A CORE VARIETY APPROACH TO THE PURE $Y = X^d$ TRUNCATED MOMENT PROBLEM: PART 1

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ABSTRACT. Let $\beta \equiv \beta^{(2n)}$ be a real bivariate sequence of degree $2n$. We study the existence of representing measures for β supported in the curve $y = x^d$ $(d \geq 1)$ in the case when all column dependence relations in the moment matrix $M_n(\beta)$ are generated by the relation $Y = X^d$. We prove that the core variety of β, $\mathcal{CV}(L_\beta)$, is nonempty (equivalently, representing measures exist) if and only if C, the partially defined *core matrix* of β, admits a positive, recursively generated completion $C[A]$. Moreover, $\mathcal{CV}(L_\beta)$ is the entire curve $y = x^d$ if and only if there is a positive definite completion $C[A]$. In the remaining case, if there is a measure, it is unique and finitely atomic. For $d = 3$, we use these results to compute the core variety of β and give new characterizations of the existence of representing measures, which complement a result of [F2].

1. INTRODUCTION.

Given a bivariate sequence of degree $2n$,

$$\beta \equiv \beta^{(2n)} = \{\beta_{ij} : i, j \geq 0, i + j \leq 2n\}, \ \beta_{00} = 1, \tag{1.1}$$

and a closed set $K \subseteq \mathbb{R}^2$, the Truncated K-Moment Problem (TKMP) seeks conditions on β such that there exists a positive Borel measure μ on $\mathbb{R}^2$, with $\operatorname{supp} \mu \subseteq K$, satisfying

$$\beta_{ij} = \int_{\mathbb{R}^2} x^i y^j d\mu(x, y) \ (i, j \geq 0, i + j \leq 2n);$$

μ is a *K-representing measure* for β. A comprehensive reference for all aspects of the Moment Problem is the recent treatise of K. Schmüdgen [Sch]. Apart from solutions based on semidefinite programming and optimization, several different *abstract* solutions to TKMP appear in the literature, including the Flat Extension Theorem [CF5], the Truncated Riesz-Haviland Theorem [CF7], the idempotent approach of [Vas], and, more recently, the Core Variety Theorem [BF]. By a *concrete* solution to TKMP we mean an implementation of one of the abstract theories involving only basic linear algebra and solving algebraic equations (or estimating the size of the solution set). The ease with which any of the abstract results can be applied to solve particular moment problems in concrete terms varies considerably depending on the problem, with most concrete results attributable to the Flat Extension Theorem and very few

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to the other approaches. In the sequel we show how the Core Variety Theorem (Theorem 2.5 below) can indeed be applied to certain concrete moment problems, namely when K is the planar curve $y = x^d$ $(d \geq 1)$.

In the classical literature TKMP has been solved concretely in terms of positive Hankel matrices when K is the real line, the half-line $[0, +\infty)$, or the closed interval $[a, b]$ (cf. [ST, CF1]). For the case when K is a planar curve $p(x, y) = 0$ with $\deg p \leq 2$, TKMP has been solved concretely in terms of moment matrix extensions (see Theorem 2.1 below, [CF3, CF4, CF6, F3]). In [F2] moment matrix extensions are used to concretely solve the truncated moment problem for $y = x^3$ and to solve (in a less concrete sense) truncated moment problems on curves of the form $y = g(x)$ and $yg(x) = 1$ $(g \in \mathbb{R}[x])$. More recently, several authors have intensively studied TKMP on certain planar curves of higher degree, using moment matrix extensions and a "reduction of degree" technique to improve and extend the results of [F2] (cf. [Z1, Z2, Z3, Z4, YZ]). We also note that for closed planar sets K that are merely semi-algebraic, such as the closed unit disk, very little is known concerning concrete solutions to TKMP (cf. [CF2]).

The results cited just above do not provide concrete solutions to TKMP for planar curves of the form $y = x^d$ $(d \geq 4)$. The aim of this note is to illustrate how the core variety, described in Theorem 2.5, can be used to study TKMP for $K = \Gamma$, the planar curve $y = x^d$ $(d \geq 1)$, when the associated moment matrix $M_n(\beta)$ is $(y - x^d)$-*pure*, i.e., the column dependence relations in $M_n(\beta)$ are precisely those that can be derived from the column relation $Y = X^d$ by *recursiveness* and linearity (see just below for terminology and notation). The core variety of β coincides with the union of supports of all representing measures for β, and in Section 3 we develop a core variety framework for studying TKMP in the $(y - x^d)$-pure case. In Theorem 3.10 we prove that β has a representing measure if and only if C, the partially defined *core matrix* for β, admits a positive semidefinite, recursively generated completion $C[A]$. The core variety of β coincides with the entire curve $y = x^d$ if and only if there exists positive definite completion $C[A]$. In the remaining case of a measure, it is unique, with support a finite subset of Γ. In Section 4 we apply the results of Section 3 to compute the core variety of β in the $(y - x^3)$-pure truncated moment problem (see Theorem 4.1); this result subsumes a result of [F2] which used a lengthy flat extension construction to give a necessary and sufficient condition for the existence of a representing measure.

2. Preliminaries

Although our focus in the sequel is TKMP for the planar curves $y = x^d$, we note that the following discussion, and the results we cite from [B, BF, CF5, CF7, F4], generalize to the *multivariable* truncated moment problem.

Let $\mathcal{P} := \mathbb{R}[x, y]$ and let $\mathcal{P}_k := \{q \in \mathcal{P} \colon \deg q \leq k\}$. Given $\beta \equiv \beta^{(2n)}$ as in (1.1), define the *Riesz functional* $L_\beta : \mathcal{P}_{2n} \longrightarrow \mathbb{R}$ by

$$\sum a_{ij} x^i y^j \longmapsto \sum a_{ij} \beta_{ij}.$$

For a sequence $\beta \equiv \beta^{(2n)}$ with Riesz functional L_β, the *moment matrix* M_n has rows and columns indexed by the monomials in $\mathcal{P}_n$ in degree-lexicographic order, i.e., $\mathit{1}, X, Y, X^2, XY, Y^2, \ldots, X^n, \ldots, Y^n$. In this case, the element of M_n in row X^iY^j, column X^kY^l is $\beta_{i+k,j+l}$. More generally, for $r, s \in \mathcal{P}_n$, with coefficient vectors $\widehat{r}, \widehat{s}$ relative to the basis of monomials, we have

$$\langle M_n\widehat{r}, \widehat{s}\rangle := L_\beta(rs). \tag{2.1}$$

In the sequel, for $q \in \mathcal{P}_n$, $q = \sum a_{ij}x^iy^j$, we set

$$q(X,Y) := \sum a_{ij}X^iY^j \ (= M_n\widehat{q}). \tag{2.2}$$

If β has a K-representing measure μ, then L_β is *K-positive*, i.e., $q \in \mathcal{P}_{2n},\ q|K \geq 0 \Longrightarrow L_\beta(q) \geq 0$ (since $L_\beta(q) = \int_K q d\mu$). The converse is not true; instead, the Truncated Riesz-Haviland Theorem [CF7] shows that β admits a K-representing measure if and only if L_β admits an extension to a K-positive linear functional on $\mathcal{P}_{2n+2}$. In [B] G. Blekherman proved that if M_n is positive semidefinite and $\operatorname{rank} M_n \leq 3n-3$, then L_β is $\mathbb{R}^2$-positive, so the Truncated Riesz-Haviland Theorem then implies that $\beta^{(2n-1)}$ has a representing measure. Using special features of the proof of Theorem 2.2 (below), in [EF] C. Easwaran and the first-named author exhibited a class of Riesz functionals that are positive but have no representing measure. Apart from these results, it seems very difficult to verify positivity of Riesz functionals in examples without first proving the existence of representing measures.

Several basic *necessary* conditions for a representing measures μ can be expressed in terms related to moment matrices (cf. [CF5]); we will refer to these without further reference in the sequel:
i) $M_n(\beta)$ is *positive semidefinite*: $\langle M_n\widehat{r}, \widehat{r}\rangle = L_\beta(r^2) = \int r^2 d\mu \geq 0$ ($\forall r \in \mathcal{P}_n$).
ii) For any representing measure μ, $\operatorname{card}(\operatorname{supp}\mu) \geq \operatorname{rank} M_n$.
iii) Note that a dependence relation in the column space of M_n can be expressed as $r(X,Y) = 0$, where $r \in \mathcal{P}_n$. Define the *variety* of M_n, $\mathcal{V}(M_n)$, as the common zeros of the polynomials $r \in \mathcal{P}_n$ such that $r(X,Y) = 0$. Then $\operatorname{supp}\mu \subseteq \mathcal{V}(M_n)$, so $\operatorname{card}\mathcal{V}(M_n) \geq \operatorname{rank} M_n$.
iv) M_n is *recursively generated*: whenever r, s, and rs are in $\mathcal{P}_n$ and $r(X,Y) = 0$, then $(rs)(X,Y) = 0$.
v) M_n (or L_β) is *consistent*: for $p \in \mathcal{P}_{2n}$, $p|\mathcal{V}(M_n) \equiv 0 \Longrightarrow L_\beta(p) = 0$; consistency implies recursiveness [CFM].

The Flat Extension Theorem [CF5] shows that β admits a representing measure if and only if M_n admits a positive semidefinite moment matrix extension M_{n+k} (for some $k \geq 0$) for which there is a rank-preserving (i.e., *flat*) moment matrix extension M_{n+k+1}. Using this result, in a series of papers R. Curto and the first-named author solved TKMP for planar curves of degrees 1 and 2 as follows.

Theorem 2.1 ([CF3, CF4, CF6, F3, Degree-2 Theorem]). *Suppose $r(x,y) \in \mathcal{P}$ with $\deg r \leq 2$. For $n \geq \deg r$, M_n has a representing measure supported in the curve $r(x,y) = 0$ if and only if $r(X,Y) = 0$ and M_n is positive semidefinite, recursively generated, and satisfies* $\operatorname{card}\mathcal{V}(M_n) \geq \operatorname{rank} M_n$.

In [CFM] it was shown that this result does not extend to $\deg r > 2$. The example in [CFM] concerns an M_3 that is positive and recursively generated, with $\operatorname{card} \mathcal{V}_\beta = \operatorname{rank} M_3$, but which has no measure. In this example, there is no measure because L_β is not consistent. The results of [F2] show that positivity, the variety condition, and consistency are still not sufficient for representing measures, as we next describe.

For $M_n \succeq 0$, consider the $(y-x^3)$-*pure* case, when the column dependence relations in M_n are precisely those given by $Y = X^3$, recursiveness, and linearity, i.e., column relations of the form $(s(x,y)(y-x^3))(X,Y) = 0$ ($\deg s \leq n-3$). Thus M_n is positive, $\operatorname{rank} M_n \leq \operatorname{card} \mathcal{V}(M_n)$ ($= \operatorname{card} \Gamma = +\infty$), and it follows from Lemma 3.1 in [F2] that M_n is consistent. In [F2] we described a particular, easily computable, rational expression in the moment data, ψ, and solved the $(y-x^3)$-pure TKMP as follows.

Theorem 2.2. *If $M_n \succeq 0$ is $(y-x^3)$-pure, then β has a representing measure if and only if $\beta_{1,2n-1} > \psi$.*

In the proof of Theorem 2.2, the numerical test $\beta_{1,2n-1} > \psi$ leads to a flat extension M_{n+1}. By contrast with this result, the other existence results in [F2, Z4] generally presuppose the existence of a certain positive moment matrix extension of M_n, but do not give an explicit test for the extension. The proof of Theorem 2.2 in [F2] is quite lengthy. In the sequel we will use the *core variety* to present a shorter, more transparent proof. This approach also provides a core variety framework for studying the $(y-x^d)$-pure truncated moment problem.

The core variety provides an approach to establishing the existence of representing measures based on methods of convex analysis. For the polynomial case, this was introduced in [F4], and some of the ideas go back to [FN]. The discussion below is based on joint work of the first author with G. Blekherman [BF], which treats general Borel measurable functions, although here we only require polynomials.

Given $\beta \equiv \beta^{(2n)}$ and its Riesz functional $L \equiv L_\beta$, define $V_0 := \mathcal{V}(M_n)$ and for $i \geq 0$, define

$$V_{i+1} := \bigcap_{\substack{f \in \ker L, \\ f|V_i \geq 0}} \mathcal{Z}(f),$$

where $\mathcal{Z}(f)$ denotes the set of zeros of $f(x,y)$ in $\mathbb{R}^2$ (or, equivalently, in V_i). We define the *core variety* of L by

$$\mathcal{CV}(L) := \bigcap_{i \geq 0} V_i.$$

Proposition 2.3 ([F4]). *If μ is a representing measure for L, then* $\operatorname{supp} \mu \subseteq \mathcal{CV}(L)$.

If μ is a representing measure, then

$$\operatorname{rank} M_n(\beta) \leq \operatorname{card}(\operatorname{supp} \mu) \leq \operatorname{card} \mathcal{CV}(L_\beta) \leq \operatorname{card} V_i \quad (\text{for every } i \geq 0).$$

We thus have the following test for the nonexistence of representing measures.

Corollary 2.4 ([F4]). *If* $\operatorname{card} V_i < \operatorname{rank} M_n$ *for some i, then β has no representing measure.*

Proposition 2.3 shows that if β has a representing measure, then $\mathcal{CV}(L)$ is nonempty. The main result concerning the core variety is the following converse.

Theorem 2.5 ([BF, Core Variety Theorem])**.** *$L \equiv L_\beta$ has a representing measure if and only if $\mathcal{CV}(L)$ is nonempty. In this case, $\mathcal{CV}(L)$ coincides with the union of supports of all finitely atomic representing measures for L.*

In view of Proposition 2.3, $\mathcal{CV}(L)$ is also the union of supports of all representing measures. In general, it may be difficult to compute the core variety, due to the difficulty of characterizing the nonnegative polynomials on V_0, V_1, $V_2, \ldots$, but Theorem 2.5 leads to the following criterion for stability.

Proposition 2.6 ([BF])**.** *If V_k is finite, then $\mathcal{CV}(L) = V_k$ or $\mathcal{CV}(L) = V_{k+1}$.*

In the $(y - x^d)$–pure case for $M_n(\beta)$, V_0 is clearly the curve $y = x^d$. Since $y - x^d$ is irreducible and $\mathcal{CV}(L)$ is an algebraic set, it follows that either $V_1 = V_0$ $(= \mathcal{CV}(L))$, or V_1 is finite and Proposition 2.6 implies $\mathcal{CV}(L) = V_1$ or $\mathcal{CV}(L) = V_2$. We conclude this section by noting the case when $M_n(\beta)$ has the $Y = X^d$ column relation but is *not* $(y - x^d)$–pure. In this case there is a column relation $g(X, Y) = 0$, where $g(x, y)$ is not a multiple of $f(x, y) := y - x^d$. Since f is irreducible, it follows that f and g are relatively prime, so Bezout's Theorem implies that $\operatorname{card} \mathcal{CV}(L) \leq \operatorname{card} V_0 \leq \deg f \cdot \deg g$. Examples computing $\mathcal{CV}(L)$ in the finite-variety case can be found in [F4].

3. A core variety approach to the pure $Y = X^d$ moment problem.

Suppose $M_n(\beta)$ is positive semidefinite and $(y - x^d)$-pure, i.e., the column dependence relations in M_n are precisely the linear combinations of the column relations

$$X^r Y^{s+1} = X^{r+d} Y^s \quad \text{for } r, s \geq 0,\ r + s \leq n - d. \tag{3.1}$$

In this section we introduce a *core matrix* C associated to β; the main result of this section, Theorem 3.10, characterizes the existence of representing measures for β in terms of the positivity properties of C and "recursiveness" in its kernel. Using the Core Variety Theorem we show that the union of supports of all representing measures is the curve

$$\Gamma := \mathcal{Z}(y - x^d) = \{(x, x^d) :\ x \in \mathbb{R}\}$$

if and only if there is a positive definite completion of the core matrix. Namely, we employ the connection between the existence of representing measures for $\beta \equiv \beta^{(2n)}$ and the core variety of the Riesz functional $L \equiv L_\beta$.

Setting $V_0 = \mathcal{V}(M_n) = \Gamma$, we seek to compute

$$V_1 := \mathcal{Z}(p \in \ker L \colon p|V_0 \geq 0),$$

the common zeros of the polynomials in $\ker L$ that are nonnegative on V_0. To this end, we require a concrete description of $\ker L$.

Lemma 3.1. *Suppose $M_n(\beta)$ satisfies column relations* (3.1)*. Then the polynomials*

$$f_{ij}(x, y) = x^i y^j - \beta_{ij} \qquad \textit{for } 0 \leq i < d, j \geq 0, \textit{ and } 0 < i + j \leq 2n,$$

$$g_{kl}(x, y) = (y - x^d)x^k y^l \qquad \textit{for } k, l \geq 0,\ k + l \leq 2n - d.$$

form a basis $\mathcal{B}$ *for* $\ker L_\beta$.

Conversely, let $L : \mathcal{P}_{2n} \to \mathbb{R}$ *be a linear functional such that* $\mathcal{B}$ *is a basis for* $\ker L$. *Then the moment matrix* $M_n(\beta)$ *of the sequence* β*, such that* $L = L_\beta$*, satisfies column relations* (3.1).

Remark 3.2. In the statement of Lemma 3.1, $M_n(\beta)$ does not have to be $(y - x^d)$-pure for $\mathcal{B}$ to be the basis for $\ker L_\beta$. There may be column relations other than the linear combinations of (3.1), but $\mathcal{B}$ will still be a basis. Another choice of a basis for $\ker L_\beta$, which works for any sequence β, is $\{f_{ij}\}$ for $0 \leq i, j$, $0 < i + j \leq 2n$, where f_{ij} are defined as in the statement of the lemma. However, this basis tells us nothing about the column relations of $M_n(\beta)$. To explicitly determine column relations from the basis for $\ker L_\beta$, in addition to a "good" choice of the basis, the rank of $M_n(\beta)$ must also be given.

Proof of Lemma 3.1. Clearly, each $f_{ij} \in \ker L_\beta$. For $k, l \geq 0$ with $k + l \leq 2n - d$, $g_{kl} \in \mathcal{P}_{2n}$. If $k + l \leq n$, then

$$L_\beta(g_{kl}) = \langle M_n \widehat{(y - x^d)}, \widehat{x^k y^l}\rangle = \langle M_n \widehat{y} - M_n \widehat{x^d}, \widehat{x^k y^l}\rangle = 0,$$

so $g_{kl} \in \ker L_\beta$ in this case. In the remaining case, $n < k + l \leq 2n - d$, so there exist integers $r, s, t, u \geq 0$ such that $r + t = k$, $s + u = l$, $r + s = n - d$, and thus $t + u = (k + l) - (r + s) \leq 2n - d - (n - d) = n$. Now

$$\begin{aligned} L_\beta(g_{kl}) &= L_\beta((y - x^d)x^r y^s \cdot x^t y^u) = \langle M_n \widehat{(y - x^d)x^r y^s}, \widehat{x^t y^u}\rangle \\ &= \langle M_n \widehat{x^r y^{s+1}} - M_n \widehat{x^{d+r} y^s}, \widehat{x^t y^u}\rangle, \end{aligned}$$

so (3.1) implies $L_\beta(g_{kl}) = 0$.

To show that $\mathcal{B}$ is a linearly independent set of elements of $\mathcal{P}_{2n}$, suppose $\{a_{ij}\}$ and $\{b_{kl}\}$ are sequences of real scalars (indexed as in the statement of the lemma) such that in $\mathcal{P}_{2n}$,

$$\sum_{\substack{0 \leq i < d,\ j \geq 0, \\ 0 < i + j \leq 2n}} a_{ij} f_{ij} + \sum_{\substack{k, l \geq 0, \\ k + l \leq 2n - d}} b_{kl} g_{kl} = 0. \tag{3.2}$$

Plugging $y = x^d$ in (3.2), it follows that

$$\sum_{\substack{0 \leq i < d,\ j \geq 0, \\ 0 < i + j \leq 2n}} a_{ij}(x^{i+dj} - \beta_{ij}) \equiv 0 \tag{3.3}$$

Suppose that $0 \leq i, i' < d$, $j, j' \geq 0$, $0 < i + j, i' + j' \leq 2n$ and $i + dj = i' + dj'$. Then $|i - i'| = d|j - j'|$, and since $|i - i'| < d$, it follows that $j = j'$ and $i = i'$. Thus, the x-exponents appearing in (3.3) are distinct, and since (3.3) holds for every real x, it follows that each $a_{ij} = 0$. Now (3.2) implies

$$\sum_{\substack{k, l \geq 0, \\ k + l \leq 2n - d}} b_{kl} x^k y^l (y - x^d) \equiv 0$$

Thus, for $y \neq x^d$, $\sum b_{kl}x^k y^l = 0$, so by continuity we have $\sum b_{kl}x^k y^l = 0$ for all $x, y \in \mathbb{R}$. It now follows that each $b_{kl} = 0$, so $\mathcal{B}$ is linearly independent.

Next we show that $\mathcal{B}$ spans $\ker L_\beta$. We need to prove that $\operatorname{card}\mathcal{B} = \dim \mathcal{P}_{2n} - 1$ ($= \dim\ker L_\beta$). Recall that $\dim \mathcal{P}_{2n} = \frac{(2n+1)(2n+2)}{2}$. Note that $\mathcal{B}$ is the disjoint union of the sets $\mathcal{C}$ and $\mathcal{D}$, consisting of all f_{ij} and g_{kl} from the lemma, respectively. Clearly, $\operatorname{card}\mathcal{D} = \dim \mathcal{P}_{2n-d} = \frac{(2n-d+1)(2n-d+2)}{2}$. To compute $\operatorname{card}\mathcal{C}$, notice that $\operatorname{card}\mathcal{C} = \operatorname{card}\mathcal{E}$, where $\mathcal{E}$ is the index set equal to

$$\begin{aligned}\mathcal{E} &:= \{(i,j)\colon 0 \le i < d,\ j \ge 0,\ 0 < i+j \le 2n\}\\ &= \{\underbrace{(0,1),\ldots,(0,2n)}_{i=0}, \underbrace{(1,0),\ldots,(1,2n-1)}_{i=1}, \ldots, \underbrace{(d-1,0),\ldots,(d-1,2n-d+1)}_{i=d-1}\}.\end{aligned}$$

It follows that

$$\begin{aligned}\operatorname{card}\mathcal{C} = \operatorname{card}\mathcal{E} &= 2n + 2n + (2n-1) + \ldots + (2n-d+2)\\ &= -1 + \sum_{i=0}^{d-1}(2n+1-i) = -1 + \sum_{i=1}^{2n+1} i - \sum_{i=1}^{2n-d+1} i\\ &= -1 + \frac{(2n+1)(2n+2)}{2} - \frac{(2n-d+1)(2n-d+2)}{2}\\ &= -1 + \operatorname{card}\mathcal{P}_{2n} - \operatorname{card}\mathcal{D},\end{aligned}$$

whence

$$\operatorname{card}\mathcal{B} = \operatorname{card}\mathcal{C} + \operatorname{card}\mathcal{D} = -1 + \operatorname{card}\mathcal{P}_{2n},$$

which shows that $\mathcal{B}$ is a basis for $\ker L_\beta$.

The converse part is clear. Namely, L determines the sequence β by $\beta_{ij} = L(x^i y^j)$ for $0 \le i,j,\ i+j \le 2n$. (Note that by $f_{ij} \in \ker L$ for $0 \le i < d$, $j \ge 0$, and $0 < i+j \le 2n$, for these indices the β_{ij} are precisely the constant terms in the respective f_{ij}.) Recall from (2.1)–(2.2) that for $q \in \mathcal{P}_n$,

$$q(X,Y) = 0 \iff L_\beta(qx^i y^j) = 0 \quad \text{for all } i,j \ge 0,\ i+j \le n.$$

Since $g_{kl} \in \ker L$ for $k,l \ge 0$, $k+l \le 2n-d$, it now follows that all of the relations of (3.1), as well as their linear combinations, are column relations of $M_n(\beta)$. $\square$

Returning to the computation of V_1, suppose $p \in \ker L$ satisfies $p|\Gamma \ge 0$, i.e., $p(x,x^d) \ge 0\ \forall x \in \mathbb{R}$. From Lemma 3.1, we may write

$$p = F + G \equiv \sum_{\substack{0\le i<d,\ j\ge 0,\\ 0<i+j\le 2n}} a_{ij}f_{ij} + \sum_{\substack{k,l\ge 0,\\ k+l\le 2n-d}} b_{kl}g_{kl}. \tag{3.4}$$

Since $p|\Gamma \ge 0$ and $G|\Gamma \equiv 0$, then

$$Q(x) := F(x,x^d) = \sum_{\substack{0\le i<d,\ j\ge 0,\\ 0<i+j\le 2n}} a_{ij}(x^{i+dj} - \beta_{ij}) \tag{3.5}$$

satisfies $Q(x) \ge 0\ \forall x \in \mathbb{R}$. Since $\deg Q \le 2nd$, there exist

$$\widehat{r} \equiv (r_0,\ldots,r_{nd}) \in \mathbb{R}^{nd+1}, \quad \widehat{s} \equiv (s_0,\ldots,s_{nd}) \in \mathbb{R}^{nd+1}$$

such that

$$R(x) := r_0 + r_1 x + \cdots + r_{nd} x^{nd} \quad \text{and} \quad S(x) := s_0 + s_1 x + \cdots + s_{nd} x^{nd} \tag{3.6}$$

satisfy

$$Q(x) = R(x)^2 + S(x)^2. \tag{3.7}$$

In the sequel (and moreso in Part 2 [FZ-]) we will require detailed information about the coefficients of F, R and S. By comparing coefficients on both sides of (3.8), we see that each a_{ij}, which is the coefficient in Q of x^{i+dj}, admits a unique expression as a homogeneous quadratic polynomial in the r_k and s_l. Indeed,

$$a_{ij} = h_{i,j}(\widehat{r}, \widehat{s}) := \sum_{\substack{0 \le k,l \le nd, \\ 0 < k+l = i+dj}} r_k r_l + s_k s_l, \qquad i, j \ge 0,\ i < d,\ i + j \le 2n. \tag{3.8}$$

Moreover, a comparison of the constant terms in (3.7) gives

$$-\sum_{\substack{0 \le i < d,\ j \ge 0, \\ 0 < i+j \le 2n}} a_{ij} \beta_{ij} = r_0^2 + s_0^2. \tag{3.9}$$

Note also that if $i, j \ge 0$, $i < d$, $i + dj \le 2nd$, but $i + j > 2n$, then since there is no moment β_{ij}, the coefficient of x^{i+dj} in $R(x)^2 + S(x)^2$ must be 0. Let $\mathcal{F}$ denote the set of all such pairs (i, j). It is convenient to extend the definition of $h_{i,j}$ in (3.8) to include these cases, together with the requirements

$$0 = h_{i,j}(\widehat{r}, \widehat{s}) \quad \text{whenever } (i, j) \in \mathcal{F}. \tag{3.10}$$

We call each such requirement an *auxiliary requirement.* Also, we introduce an arbitrary constant A_{ij} for each $(i, j) \in F$ to denote the moment β_{ij}, which is not present in $\beta^{(2n)}$. We refer to A_{ij} as an *auxiliary moment.* In the sequel (particularly in Part 2 [FZ-]) we require the number and location of the A_{ij}. To this end, note that:

$$\begin{aligned} \mathcal{F} &:= \{(i,j) \colon i, j \ge 0,\ i < d,\ i + dj \le 2nd,\ i + j > 2n\} \\ &= \{(i,j) \colon 2n - (d-2) \le j \le 2n - 1,\ 2n + 1 - j \le i \le d - 1\} = \bigcup_{j=1}^{d-2} \mathcal{F}_j \end{aligned} \tag{3.11}$$

where each $\mathcal{F}_j$ is equal to

$$\mathcal{F}_j = \begin{cases} \{(j+1, 2n-j), \ldots, (d-1, 2n-j)\}, & \text{if } j + 1 \le d - 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Hence, $\operatorname{card} \mathcal{F} = \sum_{i=1}^{d-2} i = \frac{(d-1)(d-2)}{2}$. Note that $\mathcal{F} = \emptyset$ for $n = 1, 2$.

Example 3.3. Let $n = d = 3$. Then Q (cf. (3.8)) is of the form

$$Q(x) = \sum_{\substack{0 \le i < 3,\ j \ge 0, \\ 0 < i+j \le 6}} a_{ij} (x^{i+3j} - \beta_{ij}) =: \sum_{\ell=0}^{18} q_\ell x^\ell \in \mathcal{P}_{18}.$$

To illustrate (3.8), note that q_4, which is equal to a_{11}, may be expressed as

$$\begin{aligned} h_{1,1}(\widehat{r},\widehat{s}) &= r_0r_4 + r_1r_3 + r_2r_2 + r_3r_1 + r_4r_0 + s_0s_4 + s_1s_3 + s_2s_2 + s_3s_1 + s_4s_0 \\ &= 2(r_0r_4 + s_0s_4 + r_1r_3 + s_1s_3) + r_2^2 + s_2^2. \end{aligned}$$

Note that $\mathcal{F} = \{(2,5)\}$, since for $i = 2$ and $j = 5$, we have $i+3j = 17 < 2nd = 18$, but $7 = i + j > 2n = 6$. Thus x^{17} does not appear in $Q(x)$, so, from (3.10), using $h_{2,5}(\widehat{r},\widehat{s}) = r_8r_9 + s_8s_9$, it follows that $0 = r_8r_9 + s_8s_9 = q_{17}$. The auxiliary moment in this case is $\beta_{2,5}$, which we denote by $A_{2,5}$.

For $d = 3$ and arbitrary $n \in \mathbb{N}$, which we study in Section 4, we have $\mathcal{F} = \{(2, 2n-1)\}$ and the auxiliary requirement (cf. (3.10)) is equal to

$$0 = h_{2,2n-1}(\widehat{r},\widehat{s}) = 2(r_{3n}r_{3n-1} + s_{3n}s_{3n-1}), \tag{3.12}$$

with the "missing" monomial in $Q(x)$ being $x^{2+3(2n-1)} = x^{6n-1}$. △

We next introduce the *core matrix* $C \equiv C_\beta$; in the sequel we show that positivity properties of C determine the core variety of β. Our immediate goal is to use (3.8) and the core matrix to derive an inner product expression (see (3.25)) which can be used to characterize whether (3.9) holds. This will permit us to provide a sufficient condition for representing measures via the core variety.

Let $\lfloor\cdot\rfloor$ denote the floor function. Namely, $\lfloor k\rfloor$ is the greatest integer not larger than k. For $1 \le i, j \le dn+1$, let

$$K_{ij} := (i+j-2) \bmod d \quad \text{and} \quad L_{ij} := \lfloor (i+j-2)/d \rfloor, \tag{3.13}$$

The core matrix, a $(dn+1)\times(dn+1)$ matrix, is defined by

$$C \equiv (C_{ij})_{i,j=1}^{dn+1} := (\beta_{K_{ij},L_{ij}})_{i,j=1}^{dn+1}. \tag{3.14}$$

However, if $\beta_{K_{ij},L_{ij}}$ is an auxiliary moment because $(K_{ij}, L_{ij}) \in \mathcal{F}$, we redefine $\beta_{K_{ij},L_{ij}}$ as

$$\beta_{K_{ij},L_{ij}} := A_{K_{ij},L_{ij}},$$

where $A_{K_{ij},L_{ij}}$ is an arbitrary constant. To emphasize the dependence of C on the choice of the constants A_{ij} for $(i,j) \in \mathcal{F}$, we sometimes denote C by

$$C \equiv C[\{A_{ij}\}_{(i,j)\in\mathcal{F}}].$$

From (3.14), C is clearly a Hankel matrix.

Example 3.4. For $n = d = 4$ the core matrix

$$C \equiv C[\mathbf{A}_{3,2n-2}, \mathbf{A}_{2,2n-1}, \mathbf{A}_{3,2n-1}]$$

is the following

$$\begin{pmatrix}
\beta_{00} & \beta_{10} & \beta_{20} & \beta_{30} & \beta_{01} & \beta_{11} & \beta_{21} & \beta_{31} & \beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} \\
\beta_{10} & \beta_{20} & \beta_{30} & \beta_{01} & \beta_{11} & \beta_{21} & \beta_{31} & \beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} \\
\beta_{20} & \beta_{30} & \beta_{01} & \beta_{11} & \beta_{21} & \beta_{31} & \beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} \\
\beta_{30} & \beta_{01} & \beta_{11} & \beta_{21} & \beta_{31} & \beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} \\
\beta_{01} & \beta_{11} & \beta_{21} & \beta_{31} & \beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} \\
\beta_{11} & \beta_{21} & \beta_{31} & \beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} \\
\beta_{21} & \beta_{31} & \beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} \\
\beta_{31} & \beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} \\
\beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} \\
\beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} & \beta_{16} \\
\beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} & \beta_{16} & \beta_{26} \\
\beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} & \beta_{16} & \beta_{26} & \mathbf{A36} \\
\beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} & \beta_{16} & \beta_{26} & \mathbf{A36} & \beta_{07} \\
\beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} & \beta_{16} & \beta_{26} & \mathbf{A36} & \beta_{07} & \beta_{17} \\
\beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} & \beta_{16} & \beta_{26} & \mathbf{A36} & \beta_{07} & \beta_{17} & \mathbf{A27} \\
\beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} & \beta_{16} & \beta_{26} & \mathbf{A36} & \beta_{07} & \beta_{17} & \mathbf{A27} & \mathbf{A37} \\
\beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} & \beta_{16} & \beta_{26} & \mathbf{A36} & \beta_{07} & \beta_{17} & \mathbf{A27} & \mathbf{A37} & \beta_{08}
\end{pmatrix}$$

The rows and columns of C are indexed by the ordered set

$$\{\mathit{1}, X, X^2, X^3, Y, XY, X^2Y, X^3Y, \ldots, Y^k, XY^k, X^2Y^k, X^3Y^k, \ldots,\\ Y^{n-1}, XY^{n-1}, X^2Y^{n-1}, X^3Y^{n-1}, Y^n\}.$$

Note that the columns $X^3Y^{n-2}, X^2Y^{n-1}, X^3Y^{n-1}$ are not among the columns of M_n but rather of its extension M_{n+2}. So these columns are auxiliary ones in C and contain auxiliary moments. $\triangle$

The next two results provide an alternate description of the core matrix in terms of moment matrix extensions. Let $d \geq 2$ and let M_{n+d-2} be some recursively generated extension of the positive $(y-x^d)$-pure moment matrix M_n. Let $\widetilde{\beta} \equiv \widetilde{\beta}^{(2n+2d-4)}$ be the extended sequence and let $L_{\widetilde{\beta}} : \mathcal{P}_{2(n+d-2)} \to \mathbb{R}$ be the corresponding Riesz functional. Define the ordered set of monomials

$$\mathcal{M} := \{1, x, \ldots, x^{d-1}, y, xy, \ldots, x^{d-1}y, \ldots, y^i, xy^i, \ldots, x^{d-1}y^i,\\ y^{n-1}, xy^{n-1}, \ldots, x^{d-1}y^{n-1}, y^n\}, \tag{3.15}$$

and the vector space

$$\mathcal{U} := \operatorname{Span}\{s\colon s \in \mathcal{M}\} \subset \mathcal{P}_{n+d-2}, \tag{3.16}$$

We next define an $(nd+1) \times (nd+1)$ matrix $M[\widetilde{\beta}, \mathcal{U}]$ with rows and columns indexed by the monomials in $\mathcal{M}$ in the order

$$\mathit{1}, X, \ldots, X^{d-1}, Y, XY, \ldots, X^{d-1}Y, \ldots Y^{n-1}, XY^{n-1}, \ldots, X^{d-1}Y^{n-1}, Y^n \tag{3.17}$$

i.e., for $1 \leq k \leq nd+1$, the k-th element of this order is equal to $X^{I_k}Y^{J_k}$ where

$$I_k := (k-1) \bmod d \quad \text{and} \quad J_k := \lfloor (k-1)/d \rfloor. \tag{3.18}$$

The (i,j)-th entry of $M[\widetilde{\beta}, \mathcal{U}]$ is defined to be equal to

$$L_{\widetilde{\beta}}(x^{I_i+I_j}y^{J_i+J_j}) = \widetilde{\beta}_{I_i+I_j, J_i+J_j}. \tag{3.19}$$

More generally, for $r, s \in \mathcal{U}$ (cf. (3.16)), with coefficient vectors $\widehat{r}, \widehat{s}$ relative to the ordered basis of monomials in $\mathcal{M}$ (cf. (3.15)), we have

$$\left\langle M[\widetilde{\beta}, \mathcal{U}]\, \widehat{r}, \widehat{s} \right\rangle := L_{\widetilde{\beta}}(rs). \tag{3.20}$$

Lemma 3.5. *For $1 \le i, j \le nd + 1$ the following holds:*

$$L_{\widetilde{\beta}}(x^{I_i+I_j} y^{J_i+J_j}) = \widetilde{\beta}_{K_{ij}, L_{ij}}, \tag{3.21}$$

where K_{ij}, L_{ij} are as in (3.13).

Proof. We have that

$$K_{ij} + dL_{ij} = i + j - 2 = I_i + I_j + d(J_i + J_j), \tag{3.22}$$

where in the second equality we used $i + j - 2 = (i - 1) + (j - 1)$. We separate two cases according to the value of the sum $I_i + I_j$:

Case a): $I_i + I_j < d$. Then (3.22) implies that $K_{ij} = I_i + I_j$ and $L_{ij} = J_i + J_j$. Using this in (3.19), (3.21) follows.

Case b): $I_i + I_j \ge d$. Then (3.22) implies that

$$K_{ij} = I_i + I_j - d \quad \text{and} \quad L_{ij} = J_i + J_j + 1. \tag{3.23}$$

Since M_{n+d-2} is recursively generated, we have $X^{r+d}Y^s = X^rY^{s+1}$ in the rows and columns, and therefore $\widetilde{\beta}_{r+d,s} = \widetilde{\beta}_{r,s+1}$. The assumption of Case b), and (3.23) used in (3.19), together with M_{n+d-2} being recursively generated, therefore imply that

$$\widetilde{\beta}_{I_i+I_j, J_i+J_j} = \widetilde{\beta}_{K_{ij}+d, J_i+J_j} = \widetilde{\beta}_{K_{ij}, J_i+J_j+1} \underbrace{=}_{(3.23)} \widetilde{\beta}_{K_{ij}, L_{ij}}.$$

proving (3.21). □

Proposition 3.6. Assume the notation above. Then:

(i) If the sequence $\widetilde{\beta}$ has a representing measure, then $M[\widetilde{\beta}, \mathcal{U}]$ is positive semidefinite.

(ii) Let $\widetilde{M}[\widetilde{\beta}, \mathcal{U}]$ be obtained from $M[\widetilde{\beta}, \mathcal{U}]$ by replacing each $\widetilde{\beta}_{ij}$ satisfying $i \bmod d + j + \lfloor \frac{i}{d} \rfloor > 2n$ with the auxiliary moment A_{ij}. Then

$$C[\{A_{ij}\}_{(i,j)\in\mathcal{F}}] = \widetilde{M}[\widetilde{\beta}, \mathcal{U}].$$

Proof. Part (i) follows from the equality (3.20) and $L_{\widetilde{\beta}}(r^2) = \int r^2 d\mu \ge 0$, where μ is a representing measure for β. For part (ii) first note that not all $\widetilde{\beta}_{ij}$ with $i + j > 2n$ are auxiliary moments. By recursive generation we have $\widetilde{\beta}_{ij} = \widetilde{\beta}_{i-d,j+1}$ if $d \le i < 2d - 1$ (observe that i is at most $2d - 2$) and so $\widetilde{\beta}_{ij}$ is auxiliary only if $i \bmod d + j + \lfloor \frac{i}{d} \rfloor = i - d + j + 1 > 2n$ in these cases. If $i < d$, then the condition $i \bmod d + j + \lfloor \frac{i}{d} \rfloor > 2n$ reduces to $i + j > 2n$. Now part (ii) follows from (3.14) and Lemma 3.5. □

If $H \equiv (h_{i+j-1})_{i,j=1}^{m}$ is any $m \times m$ Hankel matrix and $\widehat{t} := (t_1, \dots, t_m) \in \mathbb{R}^m$, then

$$\langle H\widehat{t}, \widehat{t}\rangle = \sum_{i=1}^{m}\sum_{j=1}^{m} t_i h_{i+j-1} t_j = \sum_{k=1}^{2m-1} \Big(h_k \cdot \sum_{\substack{1\le i,j\le m,\\ i+j=k+1}} t_i t_j \Big). \tag{3.24}$$

Lemma 3.7. *Let $\widehat{r} \equiv (r_0, \dots, r_{nd}) \in \mathbb{R}^{nd+1}$, $\widehat{s} \equiv (s_0, \dots, s_{nd}) \in \mathbb{R}^{nd+1}$ satisfy* (3.10). *For $i, j \ge 0$, with $i < d$ and $0 < i + j \le 2n$, define a_{ij} by* (3.8). *Then*

$$\langle C\widehat{r}, \widehat{r}\rangle + \langle C\widehat{s}, \widehat{s}\rangle = 0 \quad \Longleftrightarrow \quad (3.9) \textit{ holds.} \tag{3.25}$$

Proof. Let I_k, J_k be as in (3.18). Further, let

$$h_k := \begin{cases} \beta_{I_k,J_k}, & \text{if } I_k + J_k \le 2n, \\ A_{I_k,J_k}, & \text{if } I_k + J_k > 2n. \end{cases} \tag{3.26}$$

We now apply (3.24) with $m = nd+1$, $H = C$ with h_k as in (3.26), and with $t_p = r_{p-1}$ or $t_p = s_{p-1}$ $(1 \le p \le nd + 1)$:

$$\begin{aligned}
&\langle C\widehat{r}, \widehat{r}\rangle + \langle C\widehat{s}, \widehat{s}\rangle = \\
&= \sum_{k=1}^{2nd+1} \Big(h_k \cdot \sum_{\substack{1\le p,q\le nd+1,\\ p+q=k+1}} (r_{p-1}r_{q-1} + s_{p-1}s_{q-1}) \Big) \\
&= r_0^2 + s_0^2 + \sum_{\substack{0\le i<d,\ j\ge 0,\\ 0<i+j\le 2n}} \Big(\beta_{ij} \cdot \sum_{\substack{0\le p,q\le nd,\\ 0<p+q=i+dj}} (r_p r_q + s_p s_q) \Big) + \\
&\qquad\qquad + \sum_{\substack{0\le i<d,\ j\ge 0,\\ i+j>2n}} \Big(A_{ij} \cdot \sum_{\substack{0\le p,q\le nd,\\ 0<p+q=i+dj}} (r_p r_q + s_p s_q) \Big) \\
&= r_0^2 + s_0^2 + \sum_{\substack{0\le i<d,\ j\ge 0,\\ 0<i+j\le 2n}} a_{ij}\beta_{ij}
\end{aligned}$$

where we used the assumption of the lemma in the last equality. Now the equivalence of the lemma easily follows. □

Remark 3.8. It is important for the sequel to note that the implication $(\Leftarrow)$ of Lemma 3.7 may be used in order to construct elements p of $\ker L$ satisfying $p|\Gamma \ge 0$, so that $\mathcal{CV}(L) \subseteq \mathcal{Z}(p|\Gamma)$. For suppose $\widehat{r},\ \widehat{s} \in \mathbb{R}^{nd+1}$ satisfy $h_{i,j}(\widehat{r}, \widehat{s}) = 0$ for every $(i, j) \in \mathcal{F}$ and $\langle C\widehat{r}, \widehat{r}\rangle + \langle C\widehat{s}, \widehat{s}\rangle = 0$. Now define $a_{ij} = h_{ij}(\widehat{r}, \widehat{s})$ $(i, j \ge 0,\ i < d,$ $0 < i + j \le 2n)$. Then $p := \sum a_{ij} f_{ij} \in \ker L$ satisfies $p(x, x^d) = R(x)^2 + S(x)^2$, where $R(x) := r_0 + r_1 x + \cdots + r_{nd}x^{dn}$ and $S(x) := s_0 + s_1 x + \cdots + s_{nd}x^{dn}$. Now we have $\mathcal{CV}(L) \subseteq \{(x, x^d) : R(x) = S(x) = 0\}$ and $\operatorname{card} \mathcal{CV}(L) \le \min\{\deg R, \deg S\}$.

Let $A \equiv \{A_{ij}\}_{(i,j)\in\mathcal{F}}$ with $A_{ij} \in \mathbb{R}$. We say that the core matrix $C[A]$ is *recursively generated* if for every $v \in \mathbb{R}^{nd}$ satisfying $\begin{pmatrix} v \\ 0 \end{pmatrix} \in \ker C[A]$, it follows that

$\begin{pmatrix} 0 \\ v \end{pmatrix} \in \ker C[A]$.

Remark 3.9. Note that the definition above is equivalent to the definition of a "recursively generated" Hankel matrix given in [CF1]. However, it does not encompass the notion of recursiveness for a general multivariable moment matrix given in item iv) preceding Theorem 2.1.

Let $0_{k\times 1} \in \mathbb{R}^k$ stand for a zero column vector. By Remark 3.9 and properties of recursively generated Hankel matrices [CF1], for every singular, recursively generated $C[A]$ there exists $r \leq nd+1$ and a vector $v := (v_i)_{i=1}^r \in \mathbb{R}^r$ with $v_r \neq 0$, such that

$$\ker C[A] = \operatorname{span}\left\{ \begin{pmatrix} v \\ 0_{(nd+1-r)\times 1} \end{pmatrix}, \begin{pmatrix} 0 \\ v \\ 0_{(nd-r)\times 1} \end{pmatrix}, \dots, \begin{pmatrix} 0_{(nd+1-r)\times 1} \\ v \end{pmatrix} \right\}.$$

If we normalize v so that $v_r = 1$, then v is uniquely determined. We call this unique v the *generating kernel vector* of $C[A]$.

Because $y - x^d$ is irreducible, the core variety is either the entire curve or a finite set of points in the curve. The following theorem characterizes the existence of a representing measure for β in terms of the existence of auxiliary moments such that the core matrix is positive and recursively generated. It also characterizes the type of core variety in terms of positive completions of the core matrix.

Theorem 3.10. *Let $\beta \equiv \beta^{(2n)}$ be a given sequence such that $M_n \equiv M_n(\beta)$ is positive semidefinite and $(y - x^d)$-pure. Let $\Gamma := \mathcal{Z}(y - x^d)$. The following statements are equivalent:*

(i) β admits a representing measure (necessarily supported in Γ).

(ii) β admits a finitely atomic representing measure (necessarily supported in Γ).

(iii) There exist auxiliary moments $A \equiv \{A_{ij}\}_{(i,j)\in\mathcal{F}}$, such that the core matrix $C[A] \equiv C[\{A_{ij}\}_{(i,j)\in\mathcal{F}}]$ is positive semidefinite and recursively generated.

Moreover, if β has a representing measure, the core variety coincides with Γ if and only if there is some choice of auxiliary moments A such that $C[A]$ is positive definite. Further, the following are equivalent:

(iv) $\mathcal{CV}(L)$ is a nonempty finite subset of Γ.

(v) β has a unique representing measure, which is necessarily finitely atomic.

(vi) There is a unique positive semidefinite, recursively generated completion $C[A]$, which is necessarily singular.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ follows from Richter's Theorem [Ric] (or by Theorem 2.5).

Next we establish the implication $(ii) \Rightarrow (iii)$. Suppose $M_n(\beta)$ is $(y-x^d)$-pure and that β has a finitely atomic representing measure μ supported in Γ. Thus, μ is of the form

$$\mu = \sum_{k=1}^{m} a_k \delta_{(x_k,y_k)}, \tag{3.27}$$

where $m > 0$, each $a_k > 0$, and $y_k = x_k^d$ for each k. Since μ has moments of all orders, we may consider the moment matrix $M_{n+t}[\mu]$, containing μ-moments up to degree $2n + 2t$, where $t = \lceil \frac{d-2}{2} \rceil$. Here, $\lceil \cdot \rceil$ denotes the ceiling function, i.e., the smallest integer greater than or equal to its argument. Using the moment data $\widetilde{\beta}^{(2(n+t))}$ from $M_{n+t}[\mu]$, i.e., $\widetilde{\beta_{ij}} = \int x^i y^j d\mu$, $(i, j \geq 0,\ i + j \leq 2(n + t))$, let

$$\gamma_p = \widetilde{\beta}_{p \bmod d, \lfloor \frac{p}{d} \rfloor}\ (0 \leq p \leq 2nd). \tag{3.28}$$

Since $M_n[\mu] = M_n(\beta)$, we have

$$\gamma_p = \beta_{p \bmod d, \lfloor \frac{p}{d} \rfloor} \quad \text{if } 0 \leq p \leq 2nd \quad \text{and} \quad p \bmod d + \lfloor \frac{p}{d} \rfloor \leq 2n.$$

We next show that

$$\widetilde{\mu} := \sum_{k=1}^{m} a_k \delta_{x_k} \tag{3.29}$$

is a representing measure for $\gamma := \{\gamma_p\}_{0 \leq p \leq 2nd}$. Indeed, for $0 \leq p \leq 2nd$ we have

$$\sum a_k x_k^p = \sum a_k x_k^{p \bmod d + d \lfloor \frac{p}{d} \rfloor} = \sum a_k x_k^{p \bmod d} y_k^{\lfloor \frac{p}{d} \rfloor} = \widetilde{\beta}_{p \bmod d, \lfloor \frac{p}{d} \rfloor} = \gamma_p.$$

It now follows that the moment matrix for γ, which is the Hankel matrix $H(\gamma) \equiv (\gamma_{i+j})_{0 \leq i,j \leq nd}$, is positive semidefinite and recursively generated (cf. Section 2). If, in the core matrix $C[A]$, for each $(i, j) \in \mathcal{F}$ we set $A_{ij} = \gamma_{i+dj} = \widetilde{\beta}_{ij}$, then $C[A]$ coincides with $H(\gamma)$, and is thus positive semidefinite and recursively generated. This is precisely (iii).

Next we establish the implication $(iii) \Rightarrow (ii)$. Suppose there exist auxiliary moments A such that $C[A]$ is positive semidefinite and recursively generated. We will prove that β has a finitely atomic representing measure. Define a univariate sequence $\gamma \equiv \{\gamma_p\}_{0 \leq p \leq 2nd}$ as in (3.28) above, where $\widetilde{\beta}_{ij}$ is either β_{ij} or A_{ij}. Since the Hankel matrix $H(\gamma) \equiv (\gamma_{i+j})_{0 \leq i,j \leq nd}$ coincides with $C[A]$ (by definition of γ), it follows that it is positive semidefinite and recursively generated. By [CF1, Theorem 3.9], γ has a finitely atomic representing measure $\widetilde{\mu} := \sum_{k=1}^{m} a_k \delta_{x_k}$. But then $\mu = \sum_{k=1}^{m} a_k \delta_{(x_k,y_k)}$ is a representing measure for β. Indeed, for $0 \leq i, j \leq 2n, i + j \leq 2n$ we have

$$\sum a_k x_k^i y_k^j = \sum a_k x_k^{i+dj} = \gamma_{i+dj} = \beta_{i \bmod d, j + \lfloor \frac{i}{d} \rfloor} = \beta_{i,j},$$

where in the last equality we used that $\beta_{r+d,s} = \beta_{r,s+1}$ for $0 \leq r, s$ such that $r+s+d \leq 2n$.

It remains to address the core variety. First assume that $C[A]$ is positive definite for some choice of auxiliary moments A. Concerning the core variety of $L \equiv L_\beta$, we have $V_0 = \mathcal{V}(M_n) = \Gamma$, and we now consider $V_1 := \mathcal{Z}(p \in \ker L \colon p|V_0 \geq 0)$. For

$p \in \ker L$ with $p|V_0 \geq 0$, we have $p = F + G$ as in (3.4). The discussion following the proof of Lemma 3.1 shows that $Q(x) := F(x, x^d)$ satisfies $Q(x) = R(x)^2 + S(x)^2$, where $\widehat{r}$ and $\widehat{s}$ satisfy the conditions of (3.8), (3.9) and (3.10). Lemma 3.7 now shows that $\langle C\widehat{r}, \widehat{r}\rangle + \langle C\widehat{s}, \widehat{s}\rangle = 0$, and since C is positive definite, it follows that $\widehat{r} = \widehat{s} = 0$. Thus (3.8) implies that each $a_{ij} = 0$, so $F = 0$. Since $\mathcal{Z}(G|\Gamma) = \Gamma$, we now have $\mathcal{Z}(p|\Gamma) = \Gamma$. It follows that $V_1 = V_0 = \Gamma$, so $\mathcal{CV}(L) = \Gamma$ and the Core Variety Theorem implies that β has finitely atomic representing measures whose union of supports is Γ.

Assume next that $\mathcal{CV}(L) = \Gamma$. We need to prove that there exists a choice of auxiliary moments A such that $C[A]$ is positive definite. We first show that if there exist distinct completions $C[A_1]$ and $C[A_2]$ that are positive semidefinite, recursively generated and *singular*, then there is a positive definite completion $C[A]$.

Claim. $C\left[\frac{1}{2}A_1 + \frac{1}{2}A_2\right]$ is positive definite.

Proof. We have

$$C\left[\tfrac{1}{2}A_1 + \tfrac{1}{2}A_2\right] = \tfrac{1}{2}C[A_1] + \tfrac{1}{2}C[A_2].$$

Since all three matrices are positive semidefinite, it follows that

$$\ker\left(C\left[\tfrac{1}{2}A_1 + \tfrac{1}{2}A_2\right]\right) = \ker(C[A_1]) \cap \ker(C[A_2]). \tag{3.30}$$

Assume that $C\left[\frac{1}{2}A_1 + \frac{1}{2}A_2\right]$ is not positive definite. Let $v \in \mathbb{R}^r$, $r \leq nd + 1$, be its generating kernel vector. By (3.30), the vector

$$u := \begin{pmatrix} 0_{(nd+1-r)\times 1} \\ v \end{pmatrix}$$

lies in $\ker(C[A_1]) \cap \ker(C[A_2])$. Now examine the last column of each of the matrices $C[A_1]$, $C[A_2]$, $C\left[\frac{1}{2}A_1 + \frac{1}{2}A_2\right]$, proceeding from the top to the bottom row. At the first occurrence of an auxiliary moment, the corresponding entry must be identical in all three matrices, because u is a common kernel vector and the other entries in the row of the auxiliray moment coincide in all three matrices. Proceeding to the second auxiliary moment, we again conclude (using the Hankel structures and that the first auxiliary moments already coincide) that this entry must also agree in all three matrices. Continuing inductively, we find that all auxiliary moments coincide, i.e., $A_1 = A_2$. This contradicts the assumption $A_1 \neq A_2$, completing the proof of the claim.

Now let μ_1 be a finitely atomic representing measure for β as given by (3.27). As in the proof of $(ii) \Rightarrow (iii)$, we associate to μ_1 the univariate sequence γ_1 with Hankel matrix $H(\gamma_1)$ and representing measure $\widetilde{\mu}_1$ as in (3.29), and use these to define the positive semidefinite and recursively generated completion $C[A_1] := H(\gamma_1)$. If $C[A_1]$ is positive definite, we are done, so we may assume $C[A_1]$ is singular, in which case γ_1 has a unique representing measure by [CF1, Theorem 3.10], namely $\widetilde{\mu}_1$. Since $\mathcal{CV}(L) = \Gamma$, there exists a finitely atomic representing measure μ_2 for β that is distinct from μ_1. As above, we may associate to μ_2 its univariate sequence γ_2 with Hankel matrix $H(\gamma_2)$ and representing measure $\widetilde{\mu}_2$, and use these to define the positive semidefinite and recursively generated completion $C[A_2] := H(\gamma_2)$. If $C[A_2]$ is positive definite, we are done, so we may assume $C[A_2]$ is singular. Now, if $C[A_1]$ and $C[A_2]$ are distinct we may apply the Claim to conclude that there is a positive definite completion, as desired. So we may assume that $C[A_2] = C[A_1]$, whence $H(\gamma_2) = H(\gamma_1)$. Since $H(\gamma_1)$ has the unique representing measure $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ is a representing measure for $H(\gamma_2)$, we have $\widetilde{\mu}_2 = \widetilde{\mu}_1$. It follows readily that $\mu_2 = \mu_1$, a contradiction. Thus $C[A_1]$ and $C[A_2]$ are distinct, which implies that there is a positive definite completion $C[A]$.

The equivalences among $(iv), (v), (vi)$ follow directly from the reasoning above. $\square$

Remark 3.11. Note that if there is a positive definite completion $C[A]$, there may also be positive semidefinite, recursively generated but singular completions. Thus, for $d = 3$, the set of A for which $C[A]$ is positive definite forms an open interval, and at the interval endpoints the completions are positive semidefinite, recursively generated, but singular.

The rest of this paper and its sequel [FZ-] are primarily devoted to developing *concrete* conditions for the existence or nonexistence of auxiliary moments satisfying condition (iii) of Theorem 3.10. We conclude this section with examples which illustrate cases where the core variety is either the entire curve $y = x^d$ or is empty. These examples suggest the following question.

Question 3.12. *Let $\beta \equiv \beta^{(2n)}$ be such that $M_n(\beta)$ is positive semidefinite and $(y-x^d)$-pure. Is it possible for β to have a unique representing measure (cf. conditions* (iv)-(vi) *of Theorem 3.10)?*

In Example 3.13 (just below) we show that the answer is negative for $d = 1$ and $d = 2$. In Section 4 we prove a negative answer for $d = 3$. This provides a new proof of Theorem 2.2. Nevertheless, for $d = 4$ we can establish a positive answer. The example illustrating this is beyond the scope of this note, as it requires special techniques, but will appear in [FZ-].

In the sequel, for $M_n \succeq 0$ and $(y-x^d)$-pure, we denote by $\widehat{M_n}$ the central compression of M_n obtained by deleting all rows and columns $X^{d+p}Y^q$ $(p, q \geq 0,\ p+q \leq n-d)$. The number of rows and columns in $\widehat{M_n}$ is thus $\dim \mathcal{P}_n - \dim \mathcal{P}_{n-d} = \frac{d(2n-d+3)}{2}$. Since M_n is positive and $(y-x^d)$-pure, it follows immediately that $\widehat{M_n}$ is positive definite and

$$\operatorname{rank} M_n = \operatorname{rank} \widehat{M_n} = \frac{d(2n-d+3)}{2}. \tag{3.31}$$

Example 3.13. i) For $d = 1$, we have $C = \widehat{M_n} = \left(\beta_{0,i+j-2}\right)_{1 \leq i,j \leq n+1} \succ 0$, so the existence of representing measures whose union of supports is the line $y = x$ now follows from Theorem 3.10. Alternately, using flat extensions, the existence of measures in this case follows from the solution to the truncated moment problem on a line in [CF3].

ii) For $d = 2$, the core matrix C for M_n is $(2n+1) \times (2n+1)$, with

$$C_{ij} = \beta_{(i+j-2) \bmod 2, \lfloor \frac{i+j-2}{2} \rfloor}. \tag{3.32}$$

In $\widehat{M_n}$, column j is the truncation to $\widehat{M_n}$ of column $X^{(j-1) \bmod 2} Y^{\lfloor (j-1)/2 \rfloor}$ in M_n. Likewise, row i of $\widehat{M_n}$ is the truncation to $\widehat{M_n}$ of row $X^{(i-1) \bmod 2} Y^{\lfloor (i-1)/2 \rfloor}$ in M_n. Thus, using the structure of moment matries, we have

$$\widehat{M}_{ij} = \beta_{(i-1) \bmod 2 + (j-1) \bmod 2, \lfloor (i-1)/2 \rfloor + \lfloor (j-1)/2 \rfloor}. \tag{3.33}$$

By Proposition 3.6 (or using calculations based on (3.32) and (3.33)), we have $C = \widehat{M_n} \succ 0$. Since C is positive definite, Theorem 3.10 now implies that β has representing measures whose union of supports is the parabola $y = x^2$. The existence of representing measures also follows from the solution to the Parabolic Truncated Moment Problem in [CF4], based on flat extensions. △

Remark 3.14. Note that for $d \geq 3$, $C \equiv C[\{A_{ij}\}_{(i,j) \in \mathcal{F}}]$ does not coincide with $\widehat{M_n}$. However, its central compression $\widehat{C}$, obtained by deleting row k and column k from C in those cases where row k ends with an auxiliary moment, is orthogonally equivalent to $\widehat{M_n}$, and is therefore positive definite. The details of this will appear in [FZ-].

Here we only explain the case $d = 3$, since this will be needed in the next section to provide a core variety-based solution to the $(y - x^3)$-pure TMP. The core matrix C for M_n is $(3n+1) \times (3n+1)$ with

$$C_{ij} = \beta_{(i+j-2) \bmod 3, \lfloor \frac{i+j-2}{3} \rfloor}. \tag{3.34}$$

Let $\widehat{C}$ be a $3n \times 3n$ principal submatrix of C obtained by deleting nd-th row and column. Recall that the rows and columns of $\widehat{M_n}$, which is also of size $3n \times 3n$, are labelled in degree-lexicographic order,

$$1, X, Y, X^2, XY, Y^2, X^2Y, XY^2, Y^3, \ldots, X^2Y^{n-2}, XY^{n-1}, Y^n$$

(there is no row or column X^iY^j with $i \geq 3$). Let us permute these to the order

$$1, X, X^2, Y, XY, X^2Y, Y^2, XY^2, X^2Y^2, \ldots, Y^{n-1}, XY^{n-1}, Y^n. \tag{3.35}$$

Then there exists a permutation matrix U of size $3n \times 3n$ such that $U^T \widehat{M_n} U$ has rows and columns indexed by (3.35). Note that

$$\begin{aligned} (U^T \widehat{M_n} U)_{ij} &= \beta_{(i-1) \bmod 3 + (j-1) \bmod 3, \lfloor (i-1)/3 \rfloor + \lfloor (j-1)/3 \rfloor} \\ &= \beta_{(i+j-2) \bmod 3, \lfloor \frac{i+j-2}{3} \rfloor}. \end{aligned} \tag{3.36}$$

where we used Lemma 3.5 in the second equality. By (3.34) and (3.36), $\widehat{C}$ is orthogonally equivalent to $\widehat{M_n}$.

We conclude this section with some examples that illustrate Theorem 3.10 for a positive semidefinite $(y - x^d)$-pure $M_n(\beta)$. Let $\widehat{C}$ denote the compression of $C \equiv C[A]$ obtained by deleting each row and each column of C that ends in some auxiliary moment A_{ij}. In the sequel, for $1 \leq k \leq dn+1$, C_k denotes the compression of C to the first k rows and columns.

Example 3.15. Consider the moment matrix

$$M_3(\beta) = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 2 & 5 & 14 & 42 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 1 & 0 & 0 & 2 & 5 & 14 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 5 & 14 & 42 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 14 & 42 & 132 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 0 & 5 & 14 & 0 & 0 & 0 & 14 & 42 & 132 & 429 \\ 0 & 14 & 42 & 0 & 0 & 0 & 42 & 132 & 429 & s \\ 0 & 42 & 132 & 0 & 0 & 0 & 132 & 429 & s & t \end{pmatrix}. \tag{3.37}$$

A calculation with nested determinants shows that M_3 is positive semidefinite and $(y-x^3)$-pure if and only if $s \equiv \beta_{15}$ and $t \equiv \beta_{06}$ satisfy

$$t > s^2 - 2844s + 2026881. \tag{3.38}$$

The core matrix is

$$C[A] = \begin{pmatrix} 1 & 0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 & 0 \\ 0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 & 0 & 42 \\ 1 & 0 & 2 & 0 & 5 & 0 & 14 & 0 & 42 & 0 \\ 0 & 2 & 0 & 5 & 0 & 14 & 0 & 42 & 0 & 132 \\ 2 & 0 & 5 & 0 & 14 & 0 & 42 & 0 & 132 & 0 \\ 0 & 5 & 0 & 14 & 0 & 42 & 0 & 132 & 0 & 429 \\ 5 & 0 & 14 & 0 & 42 & 0 & 132 & 0 & 429 & 0 \\ 0 & 14 & 0 & 42 & 0 & 132 & 0 & 429 & 0 & s \\ 14 & 0 & 42 & 0 & 132 & 0 & 429 & 0 & s & A \\ 0 & 42 & 0 & 132 & 0 & 429 & 0 & s & A & t \end{pmatrix}. \tag{3.39}$$

i) Let $s = 1430$ and $t = 4862$, so (3.38) is satisfied. Calculations with nested determinants show that $C_9 \succ 0$, and therefore a calculation of $\det C[A]$ shows that $C[A] \succ 0$ if and only if $-1 < A < 1$. Theorem 3.10 now shows that β has representing measures and that $\mathcal{CV}(L_\beta)$ is the curve $y = x^3$.

ii) Consider next $s = 1422$, $t = 4798$. Condition (3.38) is satisfied and nested determinants show that $\widehat{C} \succ 0$. In particular, $C_8 \succ 0$, but we have $\det C_9 = -7$, so for no value of A will $C[A]$ be positive semidefinite. By Theorem 3.10, β has no measure.

iii) Now let $s = 1429$, $t = 4847$. Then (3.38) holds, and we have $C_8 \succ 0$; however, $\det C_9 = 0$, so there exists $x \in \mathbb{R}^9$ such that $C_9 x = 0$. Now $\widehat{r} := (x^t, 0) \equiv (r_0, \ldots, r_8, 0)$ satisfies $\langle C\widehat{r}, \widehat{r}\rangle = 0$ and, with $\widehat{s} \equiv 0$, also satisfies the consistency requirement $r_8 r_9 + s_8 s_9 = 0$ (cf. (3.12)). Remark 3.8 now implies that there exists $p \in \ker L_\beta$ such that $Q(x) := p(x, x^3) = r(x)^2$. Therefore, $\operatorname{card} \mathcal{CV}(L) \le \deg r \le 8 < 9 = \operatorname{rank} M_3$, so $\mathcal{CV}(L_\beta) = \emptyset$ by Corollary 2.4, and thus β has no measure. $\triangle$

In next section we will prove that the method of the preceding example applies to any positive semidefinite $M_n(\beta)$ that is $(y-x^3)$-pure. We may therefore formulate one solution to the $(y-x^3)$-pure truncated moment problem as follows (see Corollary 4.3 and its proof).

Theorem 3.16. *Suppose $M_n(\beta)$ is positive semidefinite and $(y-x^3)$-pure. Then β has a representing measure if and only if $\det C_{3n} > 0$, in which case $\mathcal{CV}(L_\beta)$ is the curve $y = x^3$.*

Example 3.17. Consider next the sequence $\beta^{(8)}$, with M_3 given by

$$\begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 14 & 0 & 5 & 0 & 132 \\ 0 & 1 & 0 & 0 & 5 & 0 & 2 & 0 & 42 & 0 \\ 2 & 0 & 14 & 5 & 0 & 132 & 0 & 42 & 0 & 1430 \\ 1 & 0 & 5 & 2 & 0 & 42 & 0 & 14 & 0 & 429 \\ 0 & 5 & 0 & 0 & 42 & 0 & 14 & 0 & 429 & 0 \\ 14 & 0 & 132 & 42 & 0 & 1430 & 0 & 429 & 0 & 16796 \\ 0 & 2 & 0 & 0 & 14 & 0 & 5 & 0 & 132 & 0 \\ 5 & 0 & 42 & 14 & 0 & 429 & 0 & 132 & 0 & 4862 \\ 0 & 42 & 0 & 0 & 429 & 0 & 132 & 0 & 4862 & 0 \\ 132 & 0 & 1430 & 429 & 0 & 16796 & 0 & 4862 & 0 & 208012 \end{pmatrix}$$

and the degree 7 and degree 8 blocks given by

$$\begin{pmatrix} 0 & 42 & 0 & 1430 & 0 \\ 42 & 0 & 1430 & 0 & 58786 \\ 0 & 1430 & 0 & 58786 & 0 \\ 1430 & 0 & 58786 & 0 & 2674440 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 0 & 429 & 0 & 16796 \\ 0 & 429 & 0 & 16796 & 0 \\ 429 & 0 & 16796 & 0 & 742900 \\ 0 & 16796 & 0 & 742900 & 0 \\ 16796 & 0 & 742900 & 0 & 353576708 \end{pmatrix}$$

The core matrix is a Hankel matrix (see Example 3.4) with anti-diagonals completely determined in the first row by

$$\begin{array}{llll} \beta_{00} = 1, & \beta_{01} = 2, & \beta_{02} = 14, & \beta_{03} = 132, \\ \beta_{10} = 0, & \beta_{11} = 0, & \beta_{12} = 0, & \beta_{13} = 0, \\ \beta_{20} = 1, & \beta_{21} = 5, & \beta_{22} = 42, & \beta_{23} = 429, \\ \beta_{30} = 0, & \beta_{31} = 0, & \beta_{32} = 0, & \beta_{33} = 0, \end{array}$$

and the last column by

$$\begin{array}{llll} \beta_{04} = 1430, & \beta_{05} = 16796 & \beta_{06} = 208012, & \beta_{07} = 2674440, \\ \beta_{14} = 0, & \beta_{15} = 0, & \beta_{16} = 0, & \beta_{17} = 0, \\ \beta_{24} = 4862, & \beta_{25} = 58786, & \beta_{26} = 742900, & \beta_{27} = A_{27}, \\ \beta_{34} = 0, & \beta_{35} = 0, & \beta_{36} = A_{36}, & \beta_{37} = A_{37} \\ & & & \beta_{08} = 353576708. \end{array}$$

It is straightforward to verify that M_4 is positive semidefinite and $(y-x^4)$–pure. Using nested determinants, it is easy to show that $C_{14} \succ 0$. A further calculation shows that $C_{15} \succ 0$ if and only if $-1 < A_{36} < 1$. Setting $A_{36} = 0$, we see that $C_{16} \succ 0$ if and only if $A_{27} = 9694844 + f$ for $f > 0$. Now $\det C = f(318219068 - 28f - f^2) - A_{37}^2$, so there exists A_{37} such that $C[A] \succ 0$ if and only if $0 < f < 96\sqrt{34529} - 14$ (≈ 17824.7). In this case, since $C[A] \succ 0$, the core variety coincides with the curve $y = x^4$.

Example 3.18. Consider next the sequence $\beta^{(8)}$, defined as in Example 3.17, except for the following 5 differences:

$$\beta_{25} = 0, \qquad \beta_{06} = 3454708516 \qquad \beta_{26} = 3448894372, \qquad \beta_{07} = 0,$$
$$\beta_{08} = 26405033821733706989067766957 25.$$

It is straightforward to verify that M_4 is positive semidefinite and $(y - x^4)$–pure. Moreover, $C[A]$ can never be positive semidefinite, since $\beta_{25} = 0$ is its 12th diagonal element, but there are nonzero entries in the 12th row and column. By the converse in Theorem 3.10, $\beta^{(8)}$ does not admit a representing measure.

4. The $(y - x^3)$-pure truncated moment problem.

In this section we apply the previous results to the moment problem for $\beta \equiv \beta^{(2n)}$ where M_n is positive semidefinite and $(y - x^3)$-pure. In particular, Theorem 4.1 provides a positive answer to Question 3.12 for $d = 3$. Let Γ stand for the curve $y = x^3$. Note that in the core matrix C, since $Y = X^d$ with $d = 3$, there is exactly 1 auxiliary moment, namely $\beta_{2,2n-1}$, which we denote by $A \equiv A_{2,2n-1}$ (cf. Example 3.3). Let $\widehat{C}$ be the principal submatrix of C obtained by deleting row and column nd. Recall from Remark 3.14 that $\widehat{C} \succ 0$. Let $H \equiv H[A]$ denote the matrix obtained from $C \equiv C[A]$ by interchanging rows and columns nd and $nd + 1$ (the last 2 rows and columns), so that H is orthogonally equivalent to C, i.e.,

$$H = P^T CP, \tag{4.1}$$

where P is a permutation matrix defined on the standard orthonormal basis $e_1, \ldots, e_{nd+1}$ for $\mathbb{R}^{nd+1}$ by

$$Pe_i = \begin{cases} e_i, & i \leq nd - 1, \\ e_{nd+1}, & i = nd, \\ e_{nd}, & i = nd + 1. \end{cases}$$

We may thus represent H as

$$H = \begin{pmatrix} \widehat{C} & v \\ v^t & \beta_{1,2n-1} \end{pmatrix}, \tag{4.2}$$

with $\widehat{C} \succ 0$ and where v is of the form

$$v = \begin{pmatrix} h \\ A \end{pmatrix}. \tag{4.3}$$

(Here $h \in \mathbb{R}^{dn-1}$ and v^t denotes the row vector transpose of v.) As in Section 3, for $1 \leq j \leq dn + 1$, let C_j denote the compression of C to the first j rows and columns. Write

$$\widehat{C} = \begin{pmatrix} C_{dn-1} & z \\ z^t & \beta_{0,2n} \end{pmatrix}, \tag{4.4}$$

where $z \in \mathbb{R}^{dn-1}$ is of the form

$$z = \begin{pmatrix} k \\ \beta_{1,2n-1} \end{pmatrix}$$

for some $k \in \mathbb{R}^{dn-2}$. We now have

$$H[A] = \begin{pmatrix} C_{dn-1} & z & h \\ z^t & \beta_{0,2n} & A \\ h^t & A & \beta_{1,2n-1} \end{pmatrix}. \tag{4.5}$$

Since $\widehat{C} \succ 0$, $\widehat{C}^{-1}$ exists and has the form

$$\widehat{C}^{-1} = \begin{pmatrix} \mathcal{C} & w \\ w^t & \epsilon \end{pmatrix}, \tag{4.6}$$

where (see e.g., [F2, p. 3144])

$$\begin{aligned} \epsilon &= \frac{1}{\beta_{0,2n} - z^t C_{dn-1}^{-1} z} > 0, \\ w &= -\epsilon C_{dn-1}^{-1} z \in \mathbb{R}^{dn-1}, \\ \mathcal{C} &= C_{dn-1}^{-1}(1 + \epsilon z z^t C_{dn-1}^{-1}) \in \mathbb{R}^{(dn-1)\times(dn-1)}. \end{aligned} \tag{4.7}$$

Now

$$\widehat{C}^{-1} v = \begin{pmatrix} \mathcal{C}h + Aw \\ w^t h + A\epsilon \end{pmatrix},$$

and we set

$$A \equiv A_0 := -\frac{w^t h}{\epsilon}, \tag{4.8}$$

so that

$$\widehat{C}^{-1} v = \begin{pmatrix} \mathcal{C}h - \frac{w^t h}{\epsilon} w \\ 0 \end{pmatrix}. \tag{4.9}$$

With this value of A in C, and thus also in v, let

$$\begin{aligned} \phi := v^t \widehat{C}^{-1} v &= h^t \mathcal{C} h - \frac{w^t h h^t w}{\epsilon} \\ &= (h^t C_{dn-1}^{-1} h + \epsilon h^t C_1^{-1} z z^t C_{dn-1}^{-1} h) - \epsilon z^t C_{dn-1}^{-1} h h^t C_{dn-1}^{-1} z \\ &= h^t C_{dn-1}^{-1} h, \end{aligned} \tag{4.10}$$

where we used (4.7) in the second equality.

To emphasize the dependence of ϕ on β, we sometimes denote ϕ as $\phi[\beta]$. In Example 4.5 (below) we will use the fact that ϕ is independent of $\beta_{1,2n-1}$ and $\beta_{0,2n}$. To see this, note that $\beta_{1,2n-1}$ is an element of vectors z and z^t, so (4.5) shows that C_{dn-1} and h are independent of $\beta_{1,2n-1}$ and $\beta_{0,2n}$. It now follows from (4.10) that ϕ is independent of $\beta_{1,2n-1}$ and $\beta_{0,2n}$ as well. Thus, if $\widetilde{\beta}^{(2n)}$ has the property that $M_n(\widetilde{\beta})$ is positive semidefinite and $(y - x^3)$-pure, and if $\beta_{ij} = \widetilde{\beta}_{ij}$ for all $(i, j) \neq (1, 2n - 1)$ and $(i, j) \neq (0, 2n)$, then $\phi[\widetilde{\beta}] = \phi[\beta]$. Note that ϕ would depend on $\beta_{1,2n-1}$ and $\beta_{0,2n}$ if A_0 in (4.8) was chosen differently. This is due to the fact that the last row of $\widehat{C}^{-1} v$ in (4.9) would be non-zero.

Theorem 4.1. *Suppose M_n is positive semidefinite and $(y - x^3)$-pure. $\beta \equiv \beta^{(2n)}$ has a representing measure if and only if $\beta_{1,2n-1} > \phi$ (equivalently, $C[A_0] \succ 0$). In this case, $\mathcal{CV}(L_\beta) = \Gamma$, which coincides with the union of supports of all representing measures (respectively, all finitely atomic representing measures).*

Proof. Recall from Remark 3.14 that $\widehat{C}$ is positive definite. Consider first the case $\beta_{1,2n-1} > \phi$. It follows from (4.2) and [A, Theorem 1] that H is positive definite. Since C is orthogonally equivalent to H, we see that C is positive definite, so the existence of representing measures and the conclusion concerning supports follow from Theorem 3.10.

We next consider the case when $\beta_{1,2n-1} = \phi$, so that by [A, Theorem 1], H is positive semidefinite, but singular. Since $\widehat{C} \succ 0$, it follows from (4.2) and (4.9) that $\ker H$ contains the vector

$$\widehat{u} := \begin{pmatrix} \widehat{C}^{-1}v \\ -1 \end{pmatrix} \equiv \begin{pmatrix} \mathcal{C}h - \frac{w^t h}{\epsilon} w \\ 0 \\ -1 \end{pmatrix} \equiv (r_0,\ r_1, \ldots,\ r_{dn-2},\ u_{dn-1},\ u_{dn})^t, \tag{4.11}$$

where $u_{dn-1} = 0$ and $u_{dn} = -1$. From the orthogonal equivalence between H and C, based on the interchange of rows and columns nd and $nd+1$, it follows that C is positive semidefinite and that $\ker C$ contains the vector

$$\widehat{r} = (r_0,\ r_1, \ldots,\ r_{dn-2},\ r_{dn-1},\ r_{dn})^t, \tag{4.12}$$

where

$$r_{dn-1} = u_{dn} = -1 \quad \text{and} \quad r_{dn} = u_{dn-1} = 0. \tag{4.13}$$

Let $\widehat{s} \equiv (s_0, \ldots,\ s_{dn})^t$ denote the 0 vector, so that $\langle C\widehat{r}, \widehat{r}\rangle + \langle C\widehat{s}, \widehat{s}\rangle = 0$ and the auxiliary requirement of (3.10), $r_{dn-1}r_{dn} + s_{dn-1}s_{dn} = 0$, is satisfied. Now, following Remark 3.8, define $a_{ij} = h_{ij}(\widehat{r}, \widehat{s})$ $(0 \le i \le 2,\ j \ge 0,\ 0 < i+j \le 2n)$. Then $p := \sum a_{ij} f_{ij}$ is an element of $\ker L_\beta$ which satisfies $Q(x) := p(x, x^3) = R(x)^2$, where

$$R(x) := r_0 + r_1 x + \cdots + r_{dn-1}x^{dn-1} + r_{dn}x^{dn}.$$

Since $r_{dn} = 0$, $R(x)$ has at most $dn-1$ real zeros, so p has at most $dn-1$ zeros in the curve $y = x^3$. Now $p \in \ker L_\beta$ satisfies $p|\Gamma \ge 0$ and $\operatorname{card} \mathcal{Z}(p|\Gamma) \le dn - 1 < \frac{d(2n-d+3)}{2} = \operatorname{rank} M_n$ (since $d = 3$), so Corollary 2.4 implies that β has no representing measure.

To complete the proof, we consider the case when $\beta_{1,2n-1} < \phi$. From (4.2) and (4.11) we have

$$\begin{aligned}\langle H\widehat{u},\ \widehat{u}\rangle &= \langle \begin{pmatrix} 0_{dn\times 1} \\ v^t\widehat{C}^{-1}v - \beta_{1,2n-1} \end{pmatrix}, \begin{pmatrix} *_{dn\times 1} \\ -1 \end{pmatrix} \rangle \\ &= \beta_{1,2n-1} - v^t\widehat{C}^{-1}v \\ &= \beta_{1,2n-1} - \phi < 0.\end{aligned}$$

Recall that $H = P^T C P$ (cf. (4.1)). Setting $\widehat{r} := P\widehat{u}$, we have

$$\langle C\widehat{r},\ \widehat{r}\rangle = \langle H\widehat{u}, \widehat{u}\rangle < 0,$$

where $\widehat{r}$ is as in (4.12), (4.13). Let $\epsilon = (\phi - \beta_{1,2n-1})^{1/2}$. Since $\langle \widehat{C}e_1,\ e_1\rangle = \beta_{00} = 1$, then the constant polynomial $S(x) = \epsilon$, with coefficient vector $\widehat{s} = (\epsilon, 0, \ldots, 0)^t$, satisfies $s_{dn-1}s_{dn} = 0$, and we have $\langle C\widehat{r},\ \widehat{r}\rangle + \langle C\widehat{s},\ \widehat{s}\rangle = 0$. So $\widehat{r}$ and $\widehat{s}$ together satisfy the auxiliary requirement of (3.10). Constructing $p(x,y)$ as in Remark 3.8, we have that

$p \in \ker L_\beta$. Now, $p(x, x^d) = R(x)^2 + S(x)^2 \geq \epsilon^2 > 0$. Since p is strictly positive on Γ, then $\mathcal{CV}(L_\beta) = \emptyset$, and therefore β has no representing measure. $\square$

Remark 4.2. In Theorem 4.1, an alternative proof of the case $\beta_{1,2n-1} < \phi$ can be based on Theorem 3.10, as follows. Let A_0 be as in (4.8). If $\beta_{1,2n-1} < \phi[A_0]$, then (4.10) implies that $\beta_{1,2n-1} < h^t C_{dn-1}^{-1} h$. It therefore follows from (4.5) that for every $A \in \mathbb{R}$, the matrix $\begin{pmatrix} C_{dn-1} & h \\ h^t & \beta_{1,2n-1} \end{pmatrix}$ is a principal submatrix of $H[A]$ that is not positive semidefinite. Thus, for every A, $H[A]$, and hence $C[A]$, is not positive semidefinite, so Theorem 3.10 implies that β has no representing measure.

Corollary 4.3. *Suppose $M_n(\beta)$ is positive semidefinite and $(y - x^3)$-pure. Then β has a representing measure if and only if $\det C_{dn} > 0$, in which case $\mathcal{CV}(L_\beta)$ is the curve $y = x^3$.*

Proof. Note that C_{dn} is equal to $\begin{pmatrix} C_{dn-1} & h \\ h^t & \beta_{1,2n-1} \end{pmatrix}$ (cf. (4.5)). From $\widehat{C} \succ 0$ it follows that $C_{dn-1} \succ 0$ (cf. (4.4)). Using [A], we have that

$$C_{dn} \succ 0 \iff \beta_{1,2n-1} > h^t C_{dn-1}^{-1} h \underbrace{\iff}_{(4.10)} \beta_{1,2n-1} > \phi.$$

Now the statement of the corollary follows from Theorem 4.1. $\square$

In [F2] a rather lengthy construction with moment matrices is used to derive a certain rational expression in the moment data, denoted by ψ in [F2], such that β has a representing measure if and only if $\beta_{1,2n-1} > \psi$, in which case M_n admits a flat extension M_{n+1}. In view of Theorem 4.1, it is clear that $\psi = \phi$ (although this is not at all apparent from the definitions of these expressions).

Corollary 4.4. *Suppose $M_n(\beta)$ is positive semidefinite and $(y - x^3)$-pure. The following are equivalent:*

(i) β has a representing measure;

(ii) β has a finitely atomic measure;

(iii) $M_n(\beta)$ has a flat extension M_{n+1};

(iv) $\mathcal{CV}(L_\beta) \neq \emptyset$;

(v) With A defined by (4.8) and ϕ defined by (4.10), $\beta_{1,2n-1} > \phi$;

(vi) $\mathcal{CV}(L_\beta) = \Gamma$.

Proof. The implications $(i) \implies (iv) \implies (ii) \implies (i)$ follow from the Core Variety Theorem and its proof. The equivalence of (i) and (iii) is established in [F2], and the equivalence of (i), (v), and (vi) is Theorem 4.1. $\square$

In [EF] the authors used the results of [F1] to exhibit a family of positive $(y - x^3)$-pure moment matrices $M_3(\beta^{(6)})$ such that $\beta^{(6)}$ has no representing measure but the Riesz functional is positive (cf. Section 2). Here, positivity of the functional cannot be derived from positivity of M_3 using an argument such as $L(p) = L(\sum p_i^2) =$

$\sum\langle M_3\widehat{p_i},\widehat{p_i}\rangle \geq 0$, because, by the theorem of Hilbert, not every nonnegative polynomial $p(x,y)$ of degree 6 can be represented as a sum of squares. Using Theorem 4.1 we can extend this example to a family of $(y-x^3)$-pure matrices M_n, for $n \geq 3$ as follows.

Example 4.5. Suppose $M \equiv M_n(\beta)$ is positive semidefinite and $(y-x^3)$-pure. Let $\phi \equiv \phi[\beta]$ be as in (4.10) and suppose $\phi = \beta_{1,2n-1}$, so that β has no representing measure by Theorem 4.1. We claim that the Riesz functional L_β is positive. Let $\widehat{M}$ denote the central compression of M to rows and columns that are of the form X^iY^j with $0 \leq i < 3$, so that $\operatorname{rank} M = \operatorname{rank} \widehat{M}$ and $\widehat{M} \succ 0$. Now let $\widetilde{\beta}$ be defined to coincide with β, except possibly in the $\beta_{1,2n-1}$ position. It follows from the structure of positive matrices that there exists $\delta > 0$ such that if $|\widetilde{\beta}_{1,2n-1} - \beta_{1,2n-1}| < \delta$, then $\widehat{M_n}(\widetilde{\beta})$ is positive definite. The structure of positive $(y-x^3)$-pure moment matrices now implies that $M_n(\widetilde{\beta})$ is positive semidefinite and $(y-x^3)$-pure. Now consider the sequence $\beta^{[m]}$ which coincides with β except that $\beta^{[m]}_{1,2n-1} = \beta_{1,2n-1} + 1/m$. It follows that there exists $m_0 > 0$ such that if $m > m_0$, then $M^{[m]} \equiv M_n(\beta^{[m]})$ is positive semidefinite and $(y-x^3)$-pure. By the remarks preceding Theorem 4.1, we have $\beta^{[m]}_{1,2n-1} = \beta_{1,2n-1} + 1/m > \beta_{1,2n-1} = \phi[M] = \phi[M_n(\beta^{[m]})]$, so Theorem 4.1 implies that $\beta^{[m]}$ has a representing measure. Thus, $L_{\beta^{[m]}}$ is positive, and since the cone of sequences with positive functionals is closed, it follows that L_β is positive. $\triangle$

To exhibit $M_n(\beta)$ as in Example 4.5, we may start with any positive semidefinite $(y-x^3)$-pure $M_n(\beta')$. Define β so that it coincides with β' except that $\beta_{1,2n-1} = \phi[\beta']$. If necessary, increase $\beta_{0,2n}$ to insure positivity of $M_n(\beta)$. Then $M_n(\beta)$ is positive semidefinite, $(y-x^3)$-pure, and $\beta_{1,2n-1} = \phi[\beta'] = \phi[\beta]$ by the remarks preceding Theorem 4.1.

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