THE PURE $Y = X^d$ TRUNCATED MOMENT PROBLEM

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Abstract. Let $\beta \equiv \beta^{(2n)}$ be a real bivariate sequence of degree $2n$. We study the existence of representing measures for β supported in the planar curve $y = x^d$ in the case where the associated moment matrix $M_n(\beta)$ is $(y - x^d)$-*pure*. In Section 2 we provide a general necessary and sufficient condition for representing measures in terms of positive semidefiniteness and recursive generation of the associated *core matrix*. In particular, if the core matrix is positive definite, then the *core variety* for β, i.e., the union of supports of all finitely atomic representing measures, is the whole curve $y = x^d$. In later sections, we provide various other *concrete* necessary or sufficient conditions for measures. For $d = 3$, we provide a core-variety proof of the result of [F2] characterizing the existence of representing measures. For $d \geq 4$ we develop a sufficient condition for the core variety to be finite or empty. For $d = 4$, we adapt the technique of [Z1], involving positive completions of partially-defined Hankel matrices, to provide necessary and sufficient numerical conditions for representing measures. We conclude with an example showing that in the $d = 4$ case, the core variety can be finite, with a unique representing measure, which cannot occur for $d < 4$.

1. Introduction.

Given a bivariate sequence of degree $2n$,

$$\beta \equiv \beta^{(2n)} = \{\beta_{ij} : i, j \geq 0, i + j \leq 2n\},\ \beta_{00} = 1,$$

and a closed set $K \subseteq \mathbb{R}^2$, the Truncated K-Moment Problem (TKMP) seeks conditions on β such that there exists a positive Borel measure μ on $\mathbb{R}^2$, with $\operatorname{supp}\mu \subseteq K$, satisfying

$$\beta_{ij} = \int_{\mathbb{R}^2} x^i y^j d\mu(x, y)\ (i, j \geq 0, i + j \leq 2n);$$

μ is a K-*representing measure* for β. A comprehensive reference for all aspects of the Moment Problem is the recent treatise of K. Schmüdgen [Sch]. Apart from solutions based on semidefinite programming and optimization, several different *abstract* solutions to TKMP appear in the literature, including the Flat Extension Theorem [CF5], the Truncated Riesz-Haviland Theorem [CF7], the idempotent approach of [Vas], and, more recently, the Core Variety Theorem [BF]. By a *concrete* solution to TKMP we mean an implementation of one of the abstract theories involving only basic linear

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algebra and solving algebraic equations (or estimating the size of the solution set). The ease with which any of the abstract results can be applied to solve particular moment problems in concrete terms varies considerably depending on the problem, with most concrete results attributable to the Flat Extension Theorem and very few to the other approaches. In this note we show how the Core Variety Theorem (Theorem 1.5 below) can indeed be applied to certain concrete moment problems, namely when K is the planar curve $y = x^d$ ($d \geq 1$).

In the classical literature TKMP has been solved concretely in terms of positive Hankel matrices when K is the real line, the half-line $[0, +\infty)$, or the closed interval $[a, b]$ (cf. [ST, CF1]). For the case when K is a planar curve $p(x, y) = 0$ with $\deg p \leq 2$, TKMP has been solved concretely in terms of moment matrix extensions (see Theorem 1.1 below, [CF3, CF4, CF6, F3]). In [F2] moment matrix extensions are used to concretely solve the truncated moment problem for $y = x^3$ and to solve (in a less concrete sense) truncated moment problems on curves of the form $y = g(x)$ and $yg(x) = 1$ ($g \in \mathbb{R}[x]$). More recently, several authors have intensively studied TKMP on certain planar curves of higher degree, using moment matrix extensions and a "reduction of degree" technique to improve and extend the results of [F2] (cf. [Z1, Z2, Z3, Z4, YZ]). We also note that for closed planar sets K that are merely semi-algebraic, such as the closed unit disk, very little is known concerning concrete solutions to TKMP (cf. [CF2]).

The results cited just above do not provide concrete solutions to TKMP for planar curves of the form $y = x^d$ ($d \geq 4$). The aim of this note is to illustrate how the core variety, described in Theorem 1.5, can be used to study TKMP for $K = \Gamma$, the planar curve $y = x^d$ ($d \geq 1$), when the associated moment matrix $M_n(\beta)$ is $(y - x^d)$-*pure*, i.e., the column dependence relations in $M_n(\beta)$ are precisely those that can be derived from the column relation $Y = X^d$ by *recursiveness* and linearity (see just below for terminology and notation). In Sections 2 and 3 we develop a core variety framework for studying TKMP in the $(y-x^d)$-pure case, and in Theorem 2.13 we present general necessary and sufficient conditions for Γ-representing measures in terms of positivity and recursiveness properties of the associated *core matrix*. In particular, if the core matrix is positive definite, then the core variety is the entire curve $y = x^d$. In Section 4 (Theorem 4.1) we apply Theorem 2.13 and core variety methods to present necessary and sufficient conditions for representing measures in the $(y - x^3)$–pure truncated moment problem. An equivalent version of this result was first proved in [F2] (see Theorem 1.2 below). The proof in [F2] entailed a lengthy construction involving flat extensions; the new core variety proof is short and more transparent. In Section 5 (Theorem 5.4), for the $(y - x^d)$–pure truncated moment problem with $d \geq 4$, we prove a concrete sufficient condition for the core variety to be finite, and also a concrete sufficient condition for the nonexistence of representing measures. In Section 6 we present concrete necessary and sufficient conditions for representing measures in the $(y-x^4)$–pure truncated moment problem. The proof of Theorem 6.3 combines core variety results with an approach adapted from [Z1] involving positive completions of partially defined Hankel matrices. Finally, we demonstrate by an example that in the $(y - x^4)$-pure case the core variety can be finite, with a unique representing measure, something that cannot occur in the $(y - x^d)$-pure truncated moment problem for $d < 4$.

Let $\mathcal{P} := \mathbb{R}[x, y]$ and let $\mathcal{P}_k := \{q \in \mathcal{P} \colon \deg q \le k\}$. Given $\beta \equiv \beta^{(2n)}$, define the *Riesz functional* $L_\beta : \mathcal{P}_{2n} \longrightarrow \mathbb{R}$ by

$$\sum a_{ij} x^i y^j \longmapsto \sum a_{ij} \beta_{ij}.$$

For a sequence $\beta \equiv \beta^{(2n)}$ with Riesz functional L_β, the *moment matrix* M_n has rows and columns indexed by the monomials in $\mathcal{P}_n$ in degree-lexicographic order, i.e., *1*, X, Y, X^2, XY, Y^2,. . . , X^n, . . ., Y^n. In this case, the element of M_n in row $X^i Y^j$, column $X^k Y^l$ is $\beta_{i+k,j+l}$. More generally, for $r, s \in \mathcal{P}_n$, with coefficient vectors $\widehat{r}, \widehat{s}$ relative to the basis of monomials, we have

$$\langle M_n \widehat{r}, \widehat{s} \rangle := L_\beta(rs).$$

In the sequel, for $q \in \mathcal{P}_n$, $q = \sum a_{ij} x^i y^j$, we set $q(X, Y) := \sum a_{ij} X^i Y^j$ $(= M_n \widehat{q})$.

If β has a K-representing measure μ, then L_β is K*-positive*, i.e., $q \in \mathcal{P}_{2n}$, $q|K \ge 0 \Longrightarrow L_\beta(q) \ge 0$ (since $L_\beta(q) = \int_K q d\mu$). The converse is not true; instead, the Truncated Riesz-Haviland Theorem [CF7] shows that β admits a K-representing measure if and only if L_β admits an extension to a K-positive linear functional on $\mathcal{P}_{2n+2}$. In [B] G. Blekherman proved that if M_n is positive semidefinite and $\operatorname{rank} M_n \le 3n - 3$, then L_β is $\mathbb{R}^2$-positive, so the Truncated Riesz-Haviland Theorem then implies that $\beta^{(2n-1)}$ has a representing measure. Using special features of the proof of Theorem 1.2 (below), in [EF] C. Easwaran and the first-named author exhibited a class of Riesz functionals that are positive but have no representing measure. Apart from these results, it seems very difficult to verify positivity of Riesz functionals in examples without first proving the existence of representing measures.

Several basic *necessary* conditions for a representing measures μ can be expressed in terms related to moment matrices (cf. [CF5]); we will refer to these without further reference in the sequel:
i) $M_n(\beta)$ is *positive semidefinite*: $\langle M_n \widehat{r}, \widehat{r} \rangle = L_\beta(r^2) = \int r^2 d\mu \ge 0$ $(\forall r \in \mathcal{P}_n)$.
ii) For any representing measure μ, $\operatorname{card}(\operatorname{supp} \mu) \ge \operatorname{rank} M_n$.
iii) Note that a dependence relation in the column space of M_n can be expressed as $r(X, Y) = 0$, where $r \in \mathcal{P}_n$. Define the *variety* of M_n, $\mathcal{V}(M_n)$, as the common zeros of the polynomials $r \in \mathcal{P}_n$ such that $r(X, Y) = 0$. Then $\operatorname{supp} \mu \subseteq \mathcal{V}(M_n)$, so $\operatorname{card} \mathcal{V}(M_n) \ge \operatorname{rank} M_n$.
iv) M_n is *recursively generated*: whenever r, s, and rs are in $\mathcal{P}_n$ and $r(X, Y) = 0$, then $(rs)(X, Y) = 0$.
v) M_n (or L_β) is *consistent*: for $p \in \mathcal{P}_{2n}$, $p|\mathcal{V}(M_n) \equiv 0 \Longrightarrow L_\beta(p) = 0$; consistency implies recursiveness [CFM].

The Flat Extension Theorem [CF5] shows that β admits a representing measure if and only if M_n admits a positive semidefinite moment matrix extension M_{n+k} (for some $k \ge 0$) for which there is a rank-preserving (i.e., *flat*) moment matrix extension M_{n+k+1}. Using this result, in a series of papers R. Curto and the first author solved TKMP for planar curves of degrees 1 and 2 as follows.

Theorem 1.1 ([CF3, CF4, CF6, F3, Degree-2 Theorem])**.** *Suppose $r(x, y) \in \mathcal{P}$ with $\deg r \leq 2$. For $n \geq \deg r$, M_n has a representing measure supported in the curve $r(x, y) = 0$ if and only if $r(X, Y) = 0$ and M_n is positive semidefinite, recursively generated, and satisfies* $\operatorname{card} \mathcal{V}(M_n) \geq \operatorname{rank} M_n$.

In [CFM] it was shown that this result does not extend to $\deg r > 2$. The example in [CFM] concerns an M_3 that is positive and recursively generated, with $\operatorname{card} \mathcal{V}_\beta = \operatorname{rank} M_3$, but which has no measure. In this example, there is no measure because L_β is not consistent. In [F2] we showed that positivity, the variety condition, and consistency are still not sufficient for representing measures, as we next describe.

For $M_n \succeq 0$, consider the $(y - x^3)$-*pure* case, when the column dependence relations in M_n are precisely those given by $Y = X^3$, recursiveness, and linearity, i.e., column relations of the form $(s(x, y)(y - x^3))(X, Y) = 0$ ($\deg s \leq n - 3$). Thus M_n is positive, $\operatorname{rank} M_n \leq \operatorname{card} \mathcal{V}(M_n)$ ($= \operatorname{card} \Gamma = +\infty$), and it follows from Lemma 3.1 in [F2] that M_n is consistent. In [F2] we described a particular, easily computable, rational expression in the moment data, ψ, and solved the $(y - x^3)$-pure TKMP as follows.

Theorem 1.2. *If $M_n \succeq 0$ is $(y - x^3)$-pure, then β has a representing measure if and only if $\beta_{1,2n-1} > \psi$.*

In the proof of Theorem 1.2, the numerical test $\beta_{1,2n-1} > \psi$ leads to a flat extension M_{n+1}. By contrast with this result, the other existence results in [F2, Z4] generally presuppose the existence of a certain positive moment matrix extension of M_n, but do not give an explicit test for the extension. The proof of Theorem 1.2 in [F2] is quite lengthy. In the sequel we will use the *core variety* to present a shorter, more transparent proof. This approach also provides a core variety framework for studying the $(y - x^d)$-pure truncated moment problem.

The core variety provides an approach to establishing the existence of representing measures based on methods of convex analysis. For the polynomial case, this was introduced in [F4], and some of the ideas go back to [FN]. The discussion below is based on joint work of the first author with G. Blekherman [BF], which treats general Borel measurable functions, although here we only require polynomials. The core variety has also been studied by P. di Dio and K. Schmüdgen [DDS] and in Schmüdgen's book [Sch].

Given $\beta \equiv \beta^{(2n)}$ and its Riesz functional $L \equiv L_\beta$, define $V_0 := \mathcal{V}(M_n)$ and for $i \geq 0$, define

$$V_{i+1} := \bigcap_{f \in \ker L,\ f|V_i \geq 0} \mathcal{Z}(f),$$

where $\mathcal{Z}(f)$ denotes the set of zeros of $f(x, y)$ in $\mathbb{R}^2$ (or, equivalently, in V_i).
We define the *core variety* of L by

$$\mathcal{CV}(L) := \bigcap_{i \geq 0} V_i.$$

Proposition 1.3 ([F4])**.** *If μ is a representing measure for L, then* $\operatorname{supp} \mu \subseteq \mathcal{CV}(L)$.

If μ is a representing measure, then

$$\operatorname{rank} M_n(\beta) \le \operatorname{card}(\operatorname{supp}\mu) \le \operatorname{card}\mathcal{CV}(L_\beta) \le \operatorname{card} V_i \quad (\text{for every } i \ge 0).$$

We thus have the following test for the nonexistence of representing measures.

Corollary 1.4 ([F4]). *If* $\operatorname{card} V_i < \operatorname{rank} M_n$ *for some* i*, then* β *has no representing measure.*

Proposition 1.3 shows that if β has a representing measure, then $\mathcal{CV}(L)$ is nonempty. The main result concerning the core variety is the following converse.

Theorem 1.5 ([BF, Core Variety Theorem]). $L \equiv L_\beta$ *has a representing measure if and only if* $\mathcal{CV}(L)$ *is nonempty. In this case,* $\mathcal{CV}(L)$ *coincides with the union of supports of all finitely atomic representing measures for* L*.*

In view of Proposition 1.3, $\mathcal{CV}(L)$ is also the union of supports of all representing measures. In general, it may be difficult to compute the core variety, due to the difficulty of characterizing the nonnegative polynomials on V_0, V_1, $V_2, \ldots$, but Theorem 1.5 leads to the following criterion for stability.

Proposition 1.6 ([BF, DDS]). *If* V_k *is finite, then* $\mathcal{CV}(L) = V_k$ *or* $\mathcal{CV}(L) = V_{k+1}$.

Although our focus in the sequel is TKMP for the planar curves $y = x^d$, we note that the results cited from [B, BF, CF5, CF7, DDS, F4] apply to the general multivariable truncated moment problem.

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2. A core variety approach to the pure $Y = X^d$ moment problem.

Suppose $M_n(\beta)$ is positive semidefinite and $(y - x^d)$-pure, i.e., the column dependence relations in M_n are precisely the linear combinations of the column relations

$$X^r Y^{s+1} = X^{r+d} Y^s \text{ for } r, s \ge 0,\ r + s \le n - d. \tag{2.1}$$

In this section we introduce a *core matrix* C associated to β; the main result of this section, Theorem 2.13, characterizes the existence of representing measures for β in terms of the positivity properties of C and "recursiveness" in its kernel. Using the Core Variety Theorem we show that in the positive definite case the union of supports of all representing measures is the curve $\Gamma := \mathcal{Z}(y - x^d) = \{(x, x^d) :\ x \in \mathbb{R}\}$. Namely, we employ the connection between the existence of representing measures for $\beta \equiv \beta^{(2n)}$ and the core variety of the Riesz functional $L \equiv L_\beta$. Setting $V_0 = \mathcal{V}(M_n) = \Gamma$, we seek to compute $V_1 := \mathcal{Z}(p \in \ker L \colon p|V_0 \ge 0)$, the common zeros of the polynomials in $\ker L$ that are nonnegative on V_0. To this end, we require a concrete description of $\ker L$.

Lemma 2.1. *Suppose $M_n(\beta)$ satisfies column relations* (2.1)*. Let $f_{ij}(x,y) = x^i y^j - \beta_{ij}$ for $0 \le i < d$, $j \ge 0$, and $0 < i+j \le 2n$. Let $g_{kl}(x,y) = (y - x^d)x^k y^l$ for $k,l \ge 0$, $k+l \le 2n-d$. Then $\mathcal{B} := \{f_{ij}\} \bigcup \{g_{kl}\}$ is a basis for* $\ker L_\beta$*.*

Conversely, let $L : \mathcal{P}_{2n} \to \mathbb{R}$ be a linear functional such that $\mathcal{B}$ is a basis for $\ker L$*. Then the moment matrix $M_n(\beta)$ of the sequence β, such that $L = L_\beta$, satisfies column relations* (2.1)*.*

Remark 2.2. In the statement of Lemma 2.1, $M_n(\beta)$ does not have to be $(y - x^d)$-pure for $\mathcal{B}$ to be the basis for $\ker L_\beta$. There may be column relations other than the linear combinations of (2.1), but $\mathcal{B}$ will still be a basis. Another choice of a basis for $\ker L_\beta$, which works for any sequence β, is $\{f_{ij}\}$ for $0 \le i,j$, $0 < i+j \le 2n$, where f_{ij} are defined as in the statement of the lemma. However, this basis tells us nothing about the column relations of $M_n(\beta)$. To explicitly determine column relations from the basis for $\ker L_\beta$, in addition to a "good" choice of the basis, the rank of $M_n(\beta)$ must also be given.

Proof of Lemma 2.1. Clearly, each $f_{ij} \in \ker L_\beta$. For $k,l \ge 0$ with $k+l \le 2n-d$, $g_{kl} \in \mathcal{P}_{2n}$. If $k+l \le n$, then $L_\beta(g_{kl}) = \langle M_n\widehat{(y - x^d)}, \widehat{x^k y^l}\rangle = \langle M_n\widehat{y} - M_n\widehat{x^d}, \widehat{x^k y^l}\rangle = 0$, so $g_{kl} \in \ker L_\beta$ in this case. In the remaining case, $n < k+l \le 2n-d$, so there exist integers $r,s,t,u \ge 0$ such that $r+t = k$, $s+u = l$, $r+s = n-d$, and thus $t+u = (k+l) - (r+s) \le 2n - d - (n-d) = n$. Now

$$\begin{aligned} L_\beta(g_{kl}) &= L_\beta((y - x^d)x^r y^s \cdot x^t y^u) = \langle M_n\widehat{(y - x^d)x^r y^s}, \widehat{x^t y^u}\rangle \\ &= \langle M_n\widehat{x^r y^{s+1}} - M_n\widehat{x^{d+r} y^s}, \widehat{x^t y^u}\rangle, \end{aligned}$$

so (2.1) implies $L_\beta(g_{kl}) = 0$.

To show that $\mathcal{B}$ is a linearly independent set of elements of $\mathcal{P}_{2n}$, suppose $\{a_{ij}\}$ and $\{b_{kl}\}$ are sequences of real scalars (indexed as in the statement of the lemma) such that in $\mathcal{P}_{2n}$,

$$\sum_{\substack{0 \le i < d,\ j \ge 0, \\ 0 < i+j \le 2n}} a_{ij} f_{ij} + \sum_{\substack{k,l \ge 0, \\ k+l \le 2n-d}} b_{kl} g_{kl} = 0. \tag{2.2}$$

For every $(x,y) \in \mathbb{R}^2$, $\sum a_{ij} f_{ij}(x,y) + \sum b_{kl} g_{kl}(x,y) = 0$, so with $y = x^d$, (2.2) implies $\sum a_{ij} F_{ij}(x) = 0$ $\forall x \in \mathbb{R}$, where $F_{ij}(x) := f_{ij}(x, x^d) = x^{i+dj} - \beta_{ij}$ ($0 \le i < d$, $j \ge 0$, $0 < i+j \le 2n$). Suppose that $0 \le i,i' < d$, $j,j' \ge 0$, $0 < i+j, i'+j' \le 2n$ and $i + dj = i' + dj'$. Then $|i - i'| = d|j - j'|$, and since $|i - i'| < d$, it follows that $j = j'$ and $i = i'$. Thus, the x-exponents appearing in

$$Q(x) \equiv \sum_{\substack{0 \le i < d,\ j \ge 0, \\ 0 < i+j \le 2n}} a_{ij}(x^{i+dj} - \beta_{ij})$$

are distinct, and since $Q(x) = 0$ for every real x, it follows that each $a_{ij} = 0$. Now (2.2) implies $\sum b_{kl} x^k y^l (y - x^d) = 0$ for all $x,y \in \mathbb{R}$. Thus, for $y \ne x^d$, $\sum b_{kl} x^k y^l = 0$, so by continuity we have $\sum b_{kl} x^k y^l = 0$ for all $x,y \in \mathbb{R}$. It now follows that each $b_{kl} = 0$, so $\mathcal{B}$ is linearly independent.

Next we show that $\mathcal{B}$ spans $\ker L_\beta$. We need to prove that $\operatorname{card}\mathcal{B} = \dim \mathcal{P}_{2n} - 1$ ($= \dim \ker L_\beta$). Recall that $\dim \mathcal{P}_{2n} = \frac{(2n+1)(2n+2)}{2}$. Note that $\mathcal{B}$ is the disjoint union of the sets $\mathcal{C} := \{f_{ij}\}$ and $\mathcal{D} := \{g_{kl}\}$. Clearly, $\operatorname{card}\mathcal{D} = \dim \mathcal{P}_{2n-d} = \frac{(2n-d+1)(2n-d+2)}{2}$. To compute $\operatorname{card}\mathcal{C}$, notice that $\operatorname{card}\mathcal{C} = \operatorname{card}\mathcal{E}$, where $\mathcal{E}$ is the index set equal to

$$\begin{aligned}\mathcal{E} &:= \{(i,j) \colon 0 \le i < d,\ j \ge 0,\ 0 < i+j \le 2n\} \\ &= \{\underbrace{(0,1),\dots,(0,2n)}_{i=0}, \underbrace{(1,0),\dots,(1,2n-1)}_{i=1}, \dots, \underbrace{(d-1,0),\dots,(d-1,2n-d+1)}_{i=d-1}\}.\end{aligned}$$

It follows that

$$\begin{aligned}\operatorname{card}\mathcal{C} = \operatorname{card}\mathcal{E} &= 2n + 2n + (2n-1) + \ldots + (2n-d+2) \\ &= -1 + \sum_{i=0}^{d-1}(2n+1-i) = -1 + \sum_{i=1}^{2n+1} i - \sum_{i=1}^{2n-d+1} i \\ &= -1 + \frac{(2n+1)(2n+2)}{2} - \frac{(2n-d+1)(2n-d+2)}{2} \\ &= -1 + \operatorname{card}\mathcal{P}_{2n} - \operatorname{card}\mathcal{D},\end{aligned}$$

whence

$$\operatorname{card}\mathcal{B} = \operatorname{card}\mathcal{C} + \operatorname{card}\mathcal{D} = -1 + \operatorname{card}\mathcal{P}_{2n},$$

which shows that $\mathcal{B}$ is a basis for $\ker L_\beta$.

The converse part is clear. Namely, L determines the sequence β by $\beta_{ij} = L(x^i y^j)$ for $0 \le i,j,\ i+j \le 2n$. (Note that by $f_{ij} \in \ker L$ for $0 \le i < d$, $j \ge 0$, and $0 < i+j \le 2n$, for these indices, β_{ij} are precisely constant terms in f_{ij}.) By $g_{kl} \in \ker L$ for $k,l \ge 0$, $k+l \le 2n-d$, all linear combinations of (2.1) are column relations of $M_n(\beta)$. ☐

Suppose $p \in \ker L$ satisfies $p|\Gamma \ge 0$, i.e., $p(x,x^d) \ge 0\ \forall x \in \mathbb{R}$. From Lemma 2.1, we may write

$$p = F + G \equiv \sum_{\substack{0\le i<d,\ j\ge 0,\\ 0<i+j\le 2n}} a_{ij} f_{ij} + \sum_{\substack{k,l\ge 0,\\ k+l\le 2n-d}} b_{kl} g_{kl}. \tag{2.3}$$

Since $p|\Gamma \ge 0$ and $G|\Gamma \equiv 0$, then

$$Q(x) := F(x,x^d) = \sum_{\substack{0\le i<d,\ j\ge 0,\\ 0<i+j\le 2n}} a_{ij}(x^{i+dj} - \beta_{ij}) \tag{2.4}$$

satisfies $Q(x) \ge 0\ \forall x \in \mathbb{R}$. Since $\deg Q \le 2nd$, there exist

$$\begin{aligned}&\widehat{r} \equiv (r_0,\dots,r_{nd}),\ \widehat{s} \equiv (s_0,\dots,s_{nd}) \in \mathbb{R}^{nd+1} \quad \text{such that} \\ &R(x) := r_0 + r_1 x + \cdots + r_{nd}x^{nd} \quad \text{and} \quad S(x) := s_0 + s_1 x + \cdots + s_{nd}x^{nd}\end{aligned} \tag{2.5}$$

satisfy

$$Q(x) = R(x)^2 + S(x)^2. \tag{2.6}$$

Comparing coefficients on both sides of (2.6), we see that each a_{ij}, which is the coefficient in Q of x^{i+dj}, admits a unique expression as a homogeneous quadratic polynomial $h(\widehat{r}, \widehat{s})$ in the r_k and s_l. Indeed, for $i, j \geq 0$, with $i < d$ and $0 < i+j \leq 2n$, we have

$$a_{ij} = h_{i,j}(\widehat{r}, \widehat{s}) := \sum_{\substack{0 \leq k,l \leq nd, \\ 0 < k+l = i+dj}} r_k r_l + s_k s_l. \tag{2.7}$$

Moreover, a comparison of the constant terms in (2.6) gives

$$-\sum_{\substack{0 \leq i < d,\ j \geq 0, \\ 0 < i+j \leq 2n}} a_{ij} \beta_{ij} = r_0^2 + s_0^2. \tag{2.8}$$

Example 2.3. Let $n = d = 3$. Then Q (cf. (2.6)) is of the form

$$Q(x) = \sum_{\substack{0 \leq i < 3,\ j \geq 0, \\ 0 < i+j \leq 6}} a_{ij}(x^{i+3j} - \beta_{ij}) =: \sum_{\ell=0}^{18} q_\ell x^\ell \in \mathcal{P}_{18}.$$

Note that $q_{17} = 0$ since $17 \neq i + 3j$ for some $0 \leq i < 3, j > 0, i + j \leq 6$. For example, the coefficient q_4, which is equal to a_{11}, may be expressed by (2.7) as

$$\begin{aligned} h_{1,1}(\widehat{r}, \widehat{s}) &= r_0 r_4 + r_1 r_3 + r_2 r_2 + r_3 r_1 + r_4 r_0 + s_0 s_4 + s_1 s_3 + s_2 s_2 + s_3 s_1 + s_4 s_0 \\ &= 2(r_0 r_4 + s_0 s_4 + r_1 r_3 + s_1 s_3) + r_2^2 + s_2^2. \end{aligned}$$

Now suppose $i, j \geq 0$, with $i < d$ and $i + dj \leq 2nd$, but $i + j > 2n$. The index set $\mathcal{F}$ of all such pairs is equal to

$$\mathcal{F} := \{(i,j) \colon 2n - (d-2) \leq j \leq 2n - 1,\ 2n + 1 - j \leq i \leq d - 1\} = \bigcup_{j=1}^{d-2} \mathcal{F}_j \tag{2.9}$$

where each $\mathcal{F}_j$ is equal to

$$\mathcal{F}_j = \begin{cases} \{(j+1, 2n-j), \ldots, (d-1, 2n-j)\}, & \text{if } j + 1 \leq d - 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Hence $\operatorname{card} \mathcal{F} = \sum_{i=1}^{d-2} i = \frac{(d-1)(d-2)}{2}$. Note that $\mathcal{F} = \emptyset$ for $n = 1,\ 2$. Although for $(i,j) \in \mathcal{F}$, terms x^{i+dj} appear in $R(x)^2$ and $S(x)^2$, since $i + j > 2n$, x^{i+dj} cannot appear in Q with a nonzero coefficient. For the sequel, it is convenient to extend the definition of h_{ij} in (2.7) to include these cases, together with the requirement

$$0 = h_{i,j}(\widehat{r}, \widehat{s}) \quad \text{whenever } (i,j) \in \mathcal{F}. \tag{2.10}$$

Also, we introduce an arbitrary constant A_{ij} for each $(i,j) \in \mathcal{F}$ to denote the moment β_{ij}, which is not present in $\beta^{(2n)}$. We will call every such moment an *auxiliary moment* in the sequel.

Example 2.4. Let $n = d = 3$. Note that $\mathcal{F} = \{(2,5)\}$, since for $i = 2$ and $j = 5$, we have $i + 3j = 17 < 2nd = 18$ but $7 = i + j > 2n = 6$. Thus x^{17} does not appear in $Q(x)$, so, from (2.7), using $h_{2,5}(\widehat{r}, \widehat{s}) = r_8 r_9 + s_8 s_9$, it follows that $r_8 r_9 + s_8 s_9 = 0$. The auxiliary moment in this case is $\beta_{2,5}$, which we denote by $A_{2,5}$.

For $d = 3$ and arbitrary $n \in \mathbb{N}$, which we study in Section 4, we have $\mathcal{F} = \{(2, 2n-1)\}$ and the condition in (2.10) is equal to

$$0 = h_{2,2n-1}(\widehat{r}, \widehat{s}) = 2(r_{3n} r_{3n-1} + s_{3n} s_{3n-1}), \tag{2.11}$$

with the "missing" monomial in $Q(x)$ being $x^{2+3(2n-1)} = x^{6n-1}$. △

Let $\lfloor \cdot \rfloor$ denote the floor function. Namely, $\lfloor k \rfloor$ is the greatest integer not larger than k. The next lemma will be essential for Section 5 to justify that the conditions (2.10) are satisfied, as part of the argument that (2.6) holds (see Lemma 5.3 below).

Lemma 2.5. *Assume that $(i,j) \in \mathcal{F}$ and let $0 \le k, l \le nd$ such that $k + l = i + dj$. Let $\widetilde{k} := k + nd$ and $\widetilde{l} := l + nd$. Then at least one of the pairs $K = (\widetilde{k} \bmod d, \lfloor \frac{\widetilde{k}}{d} \rfloor)$ and $L = (\widetilde{l} \bmod d, \lfloor \frac{\widetilde{l}}{d} \rfloor)$ belongs to $\mathcal{F}$. Equivalently, at least one of the moments $\beta_{k \bmod d, \lfloor \frac{k}{d} \rfloor + n}$ and $\beta_{l \bmod d, \lfloor \frac{l}{d} \rfloor + n}$ is auxiliary.*

Proof. Recall that membership of a pair (p,q) in $\mathcal{F}$ requires $p, q \ge 0$, $p < d$, $p + dq \le 2nd$, and $p + q > 2n$. For $p = k \bmod d$, $q = \lfloor \frac{k}{d} \rfloor + n$, the first two reqirements for membership in $\mathcal{F}$ are clearly satisfied. Now $k = k \bmod d + d\lfloor \frac{k}{d} \rfloor$, and since $k \le nd$, it follows that $p + dq \le 2nd$. Therefore, if we assume that $K \notin \mathcal{F}$, then $k \bmod d + \lfloor \frac{k}{d} \rfloor + n \le 2n$. A similar argument holds for L. If we now assume that $K \notin \mathcal{F}$ and $L \notin \mathcal{F}$, then

$$k \bmod d + \left\lfloor \frac{k}{d} \right\rfloor \le n \quad \text{and} \quad l \bmod d + \left\lfloor \frac{l}{d} \right\rfloor \le n. \tag{2.12}$$

If $k \bmod d + l \bmod d < d$, then

$$i = (k+l) \bmod d = k \bmod d + l \bmod d \quad \text{and} \quad j = \left\lfloor \frac{k+l}{d} \right\rfloor = \left\lfloor \frac{k}{d} \right\rfloor + \left\lfloor \frac{l}{d} \right\rfloor.$$

Hence,

$$i + j = k \bmod d + l \bmod d + \left\lfloor \frac{k}{d} \right\rfloor + \left\lfloor \frac{l}{d} \right\rfloor \underbrace{\le}_{(2.12)} 2n$$

which is a contradiction to the assumption $(i,j) \in \mathcal{F}$.

If $k \bmod d + l \bmod d \ge d$, then

$$i = (k+l) \bmod d = k \bmod d + l \bmod d - d \quad \text{and} \quad j = \left\lfloor \frac{k+l}{d} \right\rfloor = \left\lfloor \frac{k}{d} \right\rfloor + \left\lfloor \frac{l}{d} \right\rfloor + 1.$$

Hence,

$$i + j = k \bmod d + l \bmod d + \left\lfloor \frac{k}{d} \right\rfloor + \left\lfloor \frac{l}{d} \right\rfloor - d + 1 \underbrace{\le}_{(2.12)} 2n$$

which is a contradiction to the assumption $(i,j) \in \mathcal{F}$. □

Remark 2.6. Another way of stating Lemma 2.5 is by saying that at least one of the monomials $x^{k \bmod d} y^{\lfloor \frac{k}{d} \rfloor}$ and $x^{l \bmod d} y^{\lfloor \frac{l}{d} \rfloor}$ has total degree more than n.

We next introduce the *core matrix* $C \equiv C_\beta$; in the sequel we show that positivity properties of C determine the core variety of β. Our immediate goal is to use (2.7) and the core matrix to derive an inner product expression (see (2.25)) which can be used to characterize whether (2.8) holds. This will permit us to provide a sufficient condition for representing measures via the core variety. The core matrix, a $(dn+1) \times (dn+1)$ matrix $C \equiv (C_{ij})_{i,j}$, is defined by

$$C_{ij} = \beta_{(i+j-2) \bmod d, \lfloor (i+j-2)/d \rfloor} \ (1 \le i, j \le dn+1). \tag{2.13}$$

Let

$$K_{ij} := (i+j-2) \bmod d \text{ and } L_{ij} := \lfloor (i+j-2)/d \rfloor, \tag{2.14}$$

so that $C_{ij} = \beta_{K_{ij}, L_{ij}}$; however, if $\beta_{K_{ij}, L_{ij}}$ is an auxiliary moment because $(K_{ij}, L_{ij}) \in \mathcal{F}$, we redefine $\beta_{K_{ij}, L_{ij}}$ as $\beta_{K_{ij}, L_{ij}} = A_{K_{ij}, L_{ij}}$, where $A_{K_{ij}, L_{ij}}$ is an arbitrary constant.

To emphasize the dependence of C on the choice of the constants A_{ij} for $(i,j) \in \mathcal{F}$, we sometimes denote C by $C[\{A_{ij}\}_{(i,j) \in \mathcal{F}}]$. From (2.13), C is clearly a Hankel matrix.

Example 2.7. For $n = d = 4$ the core matrix

$$C \equiv C[\mathbf{A}_{3,2n-2}, \mathbf{A}_{2,2n-1}, \mathbf{A}_{3,2n-1}]$$

is the following

$$\begin{pmatrix}
\beta_{00} & \beta_{10} & \beta_{20} & \beta_{30} & \beta_{01} & \beta_{11} & \beta_{21} & \beta_{31} & \beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} \\
\beta_{10} & \beta_{20} & \beta_{30} & \beta_{01} & \beta_{11} & \beta_{21} & \beta_{31} & \beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} \\
\beta_{20} & \beta_{30} & \beta_{01} & \beta_{11} & \beta_{21} & \beta_{31} & \beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} \\
\beta_{30} & \beta_{01} & \beta_{11} & \beta_{21} & \beta_{31} & \beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} \\
\beta_{01} & \beta_{11} & \beta_{21} & \beta_{31} & \beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} \\
\beta_{11} & \beta_{21} & \beta_{31} & \beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} \\
\beta_{21} & \beta_{31} & \beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} \\
\beta_{31} & \beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} \\
\beta_{02} & \beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} \\
\beta_{12} & \beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} & \beta_{16} \\
\beta_{22} & \beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} & \beta_{16} & \beta_{26} \\
\beta_{32} & \beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} & \beta_{16} & \beta_{26} & \mathbf{A}36 \\
\beta_{03} & \beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} & \beta_{16} & \beta_{26} & \mathbf{A}36 & \beta_{07} \\
\beta_{13} & \beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} & \beta_{16} & \beta_{26} & \mathbf{A}36 & \beta_{07} & \beta_{17} \\
\beta_{23} & \beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} & \beta_{16} & \beta_{26} & \mathbf{A}36 & \beta_{07} & \beta_{17} & \mathbf{A}27 \\
\beta_{33} & \beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} & \beta_{16} & \beta_{26} & \mathbf{A}36 & \beta_{07} & \beta_{17} & \mathbf{A}27 & \mathbf{A}37 \\
\beta_{04} & \beta_{14} & \beta_{24} & \beta_{34} & \beta_{05} & \beta_{15} & \beta_{25} & \beta_{35} & \beta_{06} & \beta_{16} & \beta_{26} & \mathbf{A}36 & \beta_{07} & \beta_{17} & \mathbf{A}27 & \mathbf{A}37 & \beta_{08}
\end{pmatrix}$$

The rows and columns of C are indexed by the ordered set

$$\{\mathit{1}, X, X^2, X^3, Y, XY, X^2Y, X^3Y, \ldots, Y^k, XY^k, X^2Y^k, X^3Y^k, \ldots, \\ Y^{n-1}, XY^{n-1}, X^2Y^{n-1}, X^3Y^{n-1}, Y^n\}.$$

Note that the columns $X^3Y^{n-2}, X^2Y^{n-1}, X^3Y^{n-1}$ are not among columns of M_n but of its extension M_{n+2}. So these columns are auxiliary ones in C and contain auxiliary moments.

The next two results provide an alternate description of the core matrix in terms of moment matrix extensions. Let $d \geq 2$ and M_{n+d-2} be some recursively generated extension of the positive $(y-x^d)$-pure moment matrix M_n. Let $\widetilde{\beta} \equiv \widetilde{\beta}^{(2n+2d-4)}$ be the extended sequence and let $L_{\widetilde{\beta}} : \mathcal{P}_{2(n+d-2)} \to \mathbb{R}$ be the corresponding Riesz functional. Define the ordered set of monomials

$$\begin{aligned} \mathcal{M} := \{1, x, \ldots, x^{d-1}, y, xy, \ldots, x^{d-1}y, \ldots, y^i, yx^i, \ldots, x^{d-1}y^i, \\ y^{n-1}, xy^{n-1}, \ldots, x^{d-1}y^{n-1}, y^n\}, \end{aligned} \tag{2.15}$$

and the vector space

$$\mathcal{U} := \mathrm{Span}\, \{s\colon s \in \mathcal{M}\} \subset \mathcal{P}_{n+d-2}, \tag{2.16}$$

We next define an $(nd+1) \times (nd+1)$ matrix $M[\widetilde{\beta}, \mathcal{U}]$ with rows and columns indexed by the monomials in $\mathcal{M}$ in the order

$$\mathit{1}, X, \ldots, X^{d-1}, Y, XY, \ldots, X^{d-1}Y, \ldots Y^{n-1}, XY^{n-1}, \ldots, X^{d-1}Y^{n-1}, Y^n \tag{2.17}$$

(i.e., for $1 \leq k \leq nd+1$, the k-th element of this order is equal to $X^{I_k}Y^{J_k}$ where

$$I_k := (k-1) \bmod d \quad \text{and} \quad J_k := \lfloor \frac{k-1}{d} \rfloor).$$

The (i,j)-th entry of $M[\widetilde{\beta}, \mathcal{U}]$ is defined to be equal to

$$L_{\widetilde{\beta}}(x^{I_i+I_j}y^{J_i+J_j}) = \widetilde{\beta}_{I_i+I_j, J_i+J_j} = \widetilde{\beta}_{(i-1) \bmod d + (j-1) \bmod d, \lfloor \frac{i-1}{d} \rfloor + \lfloor \frac{j-1}{d} \rfloor}. \tag{2.18}$$

More generally, for $r, s \in \mathcal{U}$ (cf. (2.16)), with coefficient vectors $\widehat{r}, \widehat{s}$ relative to the ordered basis of monomials in $\mathcal{M}$ (cf. (2.15)), we have

$$\langle M[\widetilde{\beta}, \mathcal{U}]\, \widehat{r}, \widehat{s} \rangle := L_{\widetilde{\beta}}(rs). \tag{2.19}$$

Lemma 2.8. *For $1 \leq i, j \leq nd+1$ the following holds:*

$$L_{\widetilde{\beta}}(x^{I_i+I_j}y^{J_i+J_j}) = \widetilde{\beta}_{(i+j-2) \bmod d, \lfloor \frac{i+j-2}{d} \rfloor}. \tag{2.20}$$

Proof. From $i+j-2 = (i-1)+(j-1)$ it follows that

$$\begin{aligned} i+j-2 &= (i+j-2) \bmod d + d \Big\lfloor \frac{i+j-2}{d} \Big\rfloor \\ \text{and } i+j-2 &= ((i-1) \bmod d + (j-1) \bmod d) + d \left(\Big\lfloor \frac{i-1}{d} \Big\rfloor + \Big\lfloor \frac{j-1}{d} \Big\rfloor \right). \end{aligned} \tag{2.21}$$

We separate two cases:

Case a): $(i-1) \bmod d + (j-1) \bmod d < d$. Then (2.21) implies that

$$\begin{aligned} (i+j-2) \bmod d &= (i-1) \bmod d + (j-1) \bmod d, \\ \Big\lfloor \frac{i+j-2}{d} \Big\rfloor &= \Big\lfloor \frac{i-1}{d} \Big\rfloor + \Big\lfloor \frac{j-1}{d} \Big\rfloor. \end{aligned} \tag{2.22}$$

Using (2.22) in (2.18), (2.20) follows.

Case b): $(i-1) \bmod d + (j-1) \bmod d \geq d$. Then (2.21) implies that

$$(i+j-2) \bmod d = ((i-1) \bmod d + (j-1) \bmod d) - d,$$
$$\left\lfloor \frac{i+j-2}{d} \right\rfloor = \left\lfloor \frac{i-1}{d} \right\rfloor + \left\lfloor \frac{j-1}{d} \right\rfloor + 1. \tag{2.23}$$

Since M_{n+d-2} is recursively generated, we have $X^{r+d}Y^s = X^rY^{s+1}$ in the rows and columns, and therefore $\widetilde{\beta}_{r+d,s} = \widetilde{\beta}_{r,s+1}$. The assumption of Case b), and (2.23) used in (2.18), together with M_{n+d-2} being recursively generated, therefore imply that

$$\begin{aligned}\widetilde{\beta}_{(i-1) \bmod d+(j-1) \bmod d, \lfloor \frac{i-1}{d} \rfloor + \lfloor \frac{j-1}{d} \rfloor} &= \widetilde{\beta}_{(i+j-2) \bmod d\ +d, \lfloor \frac{i-1}{d} \rfloor + \lfloor \frac{j-1}{d} \rfloor} \\ &= \widetilde{\beta}_{(i+j-2) \bmod d, \lfloor \frac{i-1}{d} \rfloor + \lfloor \frac{j-1}{d} \rfloor + 1} \\ &= \widetilde{\beta}_{(i+j-2) \bmod d, \lfloor \frac{i+j-2}{d} \rfloor} \text{ (using (2.23))},\end{aligned}$$

so (2.20) follows. □

Proposition 2.9. Assume the notation above. Then:

(i) If the sequence $\widetilde{\beta}$ has a representing measure, then $M[\widetilde{\beta}, \mathcal{U}]$ is positive semidefinite.
(ii) Let $\widetilde{M}[\widetilde{\beta}, \mathcal{U}]$ be obtained from $M[\widetilde{\beta}, \mathcal{U}]$ by replacing each $\widetilde{\beta}_{ij}$ satisfying $i \bmod d + j + \lfloor \frac{i}{d} \rfloor > 2n$ with the auxiliary moment A_{ij}. Then

$$C = \widetilde{M}[\widetilde{\beta}, \mathcal{U}].$$

Proof. Part (i) follows from the equality (2.19) and $L_{\widetilde{\beta}}(r^2) = \int r^2 d\mu \geq 0$. For part (ii) first note that not all $\widetilde{\beta}_{ij}$ with $i + j > 2n$ are auxiliary moments. By recursive generation we have $\widetilde{\beta}_{ij} = \widetilde{\beta}_{i-d,j+1}$ if $d \leq i < 2d-1$ (observe that i is at most $2d-2$) and so $\widetilde{\beta}_{ij}$ is auxiliary only if $i \bmod d + j + \lfloor \frac{i}{d} \rfloor = i - d + j + 1 > 2n$ in these cases. If $i < d$, then the condition $i \bmod d + j + \lfloor \frac{i}{d} \rfloor > 2n$ reduces to $i + j > 2n$. Now part (ii) follows from (2.13) and Lemma 2.8. □

If $H \equiv (h_{i+j-1})_{1 \leq i,j \leq m}$ is any $m \times m$ Hankel matrix and $\widehat{t} := (t_1, \ldots, t_m) \in \mathbb{R}^m$, then $\langle H\widehat{t}, \widehat{t} \rangle = \sum_{i=1}^{m} \sum_{j=1}^{m} t_i h_{i+j-1} t_j$, and, after rearranging terms, we have

$$\langle H\widehat{t}, \widehat{t} \rangle = \sum_{k=1}^{2m-1} \Big(h_k \cdot \sum_{\substack{1 \leq i,j \leq m, \\ i+j=k+1}} t_i t_j \Big) \tag{2.24}$$

$(= h_1 t_1^2 + h_2(2t_1t_2) + h_3(2t_1t_3 + t_2^2) + \cdots + h_{2m-2}(2t_{m-1}t_m) + h_{2m-1}t_m^2)$.

Lemma 2.10. *Let $\widehat{r} \equiv (r_0, \ldots, r_{nd}), \widehat{s} \equiv (s_0, \ldots, s_{nd}) \in \mathbb{R}^{nd+1}$ satisfy (2.10). For $i, j \geq 0$, with $i < d$ and $0 < i + j \leq 2n$, define a_{ij} by (2.7). Then*

$$\langle C\widehat{r}, \widehat{r} \rangle + \langle C\widehat{s}, \widehat{s} \rangle = 0 \iff \text{(2.8) holds.} \tag{2.25}$$

Proof. Let

$$u_k = (k-1) \bmod d \quad \text{and} \quad v_k = \lfloor (k-1)/d \rfloor \quad (1 \le k \le 2nd+1).$$

Further, let

$$h_k := \begin{cases} \beta_{u_k v_k}, & \text{if } u_k + v_k \le 2n, \\ A_{u_k v_k}, & \text{if } u_k + v_k > 2n. \end{cases} \tag{2.26}$$

We now apply (2.24) with $m = nd+1$, $H = C$ with h_k as in (2.26), and with $t_p = r_{p-1}$ or $t_p = s_{p-1}$ $(1 \le p \le nd+1)$:

$$\begin{aligned}
&\langle C\widehat{r}, \widehat{r}\rangle + \langle C\widehat{s}, \widehat{s}\rangle = \\
&= \sum_{k=1}^{2nd+1} \Big(h_k \cdot \sum_{\substack{1 \le p,q \le nd+1, \\ p+q=k+1}} (r_{p-1}r_{q-1} + s_{p-1}s_{q-1}) \Big) \\
&= r_0^2 + s_0^2 + \sum_{\substack{0 \le i < d,\ j \ge 0, \\ 0 < i+j \le 2n}} \Big(\beta_{ij} \cdot \sum_{\substack{0 \le p,q \le nd, \\ 0 < p+q = i+dj}} (r_p r_q + s_p s_q) \Big) + \\
&\qquad + \sum_{\substack{0 \le i < d,\ j \ge 0, \\ i+j > 2n}} \Big(A_{ij} \cdot \sum_{\substack{0 \le p,q \le nd, \\ 0 < p+q = i+dj}} (r_p r_q + s_p s_q) \Big) \\
&= r_0^2 + s_0^2 + \sum_{\substack{0 \le i < d,\ j \ge 0, \\ 0 < i+j \le 2n}} a_{ij}\beta_{ij}
\end{aligned}$$

where we used (2.7) and (2.10) in the last equality. Now the equivalence of the lemma easily follows. □

Remark 2.11. It is important for the sequel to note that the implication $(\Leftarrow)$ of Lemma 2.10 may be used in order to construct elements p of $\ker L$ satisfying $p|\Gamma \ge 0$, so that $\mathcal{CV}(L) \subseteq \mathcal{Z}(p|\Gamma)$. For suppose $\widehat{r},\ \widehat{s} \in \mathbb{R}^{nd+1}$ satisfy (2.10) and $\langle C\widehat{r}, \widehat{r}\rangle + \langle C\widehat{s}, \widehat{s}\rangle = 0$. Now define $a_{ij} = h_{ij}(\widehat{r}, \widehat{s})$ $(i, j \ge 0,\ i < d,\ 0 < i+j \le 2n)$. Then $p := \sum a_{ij} f_{ij} \in \ker L$ satisfies $p(x, x^d) = R(x)^2 + S(x)^2$, where $R(x) := r_0 + r_1 x + \cdots + r_{nd}x^{dn}$ and $S(x) := s_0 + s_1 x + \cdots + s_{nd}x^{dn}$. Now we have $\mathcal{CV}(L) \subseteq \{(x, x^d) : R(x) = S(x) = 0\}$ and $\operatorname{card} \mathcal{CV}(L) \le \min\{\deg R, \deg S\}$.

Let $A \equiv \{A_{ij}\}_{(i,j)\in\mathcal{F}}$ with $A_{ij} \in \mathbb{R}$. We say that the core matrix $C[A]$ is *recursively generated* if for every $v \in \mathbb{R}^{nd}$ satisfying $\begin{pmatrix} v \\ 0 \end{pmatrix} \in \ker C[A]$, it follows that $\begin{pmatrix} 0 \\ v \end{pmatrix} \in \ker C[A]$.

Remark 2.12. Note that the definition above is equivalent to the definition of a "recursively generated" Hankel matrix given in [CF1]. However, it does not encompass the notion of recursiveness for a general multivariable moment matrix given in item iv) preceding Theorem 1.1.

We will show in the next section that C inherits a substantial amount of positivity from M_n (see Theorem 3.6). The following theorem characterizes the existence of a representing measure for β in terms of the existence of auxiliary moments such that the core matrix is positive and recursively generated.

Theorem 2.13. *Let $\beta \equiv \beta^{(2n)}$ be a given sequence such that $M_n \equiv M_n(\beta)$ is positive semidefinite and $(y - x^d)$-pure. The following statements are equivalent:*

(i) β admits a representing measure (necessarily supported in Γ).

(ii) β admits a finitely atomic representing measure (necessarily supported in Γ).

(iii) There exist auxiliary moments $A \equiv \{A_{ij}\}_{(i,j)\in\mathcal{F}}$, such that the core matrix $C[A] \equiv C[\{A_{ij}\}_{(i,j)\in\mathcal{F}}]$ is positive semidefinite and recursively generated.

Moreover, if the core matrix $C[A]$ is positive definite for some choice of auxiliary moments A, then $\beta \equiv \beta^{(2n)}$ admits finitely atomic representing measures whose union of supports coincides with Γ.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from Richter's Theorem [Ric] (or by Theorem 1.5),

Next we establish the implication (ii) $\Rightarrow$ (iii). Suppose $M_n(\beta)$ is $(y - x^d)$-pure and that β has a finitely atomic representing measure μ supported on Γ. Thus, μ is of the form

$$\mu = \sum_{k=1}^{m} a_k \delta_{(x_k, y_k)},$$

where $m > 0$, each $a_k > 0$, and $y_k = x_k^d$ for each k. Since μ has moments of all orders, we may consider the moment matrix $M_{n+t}[\mu]$, containing μ-moments up to degree $2n + 2t$, where $t = \lceil \frac{d-2}{2} \rceil$. Using the moment data $\widetilde{\beta}^{(2(n+t))}$ from $M_{n+t}[\mu]$, i.e., $\widetilde{\beta_{ij}} = \int x^i y^j \, d\mu$, $(i, j \geq 0,\ i + j \leq 2(n + t))$, let

$$\gamma_p = \widetilde{\beta}_{p \bmod d, \lfloor \frac{p}{d} \rfloor} \ (0 \leq p \leq 2nd). \tag{2.27}$$

Since $M_n[\mu] = M_n(\beta)$, we have

$$\gamma_p = \beta_{p \bmod d, \lfloor \frac{p}{d} \rfloor} \quad \text{if } 0 \leq p \leq 2nd \quad \text{and} \quad p \bmod d + \lfloor \frac{p}{d} \rfloor \leq 2n.$$

We next show that $\widetilde{\mu} := \sum_{k=1}^{m} a_k \delta_{x_k}$ is a representing measure for $\gamma := \{\gamma_p\}_{0 \leq p \leq 2nd}$. Indeed, for $0 \leq p \leq 2nd$ we have

$$\sum a_k x_k^p = \sum a_k x_k^{p \bmod d + d \lfloor \frac{p}{d} \rfloor} = \sum a_k x_k^{p \bmod d} y_k^{\lfloor \frac{p}{d} \rfloor} = \widetilde{\beta}_{p \bmod d, \lfloor \frac{p}{d} \rfloor} = \gamma_p.$$

It now follows that the moment matrix for γ, which is the Hankel matrix $H(\gamma) \equiv (\gamma_{i+j})_{0 \leq i,j \leq nd}$, is positive semidefinite and recursively generated (cf. Section 1). If, in the core matrix $C[A]$, for each $(i, j) \in \mathcal{F}$ we set $A_{ij} = \gamma_{i+dj} = \widetilde{\beta}_{ij}$, then $C[A]$ coincides with $H(\gamma)$, and is thus positive semidefinite and recursively generated. This is precisely (iii).

Next we establish the implication (iii) $\Rightarrow$ (ii). Suppose the there exist auxiliary moments A such that $C[A]$ is positive semidefinite and recursively generated. We will prove that β has a finitely atomic representing measure. Define a univariate sequence $\gamma \equiv \{\gamma_p\}_{0 \le p \le 2nd}$ as in (2.27) above, where $\widetilde{\beta}_{ij}$ is either β_{ij} or A_{ij}. Since the Hankel matrix $H(\gamma) \equiv (\gamma_{i+j})_{0 \le i,j \le nd}$ coincides with $C[A]$ (by definition of γ), it follows that it is positive semidefinite and recursively generated. By [CF1, Theorem 3.9], γ has a finitely atomic representing measure $\widetilde{\mu} := \sum_{k=1}^{m} a_k \delta_{x_k}$. But then $\mu = \sum_{k=1}^{m} a_k \delta_{(x_k, y_k)}$ is a representing measure for β. Indeed, for $0 \le i, j \le 2n, i + j \le 2n$ we have

$$\sum a_k x_k^i y_k^j = \sum a_k x_k^{i+dj} = \gamma_{i+dj} = \beta_{i \bmod d, j + \lfloor \frac{i}{d} \rfloor} = \beta_{i,j},$$

where in the last equality we used that $\beta_{r+d,s} = \beta_{r,s+1}$ for $0 \le r, s$ such that $r+s+d \le 2n$.

It remains to address the case when $C[A]$ is positive definite. Concerning the core variety of $L \equiv L_\beta$, we have $V_0 = \mathcal{V}(M_n) = \Gamma$, and we now consider $V_1 := \mathcal{Z}(p \in \ker L \colon p|V_0 \ge 0)$. For $p \in \ker L$ with $p|V_0 \ge 0$, we have $p = F + G$ as in (2.3). The preceding discussion shows that $Q(x) := F(x, x^d)$ satisfies $Q(x) = R(x)^2 + S(x)^2$, where $\widehat{r}$ and $\widehat{s}$ satisfy the conditions of (2.7), (2.8) and (2.10). Lemma 2.10 now shows that $\langle C\widehat{r}, \widehat{r} \rangle + \langle C\widehat{s}, \widehat{s} \rangle = 0$, and since C is positive definite, it follows that $\widehat{r} = \widehat{s} = 0$. Thus (2.7) implies that each $a_{ij} = 0$, so $F = 0$. Since $\mathcal{Z}(G|\Gamma) = \Gamma$, we now have $\mathcal{Z}(p|\Gamma) = \Gamma$. It follows that $V_1 = V_0 = \Gamma$, so $\mathcal{CV}(L) = \Gamma$ and the Core Variety Theorem implies that β has finitely atomic representing measures whose union of supports is Γ. ☐

The rest of the paper is primarily devoted to developing *concrete* conditions for the existence or nonexistence of auxiliary moments satisfying condition (iii).

Theorem 2.13 suggests the following question.

Question 2.14. *If $\beta \equiv \beta^{(2n)}$ has a representing measure, is there some choice of auxiliary moments $A \equiv \{A_{ij}\}_{(i,j) \in \mathcal{F}}$ such that $C[A] \succ 0$?*

In Example 2.15 (just below) we show that the answer is affirmative for $d = 1$ and $d = 2$. In Section 4, we prove an affirmative answer for $d = 3$. This provides a new proof of Theorem 1.2. Finally, in Section 6, we prove that the answer is already negative for $d = 4$ (see Example 6.6(v)).

In the sequel, for $M_n \succeq 0$ and $(y - x^d)$-pure, we denote by $\widehat{M_n}$ the central compression of M_n obtained by deleting all rows and columns $X^{d+p}Y^q$ ($p, q \ge 0$, $p + q \le n - d$). The number of rows and columns in $\widehat{M_n}$ is thus $\dim \mathcal{P}_n - \dim \mathcal{P}_{n-d} = \frac{d(2n-d+3)}{2}$. Since M_n is positive and $(y - x^d)$-pure, it follows immediately that $\widehat{M_n}$ is positive definite and

$$\text{rank}\, M_n = \text{rank}\, \widehat{M_n} = \frac{d(2n - d + 3)}{2}. \tag{2.28}$$

Example 2.15. i) For $d = 1$, we have $C = \widehat{M_n} = \left(\beta_{0,i+j-2}\right)_{1\leq i,j\leq n+1} \succ 0$, so the existence of representing measures whose union of supports is the line $y = x$ now follows from Theorem 2.13. Alternately, using flat extensions, the existence of measures in this case follows from the solution to the truncated moment problem on a line in [CF3].

ii) For $d = 2$, the core matrix C for M_n is $(2n + 1) \times (2n + 1)$, with

$$C_{ij} = \beta_{(i+j-2) \bmod 2, \lfloor \frac{i+j-2}{2} \rfloor}. \tag{2.29}$$

In $\widehat{M_n}$, column j is the truncation to $\widehat{M_n}$ of column $X^{(j-1) \bmod 2}Y^{\lfloor (j-1)/2 \rfloor}$ in M_n. Likewise, row i of $\widehat{M_n}$ is the truncation to $\widehat{M_n}$ of row $X^{(i-1) \bmod 2}Y^{\lfloor (i-1)/2 \rfloor}$ in M_n. Thus, using the structure of moment matries, we have

$$\widehat{M}_{ij} = \beta_{(i-1) \bmod 2+(j-1) \bmod 2, \lfloor (i-1)/2 \rfloor + \lfloor (j-1)/2 \rfloor}. \tag{2.30}$$

By Proposition 2.9 (or using calculations based on (2.29)), we have $C = \widehat{M_n} \succ 0$.

Since C is positive definite, Theorem 2.13 now implies that β has representing measures whose union of supports is the parabola $y = x^2$. The existence of representing measures also follows from the solution to the Parabolic Truncated Moment Problem in [CF4], based on flat extensions. △

As we show in the sequel, for $d \geq 3$, $C \equiv C[\{A_{ij}\}_{(i,j)\in\mathcal{F}}]$ does not coincide with $\widehat{M_n}$ and is not necessarily positive definite; nevertheless, we will relate positivity properties of $C[\{A_{ij}\}_{(i,j)\in\mathcal{F}}]$ to the existence of representing measures.

We conclude this section with some examples that illustrate Theorem 2.13 for a positive semidefinite $(y-x^d)$-pure $M_n(\beta)$. Let $\widehat{C}$ denote the compression of $C \equiv C[A]$ obtained by deleting each row and each column of C that ends in some auxiliary moment A_{ij}. In the sequel, for $1 \leq k \leq dn + 1$, C_k denotes the compression of C to the first k rows and columns.

Example 2.16. Consider the moment matrix

$$M_3(\beta) = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 2 & 5 & 14 & 42 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 1 & 0 & 0 & 2 & 5 & 14 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 5 & 14 & 42 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 14 & 42 & 132 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 0 & 5 & 14 & 0 & 0 & 0 & 14 & 42 & 132 & 429 \\ 0 & 14 & 42 & 0 & 0 & 0 & 42 & 132 & 429 & s \\ 0 & 42 & 132 & 0 & 0 & 0 & 132 & 429 & s & t \end{pmatrix}. \tag{2.31}$$

A calculation with nested determinants shows that M_3 is positive semidefinite and $(y - x^3)$-pure if and only if $s \equiv \beta_{15}$ and $t \equiv \beta_{06}$ satisfy

$$t > s^2 - 2844s + 2026881. \tag{2.32}$$

The core matrix is

$$C[A]=\begin{pmatrix}
1 & 0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 & 0\\
0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 & 0 & 42\\
1 & 0 & 2 & 0 & 5 & 0 & 14 & 0 & 42 & 0\\
0 & 2 & 0 & 5 & 0 & 14 & 0 & 42 & 0 & 132\\
2 & 0 & 5 & 0 & 14 & 0 & 42 & 0 & 132 & 0\\
0 & 5 & 0 & 14 & 0 & 42 & 0 & 132 & 0 & 429\\
5 & 0 & 14 & 0 & 42 & 0 & 132 & 0 & 429 & 0\\
0 & 14 & 0 & 42 & 0 & 132 & 0 & 429 & 0 & s\\
14 & 0 & 42 & 0 & 132 & 0 & 429 & 0 & s & A\\
0 & 42 & 0 & 132 & 0 & 429 & 0 & s & A & t
\end{pmatrix}. \tag{2.33}$$

i) Let $s=1430$ and $t=4862$, so (2.32) is satisfied. Calculations with nested determinants show that $C_9\succ 0$, and therefore a calculation of $\det C[A]$ shows that $C[A]\succ 0$ if and only if $-1<A<1$. Theorem 2.13 now shows that β has representing measures and that $\mathcal{CV}(L_\beta)$ is the curve $y=x^3$.

ii) Consider next $s=1422$, $t=4798$. Condition (2.32) is satisfied and nested determinants show that $\widehat{C}\succ 0$. In particular, $C_8\succ 0$, but we have *det* $C_9=-7$, so for no value of A will $C[A]$ be positive semidefinite. By Theorem 2.13, β has no measure.

iii) Now let $s=1429$, $t=4847$. Then (2.32) holds, and we have $C_8\succ 0$; however, $\det C_9=0$, so there exists $x\in\mathbb{R}^9$ such that $C_9x=0$. Now $\widehat{r}:=(x^t,0)\equiv(r_0,\dots,r_8,0)$ satisfies $\langle C\widehat{r},\widehat{r}\rangle=0$ and, with $\widehat{s}\equiv 0$, also satisfies the consistency requirement $r_8r_9+s_8s_9=0$ (cf. (2.11)). Remark 2.11 now implies that there exists $p\in\ker L_\beta$ such that $Q(x):=p(x,x^3)=r(x)^2$. Therefore, $\operatorname{card}\mathcal{CV}(L)\le\deg r\le 8<9=\operatorname{rank}M_3$, so $\mathcal{CV}(L_\beta)=\emptyset$ by Corollary 1.4, and thus β has no measure. $\triangle$

In the next section we will prove that if $M_n(\beta)$ is positive semidefinite and $(y-x^d)$-pure, then $\widehat{C}$ is positive definite. In particular, for $d=3$, $C_{3n-1}\succ 0$. It follows that the method of the preceding example applies to any positive semidefinite $M_n(\beta)$ that is $(y-x^3)$-pure, as follows.

Theorem 2.17. *Suppose $M_n(\beta)$ is positive semidefinite and $(y-x^3)$-pure. Then β has a representing measure if and only if $\det C_{3n}>0$, in which case $\mathcal{CV}(L_\beta)$ is the curve $y=x^3$.*

Example 2.18. Consider next the sequence $\beta^{(8)}$, with M_3 given by

$$\begin{pmatrix}
1 & 0 & 2 & 1 & 0 & 14 & 0 & 5 & 0 & 132\\
0 & 1 & 0 & 0 & 5 & 0 & 2 & 0 & 42 & 0\\
2 & 0 & 14 & 5 & 0 & 132 & 0 & 42 & 0 & 1430\\
1 & 0 & 5 & 2 & 0 & 42 & 0 & 14 & 0 & 429\\
0 & 5 & 0 & 0 & 42 & 0 & 14 & 0 & 429 & 0\\
14 & 0 & 132 & 42 & 0 & 1430 & 0 & 429 & 0 & 16796\\
0 & 2 & 0 & 0 & 14 & 0 & 5 & 0 & 132 & 0\\
5 & 0 & 42 & 14 & 0 & 429 & 0 & 132 & 0 & 4862\\
0 & 42 & 0 & 0 & 429 & 0 & 132 & 0 & 4862 & 0\\
132 & 0 & 1430 & 429 & 0 & 16796 & 0 & 4862 & 0 & 208012
\end{pmatrix}$$

and the degree 7 and degree 8 blocks given by

$$\begin{pmatrix}
0 & 42 & 0 & 1430 & 0\\
42 & 0 & 1430 & 0 & 58786\\
0 & 1430 & 0 & 58786 & 0\\
1430 & 0 & 58786 & 0 & 2674440
\end{pmatrix}$$

$$\begin{pmatrix} 14 & 0 & 429 & 0 & 16796 \\ 0 & 429 & 0 & 16796 & 0 \\ 429 & 0 & 16796 & 0 & 742900 \\ 0 & 16796 & 0 & 742900 & 0 \\ 16796 & 0 & 742900 & 0 & 353576708 \end{pmatrix}$$

The core matrix is a Hankel matrix (see Example 2.7) with anti-diagonals completely determined in the first row by

$$\begin{array}{llll} \beta_{00} = 1, & \beta_{01} = 2, & \beta_{02} = 14, & \beta_{03} = 132, \\ \beta_{10} = 0, & \beta_{11} = 0, & \beta_{12} = 0, & \beta_{13} = 0, \\ \beta_{20} = 1, & \beta_{21} = 5, & \beta_{22} = 42, & \beta_{23} = 429, \\ \beta_{30} = 0, & \beta_{31} = 0, & \beta_{32} = 0, & \beta_{33} = 0, \end{array}$$

and the last column by

$$\begin{array}{llll} \beta_{04} = 1430, & \beta_{05} = 16796 & \beta_{06} = 208012, & \beta_{07} = 2674440, \\ \beta_{14} = 0, & \beta_{15} = 0, & \beta_{16} = 0, & \beta_{17} = 0, \\ \beta_{24} = 4862, & \beta_{25} = 58786, & \beta_{26} = 742900, & \beta_{27} = A_{27}, \\ \beta_{34} = 0, & \beta_{35} = 0, & \beta_{36} = A_{36}, & \beta_{37} = A_{37} \\ & & & \beta_{08} = 353576708. \end{array}$$

It is straightforward to verify that M_4 is positive semidefinite and $(y - x^4)$–pure. Using nested determinants, it is easy to show that $C_{14} \succ 0$. A further calculation shows that $C_{15} \succ 0$ if and only if $-1 < A_{36} < 1$. Setting $A_{36} = 0$, we see that $C_{16} \succ 0$ if and only if $A_{27} = 9694844 + f$ for $f > 0$. Now $\det C = f(318219068 - 28f - f^2) - A_{37}^2$, so there exists A_{37} such that $C[A] \succ 0$ if and only if $0 < f < 96\sqrt{34529} - 14$ (≈ 17824.7). In this case, since $C[A] \succ 0$, the core variety coincides with the curve $y = x^4$.

Example 2.19. Consider next the sequence $\beta^{(8)}$, defined as in Example 2.18, except for the following 5 differences:

$$\beta_{25} = 0, \qquad \beta_{06} = 3454708516 \qquad \beta_{26} = 3448894372, \qquad \beta_{07} = 0,$$

$$\beta_{08} = 26405033821733706989067766695725.$$

It is straightforward to verify that M_4 is positive semidefinite and $(y - x^4)$–pure. Moreover, $C[A]$ can never be positive semidefinite, since $\beta_{25} = 0$ is its 12th diagonal element, but there are nonzero entries in the 12th row and column. By the converse in Theorem 2.13, $\beta^{(8)}$ does not admit a representing measure.

3. A central compression of the core matrix equivalent to $\widehat{M_n}$.

In this section, we describe a central compression of the core matrix C that is orthogonally equivalent to $\widehat{M_n}$, and is thus positive definite. We will show in Sections 4-6 that this provides a useful tool for studying the $(y - x^d)$-pure truncated moment problem. Further, for an $(nd + 1) \times (nd + 1)$ Hankel matrix H, we identify a central compression $\widehat{H}$ that uniquely determines M_n of some β satisfying at least the column relations which are linear combinations of the relations coming from $Y = X^d$ by recursive generation; additionally, $\widehat{H} \succ 0$ if and only if $M_n \succeq 0$ and M_n is $(y - x^d)$-pure.

We first require a brief discussion of permutation matrices and orthogonal equivalence. Recall that a real $m \times m$ matrix U is an orthogonal matrix if $U^t = U^{-1}$; equivalently, U maps an orthonormal basis into an orthonormal basis. Let $e_1, \ldots, e_m$ denote the standard orthonormal basis for $\mathbb{R}^m$, and let σ denote a permutation of $\{1, \ldots, m\}$. The *permutation matrix* U_σ is defined by $U_\sigma(e_i) = e_{\sigma(i)}$. Clearly, U_σ is invertible, with $U_\sigma^{-1} = U_{\sigma^{-1}}$, and we note that $U_\sigma^{-1} = U_\sigma^t$, so that U_σ is real orthogonal. To see this, it suffices to check that for $1 \le j, k \le m$, $\langle U_\sigma^{-1} e_j,\ e_k\rangle = \langle U_\sigma^t e_j,\ e_k\rangle$. Setting $i = \sigma^{-1}(j)$, so that $\sigma(i) = j$, we have $\langle U_\sigma^{-1} e_j,\ e_k\rangle = \langle e_{\sigma^{-1}(j)},\ e_k\rangle = \langle e_i,\ e_k\rangle = \delta_{ik}$ (where δ denotes the Kroneker delta). Now, $\langle U_\sigma^t e_j,\ e_k\rangle = \langle e_j,\ U_\sigma e_k\rangle = \langle e_{\sigma(i)},\ e_{\sigma(k)}\rangle = \delta_{\sigma(i)\sigma(k)}$. Since $\sigma(i) = \sigma(k)$ if and only if $i = k$, it follows that $U_\sigma^{-1} = U_\sigma^t$, so U_σ is real orthogonal. We note for the sequel (in Sections 4 and 5) that

$$U_\sigma^{-1}\Big(\sum_{i=1}^{m} x_i e_i\Big) = \sum_{i=1}^{m} x_i e_{\sigma^{-1}(i)} \tag{3.1}$$

Recall that if H and J are $m \times m$ real matrices, H and J are *orthogonally equivalent* if $H = UJU^t$ for some real orthogonal matrix U; clearly, H is positive semidefinite (respectively, positive definite) if and only if J is.

We label the rows and columns of C sequentially from 1 to $nd+1$. Corresponding to column k, let

$$I_k := (k-1) \bmod d \quad \text{and} \quad J_k := \lfloor (k-1)/d \rfloor, \tag{3.2}$$

so that $C_{1,k} = \beta_{I_k,J_k}$ (cf. (2.13)). Now suppose $i, j \ge 0$, with $i < d$ and $i + dj \le nd$, but $i + j > n$. The index set $\widehat{\mathcal{F}}$ of all such pairs is equal to

$$\widehat{\mathcal{F}} := \{(i,j) \colon n - (d-2) \le j \le n-1,\ n+1-j \le i \le d-1\} = \bigcup_{j=1}^{d-2} \widehat{\mathcal{F}}_j \tag{3.3}$$

where each $\widehat{\mathcal{F}}_j$ is equal to

$$\widehat{\mathcal{F}}_j = \begin{cases} \{(j+1, n-j), \ldots, (d-1, n-j)\}, & \text{if } j+1 \le d-1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Hence, $\operatorname{card} \widehat{\mathcal{F}} = \sum_{i=1}^{d-2} i = \frac{(d-1)(d-2)}{2}$. We consider the compression $\widehat{C}$ of C that is obtained by deleting row k and column k from C in those cases where $(I_k, J_k) \in \widehat{\mathcal{F}}$. There are $\frac{(d-2)(d-1)}{2}$ (=card $\widehat{\mathcal{F}}$) such cases, so $\widehat{C}$ has $\psi_{n,d} := \frac{d(2n-d+3)}{2}$ $(= (nd+1) - \frac{(d-2)(d-1)}{2})$ rows and columns, and therefore $\widehat{C}$ has the same size as $\widehat{M_n}$. In $\widehat{C}$, the compressed rows and columns retain the same row and column numbers as the corresponding uncompressed rows and columns in C.

Example 3.1. Let $n = d = 4$. Then C is 17×17, and we delete rows and columns 12, 15, and 16, since $k = 12$ yields $(I_{12}, J_{12}) = (3, 2)$, $k = 15$ has $(I_{15}, J_{15}) = (2, 3)$, $k = 16$ has $(I_{16}, J_{16}) = (3, 3)$, and in each case $I_k + J_k > 4$. The 14 rows and columns in $\widehat{C}$ are numbered $1, \ldots, 11,\ 13,\ 14,\ 17$.

We next let $\widehat{C}_1$ denote a copy of $\widehat{C}$, but with the row and column numbers inherited from C replaced by a sequential relabeling, as follows. For each undeleted column of C, say column k, let $\mathrm{del}(k) \equiv \mathrm{del}_{n,d}(k)$ denote the number of columns of C to the left of column k which are deleted to create $\widehat{C}$. Namely,

$$\mathrm{del}(k) = \mathrm{card}\left(\{j\colon 1 \le j < k,\ (I_j, J_j) \in \widehat{\mathcal{F}}\}\right).$$

Note that the largest k with $\mathrm{del}(k) = 0$ satisfies $(I_k, J_k) = (d-2, n-d+2)$, and is thus equal to

$$d - 2 + d(n-d+2) + 1 = nd - (d-2)(d-1) + 1 =: m_{n,d}.$$

Moreover, $k \le m_{n,d}$ implies that $\mathrm{del}(k) = 0$. For $k > m_{n,d}$ we have that $J_k > n-d+2$ and $I_k \le n - J_k$ (since column k is undeleted) and thus

$$\begin{aligned}\mathrm{del}(k) &= \sum_{i=1}^{J_k-(n-d+2)} i = \frac{(J_k - (n-d+2))(J_k - (n-d+2)+1)}{2}\\ &= \frac{(\lfloor \frac{k-1}{d} \rfloor - n + d - 2))(\lfloor \frac{k-1}{d} \rfloor - n + d - 1)}{2}\\ &= \frac{(\lfloor \frac{k}{d} \rfloor - n + d - 2))(\lfloor \frac{k}{d} \rfloor - n + d - 1)}{2}.\end{aligned}$$

Note that in the last equality we used that $\lfloor \frac{k-1}{d} \rfloor = \lfloor \frac{k}{d} \rfloor$ for every undeleted column k with $k > m_{n,d}$. Indeed, $\lfloor \frac{k-1}{d} \rfloor \ne \lfloor \frac{k}{d} \rfloor$ if and only if $k = dk'$ for some $k' \in \mathbb{N}$. Since $k > m_{n,d}$, we have that k' is at least $n-d+3$. But every column dk' with $k' \ge n-d+3$ is deleted, since $I_{dk'} = d-1$, $J_{dk'} = k'-1 \ge n-d+2$ and hence $I_{dk'} + J_{dk'} > n$.

The compression of column k of C is now used as column $\widehat{k}$ of $\widehat{C}_1$, where $\widehat{k} = k - \mathrm{del}(k)$; we also write $k = \phi(\widehat{k})$ (a relation we will refer to in the sequel). In this way, the columns of $\widehat{C}_1$ are numbered sequentially from 1 to $\psi_{n,d}$, and we also renumber the rows of $\widehat{C}_1$ in a similar sequential manner.

Example 3.2. Let $n = d = 4$. We have $\widehat{k} = k$ for $1 \le k \le 11$, $\widehat{k} = 12$ for $k = 13$, $\widehat{k} = 13$ for $k = 14$, and $\widehat{k} = 14$ for $k = 17$.

We next describe a 2-step transformation of $\widehat{C}_1$ into a matrix $\widetilde{C}$. For $1 \le \widehat{k} \le \psi_{n,d}$, note that $\binom{I_k+J_k+1}{2} + J_k + 1$ is the column number corresponding to $X^{I_k}Y^{J_k}$ in M_n. The column number corresponding to $X^{I_k}Y^{J_k}$ in $\widehat{M_n}$ must, however, take into account any columns of the form $X^{d+r}Y^s$ which precede $X^{I_k}Y^{J_k}$ in the degree-lexicographic ordering of the columns of M_n, since every such is deleted from M_n when forming $\widehat{M_n}$. There are $\binom{I_k+J_k-d+2}{2}$ such columns, so we define

$$K_k := \binom{I_k+J_k+1}{2} + J_k + 1 \qquad \text{if } I_k + J_k < d, \tag{3.4}$$

$$K_k := \binom{I_k+J_k+1}{2} + J_k + 1 - \binom{I_k+J_k-d+2}{2} \qquad \text{if } I_k + J_k \ge d. \tag{3.5}$$

We now define column K_k of $\widehat{C}_2$ to be column $\widehat{k}$ of $\widehat{C_1}$. We will show just below that the mapping from $\widehat{k}$ to K_k defines a permutation of the integers $1, \ldots, \psi_{n,d}$, so that the columns of $\widehat{C_2}$ comprise a permutation of the columns of $\widehat{C_1}$. Finally, we transform $\widehat{C_2}$ into $\widetilde{C}$ by applying the same permutation to the rows of $\widehat{C_2}$ that we just applied to the columns of $\widehat{C_1}$.

Example 3.3. Let $d = 4$ and $n \geq d$. There are always 3 auxiliary moments, $A_{3,2n-1}$, $A_{2,2n-1}$, $A_{3,2n-2}$. With $n = d = 4$, the permutation may thus be described as

$$\begin{pmatrix} \text{row/col } \widehat{C_1} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ \text{row/col } \widetilde{C} & 1 & 2 & 4 & 7 & 3 & 5 & 8 & 11 & 6 & 9 & 12 & 10 & 13 & 14 \end{pmatrix}.$$

To validate the preceding argument we must verify that the mapping from $\widehat{k}$ to K_k is a permutation of the integers $1, \dots, \psi_{n,d}$. For this aim we first establish an auxiliary lemma.

Lemma 3.4. *Let $f : \{0, 1, \dots, n\} \mapsto \mathbb{Z}_+$ be a map defined by*

$$f(p) := \begin{cases} \binom{p+1}{2} + 1, & p < d, \\ \binom{p+1}{2} - \binom{p-d+2}{2} + 1, & p \geq d. \end{cases}$$

Then for $p \in \{0, 1, \dots, n-1\} \setminus \{d-1\}$ we have

$$f(p+1) > f(p) + \min(p, d-1) =: g(p),$$

while

$$f(d) = f(d-1) + d - 1.$$

Proof. We separate three cases:

Case a) $0 \leq p < d - 1$. Then

$$f(p+1) = \frac{(p+1)(p+2)}{2} + 1 > \frac{p(p+3)}{2} + 1 = f(p) + p = g(p).$$

Case b) $p = d - 1$. Then

$$\begin{aligned} f(p+1) = f(d) &= \frac{d(d+1)}{2} - \binom{2}{2} + 1 = \frac{d(d+1)}{2} \\ &= \frac{(d-1)d}{2} + d = f(d-1) + d - 1. \end{aligned}$$

Case c) $d \leq p \leq n - 1$. Then

$$\begin{aligned} f(p+1) &= \frac{(p+1)(p+2)}{2} - \frac{(p-d+2)(p-d+3)}{2} + 1 \\ &= \frac{p(p+1)}{2} - \frac{(p-d+1)(p-d+2)}{2} + d \\ &= f(p) + d - 1 = g(p), \end{aligned}$$

which concludes the proof of the lemma. □

Lemma 3.5. *The map π, defined by $\widehat{k} \mapsto K_k$ is a permutation of $S := (1, \dots, \psi_{n,d})$.*

Proof. For each $\widehat{k}$ $(1 \le \widehat{k} \le \psi_{n,d})$, $k := \phi(\widehat{k})$ satisfies $I_k + J_k \le n$ by the construction of $\widehat{C}$. Let f be defined as in Lemma 3.4. Note that $K_k = f(I_k + J_k) + J_k$. We will prove the following two facts:

(i) $I_k + J_k < I_l + J_l$ implies that $K_k < K_l$.

(ii) $I_k + J_k = I_l + J_l$ and $k \ne l$ implies that $K_k \ne K_l$.

Using (i) and (ii) it is clear that π is 1-to-1 and the largest K_k corresponds to $I_k + J_k = n$, $J_k = n$. But then $k = nd + 1$ and $\widehat{nd+1} = \psi_{n,d} = K_{nd+1}$. So π is a permuation of S.

It remains to prove (i) and (ii). The latter is clear since $I_k + J_k = I_l + J_l$ and $I_k + dJ_k = k - 1 \ne l - 1 = I_l + dJ_l$, implies that $J_k \ne J_l$. But then by the definition of K_k (cf. (3.4), (3.5)) it follows that $K_k \ne K_l$. To prove (i) it is enough to consider the case $I_l + J_l = I_k + J_k + 1$, since then (i) follows inductively. Assume that $I_l + J_l = I_k + J_k + 1$. We separate three cases according to the value of $I_k + J_k$:

Case a) $I_k + J_k \le d - 2$. We have that

$$\begin{aligned} K_k &= f(I_k + J_k) + J_k \le f(I_k + J_k) + (I_k + J_k) < f(I_k + J_k + 1) \\ &= f(I_l + J_l) \le f(I_l + J_l) + J_l = K_l, \end{aligned}$$

where we used Lemma 3.4 in the second inequality of the first line.

Case b) $I_k + J_k = d - 1$. Then

$$\begin{aligned} K_k &= f(d-1) + J_k \le f(d-1) + d - 1 = f(d) = f(I_l + J_l) \\ &\underbrace{<}_{J_l \ge 1} f(I_l + J_l) + J_l = K_l, \end{aligned}$$

where we used Lemma 3.4 in the second equality. Note also that $J_l \ge 1$, since otherwise $I_l + J_l = I_l \le d - 1$, which is a contradiction with the assumption of this case.

Case c) $d \le I_k + J_k$. Then

$$\begin{aligned} K_k &= f(I_k + J_k) + J_k < f(I_k + J_k + 1) - (d-1) + J_k \\ &= f(I_l + J_l) - (d-1) + J_k < f(I_l + J_l) + J_l = K_l, \end{aligned}$$

where we used Lemma 3.4 in the first inequality, while in the second inequality we used that

$$J_l = J_k + \underbrace{(I_l + J_l - I_k - J_k)}_{=1} - (I_l - I_k) \underbrace{>}_{I_l - I_k \le d-1} J_k - (d-1).$$

This concludes the proof of the lemma. ☐

Theorem 3.6. *$\widehat{C}$ is orthogonally equivalent to $\widehat{M_n}$; in particular, $\widehat{C} \succ 0$.*

Proof. The renumbering of the rows and columns of $\widehat{C}$ to form $\widehat{C_1}$, followed by the orthogonal equivalence induced by permutation π (cf. Lemma 3.5), shows that $\widehat{C}$ is orthogonally equivalent to $\widetilde{C}$, so it suffices to verify that $\widetilde{C}$ coincides with $\widehat{M_n}$. Recall that the rows and columns of $\widehat{M_n}$ are labelled in degree-lexicographic order, $1,\ X,\ Y,\dots,X^{d-1},\dots,\ Y^{d-1},\ X^{d-1}Y,\dots,\ Y^d,\dots,X^{d-1}Y^{n-d+1},\dots,Y^n$ (there is no row or column X^iY^j with $i \geq d$), so we label the rows and columns of $\widetilde{C}$ in the same way. From the structure of M_n, the entry in row X^lY^m, column X^iY^j of $\widehat{M_n}$ is $\beta_{i+l,j+m}$, so we seek to show that the entry in row X^lY^m, column X^iY^j of $\widetilde{C}$ is also $\beta_{i+l,j+m}$.

Since $0 \leq i < d$, then $k := i + dj + 1$ ($\leq nd + 1$) is the unique column number of C satisfying $i = (k-1) \bmod d$ and $j = \lfloor (k-1)/d \rfloor$. Thus, column k is the unique column of C (or of $\widehat{C}$) that is transformed by compression and permutation π into column X^iY^j in the degree-lexicographic ordering of the columns of $\widetilde{C}$. Since $\pi(1) = 1$, column X^iY^j in $\widetilde{C}$ starts with $C_{1,k} = \beta_{ij}$, and the other components of column X^iY^j in $\widetilde{C}$ are components of column k in C rearranged according to compression and permutation π. Since $1 \leq l < d$, then, exactly as above, row X^lY^m in $\widetilde{C}$ originates from row $p := l + dm + 1$ in C. Therefore, the row X^lY^m, column X^iY^j entry of $\widetilde{C}$ is equal to $C_{p,k}$. From (2.13), we have

$$\begin{aligned} C_{p,k} &= \beta_{(p+k-2) \bmod d,\ \lfloor (p+k-2)/d \rfloor} \\ &= \beta_{((d(j+m)+i+l+2)-2) \bmod d, \lfloor ((d(j+m)+i+l+2)-2)/d \rfloor} \\ &= \beta_{(i+l) \bmod d, j+m+\lfloor (i+l)/d \rfloor}. \end{aligned}$$

Since $Y = X^d$ in M_n, we have $\beta_{a+db,c} = \beta_{a,b+c}$ whenever $a,\ b,\ c \geq 0$ and $a+db+c \leq 2n$. Since $i + l = rd + s$ with $r = \lfloor (i+l)/d \rfloor$ and $s = (i+l) \bmod d$, it follows that

$$\beta_{(i+l) \bmod d, j+m+\lfloor (i+l)/d \rfloor} = \beta_{s+dr,j+m} = \beta_{i+l,j+m}.$$

Thus, the row X^lY^m, column X^iY^j entries of $\widetilde{C}$ and $\widehat{M_n}$ coincide. □

We conclude the section with a result which, together with Theorem 3.6, shows that positive definite central compressions of Hankel $(nd+1) \times (nd+1)$ matrices are in bijection with positive semidefinite, $(y - x^d)$-pure moment matrices $M_n(\beta)$ and can be used as a simple tool to generate examples for these. In Examples 2.18, 2.19, instead of forming the entire matrix $M_4(\beta)$ and checking positive semidefiniteness and $(y - x^4)$-pureness, it is sufficient to form only $\widehat{C}$ using β_{ij}, $0 \leq i < 4$, $0 \leq j$, $0 \leq i + j \leq 8$ and check whether it is positive definite. The following proposition then uniquely determines $M_4(\beta)$ with the desired properties.

Proposition 3.7. Let $H := (H_{ij})_{i,j} = (h_{i+j-2})_{i,j}$ be a $(nd+1) \times (nd+1)$ Hankel matrix, where $h_l \in \mathbb{R}$ for $0 \leq l \leq 2nd$. Let $\widehat{H}$ be obtained from H by deleting row k and column k from H if $(I_k, J_k) \in \widehat{\mathcal{F}}$, where $I_k, J_k, \widehat{\mathcal{F}}$ are as in (3.2), (3.3), respectively. Assume that $\widehat{H}$ is positive definite. Then there is a unique β such that $M_n(\beta)$ is positive semidefinite, $(y - x^d)$-pure and $\widehat{C} = \widehat{H}$.

Proof. For $i,j \geq 0$ and $0 \leq i+j \leq 2n$ define $\beta_{ij} := h_{i+dj}$. Let $L : \mathcal{P}_{2n} \to \mathbb{R}$ be a linear functional, defined by $L(x^i y^j) := \beta_{ij}$. Note that $\ker L$ contains the polynomials

- $f_{ij}(x,y) = x^i y^j - \beta_{ij}$, $0 \leq i < d, j \geq 0, 0 < i+j \leq 2n$, and
- $g_{kl} = (y - x^d)x^k y^l$ for $k,l \geq 0$, $k+l \leq 2n-d$.

Indeed, $f_{ij} \in \ker L$ is clear from the definition of L, while

$$L(g_{kl}) = L(x^k y^{l+1}) - L(x^{d+k}y^l) = \beta_{k,l+1} - \beta_{d+k,l} = h_{k+d(l+1)} - h_{(d+k)+dl} = 0.$$

Since the set $\mathcal{B} := \{f_{ij}\} \cup \{g_{kl}\}$ is linearly independent, $\operatorname{card}\mathcal{B} = \dim \mathcal{P}_{2n} - 1$ (see the proof of Lemma 2.1) and $\operatorname{Lin}\mathcal{B} \subseteq \ker L$, it follows that $\mathcal{B}$ is a basis for $\ker L$. By the converse part in Lemma 2.1, $M_n(\beta)$ satisfies the column relations (2.1). By definition of the core matrix C (cf. (2.13)), for $1 \leq i,j \leq dn+1$, we have that

$$C_{ij} = \beta_{(i+j-2) \bmod d, \lfloor (i+j-2)/d \rfloor} = h_{(i+j-2) \bmod d + d\lfloor (i+j-2)/d \rfloor} = h_{i+j-2} = H_{ij},$$

and in particular, $\widehat{C} = \widehat{H}$. The assumption $\widehat{H} \succ 0$ implies that $\widehat{C} \succ 0$ and by Theorem 3.6, the central compression $\widehat{M_n}$ is positive definite, whence M_n is positive semidefinite and $(y - x^d)$-pure. □

4. The $(y - x^3)$-pure truncated moment problem.

In this section we apply the previous results to the moment problem for $\beta \equiv \beta^{(2n)}$ where M_n is positive semidefinite and $(y - x^3)$-pure. In particular, Theorem 4.1 provides a positive answer to Question 2.14 for $d = 3$. Let Γ stand for the curve $y = x^3$. Note that in the core matrix C, since $Y = X^d$ with $d = 3$, (2.10) implies that there is exactly 1 auxiliary moment, namely $\beta_{2,2n-1}$, which we denote by $A \equiv A_{2,2n-1}$. Thus, $\widehat{C}$ is obtained from C by deleting row and column nd. Let $H \equiv H[A]$ denote the matrix obtained from $C \equiv C[A]$ by interchanging rows and columns nd and $nd+1$ (the last 2 rows and columns), so that H is orthogonally equivalent to C. In the notation of Section 3, let σ denote the permutation of $\{1, \ldots, nd+1\}$ such that $\sigma(i) = i$ $(1 \leq i \leq nd-1)$, $\sigma(nd) = nd+1$, $\sigma(nd+1) = nd$; then $H = P_\sigma C P_{\sigma^{-1}}$.

Note that the compression of H to its first nd rows and columns coincides with $\widehat{C}$, and is thus positive definite by Theorem 3.6. We may thus represent H as

$$H = \begin{pmatrix} \widehat{C} & v \\ v^t & \beta_{1,2n-1} \end{pmatrix}, \tag{4.1}$$

with $\widehat{C} \succ 0$ and where v is of the form

$$v = \begin{pmatrix} h \\ A \end{pmatrix}. \tag{4.2}$$

(Here $h \in \mathbb{R}^{dn-1}$ and v^t denotes the row vector transpose of v.) Write

$$\widehat{C} = \begin{pmatrix} C_1 & z \\ z^t & \beta_{0,2n} \end{pmatrix},$$

where C_1 is of size $(dn-1)\times(dn-1)$ and $z \in \mathbb{R}^{dn-1}$ is of the form $z = (k, \beta_{1,2n-1})^t$ for some $k \in \mathbb{R}^{dn-2}$. We now have

$$H[A] = \begin{pmatrix} C_1 & z & h \\ z^t & \beta_{0,2n} & A \\ h^t & A & \beta_{1,2n-1} \end{pmatrix}. \tag{4.3}$$

Since $\widehat{C} \succ 0$, $\widehat{C}^{-1}$ exists and has the form

$$\widehat{C}^{-1} = \begin{pmatrix} \mathcal{C} & w \\ w^t & \epsilon \end{pmatrix}, \tag{4.4}$$

where (see e.g., [F2, p. 3144])

$$\begin{aligned} \epsilon &= \frac{1}{\beta_{0,2n} - z^t C_1^{-1} z} > 0, \qquad w = -\epsilon C_1^{-1} z \in \mathbb{R}^{dn-1}, \\ \mathcal{C} &= C_1^{-1}(1 + \epsilon z z^t C_1^{-1}) \in \mathbb{R}^{(dn-1)\times(dn-1)}. \end{aligned} \tag{4.5}$$

Now

$$\widehat{C}^{-1} v = \begin{pmatrix} \mathcal{C}h + Aw \\ w^t h + A\epsilon \end{pmatrix},$$

and we set

$$A \equiv A_0 := -\frac{w^t h}{\epsilon}, \tag{4.6}$$

so that

$$\widehat{C}^{-1} v = \begin{pmatrix} \mathcal{C}h - \frac{w^t h}{\epsilon} w \\ 0 \end{pmatrix}. \tag{4.7}$$

With this value of A in C, and thus also in v, let

$$\begin{aligned} \phi := v^t \widehat{C}^{-1} v = h^t \mathcal{C} h - \frac{w^t h h^t w}{\epsilon} \\ &= (h^t C_1^{-1} h + \epsilon h^t C_1^{-1} z z^t C_1^{-1} h) - \epsilon z^t C_1^{-1} h h^t C_1^{-1} z \\ &= h^t C_1^{-1} h, \end{aligned} \tag{4.8}$$

where we used (4.5) in the second equality.

To emphasize the dependence of ϕ on β, we sometimes denote ϕ as $\phi[\beta]$. In Example 4.4 (below) we will use the fact that ϕ is independent of $\beta_{1,2n-1}$ and $\beta_{0,2n}$. To see this, note that $\beta_{1,2n-1}$ is an element of vectors z and z^t, so (4.3) shows that C_1 and h are independent of $\beta_{1,2n-1}$ and $\beta_{0,2n}$. It now follows from (4.8) that ϕ is independent of $\beta_{1,2n-1}$ and $\beta_{0,2n}$ as well. Thus, if $\widetilde{\beta}^{(2n)}$ has the property that $M_n(\widetilde{\beta})$ is positive semidefinite and $(y-x^3)$-pure, and if $\beta_{ij} = \widetilde{\beta}_{ij}$ for all $(i,j) \neq (1, 2n-1)$ and $(i,j) \neq (0, 2n)$, then $\phi[\widetilde{\beta}] = \phi[\beta]$. Note that ϕ would depend on $\beta_{1,2n-1}$ and $\beta_{0,2n}$ if A_0 in (4.6) was chosen differently. This is due to the fact that the last row of $\widehat{C}^{-1}v$ in (4.7) would be non-zero.

Theorem 4.1. *Suppose M_n is positive semidefinite and $(y-x^3)$-pure. $\beta \equiv \beta^{(2n)}$ has a representing measure if and only if $\beta_{1,2n-1} > \phi$ (equivalently, $C[A_0] \succ 0$). In this case, $\mathcal{CV}(L_\beta) = \Gamma$, which coincides with the union of supports of all representing measures (respectively, all finitely atomic representing measures).*

Proof. Recall from Theorem 3.6 that $\widehat{C}$ is positive definite. Consider first the case $\beta_{1,2n-1} > \phi$. It follows from (4.1) and [A, Theorem 1] that H is positive definite. Since C is orthogonally equivalent to H, we see that C is positive definite, so the existence of representing measures and the conclusion concerning supports follow from Theorem 2.13.

We next consider the case when $\beta_{1,2n-1} = \phi$, so that by [A, Theorem 1], H is positive semidefinite, but singular. Since $\widehat{C} \succ 0$, it follows from (4.1) and (4.7) that $\ker H$ contains the vector

$$(4.9) \qquad \widehat{u} := \begin{pmatrix} \widehat{C}^{-1}v \\ -1 \end{pmatrix} \equiv \begin{pmatrix} \mathcal{C}h - \frac{w^t h}{\epsilon} w \\ 0 \\ -1 \end{pmatrix} \equiv (r_0,\ r_1, \ldots,\ r_{dn-2},\ u_{dn-1},\ u_{dn})^t,$$

where $u_{dn-1} = 0$ and $u_{dn} = -1$. From the orthogonal equivalence between H and C, based on the interchange of rows and columns nd and $nd+1$, it follows that C is positive semidefinite and that $\ker C$ contains the vector $\widehat{r} = (r_0,\ r_1, \ldots,\ r_{dn-2},\ r_{dn-1},\ r_{dn})^t$, where $r_{dn-1} = u_{dn} = -1$ and $r_{dn} = u_{dn-1} = 0$. Let $\widehat{s} \equiv (s_0, \ldots,\ s_{dn})^t$ denote the 0 vector, so that $\langle C\widehat{r}, \widehat{r}\rangle + \langle C\widehat{s}, \widehat{s}\rangle = 0$ and the auxiliary condition of (2.11), $r_{dn-1}r_{dn} + s_{dn-1}s_{dn} = 0$, is satisfied. Now, following Remark 2.11 and (2.7), define $a_{ij} = h_{ij}(\widehat{r}, \widehat{s})$ $(0 \le i \le 2,\ j \ge 0,\ 0 < i+j \le 2n)$. Then $p := \sum a_{ij} f_{ij}$ is an element of $\ker L_\beta$ which satisfies $Q(x) := p(x, x^3) = R(x)^2$, where $R(x) := r_0 + r_1 x + \cdots + r_{dn-1}x^{dn-1} + r_{dn}x^{dn}$. Since $r_{dn} = 0$, $R(x)$ has at most $dn-1$ real zeros, so p has at most $dn-1$ zeros in the curve $y = x^3$. Now $p \in \ker L_\beta$ satisfies $p|\Gamma \ge 0$ and $\operatorname{card} \mathcal{Z}(p|\Gamma) \le dn - 1 < \frac{d(2n-d+3)}{2} = \operatorname{rank} M_n$ (since $d = 3$), so Corollary 1.4 implies that β has no representing measure.

To complete the proof, we consider the case when $\beta_{1,2n-1} < \phi$. From (4.1) and (4.9) we have

$$\langle H\widehat{u},\ \widehat{u}\rangle = \langle \begin{pmatrix} 0_{dn\times 1} \\ v^t\widehat{C}^{-1}v - \beta_{1,2n-1} \end{pmatrix}, \begin{pmatrix} *_{dn\times 1} \\ -1 \end{pmatrix} \rangle = \beta_{1,2n-1} - v^t\widehat{C}^{-1}v = \beta_{1,2n-1} - \phi < 0.$$

Recall that $H = P_\sigma C P_{\sigma^{-1}}$. Setting $\widehat{r} := P_{\sigma^{-1}}\widehat{u}$, we have $\langle C\widehat{r},\ \widehat{r}\rangle = \langle H\widehat{u}, \widehat{u}\rangle < 0$, and, from (4.9), $\widehat{r}$ is of the form $\widehat{r} = (r_0, \ldots, r_{dn-1},\ r_{dn})^t \equiv \begin{pmatrix} *_{(dn-1)\times 1} \\ -1 \\ 0 \end{pmatrix}$; in particular, $r_{dn-1}r_{dn} = 0$. Let $\epsilon = (\phi - \beta_{1,2n-1})^{1/2}$. Since $\langle \widehat{C}e_1,\ e_1\rangle = \beta_{00} = 1$, then the constant polynomial $S(x) = \epsilon$, with coefficient vector $\widehat{s} = (\epsilon, 0, \ldots, 0)^t$, satisfies $s_{dn-1}s_{dn} = 0$ and we have $\langle C\widehat{r},\ \widehat{r}\rangle + \langle C\widehat{s},\ \widehat{s}\rangle = 0$. So $\widehat{r}$ and $\widehat{s}$ together satisfy the auxiliary requirements of (2.10). Constructing $p(x,y)$ as in Remark 2.11, (2.10) shows that $p \in \ker L_\beta$. Now, $p(x, x^d) = R(x)^2 + S(x)^2 \ge \epsilon^2 > 0$. Since p is strictly positive on Γ, then $\mathcal{CV}(L_\beta) = \emptyset$, and therefore β has no representing measure. $\square$

Remark 4.2. In Theorem 4.1, an alternate proof of the case $\beta_{1,2n-1} < \phi$ can be based on Theorem 2.13, as follows. Let A_0 be as in (4.6). If $\beta_{1,2n-1} < \phi[A_0]$, then (4.8) implies that $\beta_{1,2n-1} < h^t C_1^{-1} h$. It therefore follows from (4.3) that for every $A \in \mathbb{R}$, the matrix $\begin{pmatrix} C_1 & h \\ h^t & \beta_{1,2n-1} \end{pmatrix}$ is a principal submatrix of $H[A]$ that is not positive semidefinite. Thus, for every A, $H[A]$, and hence $C[A]$, is not positive semidefinite, so Theorem 2.13 implies that β has no representing measure.

In [F2] a rather lengthy construction with moment matrices is used to derive a certain rational expression in the moment data, denoted by ψ in [F2], such that β has a representing measure if and only if $\beta_{1,2n-1} > \psi$, in which case M_n admits a flat extension M_{n+1}. In view of Theorem 4.1, it is clear that $\psi = \phi$ (although this is not at all apparent from the definitions of these expressions).

Corollary 4.3. *Suppose $M_n(\beta)$ is positive semidefinite and $(y - x^3)$-pure. The following are equivalent:*

(i) β has a representing measure;
(ii) β has a finitely atomic measure;
(iii) $M_n(\beta)$ has a flat extension M_{n+1};
(iv) $\mathcal{CV}(L_\beta) \neq \emptyset$;
(v) With A defined by (4.6) and ϕ defined by (4.8), $\beta_{1,2n-1} > \phi$;
(vi) $\mathcal{CV}(L_\beta) = \Gamma$.

Proof. The implications $(i) \implies (iv) \implies (ii) \implies (i)$ follow from the Core Variety Theorem and its proof. The equivalence of (i) and (iii) is established in [F2], and the equivalence of (i), (v), and (vi) is Theorem 4.1. □

In [EF] the authors used the results of [F1] to exhibit a family of positive $(y - x^3)$-pure moment matrices $M_3(\beta^{(6)})$ such that $\beta^{(6)}$ has no representing measure but the Riesz functional is positive (cf. Section 1). Here, positivity of the functional cannot be derived from positivity of M_3 using an argument such as $L(p) = L(\sum p_i^2) = \sum \langle M_3 \widehat{p_i}, \widehat{p_i} \rangle \geq 0$, because, by the theorem of Hilbert, not every nonnegative polynomial $p(x, y)$ of degree 6 can be represented as a sum of squares. Using Theorem 4.1 we can extend this example to a family of $(y - x^3)$-pure matrices M_n, for $n \geq 3$ as follows.

Example 4.4. Suppose $M \equiv M_n(\beta)$ is positive semidefinite and $(y - x^3)$-pure. Let $\phi \equiv \phi[\beta]$ be as in (4.8) and suppose $\phi = \beta_{1,2n-1}$, so that β has no representing measure by Theorem 4.1. We claim that the Riesz functional L_β is positive. Let $\widehat{M}$ denote the central compression of M to rows and columns that are of the form $X^i Y^j$ with $0 \leq i < 3$, so that $\operatorname{rank} M = \operatorname{rank} \widehat{M}$ and $\widehat{M} \succ 0$. Now let $\widetilde{\beta}$ be defined to coincide with β, except possibly in the $\beta_{1,2n-1}$ position. It follows from the structure of positive matrices that there exists $\delta > 0$ such that if $|\widetilde{\beta}_{1,2n-1} - \beta_{1,2n-1}| < \delta$, then $\widehat{M_n}(\widetilde{\beta})$ is positive definite. The structure of positive $(y - x^3)$-pure moment matrices now implies that $M_n(\widetilde{\beta})$ is positive semidefinite and $(y - x^3)$-pure. Now consider the sequence $\beta^{[m]}$ which coincides with β except that $\beta^{[m]}_{1,2n-1} = \beta_{1,2n-1} + 1/m$. It follows that there exists $m_0 > 0$ such that if $m > m_0$, then $M^{[m]} \equiv M_n(\beta^{[m]})$ is positive semidefinite and $(y - x^3)$-pure. By the remarks preceding Theorem 4.1, we have $\beta^{[m]}_{1,2n-1} = \beta_{1,2n-1} + 1/m > \beta_{1,2n-1} = \phi[M] = \phi[M_n(\beta^{[m]})]$, so Theorem 4.1 implies that $\beta^{[m]}$ has a representing measure. Thus, $L_{\beta^{[m]}}$ is positive, and since the cone of sequences with positive functionals is closed, it follows that L_β is positive.

To exhibit $M_n(\beta)$ as in Example 4.4, we may start with any positive semidefinite $(y-x^3)$-pure $M_n(\beta')$. Define β so that it coincides with β' except that $\beta_{1,2n-1} = \phi[\beta']$. If necessary, increase $\beta_{0,2n}$ to insure positivity of $M_n(\beta)$. Then $M_n(\beta)$ is positive semidefinite, $(y - x^3)$-pure, and $\beta_{1,2n-1} = \phi[\beta'] = \phi[\beta]$ by the remarks preceding Theorem 4.1.

5. A test for finiteness of the core variety in the $(y - x^d)$-pure truncated moment problem.

In this section we extend the method of the previous section to develop a sufficient condition for finiteness of the core variety in the $(y-x^d)$-pure truncated moment problem for $d \geq 4$. For $d = 4$, this condition actually implies an empty core variety and the nonexistence of representing measures. We begin with a construction that applies to the $(y - x^d)$-pure truncated moment problem for $d \geq 4$ (so that there are at least 3 auxiliary moments).

Using (3.3) note that in the core matrix C, the $\eta \equiv \frac{(d-1)(d-2)}{2}$ antidiagonals with auxiliary moments are contained within the final 2η antidiagonals. Namely, for $1 \leq k \leq d-2$, the auxiliary moments $A_{k+1,2n-k}, \ldots, A_{d-1,2n-k}$ are contained in $d-1-k$ such antidiagonals. We divide the final column f of C into d vectors $f[\ell]$, $\ell = 0, \ldots, d-1$, such that

$$f \equiv \begin{pmatrix} f[d-1] \\ \vdots \\ f[\ell] \\ \vdots \\ f[0] \end{pmatrix} \in \mathbb{R}^{nd+1}, \tag{5.1}$$

where

$$f[0] = \left(\beta_{0,2n}\right), \quad f[\ell] = \begin{pmatrix} \beta_{0,2n-\ell} \\ \beta_{1,2n-\ell} \\ \vdots \\ \beta_{\ell,2n-\ell} \\ A_{\ell+1,2n-\ell} \\ \vdots \\ A_{d-1,2n-\ell} \end{pmatrix} \in \mathbb{R}^d \quad \text{for } 1 \leq \ell \leq d-2$$

and

$$f[d-1] = \begin{pmatrix} g[n] \\ g[n+1] \\ \vdots \\ g[2n-d+1] \end{pmatrix} \in \mathbb{R}^{(n-d+2)d} \quad \text{with} \quad g[i] = \begin{pmatrix} \beta_{0,i} \\ \beta_{1,i} \\ \vdots \\ \beta_{d-1,i} \end{pmatrix} \quad \text{for each } i.$$

So the first auxiliary moment occurs in the last coordinate of $f[d-2]$, which is the antidiagonal $(d-1)(d-2)$ of C counted from the final one backwards.

Consider the following permutation σ of the rows of C to form a matrix H_1. The first $(nd+1)-2\eta$ rows and columns of H_1 coincide with those of C. The η rows of C ending in auxiliary moments (as just described above) are shifted into the final η rows of H_1, maintaining the same relative position ordering as in C. The rows among the final 2η rows of C that do not contain auxiliary moments in the rightmost position are shifted upward into consecutive rows of H_1 beginning in row $nd+2-2\eta$ (and maintaining the same order).

Matrix H is obtained from H_1 by permuting the columns of H_1 in the same way as the rows of C were permuted to form H_1; thus $H = P_\sigma C P_{\sigma^{-1}}$, where P_σ is the permutation matrix associated with σ ($P_\sigma e_i = e_{\sigma(i)}$ $(1 \le i \le dn+1)$, cf. Section 3).

Remark 5.1. Recall from Proposition 2.9(ii) that $C = \widetilde{M}[\widetilde{\beta},\mathcal{U}]$, where

- $\widetilde{\beta} \equiv \widetilde{\beta}^{(2(n+d-2))}$ is any extension of β such that M_{n+d-2} is recursively generated,
- $\mathcal{U}$ is as in (2.16),
- $M[\widetilde{\beta},\mathcal{U}]$ is a matrix with rows and columns indexed in the order (2.17) with the entry in row X^iY^j and column X^kY^l equal to $\widetilde{\beta}_{i+k,j+l}$ (cf. (2.18)),
- $\widetilde{M}[\widetilde{\beta},\mathcal{U}]$ is obtained from $M[\widetilde{\beta},\mathcal{U}]$ by replacing each $\widetilde{\beta}_{ij}$ such that $i \bmod d + j + \lfloor \frac{i}{d} \rfloor > 2n$ with the auxiliary moment A_{ij}.

Then H is a matrix obtained from $\widetilde{M}[\widetilde{\beta},\mathcal{U}]$ by permuting its rows and columns. First, the rows and columns that are not shifted from C appear, i.e., all X^iY^j from (2.17) with $i+j \le n$ up to $X^{d-2}Y^{n-d+2}$. Then the rows and columns that are shifted upward from C, i.e., X^iY^j from (2.17) right to $X^{d-2}Y^{n-d+2}$ with $i+j \le n$. Finally, the rows and columns that are shifted downward from C follow, i.e., $(i,j) \in \widehat{\mathcal{F}}$ (cf. (3.3)) with their relative position ordering preserved.

From this construction, it is apparent that the first

$$\tau \equiv nd + 1 - \operatorname{card}\widehat{\mathcal{F}} \quad \left(= nd + 1 - \frac{(d-2)(d-1)}{2} = \frac{2nd - d^2 + 3d}{2} \right)$$

rows and columns of H coincide with $\widehat{C}$, and that H admits a decomposition

$$H = \begin{pmatrix} \widehat{C} & v & B \\ v^t & \lambda & D \\ B^t & D^t & E \end{pmatrix}, \tag{5.2}$$

where rows and columns of $\widehat{C}$ are indexed by elements X^iY^j from (2.17) with $(i,j) \notin \widehat{\mathcal{F}}$, the row $\begin{pmatrix} v^t & \lambda & D \end{pmatrix}$ has index $X^{d-1}Y^{n-d+2}$ and the rows in $\begin{pmatrix} B^t & D^t & E \end{pmatrix}$ run over X^iY^j for $(i,j) \in \widehat{\mathcal{F}} \setminus \{(d-1, n-d+2)\}$.

Note that v is of the form

$$v = \begin{pmatrix} h \\ \widetilde{A} \equiv A_{d-1,2n-d+2} \end{pmatrix}, \tag{5.3}$$

with $h \in \mathbb{R}^{\tau-1}$ and

$$\lambda = H_{nd+2-\eta,nd+2-\eta} = H_{\sigma(d(n-d+3)),\sigma(d(n-d+3))} = C_{d(n-d+3),d(n-d+3)} = \beta_{\xi \bmod d, \lfloor \frac{\xi}{d} \rfloor},$$

where $\xi = 2nd - 2d^2 + 6d - 2$ and we used (2.13) in the last equality. Note that in the second and third equality we used the fact that $d(n-d+3)$ is the number of the row in C (at level $d-2$) ending in the first auxiliary moment, $A_{d-1,2n-d+2}$, and σ moves this row to row $nd+2-\eta$ $(= \tau+1)$ in H_1 (the first row of H_1 ending in an auxiliary moment). Write

$$\widehat{C} = \begin{pmatrix} C_1 & z \\ z^t & \beta_{0,2n} \end{pmatrix},$$

where C_1 is of size $(\tau-1) \times (\tau-1)$ and $z \in \mathbb{R}^{\tau-1}$. We now have

$$H[\widetilde{A}] = \begin{pmatrix} C_1 & z & h & B_1 \\ z^t & \beta_{0,2n} & \widetilde{A} & B_2 \\ h^t & \widetilde{A} & \lambda & D \\ B_1^t & B_2^t & D^t & E \end{pmatrix}. \tag{5.4}$$

In the sequel we will provide a partial analogue to Theorem 4.1 based on the relative value of λ.

Since $\widehat{C} \succ 0$, $\widehat{C}^{-1}$ has the form

$$\widehat{C}^{-1} = \begin{pmatrix} \mathcal{C} & w \\ w^t & \epsilon \end{pmatrix}, \tag{5.5}$$

where $\mathcal{C} \succ 0$, $\epsilon > 0$, and $w \in \mathbb{R}^{\tau-1}$. Now $\widehat{C}^{-1}v = \begin{pmatrix} \mathcal{C}h + \widetilde{A}w \\ w^th + \widetilde{A}\epsilon \end{pmatrix}$, and we set

$$\widetilde{A}_0 := -\frac{w^th}{\epsilon}, \tag{5.6}$$

so that

$$\widehat{C}^{-1}v = \begin{pmatrix} \mathcal{C}h - \frac{w^thw}{\epsilon} \\ 0 \end{pmatrix}. \tag{5.7}$$

With this value of $\widetilde{A}$ in C, and thus also in v, let

$$\phi := v^t\widehat{C}^{-1}v = h^t\mathcal{C}h - \frac{w^thh^tw}{\epsilon} = h^tC_1^{-1}h, \tag{5.8}$$

where the last equality follows by the same computation as for (4.8) above. Let

$$(5.9)\qquad \widehat{u} := \begin{pmatrix} \widehat{C}^{-1}v \\ -1 \\ 0_{(nd-\tau)\times 1} \end{pmatrix} = \begin{pmatrix} \mathcal{C}h - \frac{w^t h h^t w}{\epsilon} \\ 0 \\ -1 \\ 0_{(nd-\tau)\times 1} \end{pmatrix} \equiv \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{dn} \end{pmatrix},$$

where $u_{\tau+1} = u_{\tau+2} = \cdots = u_{dn} = 0$. A calculation shows that

$$(5.10)\qquad \langle H\widehat{u},\ \widehat{u}\rangle = \lambda - \phi.$$

We next apply the inverse permutation σ^{-1}. Let $\widehat{r} = P_{\sigma^{-1}}\widehat{u} \equiv (r_0, \ldots, r_{dn})^t$. Note that decomposing $\widehat{r}$ in the same way as f in (5.1), the $nd-\tau = \frac{(d-2)(d-1)}{2} - 1$ zeros at the bottom of $\widehat{u}$ correspond to zeros in $\widehat{r}$ as follows: at level k of $\widehat{r}$ ($1 \le k \le d-3$), zeros appear in the high-indexed $d-k-1$ positions of this level, corresponding to the positions of the auxiliary moments A_{ij} at this level of f. Note that the indices of these r_i are precisely those that satisfy

$$i \bmod d + \left\lfloor \frac{i}{d} \right\rfloor > n \quad \text{and} \quad \big(i \bmod d, n + \left\lfloor \frac{i}{d} \right\rfloor\big) \neq (d-1, 2n-d+2),$$

or equivalently, with $\mathcal{F}$ as in (2.9),

$$(5.11)\qquad \big(i \bmod d, n + \left\lfloor \frac{i}{d} \right\rfloor\big) \in \mathcal{F} \setminus \{(d-1, 2n-d+2)\}.$$

Moreover, by the choice of $\widetilde{A}$, $u_{\tau-1} = 0$, (cf. (5.6), (5.9)), and σ^{-1} shifts this 0 to the end of $\widehat{r}$, so we have

$$(5.12)\qquad r_{dn} = 0.$$

Further

$$(5.13)\qquad \langle C\widehat{r},\ \widehat{r}\rangle = \langle H\widehat{u},\ \widehat{u}\rangle\ (= \lambda - \phi).$$

Now suppose $\lambda = \phi$, so that $\langle C\widehat{r},\ \widehat{r}\rangle = 0$. We seek to apply Remark 2.11 to show that the core variety is at most finite in this case. In order to do so with a polynomial $p \in \ker L$ satisfying $p(x, x^d) = R(x)^2$, with $R(x) = r_0 + r_1 x + \cdots r_{nd}x^{nd}$ (using $\widehat{r}$ as described above), we will show that the auxiliary requirements $h_{ij}(\widehat{r}, \widehat{s}) = 0$ of (2.10) are satisfied with this $\widehat{r}$ and using $\widehat{s} \equiv 0$.

Example 5.2. Let $d = 4$. In this case the auxiliary conditions of (2.10) are

$$\begin{aligned} h_{3,2n-1} &= 2(r_{4n}r_{4n-1} + s_{4n}s_{4n-1}) = 0, \\ h_{2,2n-1} &= 2(r_{4n}r_{4n-2} + s_{4n}s_{4n-2}) + r_{4n-1}^2 + s_{4n-1}^2 = 0, \\ h_{3,2n-2} &= 2(r_{4n}r_{4n-5} + r_{4n-1}r_{4n-4} + r_{4n-2}r_{4n-3} + \\ &\qquad + s_{4n}s_{4n-5} + s_{4n-1}s_{4n-4} + s_{4n-2}s_{4n-3}) = 0. \end{aligned}$$

From (5.9) and (5.12) we have $r_{4n} = r_{4n-1} = r_{4n-2} = 0$, and since $\widehat{s} = 0$, it follows that $h_{3,2n-1} = h_{2,2n-1} = h_{3,2n-2} = 0$.

We now turn to the general case of $\lambda = \phi$ for $d \geq 4$, and we again utilize $\widehat{r}$ as described above and $\widehat{s} = 0$. For each $(i, 2n-k) \in \mathcal{F}$ (cf. (2.9)), i.e., for $1 \leq k \leq d-2$ and $k+1 \leq i \leq d-1$, we consider the auxiliary function

$$\begin{aligned} h_{i,2n-k} := r_{nd} r_{(n-k)d+i} + r_{nd-1} r_{(n-k)d+i+1} + \cdots + \\ + r_{nd-p} r_{(n-k)d+i+p} + \cdots + r_{(n-k)d+i} r_{nd}. \end{aligned} \tag{5.14}$$

(Here $0 \leq p \leq kd - i$, so that $nd - p \geq 0$ and $(n-k)d + i + p \leq nd$.) To show that $h_{i,2n-k} = 0$, we will rely on the following result.

Lemma 5.3. *For $(i, 2n-k) \in \mathcal{F}$ (cf. (2.9)) and $0 \leq p \leq kd - i$, we have*

$$r_{nd-p} r_{(n-k)d+i+p} = 0. \tag{5.15}$$

Proof. Let $(i, 2n-k) \in \mathcal{F}$ and let $r_j r_l$ be one of the terms appearing in the sum of (5.14); thus

$$j = nd - p \text{ for some } p,\ 0 \leq p \leq kd - i, \text{ and } l = (n-k)d + i + p. \tag{5.16}$$

We seek to utilize Lemma 2.5, and to this end we let i in Lemma 2.5 coincide with i and let j in Lemma 2.5 correspond to $2n - k$. Let k and l in Lemma 2.5 correspond, respectively, to j and l defined in (5.16). Note that

$$\big((j+l) \bmod d, \lfloor \frac{j+l}{d} \rfloor\big) = (i, 2n-k) \in \mathcal{F}. \tag{5.17}$$

It is not difficult to verify that the values for j and l in (5.16) satisfy the hypotheses of Lemma 2.5: $j \equiv nd - p$, $l \equiv (n-k)d + i + p \leq nd$ and $j + l = i + d(2n-k)$. By Lemma 2.5, one of $J := \big(j \bmod d, n + \left\lfloor \frac{j}{d} \right\rfloor\big) \in \mathcal{F}$ and $L := \big(l \bmod d, n + \left\lfloor \frac{l}{d} \right\rfloor\big) \in \mathcal{F}$ holds. By symmetry with respect to p in (5.14) we may assume that $J \in \mathcal{F}$. If $J \in \mathcal{F} \setminus \{(d-1, 2n-d+2)\}$, then $r_j = 0$ by (5.9) and (5.11).

Now suppose $J = (d-1, 2n-d+2)$, so that $j \bmod d = d-1$ and $\left\lfloor \frac{j}{d} \right\rfloor = n - d + 2$. An examination of (2.9) shows that for every pair $(s,t) \in \mathcal{F}$, we have $s + dt \geq d - 1 + d(2n - d + 2)$. Now, if $l < dn$, then

$$\begin{aligned} j + l = \Big((j \bmod d) + d\lfloor \frac{j}{d} \rfloor\Big) + l < (d-1) + d(n-d+2) + dn \\ = (d-1) + d(2n-d+2). \end{aligned} \tag{5.18}$$

With $s = (j+l) \bmod d$ and $t = \lfloor \frac{j+l}{d} \rfloor$, we have $s + dt = j + l$, so (5.18) implies that $\big((j+l) \bmod d, \left\lfloor \frac{j+l}{d} \right\rfloor\big) \notin \mathcal{F}$, contradicting (5.17). Therefore $l = dn$, in which case $r_l = 0$ by (5.12). $\square$

Theorem 5.4. *Let $d \geq 4$.*

(i) If $\lambda = \phi$, then $\mathcal{CV}(L_\beta)$ is finite or empty.

(ii) If $\lambda = \phi$ and $d = 4$, then there is no representing measure.

(iii) If $\lambda < \phi$, then there is no representing measure.

Proof. Let Δ denote the curve $y = x^d$. We first consider the case when $\lambda = \phi$, so that $\langle H\widehat{u},\ \widehat{u}\rangle = 0$ by (5.10). When we reverse the permutation σ described above to produce vector $\widehat{r}$, we have $\langle C\widehat{r},\ \widehat{r}\rangle = 0$ by (5.13). Let $\widehat{s} = 0$. Lemma 5.3 now shows that all of the auxiliary conditions (2.10) are satisfied. Following Remark 2.11 and (2.7), define $a_{ij} = h_{ij}(\widehat{r}, \widehat{s})$ $(0 \le i \le d-1,\ i, j \ge 0,\ 0 < i + j \le 2n)$. Then $P(x, y) := \sum a_{ij} f_{ij}$ is an element of $\ker L_\beta$ which satisfies $Q(x) := P(x, x^d) = R(x)^2$, where $R(x) := r_0 + r_1 x + \cdots + r_{dn-1} x^{dn-1} + r_{dn} x^{dn}$. Since $r_{dn} = r_{dn-1} = \cdots = r_{dn-(d-2)} = 0$, $R(x)$ has at most $dn - (d-1)$ real zeros, so P has at most $dn - (d-1)$ zeros in the curve Δ. Since $P \in \ker L_\beta$ satisfies $P|\Delta \ge 0$ and $\operatorname{card} \mathcal{Z}(P|\Delta) \le dn - (d-1)$, it follows that $\operatorname{card} \mathcal{CV}(L_\beta) \le dn - (d-1)$. In the case $d = 4$,

$$dn - (d-1) = 4n - 3 < 4n - 2 = \operatorname{rank} M_n,$$

so Corollary 1.4 implies that β has no representing measure.

In the case where $\lambda < \phi$, we may construct H, $\widehat{u}$ and then $\widehat{r}$ and $R(x)$ as in the preceding case, but now $\langle C\widehat{r},\ \widehat{r}\rangle = \langle H\widehat{u},\ \widehat{u}\rangle = \lambda - \phi < 0$. Let $\epsilon = (\phi - \lambda)^{1/2}$. Since $\langle \widehat{C} e_1,\ e_1\rangle = \beta_{00} = 1$, then the constant polynomial $S(x) = \epsilon$, with coefficient vector $\widehat{s} = (\epsilon, 0, \ldots, 0)^t$, satisfies $\langle C\widehat{r},\ \widehat{r}\rangle + \langle C\widehat{s},\ \widehat{s}\rangle = 0$. As in the first paragraph, $\widehat{r}$ satisfies (5.15), while from definition of $\widehat{s}$, the analogue of (5.15) for $\widehat{s}$ clearly holds. So $\widehat{r}$ and $\widehat{s}$ together satisfy the auxiliary requirements of (2.10). Constructing $p(x, y)$ as in Remark 2.11, (2.10) shows that $p \in \ker L_\beta$. Now, $p(x, x^d) = R(x)^2 + S(x)^2 \ge \epsilon^2 > 0$. Since p is strictly positive on Δ, then $\mathcal{CV}(L_\beta) = \emptyset$, and therefore β has no representing measure. □

Remark 5.5. (i) The content of Remark 4.2 about the alternate proof of the case $\beta_{1,2n-1} < \phi$ for $d = 3$ extends to the case $\lambda < \phi$ in Theorem 5.4. Let $\widetilde{A}_0$ be as in (5.6). Let $\phi[\widetilde{A}]$ denote ϕ as in (5.8) where we emphasize the dependence on $\widetilde{A}$. If $\lambda < \phi[\widetilde{A}_0]$, then (5.8) implies that $\lambda < h^t C_1^{-1} h$. It therefore follows from (5.4) that for every $\widetilde{A} \in \mathbb{R}$, the matrix $\begin{pmatrix} C_1 & h \\ h^t & \lambda \end{pmatrix}$ is a principal submatrix of $H[\widetilde{A}]$ that is independent of $\widetilde{A}$ and not positive semidefinite. Thus, for every $\widetilde{A}$, $H[\widetilde{A}]$, and hence $C[\widetilde{A}]$, is not positive semidefinite, so Theorem 2.13 implies that β has no representing measure.
(ii) In Section 6, we show that there exists a $(y - x^4)$–pure sequence with a unique representing measure and therefore a finite core variety (see Example 6.6.v)). By Theorem 5.4.(ii), it follows that $\lambda > \phi$ in every such example.

Theorem 5.4 suggests the following question.

Question 5.6. *For $d \ge 5$, if $\lambda = \phi$, is it possible for the core variety to be nonempty?*

6. The $(y - x^4)$-pure truncated moment problem

In this section we establish a complete solution to the moment problem for $\beta \equiv \beta^{(2n)}$ where M_n is positive semidefinite and $(y - x^4)$-pure (see Theorem 6.3). In addition to the core variety approach, we also use the method of [Z1] involving positive completions of partially defined Hankel matrices (see Lemma 6.1 and Theorem 6.2).

Note that (cf. (2.28))

$$\operatorname{rank} M_n = 4n - 2. \tag{6.1}$$

Recall (cf. Example 2.7) that the core matrix C of $\beta \equiv \beta^{(2n)}$ has three auxiliary moments, i.e., $\beta_{3,2n-2}$, $\beta_{2,2n-1}$, $\beta_{3,2n-1}$, which we denote by $\mathbf{A}_{3,2n-2}, \mathbf{A}_{2,2n-1}, \mathbf{A}_{3,2n-1}$.

Convention: In what follows we write $\mathbf{A}_{ij}$ in bold whenever the auxiliary moment is meant as a variable. When we use a non-bold notation A_{ij} we mean a specific value of the variable $\mathbf{A}_{ij}$.

Recall from Examples 2.7 that the rows and columns of the core matrix

$$C \equiv C[\mathbf{A}_{3,2n-2}, \mathbf{A}_{2,2n-1}, \mathbf{A}_{3,2n-1}]$$

are indexed by the ordered set

$$\mathcal{B} := \{\mathit{1}, X, X^2, X^3, Y, XY, X^2Y, X^3Y, \ldots, Y^k, XY^k, X^2Y^k, X^3Y^k, \ldots, \\ Y^{n-1}, XY^{n-1}, X^2Y^{n-1}, X^3Y^{n-1}, Y^n\}.$$

For $1 \le k \le 4n+1$ let

$$I_k := (k-1) \bmod 4 \quad \text{and} \quad J_k := \lfloor \frac{(k-1)}{4} \rfloor \tag{6.2}$$

(cf. (3.2) with $d = 4$). Let $A_{3,2n-2}, A_{2,2n-1}, A_{3,2n-1} \in \mathbb{R}$ be such that

$$C \equiv C[A_{3,2n-2}, A_{2,2n-1}, A_{3,2n-1}]$$

satisfies a column relation

$$X^{I_k}Y^{J_k} = \sum_{i=1}^{k-1} \varphi_i X^{I_i} Y^{J_i} \quad \text{for some} \quad 2 \le k \le 4n+1 \text{ and } \varphi_i \in \mathbb{R}. \tag{6.3}$$

We say that the column relation (6.3) *propagates through* C if the relations

$$X^{I_{k+\ell}}Y^{J_{k+\ell}} = \sum_{i=1}^{k-1} \varphi_i X^{I_{i+\ell}} Y^{J_{i+\ell}} \quad \text{for} \quad \ell = 1, \ldots, 4n+1-k \tag{6.4}$$

also represent column relations of C.

In what follows we will need a notion of a Schur complement. Let $M = \begin{pmatrix} A & a \\ a^t & \alpha \end{pmatrix}$ be a real matrix where $A \in \mathbb{R}^{(m-1)\times(m-1)}$ is invertible, $a \in \mathbb{R}^{m-1}$ and $\alpha \in \mathbb{R}$. The *Schur complement* of A in M is defined by $M/A = \alpha - a^t A^{-1} a$.

To prove the main result of this section (see Theorem 6.3 below), we will need the following two results from [Z1]. The first is about the existence of a positive completion of a partially defined Hankel matrix (see Lemma 6.1), while the other is about the existence of a measure for a univariate sequence with two missing entries (see Theorem 6.2).

Lemma 6.1 ([Z1, Special case of Lemma 2.11]). *Let $m \in \mathbb{N}$, $m \geq 3$ and*

$$A(\mathbf{x}) := \begin{pmatrix} A_1 & a & b \\ a^T & \alpha & \mathbf{x} \\ b^T & \mathbf{x} & \beta \end{pmatrix}$$

be a real symmetric $m \times m$ matrix, where A_1 is a real symmetric $(m-2) \times (m-2)$ matrix, $a, b \in \mathbb{R}^{m-2}$, $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}$ is a variable. Assume that A_1 is positive definite and the submatrices $A_2 := \begin{pmatrix} A_1 & a \\ a^T & \alpha \end{pmatrix}$, $A_3 := \begin{pmatrix} A_1 & b \\ b^T & \beta \end{pmatrix}$ of $A(\mathbf{x})$ are positive semidefinite. Let

$$x_\pm := b^T A_1^{-1} a \pm \sqrt{(A_2/A_1)(A_3/A_1)} \in \mathbb{R}.$$

Then:

(i) *$A(x_0)$ is positive semidefinite if and only if $x_0 \in [x_-, x_+]$.*
(ii) *If $x_0 \in \{x_-, x_+\}$, then* $\operatorname{rank} A(x_0) = \max\big\{\operatorname{rank} A_2, \operatorname{rank} A_3\big\}$.
(iii) *If $x_0 \in (x_-, x_+)$, then* $\operatorname{rank} A(x_0) = \max\big\{\operatorname{rank} A_2, \operatorname{rank} A_3\big\} + 1$.

Theorem 6.2 ([Z1, Special case of Theorem 3.5]). *Let $m \in \mathbb{N}$, $m > 3$, and*

$$\gamma(\mathbf{x}, \mathbf{y}) := (\gamma_0, \gamma_1, \ldots, \gamma_{2m-3}, \mathbf{y}, \mathbf{x}, \gamma_{2m})$$

be a sequence, where each γ_i is a real number, $\gamma_0 > 0$ and $\mathbf{x}, \mathbf{y}$ are variables. Assume that the Hankel matrices $H_1 := (\gamma_{i+j-1})_{1 \leq i,j \leq m-3}$ and $H_2 = (\gamma_{i+j-1})_{1 \leq i,j \leq m-2}$ are positive definite. Then the following statements are equivalent:

(i) *There exist $x_0, y_0 \in \mathbb{R}$ such that $\gamma(x_0, y_0)$ admits a representing measure supported in $\mathbb{R}$.*
(ii) *The matrix $\widetilde{A} := \begin{pmatrix} H_1 & u \\ u^t & \gamma_{2m} \end{pmatrix}$, where $u^T := \begin{pmatrix} \gamma_m & \cdots & \gamma_{2m-3} \end{pmatrix}$, is positive semidefinite and the inequality*

$$sH_2^{-1}s^T \leq uH_1^{-1}w^T + \sqrt{(H_2/H_1)(\widetilde{A}/H_1)} \tag{6.5}$$

holds, where $s^T := \begin{pmatrix} \gamma_{m-1} & \cdots & \gamma_{2m-3} \end{pmatrix}$, $w^T := \begin{pmatrix} \gamma_{m-2} & \cdots & \gamma_{2m-5} \end{pmatrix}$.

Next we introduce five submatrices of C which occur in the statement of the solution to the $(y - x^4)$-pure TMP. For $\mathcal{S} \subset \mathcal{B}$ we denote by $C|_{\mathcal{S}}$ the restriction of C to rows and columns indexed by elements from $\mathcal{S}$. Let

$$\begin{aligned}
\mathcal{S}_1 &:= \mathcal{B} \setminus \{XY^{n-1}, X^2Y^{n-1}, X^3Y^{n-1}, Y^n\}, \\
\mathcal{S}_2 &:= \mathcal{B} \setminus \{X^2Y^{n-1}, X^3Y^{n-1}, Y^n\}, \\
\mathcal{S}_3 &:= \mathcal{B} \setminus \{XY^{n-1}, X^3Y^{n-1}, Y^n\}, \\
\mathcal{S}_4 &:= \mathcal{B} \setminus \{X^3Y^{n-1}, Y^n\}, \\
\mathcal{S}_5 &:= \mathcal{B} \setminus \{X^2Y^{n-1}, X^3Y^{n-1}\}
\end{aligned}$$

and $C_i := C|_{\mathcal{S}_i}$ for each i. Note that C_1, C_2, C_3 are completely determined by β, and we have

$$
\begin{aligned}
C_2 &= \begin{pmatrix} C_1 & u \\ u^t & \beta_{2,2n-2} \end{pmatrix}, \quad C_3 = \begin{pmatrix} C_1 & v \\ v^t & \beta_{0,2n-1} \end{pmatrix}, \\
C_4[\mathbf{A}_{3,2n-2}] &= \begin{pmatrix} C_2 & w \\ w^t & \beta_{0,2n-1} \end{pmatrix} \quad \text{with} \quad w = \begin{pmatrix} w_1 \\ \mathbf{A}_{3,2n-2} \end{pmatrix}, \\
C_5[\mathbf{A}_{3,2n-2}] &= \begin{pmatrix} C_2 & z \\ z^t & \beta_{0,2n} \end{pmatrix} \quad \text{with} \quad z^t = \begin{pmatrix} z_1^t & \mathbf{A}_{3,2n-2} & \beta_{0,2n-1} & \beta_{1,2n-1} \end{pmatrix},
\end{aligned} \tag{6.6}
$$

where u, v, w_1, z_1 are independent of the auxiliary moments. Assume that C_2 is positive definite. (Note that by Theorem 6.3 below this is a necessary condition for the existence of a representing measure for β.) Using $C_4[\mathbf{A}_{3,2n-2}]$ as $A(\mathbf{x})$ in Lemma 6.1, it follows that $C_4[A_{3,2n-2}]$ is positive semidefinite if and only if $A_{3,2n-2} \in \big[(A_{3,2n-2})_-, (A_{3,2n-2})_+\big]$, where

$$
\begin{aligned}
(A_{3,2n-2})_- &= v^t C_1^{-1} u - \sqrt{(C_2/C_1)(C_3/C_1)}, \\
(A_{3,2n-2})_+ &= v^t C_1^{-1} u + \sqrt{(C_2/C_1)(C_3/C_1)}.
\end{aligned} \tag{6.7}
$$

Moreover, Lemma 6.1.(ii) implies that the last column of $C_4[(A_{3,2n-2})_-]$ is linearly dependent on the previous columns:

$$
\begin{aligned}
X^2Y^{n-1} &= \varphi_1^{(-)} \mathit{1} + \varphi_2^{(-)} X + \varphi_3^{(-)} X^2 + \ldots + \varphi_{4n-3}^{(-)} Y^{n-1} + \varphi_{4n-2}^{(-)} XY^{n-1} \\
&= \sum_{i=1}^{4n-2} \varphi_i^{(-)} X^{I_i} Y^{J_i} \quad \text{for some} \quad \varphi_i^{(-)} \in \mathbb{R}.
\end{aligned} \tag{6.8}
$$

Similarly, in $C_4[(A_{3,2n-2})_+]$ we have

$$
\begin{aligned}
X^2Y^{n-1} &= \varphi_1^{(+)} \mathit{1} + \varphi_2^{(+)} X + \varphi_3^{(+)} X^2 + \ldots + \varphi_{4n-3}^{(+)} Y^{n-1} + \varphi_{4n-2}^{(+)} XY^{n-1} \\
&= \sum_{i=1}^{4n-2} \varphi_i^{(+)} X^{I_i} Y^{J_i} \quad \text{for some} \quad \varphi_i^{(+)} \in \mathbb{R}.
\end{aligned} \tag{6.9}
$$

Let $[X^iY^j]_{X^kY^l}$ be the entry in the row X^kY^l of the column X^iY^j of C. Assuming (6.8) and (6.9) we also define

$$
(A_{2,2n-1})_- = \sum_{i=1}^{4n-2} \varphi_i^{(-)} [X^{I_i} Y^{J_i}]_{Y^n}, \; (A_{3,2n-1})_- = \sum_{i=1}^{4n-2} \varphi_i^{(-)} [X^{I_{i+1}} Y^{J_{i+1}}]_{Y^n} \tag{6.10}
$$

and

$$
(A_{2,2n-1})_+ = \sum_{i=1}^{4n-2} \varphi_i^{(+)} [X^{I_i} Y^{J_i}]_{Y^n}, \; (A_{3,2n-1})_+ = \sum_{i=1}^{4n-2} \varphi_i^{(+)} [X^{I_{i+1}} Y^{J_{i+1}}]_{Y^n}. \tag{6.11}
$$

Note that in the definitions of $(A_{3,2n-1})_\pm$ we used that $[X^2Y^{n-1}]_{Y^n} = (A_{2.2n-1})_\pm$, and hence $(A_{2,2n-1})_\pm$ needs to be defined before $(A_{3,2n-1})_\pm$ in (6.10), (6.11). In the sequel, for the case when $A_{3,2n-2} = (A_{3,2n-2})_-$, (6.10) is used to define $A_{2,2n-1}$ so that the relation (6.8) becomes a full column relation in C. (Similarly for $A_{3,2n-2} = (A_{3,2n-2})_+$, (6.11), (6.9).)

Recall that for $\mathcal{S} \subseteq \mathcal{B}$ we denote by $[X^iY^j]_{\mathcal{S}}$ the restriction of the column X^iY^j of C to the rows indexed by elements of $\mathcal{S}$. The solution to the $(y-x^4)$-pure TMP is the following.

Theorem 6.3. *Suppose M_n is positive semidefinite and $(y-x^4)$-pure. Assume the notaion above. $\beta \equiv \beta^{(2n)}$ has a representing measure if and only if the following conditions hold:*

(i) C_2 is positive definite.

(ii) C_3 is positive semidefinite.

(iii) One of the following statements holds:

- *(a) The relation (6.8) propagates through $C[(A_{3,2n-2})_-, (A_{2,2n-1})_-, (A_{3,2n-1})_-]$.*
- *(b) The relation (6.9) propagates through $C[(A_{3,2n-2})_+, (A_{2,2n-1})_+, (A_{3,2n-1})_+]$.*
- *(c) There exists*

$$A_{3,2n-2} \in \big((A_{3,2n-2})_-, (A_{3,2n-2})_+\big) \tag{6.12}$$

such that

$$\delta \leq \rho, \tag{6.13}$$

where

$$\begin{aligned} \delta &:= \big([X^3Y^{n-1}]_{\mathcal{S}_4}\big)^t \big(C_4[A_{3,2n-2}]\big)^{-1}[X^3Y^{n-1}]_{\mathcal{S}_4}, \\ \rho &:= ([Y^n]_{\mathcal{S}_2})^t C_2^{-1}[X^2Y^{n-1}]_{\mathcal{S}_2} + \sqrt{(C_4[A_{3,2n-2}]/C_2)(C_5[A_{3,2n-2}]/C_2)}. \end{aligned} \tag{6.14}$$

Remark 6.4. (i) Before we prove Theorem 6.3, let us briefly explain how it is related to Lemma 6.1 and Theorem 6.2. Theorem 6.3.(i) comes from the assumption that M_n is $(y-x^4)$–pure, while Theorem 6.3.(ii) from Theorem 2.13. The condition $A_{3,2n-2} \in \big[(A_{3,2n-2})_-, (A_{3,2n-2})_+\big]$ comes from Lemma 6.1 as explained in the paragraph before Theorem 6.3. For

$$A_{3,2n-2} \in \{(A_{3,2n-2})_-, (A_{3,2n-2})_+\},$$

flatness of $C_4[(A_{3,2n-2})_\pm]$ implies that β has a representing measure if and only if $C[(A_{3,2n-2})_\pm, (A_{2,2n-1})_\pm, (A_{3,2n-1})_\pm]$ is a flat extension of C_4, which is equivalent to one of Theorem 6.3.(iiia) or Theorem 6.3.(iiib). For the remaining cases $A_{3,2n-2} \in \big((A_{3,2n-2})_-, (A_{3,2n-2})_+\big)$ we use Theorem 6.2 for the univariate sequence

$$\gamma = \gamma^{(8n)} \equiv \{\gamma_k\}_{k=0}^{8n}, \quad \text{where} \quad \gamma_k := \begin{cases} \beta_{I_{k+1}J_{k+1}}, & \text{if } I_{k+1}+J_{k+1} \leq 2n, \\ A_{3,2n-2}, & \text{if } I_{k+1}=3, J_{k+1}=2n-2, \\ \mathbf{A}_{2,2n-1}, & \text{if } I_{k+1}=2, J_{k+1}=2n-1, \\ \mathbf{A}_{3,2n-1}, & \text{if } I_{k+1}=3, J_{k+1}=2n-1, \end{cases}$$

to obtain Theorem 6.3.(iiic).

(ii) Observing the proof of Theorem 6.2 in [Z1], it turns out that for $A_{3,2n-2}$ satisfying (6.12), the inequality (6.13) is equivalent to the existence of a positive semidefinite completion of $C[A_{3,2n-2}, \mathbf{A_{2,2n-1}}, \mathbf{A_{3,2n-1}}]$, and is therefore equivalent to the existence of a representing measure μ for β by Theorem 2.13. Namely, δ comes from the submatrix $C|_{\mathcal{B}\setminus\{Y^n\}}$ and is the lower bound on the auxiliary moment $A_{2,2n-1}$, so that $C|_{\mathcal{B}\setminus\{Y^n\}}$ is positive semidefinite. On the other hand ρ comes from the submatrix $C|_{\mathcal{B}\setminus\{X^3Y^{n-1}\}}$ and is the upper bound on the auxiliary moment $A_{2,2n-1}$, so that $C|_{\mathcal{B}\setminus\{X^3Y^{n-1}\}}$ is positive semidefinite. Consequently, if there is only one $A_{3,2n-2}$ satisfying (6.12) and (6.13), then μ is the unique representing measure for β arising from Theorem 6.2. We will show below that μ is actually the unique representing measure for β. Note first that for this unique choice of $A_{3,2n-2} \in \big((A_{3,2n-2})_-, (A_{3,2n-2})_+\big)$, there must be equality in (6.13) because of the continuity of the condition of being positive definite. If we had a strict inequality in (6.13) for this $A_{3,2n-2}$, then a slightly perturbed $A_{3,2n-2}$ would still satisfy (6.12) and (6.13), yielding a different measure.

Concerning uniqueness, suppose as above that $A_{3,2n-2}$ uniquely satisfies (6.12) and (6.13), and let $A_{2,2n-1}$ and $A_{3,2n-1}$ be such that

$$C_{un} := C[A_{3,2n-2}, A_{2,2n-1}, A_{3,2n-1}]$$

is positive semidefinite, with corresponding measure μ. We claim that no measure can arise as in Theorem 6.3.(iiia). Let

$$C_- := C[(A_{3,2n-2})_-, (A_{2,2n-1})_-, (A_{3,2n-1})_-].$$

If (iiia) holds, then C_- is positive semidefinite, so

$$\frac{1}{2}(C_- + C_{un}) =$$
$$= C\Big[\frac{(A_{3,2n-2})_- + A_{3,2n-2}}{2}, \frac{(A_{2,2n-1})_- + A_{2,2n-1}}{2}, \frac{(A_{3,2n-1})_- + A_{3,2n-1}}{2}\Big]$$

is positive semidefinite as well. But then

$$\delta\big(\frac{(A_{3,2n-2})_- + A_{3,2n-2}}{2}\big) \le \rho\big(\frac{(A_{3,2n-2})_- + A_{3,2n-2}}{2}\big)$$

and $\frac{(A_{3,2n-2})_- + A_{3,2n-2}}{2}$ also satisfies (6.12) and (6.13), which is a contradiction with the uniqueness of $A_{3,2n-2}$. So C_- is not positive semidefinite and does not admit a representing measure. Analogously, the same holds for

$$C_+ := C[(A_{3,2n-2})_+, (A_{2,2n-1})_+, (A_{3,2n-1})_+].$$

We now conclude that μ is the unique representing measure for β. In Example 6.6(v) below we show that there *are* pure sequences with a unique representing measure, as just described.

Proof of Theorem 6.3. Let Γ be the curve $y = x^4$. First we prove the implication $(\Rightarrow)$. Assume that β has a representing measure μ. By Theorem 1.5, β admits a finitely atomic representing measure μ, necessarily supported in Γ. Let $\widetilde{\beta} \equiv \beta^{(2n+4)} = \{\widetilde{\beta}_{ij} : i, j \ge 0, i + j \le 2n + 4\}$ be the extension of β generated by μ. By Proposition

2.9, $C \equiv C[\widetilde{\beta}_{3,2n-2}, \widetilde{\beta}_{2,2n-1}, \widetilde{\beta}_{3,2n-1}]$ is positive semidefinite, which implies that C_2 and C_3 are positive semidefinite.

Next we show that C_2 is positive definite. Assume that C_2 is not definite. Then there is a column relation in C_2 of the form $\sum_{k=0}^{4n-3} r_k X^{I_{k+1}} Y^{J_{k+1}} = \mathbf{0}$ for some $r_k \in \mathbb{R}$, not all equal to 0. By the extension principle [F1, Proposition 2.4], this column relation must also hold in M_{n+1} and in particular in C. Hence, $\widehat{r} = (r_0, r_1, \ldots, r_{4n-3}, 0, 0, 0)^t \in \ker C$. Thus, $\langle C\widehat{r}, \widehat{r}\rangle + \langle C\widehat{s}, \widehat{s}\rangle = 0$, where $\widehat{s}$ is the zero vector, and the auxiliary conditions (2.10) (see Example 5.2) are satisfied. Following Remark 2.11 and (2.7), define $a_{ij} = h_{ij}(\widehat{r}, \widehat{s})$ $(0 \le i \le 3,\ i, j \ge 0,\ 0 < i + j \le 2n)$ to obtain $P(x, y) := \sum a_{ij} f_{ij} \in \ker L_\beta$ and $P(x, x^4) = \big(\sum_{k=0}^{4n-3} r_k x^k\big)^2$. Since $P(x, x^4)$ has at most $4n-3$ real zeros, P has at most $4n-3$ zeros in the curve $y = x^4$. It follows that $\operatorname{card} \mathcal{CV}(L_\beta) \le 4n-3$, which contradicts $4n-2 = \operatorname{rank} M_n \le \operatorname{card} \mathcal{CV}(L_\beta)$, where the equality is (6.1). Hence, C_2 must be definite.

It remains to prove (iii). Let $A_{3,2n-2} = \widetilde{\beta}_{3,2n-2}$, $A_{2,2n-1} = \widetilde{\beta}_{2,2n-1}$, $A_{3,2n-1} = \widetilde{\beta}_{3,2n-1}$, where $\widetilde{\beta}$ and C are as in the first paragraph above. Since C is positive semidefinite, so also is its central compression $C_4[A_{3,2n-2}]$. Using $C_4[\mathbf{A}_{3,2n-2}]$ as $A(\mathbf{x})$ in Lemma 6.1, it follows that $A_{3,2n-2} \in \big[(A_{3,2n-2})_-, (A_{3,2n-2})_+\big]$ where $(A_{3,2n-2})_\pm$ are as in (6.7). We separate three cases according to the value of $A_{3,2n-2}$.

Case 1: $A_{3,2n-2} = (A_{3,2n-2})_-$. Then in $C_4[(A_{3,2n-2})_-]$ there is a column relation of the form (6.8). By the extension principle [F1, Proposition 2.4], this column relation also holds in C. In particular, observing row Y^n, it follows that $A_{2,2n-1} = (A_{2,2n-1})_-$ (cf. (6.10)). Since M_{n+1} is recursively generated, this column relation propagates through $C[(A_{3,2n-2})_-, (A_{2,2n-1})_-, (A_{3,2n-1})_-]$ in the sense of (6.4). Therefore $X^3Y^{n-1} = \sum_{i=1}^{4n-2} \varphi_i^{(-)} X^{I_{i+1}} Y^{L_{i+1}}$ holds. Observing row Y^n, it follows that $A_{3,2n-1} = (A_{3,2n-1})_-$ (cf. (6.10)). So $C[(A_{3,2n-2})_-, (A_{2,2n-1})_-, (A_{3,2n-1})_-]$ has the propagating relation (6.8), which is (iiia).

Case 2: $A_{3,2n-2} = (A_{3,2n-2})_+$. The proof is analogous to the case $A_{3,2n-2} = (A_{3,2n-2})_-$, implying (iiib) holds.

Case 3: (6.12) **holds.** Note that the existence of a representing measure for β is equivalent to the existence of a representing measure for a univariate sequence

$$(6.15) \qquad \gamma = \gamma^{(8n)} \equiv \{\gamma_k\}_{k=0}^{8n}, \quad \text{where} \quad \gamma_k := \begin{cases} \beta_{I_{k+1}J_{k+1}}, & \text{if } I_{k+1} + J_{k+1} \le 2n, \\ A_{I_{k+1}J_{k+1}}, & \text{if } I_{k+1} + J_{k+1} > 2n. \end{cases}$$

Indeed, $\sum_{\ell=1}^{r} \rho_\ell \delta_{(x_\ell, x_\ell^4)}$ is a representing measure for β if and only if $\sum_{\ell=1}^{r} \rho_\ell \delta_{x_\ell}$ is a representing measure for γ (cf. (6.2) and the proof of Theorem 2.13). Finally, (iiic) follows by applying Theorem 6.2 to γ. Namely, γ corresponds to $\gamma(x_0, y_0)$; $A_{2,2n-1}$, $A_{3,2n-1}$ to x_0, y_0, respectively; $C_4[A_{3,2n-2}]$ to H_2; $C_5[A_{3,2n-2}]$ to $\widetilde{A}$; $[X^3Y^{n-1}]_{\mathcal{S}_4}$ to s; $[Y^n]_{\mathcal{S}_2}$ to u^t; $[X^2Y^{n-1}]_{\mathcal{S}_2}$ to w^t; and C_2 to H_1.

This concludes the proof of the implication $(\Rightarrow)$.

It remains to prove the implication ($\Leftarrow$). Let γ be as in (6.15). By Lemma 6.1 with $C_4[\mathbf{A}_{3,2n-2}]$ as $A(\mathbf{x})$, (i) and (ii) imply that for $A_{3,2n-2} \in \big[(A_{3,2n-2})_-, (A_{3,2n-2})_+\big]$, the submatrix $C_4[A_{3,2n-2}]$ is positive semidefinite. We separate three cases acccording to the assumption in Theorem 6.3:

If (iiia) holds, then $(\gamma_{i+j-1})_{1\leq i+j\leq 8k-6}$ is positive definite and

$$\mathrm{rank}(\gamma_{i+j-1})_{1\leq i+j\leq 8k} = \mathrm{rank}(\gamma_{i+j-1})_{1\leq i+j\leq 8k-6}.$$

Therefore $C[(A_{3,2n-2})_-, (A_{2,2n-1})_-, (A_{3,2n-1})_-]$ is positive semidefinite and recursively generated, so Theorem 2.13 implies that β has a representing measure.

If (iiib) holds, then the proof is analogous to the proof in the case (iiia) above.

Finally, if (iiic) holds, then Theorem 6.2 implies the existence of $A_{3,2n-2}$, $A_{2,2n-1}$, $A_{3,2n-1}$, such that $(\gamma_{i+j-1})_{1\leq i,j\leq 8k}$ as in (6.15) is positive semidefinite and recursively generated, whence the same is true for $C[A_{3,2n-2}, A_{2,2n-1}, A_{3,2n-1}]$. By Theorem 2.13, β has a representing measure.

This concludes the proof of the implication ($\Leftarrow$). $\square$

Remark 6.5. Let us comment the type of the inequality in (6.13) when regarding $A_{3,2n-2}$ as a variable $\mathbf{A}_{3,2n-2}$ in (6.14).

First we observe the left hand side of (6.13). $[X^3Y^{n-1}]_{\mathcal{S}_4}$ has one coordinate equal to $\mathbf{A}_{3,2n-2}$, while by [F2, p. 3144], $\big(C_4[\mathbf{A}_{3,2n-2}]\big)^{-1}$ is equal to

$$\big(C_4[\mathbf{A}_{3,2n-2}]\big)^{-1} = \begin{pmatrix} C_2^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{C_4[\mathbf{A}_{3,2n-2}]/C_2}\begin{pmatrix} C_2^{-1}ww^TC_2^{-T} & -C_2^{-1}w \\ -w^TC_2^{-T} & 1 \end{pmatrix},$$

where $w = \begin{pmatrix} w_1 \\ \mathbf{A}_{3,2n-2} \end{pmatrix}$ and $C_4/C_2 = \beta_{0,2n-1} - w^TC_2^{-1}w$. So the left hand side of (6.13) is a rational function in $\mathbf{A}_{3,2n-2}$ with the numerator being of degree 4, while the denominator of degree 2.

Now observe the right hand side of (6.13). Note that all terms

$$\big([Y^n]_{\mathcal{S}_2}\big)^t C_2^{-1}[X^2Y^{n-1}]_{\mathcal{S}_2}, \quad C_4[\mathbf{A}_{3,2n-2}]/C_2 \quad \text{and} \quad C_5[\mathbf{A}_{3,2n-2}]/C_2$$

are quadratic in $\mathbf{A}_{3,2n-2}$. Hence, the right hand side is a sum of a quadratic polynomial and square root of a quartic one.

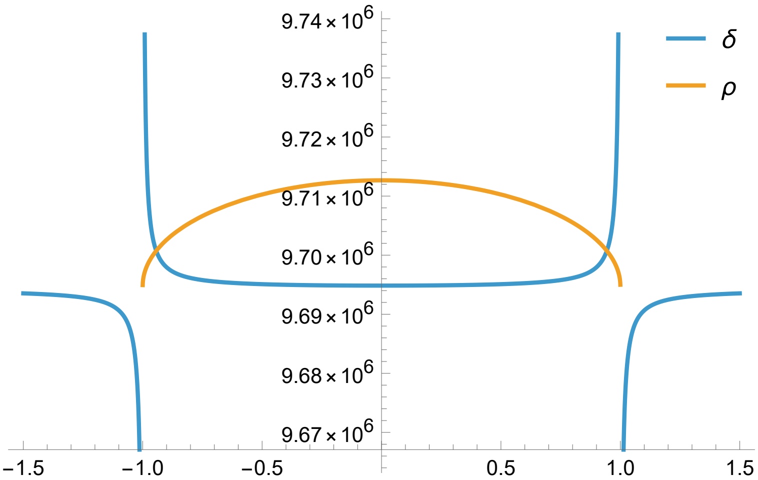
The following examples demonstrate the statements of Theorem 6.3.

Example 6.6. i) Let $\beta \equiv \beta^{(8)}$ as in Example 2.18. It is straightforward to verify that C_2 and C_3 from (6.6) are positive definite. A computation of $(A_{36})_\pm$ by (6.7) gives $(A_{36})_\pm = \pm 1$. Further, a computation of $\delta(\mathbf{A}_{36}), \rho(\mathbf{A}_{36})$ by (6.14) gives

$$\delta(\mathbf{A}_{36}) = \frac{-9694844 + 9694114\mathbf{A}_{36}^2 + \mathbf{A}_{36}^4}{-1 + \mathbf{A}_{36}^2},$$

$$\rho(\mathbf{A}_{36}) = 9694830 - 12\mathbf{A}_{36}^2 + \sqrt{(-1 + \mathbf{A}_{36}^2)(-318219264 + 145\mathbf{A}_{36}^2)}.$$

It turns out that $\delta(A_{36}) = \rho(A_{36})$ for $A_{36,l} \approx -0.943353$, $A_{36,u} \approx 0.943353$.



So the choices for A_{36} satisfying (6.12) and (6.13) are the ones lying on the interval $[A_{36,l}, A_{36,u}]$. By Theorem 6.3, this confirms the existence of a measure for β, in agreement with the conclusion in Example 2.18. Note also that $C[(A_{3,6})_-, (A_{2,7})_-, (A_{3,7})_-]$ cannot admit a representing measure, since this would imply, by a convexity argument analogous to the one from Remark 6.4.(ii), that $\delta\big(\frac{(A_{36})_- + A_{36,l}}{2}\big) \leq \rho\big(\frac{(A_{36})_- + A_{36,l}}{2}\big)$, which is not true. Similarly, $C[(A_{3,6})_+, (A_{2,7})_+, (A_{3,7})_+]$ does not admit a representing measure.

ii) Let $\beta \equiv \beta^{(8)}$ be as in Example 2.19. Since C_2 from (6.6) is not positive definite, this violates Theorem 6.3.(i), whence the measure for β does not exist, in agreement with the conclusion in Example 2.19.

iii) Let $\beta \equiv \beta^{(8)}$ be as in Example 2.18, except for changing β_{07} to $\beta_{07} = 0$ and β_{25} to $\beta_{25} = 2640503382173370698906776695725$. It is straightforward to verify that $\widehat{C}$ is positive definite, which implies, by Proposition 3.7, that $M_4(\beta)$ is positive semidefinite and $(y - x^4)$-pure. Further, C_2 from (6.6) is positive definite, while C_3 is not positive semidefinite. This violates Theorem 6.3.(ii), whence a measure for β does not exist.

iv) Let $\beta \equiv \beta^{(8)}$ be as in Example 2.18 with $\beta_{17} = 150$ instead of $\beta_{17} = 0$. Since $\widehat{C}$ is positive definite, Proposition 3.7 implies that $M_4(\beta)$ is positive semidefinite and $(y - x^4)$-pure. Further, C_2 and C_3 from (6.6) are positive definite, which satisfies (i) and (ii) of Theorem 6.3. It turns out that $(A_{36})_- = -1$, $(A_{36})_+ = 1$ $(A_{27})_- = 9694668$, $(A_{27})_+ = 9694968$, $(A_{37})_- = 2074$, $(A_{37})_+ = 2126$ and the relations (6.8) and (6.9) are

$$\begin{aligned}
X^2Y^3 &= -XY^3 + 13Y^3 + 12X^3Y^2 - 66X^2Y^2 - 55XY^2 + 165Y^2 + 120X^3Y \\
&\quad - 210X^2Y - 126XY + 126Y + 56X^3 - 28X^2 - 7X + \mathit{1}, \\
X^2Y^3 &= XY^3 + 13Y^3 - 12X^3Y^2 - 66X^2Y^2 + 55XY^2 + 165Y^2 - 120X^3Y \\
&\quad - 210X^2Y + 126XY + 126Y - 56X^3 - 28X^2 + 7X + \mathit{1},
\end{aligned}$$

respectively. However,

$$C[(A_{3,6})_-,(A_{2,7})_-,(A_{3,7})_-]=C[-1,9694668,2074],$$
$$C[(A_{3,6})_+,(A_{2,7})_+,(A_{3,7})_+]=C[1,9694968,2126]$$

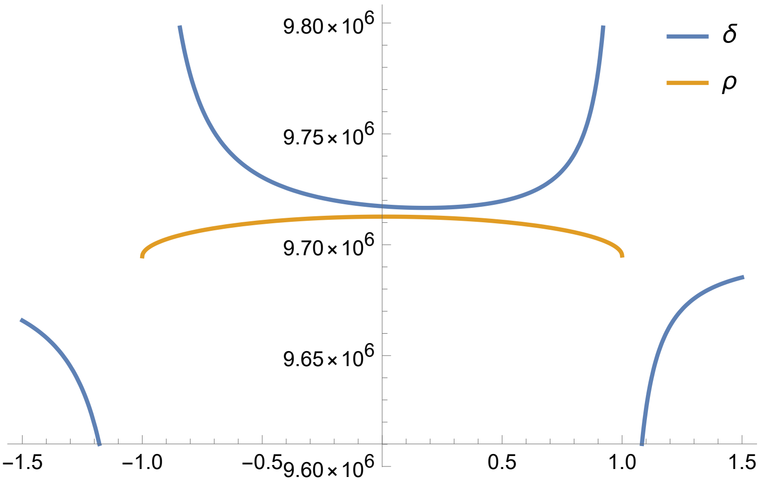
do not satisfy the relations

$$\begin{aligned} Y^4 = &-X^3Y^3+13X^2Y^3+12XY^3-66Y^3-55X^3Y^2+165X^2Y^2+120XY^2\\ &-210Y^2-126X^3Y+126X^2Y+56XY-28Y-7X^3+X^2,\\ Y^4 = &\ X^3Y^3+13X^2Y^3-12XY^3-66Y^3+55X^3Y^2+165X^2Y^2-120XY^2\\ &-210Y^2+126X^3Y+126X^2Y-56XY-28Y+7X^3+X^2, \end{aligned}$$

respectively. This violates Theorem 6.3.(iiia) and 6.3.(iiib), so the choices $A_{3,6}\in\{-1,1\}$ do not lead to a measure. It remains to consider the case $A_{36}\in(-1,1)$. A computation of $\delta(\mathbf{A}_{36}),\rho(\mathbf{A}_{36})$ by (6.14) gives

$$\begin{aligned} \delta(\mathbf{A}_{36}) &= \frac{-9717344+8100\mathbf{A_{36}}+9694114\mathbf{A36}^2+\mathbf{A}_{36}^4}{-1+\mathbf{A}_{36}^2},\\ \rho(\mathbf{A}_{36}) &= 9694830+150\mathbf{A}_{36}-12\mathbf{A}_{36}^2+\\ &\quad+\sqrt{318196764+3600\mathbf{A}_{36}-318196909\mathbf{A}_{36}^2-3600\mathbf{A}_{36}^3+145\mathbf{A}_{36}^4}. \end{aligned}$$

However, there is no $A_{36}\in(-1,1)$ such that $\delta>\rho$, which violates Theorem 6.3.(iiic).



It follows that Theorem 6.3.(iii) is violated, whence the measure for β does not exist.

v) Let $\beta\equiv\beta^{(8)}$ be as in Example 2.18 with the difference that β_{17} is a variable. Since C_2 and C_3 do not depend on β_{17}, they are positive definite as in iv) above. It turns out that $(A_{36})_-=-1$, $(A_{36})_+=1$ $(A_{27})_-=9.69468\cdot10^6$, $(A_{27})_+=9.69495\cdot10^6$, $(A_{37})_-=1869.46$, $(A_{37})_+=1921.46$. Similarly as in iv) above it is easy to check that $C_-:=C[(A_{3,6})_-,(A_{2,7})_-,(A_{3,7})_-]$ and $C_+:=C[(A_{3,6})_+,(A_{2,7})_+,(A_{3,7})_+]$ do not satisfy Theorem 6.3.(iiia) and 6.3.(iiib), respectively. (It will also follow from the uniqueness of the choice of $A_{3,6}\in((A_{3,6})_-,(A_{3,6})_+)$ such that $C[A_{3,6},A_{2,7},A_{3,7}]\succeq 0$

for some $A_{2,7}, A_{3,7} \in \mathbb{R}$, that none of the matrices C_-, C_+ can be positive semidefinite due to the convexity of the solution set of $C[\mathbf{A}_{3,6}, \mathbf{A}_{2,7}, \mathbf{A}_{3,7}] \succeq 0$.) It remains to consider the case $A_{36} \in (-1,1)$. A computation of $\delta(\mathbf{A}_{36}), \rho(\mathbf{A}_{36})$ by (6.14) gives

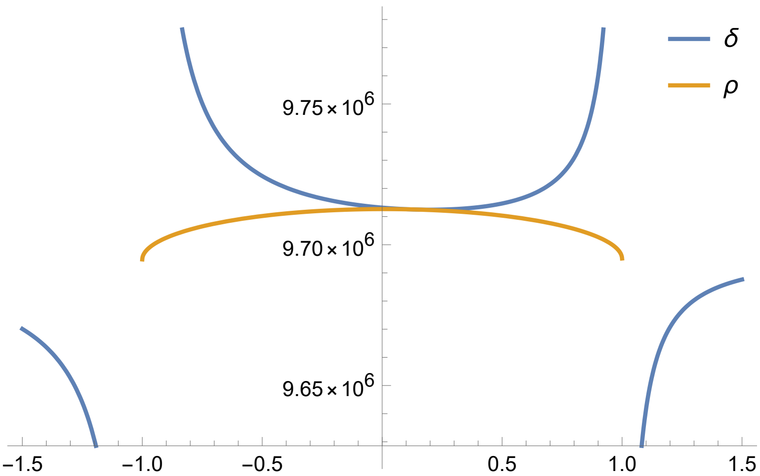
$$\delta(\mathbf{A}_{36}, \beta_{17}) = \frac{-\beta_{17}^2 - 9694844 + 9694114 + 54\beta_{17}\mathbf{A_{36}} + \mathbf{A_{36}^2} + \mathbf{A_{36}^4}}{-1 + \mathbf{A_{36}^2}}$$

$$\rho(\mathbf{A}_{36}, \beta_{17}) = 9694830 + \mathbf{A_{36}}\beta_{17} - 12\mathbf{A_{36}^2} + \sqrt{(-1 + \mathbf{A_{36}^2})(-318219264 + \beta_{17}^2 + 145\mathbf{A_{36}^2} - 24\mathbf{A_{36}}\beta_{17})}.$$

We are searching for β_{17} such that the curves δ and ρ would only touch for a unique A_{36}. Only for this A_{36} will a representing measure exist. Solving the system

$$\rho(\mathbf{A_{36}}, \beta_{17}) = \delta(\mathbf{A_{36}}, \beta_{17}) \quad \text{and} \quad \frac{\partial\rho(\mathbf{A_{36}}, \beta_{17})}{\partial\mathbf{A_{36}}} = \frac{\partial\delta(\mathbf{A_{36}}, \beta_{17})}{\partial\mathbf{A_{36}}} \tag{6.16}$$

on $\mathbf{A_{36}}$ and β_{17}, one of the solutions is $\beta_{17} \approx 135.39$. (The system (6.16) was solved in exact arithmetic using [Wol].) Choosing this solution and repeating the computations we get graphs that touch in a single point:



By Theorem 6.3.(iii) the measure for β exists and is unique as explained in Remark 6.4.(ii). Namely, there is only one good choice for $A_{36} \in (-1,1)$ such that $\delta(A_{36}) \leq \rho(A_{36})$, in which case $\delta(A_{36}) = \rho(A_{36})$. For this choice of A_{36}, there is only one choice for A_{27}, i.e., $A_{27} = \delta(A_{36}) = \rho(A_{36})$. Finally, A_{37} such that $\operatorname{rank} C = \operatorname{rank} C|_{\mathcal{B}\setminus\{Y^4\}} = \operatorname{rank} C|_{\mathcal{B}\setminus\{X^3Y^3, Y^4\}}$ is unique. Note that Theorem 1.5 implies the core variety of L_β is finite. Moreover, in the notation of Theorem 5.4, in this example we have $\lambda > \phi$.

Finally, this example also shows the answer to Question 2.14 is negative. Namely, β in this example has $(y-x^4)$-pure M_4, and for the unique representing measure $C[A]$ is not positive definite but of rank 15, since the last two columns do not increase the rank. This is due to the fact that δ is the smallest such that $C|_{\mathcal{B}\setminus\{Y^n\}}$ is positive semidefinite and the largest such that $C|_{\mathcal{B}\setminus\{X^3Y^{n-1}\}}$ is positive semidefinite (cf. Remark 6.4.(ii)).

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