linear operator pencil and

$$P = P^* \in B(\mathscr{K}) \otimes \mathbb{R}\langle x \rangle$$

a symmetric operator-valued nc polynomial such that

$$D_L \subseteq D_P.$$

The problem is to find an algebraic expression for the polynomial P in terms of the polynomial L.

Operator convex multivariate Positivstellensatz

Theorem

Let $L \in \mathbb{S}_{\mathscr{H}}\langle x\rangle$ *be a monic linear operator pencil. Then for every symmetric matrix-valued noncommutative polynomial* $P \in \mathbb{R}^{\nu\times\nu}\langle x\rangle$ *with* $P|_{D_L} \succeq 0$*, there is a separable real Hilbert space* $\mathscr{K}$*, a* $*$*-homomorphism* $\pi : B(\mathscr{H}) \to B(\mathscr{K})$*, finitely many matrix polynomials* $R_j \in \mathbb{R}^{\nu\times\nu}\langle x\rangle$ *and operator polynomials* $Q_k \in B(\mathbb{R}^{\nu}, \mathscr{K}) \otimes \mathbb{R}\langle x\rangle$ *all of degree at most* $\frac{\deg(P)+2}{2}$ *such that*

$$P = \sum_j R_j^* R_j + \sum_k Q_k^* \pi(L) Q_k.$$

Operator convex univariate Positivstellensatz

Theorem

Suppose $L = I_{\mathscr{H}} + A_1 y \in \mathbb{S}_{\mathscr{H}}\langle y\rangle$ *is a univariate monic linear operator pencil. Then for every symmetric operator-valued noncommutative polynomial* $P \in B(\mathscr{K}) \otimes \mathbb{R}\langle y\rangle$ *with* $P|_{D_L} \succeq 0$, *there exists a separable real Hilbert space* $\mathscr{G}$, *a* $*$*-homomorphism* $\pi : B(\mathscr{H}) \to B(\mathscr{G})$ *and finitely many operator polynomials* $R_j \in B(\mathscr{K}) \otimes \mathbb{R}\langle x\rangle$ *and* $Q_k \in B(\mathscr{K}, \mathscr{G}) \otimes \mathbb{R}\langle x\rangle$ *all of degree at most* $\frac{\deg(P)+2}{2}$ *such that*

$$P = \sum_j R_j^* R_j + \sum_k Q_k^* \pi(L) Q_k.$$

Monicity

Example

Let $L(y) = A_0 + A_1 y \in B(\ell^2)$ be a linear operator pencil, where

$$A_0 = \oplus_{n\in\mathbb{N}}(-\frac{1}{n}), \quad A_1 = \oplus_{n\in\mathbb{N}}(\frac{1}{n^2}).$$

Then the spectrahedron $D_L(1)$ is $\emptyset$ and $\ell(y) = -1$ is non-negative on $D_L(1)$, but there do not exist a Hilbert space $\mathscr{K}$, a unital $*$-homomorphism $\pi : B(\ell^2) \to B(\mathscr{K})$, polynomials $r_j \in \mathbb{R}\langle y\rangle$ and operator polynomials $b_k \in B(\mathbb{R}, \mathscr{K})\langle y\rangle$ such that

$$-1 = \sum_j r_j^2 + \sum_k q_k^* \pi(L) q_k. \qquad (1)$$

Thank you for your attention!