Matrix Fejér-Riesz theorem with gaps

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Notation

Let $\mathbb{C}[x]$ be the ring of complex polynomials equipped with involution

$$(\sum_{j=0}^{m} a_j x^j)^* := \sum_{j=0}^{m} \overline{a_j} x^j.$$

Let $M_n(\mathbb{C}[x])$ be the ring of matrix polynomials equipped with involution

$$G(x)^* := (\sum_{j=0}^{m} G_j x^j)^* = \sum_{j=0}^{m} \overline{G_j}^T x^j = \overline{G(x)}^T.$$

Let $\mathbb{H}_n(\mathbb{C}[x])$ be the set of hermitian matrix polynomials, i.e. $F \in \mathbb{H}_n(\mathbb{C}[x])$ iff $F^* = F$.

Let $\sum M_n(\mathbb{C}[x])^2$ be the the set of finite sums of the elements of the form $A_i^* A_i$, where $A_i \in M_n(\mathbb{C}[x])$.

Matrix Fejér-Riesz theorem

Theorem (Fejér-Riesz theorem on $\mathbb{R}$)

Let

$$F(x)=\sum_{m=0}^{2N} F_m x^m \in M_n(\mathbb{C}[x])$$

be a $n\times n$ matrix polynomial, such that $F(x)$ is positive semidefinite for every $x\in\mathbb{R}$. Then there exists a matrix polynomial $G(x)=\sum_{m=0}^{N} G_m x^m \in M_n(\mathbb{C}[x])$, such that

$$F(x)=G(x)^* G(x).$$

Main problem

Problem

Characterize univariate matrix polynomials, which are positive semidefinite on a union of points and intervals (not necessarily bounded) in $\mathbb{R}$.

Notation

A *closed semialgebraic set* $K_S \subseteq \mathbb{R}$ associated to a finite subset $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ is given by

$$K_S = \{x \in \mathbb{R} \colon g_j(x) \geq 0,\ j = 1, \ldots, s\}.$$

We define the *n-th matrix quadratic module* M_S^n by

$$\begin{aligned} M_S^n \quad := \quad & \{\sigma_0 + \sigma_1 g_1 + \ldots + \sigma_s g_s \colon \\ & \sigma_j \in \sum M_n(\mathbb{C}[x])^2 \text{ for } j = 0, \ldots, s\}. \end{aligned}$$

Notation

Let $\prod S := \{g_1^{e_1} \cdots g_s^{e_s} : e_j \in \{0,1\},\ j = 1, \ldots, s\}$. The *n-th matrix preordering* T_S^n is $M^n_{\prod S}$.

Let $\mathrm{Pos}^n_{\succeq 0}(K_S)$ be the set of all $n \times n$ hermitian matrix polynomials, which are positive semidefinite on K_S.

We say a matrix quadratic module M_S^n is *saturated* if $M_S^n = \mathrm{Pos}^n_{\succeq 0}(K_S)$.

Notation

Let $K \subseteq \mathbb{R}$ be a closed semialgebraic set.

A set $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ is the *natural description* of K, if it satisfies the following conditions:

(a) If K has the least element a, then $x - a \in S$.

(b) If K has the greatest element a, then $a - x \in S$.

(c) For every $a \neq b \in K$, if $(a, b) \cap K = \emptyset$, then $(x - a)(x - b) \in S$.

(d) These are the only elements of S.

Notation

Let $K = \cup_{j=1}^{m} [x_j, y_j] \subseteq \mathbb{R}$ be a compact semialgebraic set.

A set $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ with $K = K_S$ is the *saturated description* of K, if it satisfies the following conditions:

(a) For every left endpoint x_j there exists $k \in \{1, \ldots, s\}$, such that $g_k(x_j) = 0$ and $g_k'(x_j) > 0$.

(b) For every right endpoint y_j there exists $k \in \{1, \ldots, s\}$, such that $g_k(y_j) = 0$ and $g_k'(y_j) < 0$.

Known results - scalar case

1. (Kuhlmann, Marshall, 2002) If S is the natural description of K, then the preordering $T_S^1 = M^1_{\prod S}$ is saturated.

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1 (Kuhlmann, Marshall, 2002) If S is the natural description of K, then the preordering $T_S^1 = M^1_{\prod S}$ is saturated.

- K not compact: T_S^1 is saturated if and only if S contains each of the polynomials in the natural description of K up to scaling by positive constants.
- K compact (Scheiderer, 2003): T_S^1 is saturated if and only if S is a saturated description of K. Moreover, $T_S^1 = M_S^1$.

Known results - matrix case

1. (Gohberg, Krein, 1958) For $K = \mathbb{R}$, $M_{\emptyset}^{n}$ is saturated for every $n \in \mathbb{N}$.
2. (Dette, Studden, 2002) For $K = K_{\{x,1-x\}} = [0,1]$, $M_{\{x,1-x\}}^{n}$ is saturated for every $n \in \mathbb{N}$.
3. (Schmüdgen, Savchuk, 2012) For $K = K_{\{x\}} = [0,\infty)$, $M_{\{x\}}^{n}$ is saturated for every $n \in \mathbb{N}$.
4. (Hol, Scherer, 2006) For a finite set $S \subseteq \mathbb{R}[x]$ with a compact set $K = K_S$, M_S^n contains every $F \in M_n(\mathbb{R}[x])$ such that $F|_K \succ 0$.

New results

Theorem (Compact Nichtnegativstellensatz for $\mathbb{R}$)

Let $K \subset \mathbb{R}$ be compact. The n-th matrix quadratic module M_S^n is saturated for every $n \in \mathbb{N}$ if and only if S is a saturated description of K.

Sketch of the proof of compact Nsatz

Proposition (h^2F-proposition)

Suppose K is a non-empty closed semialgebraic set in $\mathbb{R}$ and S a saturated description of K. Then for every $F \in Pos^n_{\succeq 0}(K)$ and every $w \in \mathbb{C}$ there exists $h \in \mathbb{R}[x]$ such that $h(w) \neq 0$ and

$$h^2 F \in M^n_S.$$

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Proof of h^2F-proposition.

The proof is by induction on the size of matrix polynomials n. We write $F(x) = p(x)^m G(x)$, where

$$p(x) = \begin{cases} x - w, & w \in \mathbb{R} \\ (x-w)(x-\overline{w}), & w \notin \mathbb{R} \end{cases}, \ m \in \mathbb{N}_0, \ G(w) \neq 0.$$

Sketch of the proof of Compact Nsatz

Proof of h^2F-proposition.

Writing $G = \begin{bmatrix} a & \beta \\ \beta^* & C \end{bmatrix} \in M_n(\mathbb{C}[x])$, where $a = a^* \in \mathbb{R}[x]$, $\beta \in M_{1,n-1}(\mathbb{C}[x])$ and $C \in H_{n-1}(\mathbb{C}[x])$ it holds

(i) $a^4 \cdot G = \begin{bmatrix} a^* & 0 \\ \beta^* & a^* I_{n-1} \end{bmatrix} \begin{bmatrix} a^3 & 0 \\ 0 & a(aC - \beta^*\beta) \end{bmatrix} \begin{bmatrix} a & \beta \\ 0 & aI_{n-1} \end{bmatrix}$

(ii) $\begin{bmatrix} a^3 & 0 \\ 0 & a(aC - \beta^*\beta) \end{bmatrix} = \begin{bmatrix} a^* & 0 \\ -\beta^* & a^* I_{n-1} \end{bmatrix} \cdot G \cdot \begin{bmatrix} a & -\beta \\ 0 & aI_{n-1} \end{bmatrix}.$

Sketch of the proof of compact Nsatz

Proof of h^2F-proposition.

Therefore

$$a^4F = \begin{bmatrix} a & 0 \\ \beta^* & aI_{n-1} \end{bmatrix} \begin{bmatrix} d & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} a & \beta \\ 0 & aI_{n-1} \end{bmatrix},$$

where $d = p^m a^3 \in \mathbb{R}[x]$, $D = p^m (aC - \beta^*\beta) \in H_{n-1}(\mathbb{C}[x])$. and

$$\begin{bmatrix} d & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} a & 0 \\ -\beta^* & aI_{n-1} \end{bmatrix} F \begin{bmatrix} a & -\beta \\ 0 & aI_{n-1} \end{bmatrix}.$$

By the induction hypothesis, there exists appropriate $h_1 \in \mathbb{R}[x]$, such that $h_1^2 D \in M_S^{n-1}$ and by $h_1^2 d \in M_S^1$, it follows that $(a^2h_1)^2 F \in M_S^n$. □

Sketch of the proof of compact Nsatz

To conclude the proof we need the following:

Proposition (Scheiderer, 2006)

Suppose R is a commutative ring with 1 and $\mathbb{Q} \subseteq R$. Let $\Phi : R \to C(K, \mathbb{R})$ be a ring homomorphism, where K is a topological space which is compact and Hausdorff. Suppose $\Phi(R)$ separates points in K. Suppose $f_1, \dots, f_k \in R$ are such that $\Phi(f_j) \geq 0$, $j = 1, \dots, k$ and $(f_1, \dots, f_k) = (1)$. Then there exist $s_1, \dots, s_k \in R$ such that $s_1 f_1 + \dots + s_k f_k = 1$ and such that each $\Phi(s_j)$ is strictly positive.

Sketch of the proof of compact Nsatz

The ideal

$$I := \left\langle h^2 \colon h \in \mathbb{R}[x], h^2 F \in M_S^n \right\rangle$$

is $\mathbb{R}[x]$. Therefore there exist $s_1, \ldots, s_k \in \mathrm{Pos}^1_{\succ 0}(K)$ and $h_1, \ldots, h_k \in I$, such that

$$s_1 h_1^2 + s_2 h_2^2 + \ldots + s_k h_k^2 = 1.$$

Hence,

$$\sum_{j=1}^{k} s_j h_j^2 F = F \in M_S^n,$$

which concludes the proof.

Counterexample for the non-compact case

Example

The matrix polynomial $F(x) := \begin{bmatrix} x+2 & \sqrt{6} \\ \sqrt{6} & x^2-2x+3 \end{bmatrix}$ is positive semidefinite on $K := [-1,0] \cup [1,\infty)$, but $F \notin T_S^2 = M^2_{\prod S}$, where S is the natural description of K.

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Proof.

All the principal minors of F, $x+2$, x^2-2x+3 and $\det(F) = x^3 - x$ are non-negative on K. Suppose

$$F(x) = \sigma_0 + \sigma_1(x+1) + \sigma_2 x(x-1) + \sigma_3(x+1)x(x-1),$$

where $\sigma_i \in \sum M_2(\mathbb{C}[x])^2$. □

Counterexample for the non-compact case

Proof.

After comparing the degrees of both sides we conclude that

$$\sigma_3 = 0, \ \deg(\sigma_0) \leq 2, \ \deg(\sigma_0) = \deg(\sigma_2) = 0.$$

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Observing the monomial x^2 on both side it follows that $\begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$ for some $c \in [0,1]$.

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$$F(x) - \sigma_2 x(x-1) = \sigma_0 + \sigma_1(x+1).$$

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$$F(x) - \sigma_2 x(x-1) = \sigma_0 + \sigma_1(x+1).$$

The right-hand side is positive semidefinite on $[-1,\infty)$, while the determinant $q(x)$ of the left-hand side is not. Contradiction.

$$q(x) := -(-1+x)x(-1+2c+(-1+c)x).$$

Classification of non-compact sets K

Let K be a non-compact closed semialgebraic set with a natural description S. The classification of sets K according to the matrix preordering T_S^n being saturated is the following:

Classification of non-compact sets K

K	T_S^n sat.
an unbounded interval	Yes
a union of an unbounded interval and an isolated point	?
a union of an unbounded interval and m isolated points with $m \geq 2$	No
a union of two unbounded intervals	Yes
a union of two unbounded intervals and an isolated point	?
a union of two unbounded intervals and m isolated points with $m \geq 2$	No
includes a bounded and an unbounded interval	No

Classification of compact sets K

Let K be a compact closed semialgebraic set with a natural description S. We say that the matrix preordering T_S^n is *boundedly saturated* if every $F \in \mathrm{Pos}_{\succeq 0}^n(K_S)$ is of the form $\sum_{e\in\{0,1\}^s} \sigma_e \underline{g}^e$, where

$$\deg(\sigma_e \underline{g}^e) \leq \deg(F).$$

The classification of sets K according to T_S^n being boundedly saturated is the following:

Classification of compact sets K

K	T_S^n sat.	T_S^n bsat.
a union of at most three points	Yes	Yes
a union of m points with $m \geq 4$	Yes	No stable
a bounded interval	Yes	Yes
a union of a bounded interval and an isolated point	Yes	?
a union of a bounded interval and m isolated points with $m \geq 2$	Yes	No
a compact set containing at least two intervals	Yes	No

Non-compact Nichtnegativstellensatz

Theorem (Non-compact Nichtnegativstellensatz)

Suppose K is an unbounded closed semialgebraic set in $\mathbb{R}$ and S a saturated description of K. Then, for a hermitian $F \in M_n(\mathbb{C}[x])$, the following are equivalent:

1. $F \in Pos^n_{\succeq 0}(K)$.
2. $(1+x^2)^k F \in T^n_S$ for some $k \in \mathbb{N} \cup \{0\}$.

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1. $F \in Pos^n_{\succeq 0}(K)$.
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Thank you for your attention!