UNOBSTRUCTEDNESS OF AFFINE TERMINAL GORENSTEIN TORIC FOURFOLDS

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ABSTRACT. We prove that every affine terminal Gorenstein toric variety of dimension at most four is unobstructed. In dimension four, the obstruction space need not vanish; instead, we determine the possible homogeneous obstruction degrees and construct simultaneous two-parameter deformations which eliminate the remaining potential obstructions.

1. INTRODUCTION

Deformation theory provides a fundamental tool for relating singular algebraic varieties to smooth ones and for understanding the local structure of moduli spaces. A variety X is called *unobstructed* if its deformation functor Def_X is formally smooth, that is, if every infinitesimal deformation of X lifts across every small extension of local Artinian algebras. In particular, every first-order deformation can then be extended to a formal deformation of all orders. If T^1_X is finite-dimensional and Def_X admits a hull, unobstructedness is equivalent to the smoothness of the miniversal base. Thus unobstructedness is important both for constructing smoothings and for controlling the local geometry of deformation and moduli spaces.

Unobstructedness is known in several important geometric settings. The Bogomolov–Tian–Todorov theorem gives unobstructedness for smooth Calabi–Yau manifolds, with T^1-lifting approaches developed by Ran and Kawamata; see [Ran92b, Kaw92]. For singular Calabi–Yau varieties, unobstructedness results under various assumptions were obtained by Kawamata, Ran, Tian, and Namikawa; see [Kaw92, Ran92a, Tia92, Nam94]. More recent generalisations appear in [FL24, FL25]. On the Fano side, every smooth Fano manifold is unobstructed. More generally, Sano proved that every weak Fano manifold is unobstructed [San14].

For affine toric varieties, unobstructedness can often be established by proving the vanishing of the obstruction space T^2_X. Altmann and Sletsjøe gave a combinatorial description of the relevant André–Quillen cohomology groups in [AS98], while Altmann and van Straten subsequently established polyhedral vanishing criteria for T^2_X, yielding unobstructedness for certain classes of affine Gorenstein toric singularities in [AvS00].

The principal result of this paper is that every affine terminal Gorenstein toric variety of dimension at most four is unobstructed; see Theorem 5.3. In dimensions at most three, this follows from the vanishing of the obstruction space T^2_X; see Lemma 4.3. In dimension four, however, T^2_X need not vanish; see Example 4.4. Thus, in this case, unobstructedness cannot be deduced solely from the vanishing of the obstruction space.

2020 *Mathematics Subject Classification.* 14B07, 14M25, 13D10, 14B05.

Key words and phrases. deformation theory; toric singularities; terminal singularities; unobstructedness.

[1]Supported by Slovenian Research Agency program P1-0222 and grants J1-60011, J1-70017.

[2]Supported by Slovenian Research Agency program P1-0288 and grants J1-50002, J1-60011, and J1-70017.

This unobstructedness result is also relevant to mirror symmetry. It was shown in [Fil26], for Laurent polynomials in two variables, that if f mutates to g and $X_{\Delta(g)}$ is unobstructed, then the deformation of $X_{\Delta(f)}$ corresponding to the sequence of mutations from f to g is unobstructed. The results of the present paper therefore suggest a four-dimensional analogue. Namely, if a Laurent polynomial f in three variables can be mutated to a Laurent polynomial g such that $\Delta(g)$ is empty, or equivalently such that $X_{\Delta(g)}$ is terminal (see Definition 2.5), then one expects the corresponding mutation deformation of $X_{\Delta(f)}$ to be unobstructed. Establishing this statement requires extending the mutation–deformation comparison of [Fil26], which was proved only in affine dimension three, to the four-dimensional setting.

We now explain the strategy for proving unobstructedness in dimension four. Let X_P be a four-dimensional affine terminal Gorenstein toric variety. Then P is an empty three-dimensional lattice polytope. By Proposition 4.7, $T^2_{X_P}(-r) = 0$ for every $r \neq R^*$, where R^* denotes the Gorenstein degree. Thus $T^2_{X_P}(-R^*)$ is the only homogeneous component of the obstruction space which may be nonzero.

On the other hand, Proposition 4.5 determines all nonzero homogeneous components of the tangent space $T^1_{X_P}$. Besides the component in degree $-R^*$, they are associated with square facets $F \subset P$ and occur in degrees $-\big(R^* - ns_F\big)$, $n \geq 1$, where $s_F \in \widetilde{M}$ is the primitive generator of the ray of $\sigma^\vee$ corresponding to F.

We first consider the finite-dimensional primitive tangent subspace corresponding to $n = 1$. Its square-facet parameters have degrees $m_F := R^* - s_F$, and let t_F denote the deformation parameter corresponding to a nonzero generator $\xi_F \in T^1_{X_P}(-m_F)$. Thus $\deg(t_F) = m_F$. Lemma 5.1 determines exactly when a sum of these primitive tangent parameter degrees can reach the exceptional obstruction degree R^*. It shows that the only possibility is $m_{F_1} + m_{F_2} = R^*$ for two parallel square facets F_1 and F_2.

The cohomological vanishing criterion therefore leaves precisely one type of possible mixed obstruction. To eliminate it, in Section 3 we construct a simultaneous two-parameter deformation associated with two compatible deformation pairs. Applied to the two parallel square facets above, this construction proves that the coefficient of the possible mixed quadratic obstruction $t_{F_1}t_{F_2}$ vanishes, even when its target $T^2_{X_P}(-R^*)$ is nonzero.

A further difficulty is that X_P is generally non-isolated and thus $T^1_{X_P}$ may be infinite-dimensional. Consequently, the full deformation functor cannot in general be treated by invoking a noetherian miniversal base. We instead construct compatible formal deformations over finite-dimensional power-series rings, containing the tangent directions with $n \leq N$. The higher tangent directions are obtained from the primitive ones using the module structure on $T^1_{X_P}$, and the corresponding families are constructed by a homogeneous substitution in both the defining equations and their lifted relations. Finally, an induction over Artinian small extensions shows directly that Def_{X_P} is formally smooth. All these steps are carried out in the proof of Theorem 5.3.

The paper is organised as follows. In the preliminaries, we recall the graded tangent and obstruction spaces of an affine Gorenstein toric variety. In Section 3, we construct simultaneous two-parameter deformations associated with compatible deformation pairs. In Section 4, we compute the tangent and obstruction degrees of affine terminal Gorenstein toric varieties and prove their unobstructedness in dimensions at most three. Finally, in Section 5, we prove that every affine terminal Gorenstein toric fourfold is unobstructed; see Theorem 5.3.

2. Preliminaries

2.1. The setup. We work over the field $\mathbb{C}$, which is algebraically closed and of characteristic zero. Let $N \cong \mathbb{Z}^n$ be a lattice of rank n, and let $M := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual lattice. We write

$$N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}, \qquad M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $P \subset N_{\mathbb{R}}$ be a full-dimensional lattice polytope with vertices $v^1, \dots, v^p$. Set

$$\widetilde{N} := N \oplus \mathbb{Z}, \qquad \widetilde{M} := M \oplus \mathbb{Z}.$$

By embedding P at height 1, we obtain the strongly convex rational polyhedral cone

$$\sigma := \operatorname{cone}(a^1, \dots, a^p) = \left\{ \sum_{i=1}^{p} \lambda_i a^i \;\middle|\; \lambda_i \in \mathbb{R}_{\geq 0} \right\} \subset \widetilde{N}_{\mathbb{R}}, \tag{2.1}$$

where

$$a^i := (v^i, 1) \in \widetilde{N}, \qquad i = 1, \dots, p,$$

and $\widetilde{N}_{\mathbb{R}} := \widetilde{N} \otimes_{\mathbb{Z}} \mathbb{R}$. The dual cone is

$$\sigma^{\vee} := \left\{ r \in \widetilde{M}_{\mathbb{R}} \;\middle|\; \langle n, r \rangle \geq 0 \text{ for every } n \in \sigma \right\},$$

where $\widetilde{M}_{\mathbb{R}} := \widetilde{M} \otimes_{\mathbb{Z}} \mathbb{R}$. We consider the affine semigroup $S_P := \sigma^{\vee} \cap \widetilde{M}$ and the associated affine Gorenstein toric variety $X_P := \operatorname{Spec} \mathbb{C}[S_P]$. Conversely, every affine Gorenstein toric variety without torus factors is isomorphic to X_P for some full-dimensional lattice polytope P.

We denote the coordinate projections by

$$\pi_M : \widetilde{M} \longrightarrow M, \qquad \pi_{\mathbb{Z}} : \widetilde{M} \longrightarrow \mathbb{Z}.$$

These projections will be used throughout the paper.

Definition 2.1. Let $Q \subset N_{\mathbb{R}}$ be a polytope and let $c \in M_{\mathbb{R}}$. Choose a vertex $v_Q(c)$ of Q at which the function $\langle c, \cdot \rangle : Q \longrightarrow \mathbb{R}$ attains its minimum. We define

$$\eta_Q(c) := -\min_{v \in Q} \langle v, c \rangle = -\langle v_Q(c), c \rangle.$$

Although the vertex $v_Q(c)$ need not be unique, the value $\eta_Q(c)$ is independent of this choice.

The Hilbert basis of $S_P = \sigma^{\vee} \cap \widetilde{M}$ can be written as

$$H_P = \{ s_1 = (c_1, \eta_P(c_1)), \dots, s_r = (c_r, \eta_P(c_r)), R^* \}, \tag{2.2}$$

where $R^* := (\underline{0}, 1)$, with the convention that R^* is omitted if it is decomposable in S_P; see [Alt97, Section 4.3]. Here $c_1, \dots, c_r \in M \setminus \{\underline{0}\}$ are pairwise distinct and uniquely determined up to ordering. The element R^* is called the *Gorenstein degree*.

Example 2.2. [1] Let $N = \mathbb{Z}^3$, with coordinates (x, y, z), and let

$$\begin{aligned} P = \operatorname{conv}\{ & v^1 = (0,0,0),\ v^2 = (1,0,0),\ v^3 = (0,1,0),\ v^4 = (1,1,0), \\ & v^5 = (0,0,1),\ v^6 = (1,0,1),\ v^7 = (1,1,1),\ v^8 = (2,1,1) \}. \end{aligned}$$

[1] The file with Magma code for this example numerical computations can be found on the link https://github.com/matfilip/Unobstructedness.

A direct facet computation gives

$$\begin{gathered} x \geq 0, \qquad y \geq 0, \qquad z \geq 0, \qquad 1+x-y-z \geq 0, \\ 1-x+y \geq 0, \qquad 1-x+z \geq 0, \qquad 1-y \geq 0, \qquad 1-z \geq 0. \end{gathered}$$

These inequalities force $0 \leq y, z \leq 1$. Checking the four possible pairs $(y,z) \in \{0,1\}^2$ shows that the lattice points of P are precisely $v^1, \dots, v^8$. In particular, P is empty, cf. Definition 2.5.

The corresponding facets, listed in the same order as the inequalities above, are

$$\begin{array}{cr} x = 0 & \operatorname{conv}\{v^1, v^3, v^5\}, \\ y = 0 & F_{y,0} := \operatorname{conv}\{v^1, v^2, v^6, v^5\}, \\ z = 0 & F_{z,0} := \operatorname{conv}\{v^1, v^2, v^4, v^3\}, \\ 1+x-y-z = 0 & \operatorname{conv}\{v^3, v^5, v^7\}, \\ 1-x+y = 0 & \operatorname{conv}\{v^2, v^6, v^8\}, \\ 1-x+z = 0 & \operatorname{conv}\{v^2, v^4, v^8\}, \\ y = 1 & F_{y,1} := \operatorname{conv}\{v^3, v^4, v^8, v^7\}, \\ z = 1 & F_{z,1} := \operatorname{conv}\{v^5, v^6, v^8, v^7\}. \end{array}$$

Thus P has four triangular facets and four square facets, the latter being $F_{y,0}, F_{y,1}, F_{z,0}$, and $F_{z,1}$.

Writing ij for the edge $[v^i, v^j]$, the edge set is

$$\mathcal{E}(P) = \{12, 13, 15, 24, 26, 28, 34, 35, 37, 48, 56, 57, 68, 78\}.$$

Together with the vertices $v^1, \dots, v^8$, these are all the nonempty proper faces of P.

The primitive ray generators of $\sigma^\vee$ are

$$\begin{array}{llll} s_1 = (1,0,0,0), & s_2 = (0,1,0,0), & s_3 = (0,0,1,0), & s_4 = (1,-1,-1,1), \\ s_5 = (-1,1,0,1), & s_6 = (-1,0,1,1), & s_7 = (0,-1,0,1), & s_8 = (0,0,-1,1). \end{array}$$

Here s_i is the primitive support element corresponding to the i-th facet in the preceding list. Moreover,

$$R^* = (0,0,0,1) = s_2 + s_7 = s_3 + s_8.$$

To determine the Hilbert basis, consider the following triangulation of $\sigma^\vee$:

$$\begin{array}{lll} \operatorname{cone}(R^*, s_6, s_3, s_7), & \operatorname{cone}(R^*, s_1, s_3, s_7), & \operatorname{cone}(R^*, s_1, s_3, s_2), \\ \operatorname{cone}(s_8, R^*, s_6, s_7), & \operatorname{cone}(s_8, R^*, s_1, s_2), & \operatorname{cone}(s_4, R^*, s_1, s_7), \\ \operatorname{cone}(s_4, s_8, R^*, s_7), & \operatorname{cone}(s_4, s_8, R^*, s_1), & \operatorname{cone}(s_5, s_8, R^*, s_2), \\ \operatorname{cone}(s_5, s_8, R^*, s_6), & \operatorname{cone}(s_5, R^*, s_3, s_2), & \operatorname{cone}(s_5, R^*, s_6, s_3). \end{array}$$

The determinant of the four generators of each cone is ± 1, so this triangulation is unimodular. Hence every lattice point of $\sigma^\vee$ is generated by $s_1, \dots, s_8, R^*$. Since $R^* = s_2 + s_7 = s_3 + s_8$, it is decomposable, whereas the s_i are primitive generators of extremal rays and hence indecomposable. Therefore

$$H_P = \{s_1, \dots, s_8\}.$$

2.2. **Graded deformation theory.** We briefly recall from [Ser06, Chapters 2 and 3] the basic notions concerning deformation functors, small extensions, tangent and obstruction spaces, and formal smoothness.

Let X be an affine scheme over $\mathbb{C}$, and let Def_X denote its deformation functor. We denote by T^1_X its tangent space, so that

$$T^1_X \cong \mathrm{Def}_X\big(\mathbb{C}[\epsilon]/(\epsilon^2)\big).$$

Let $(B, \mathfrak{m}_B)$ be a complete local $\mathbb{C}$-algebra. A formal deformation of X over B is a compatible system of deformations over the Artinian quotients $B/\mathfrak{m}_B^{k+1}$. Its first-order truncation over $B/\mathfrak{m}_B^2$ determines the Kodaira–Spencer map

$$\mathrm{KS}_{\mathcal{X}} \colon (\mathfrak{m}_B/\mathfrak{m}_B^2)^* \longrightarrow T^1_X.$$

Equivalently, the image of a tangent vector is the class of the first-order deformation obtained by pulling back along the corresponding homomorphism $B/\mathfrak{m}_B^2 \to \mathbb{C}[\epsilon]/(\epsilon^2)$.

We also denote by T^2_X a natural obstruction space for Def_X. More precisely, let

$$0 \longrightarrow J \longrightarrow B' \longrightarrow B \longrightarrow 0$$

be a small extension of local Artinian $\mathbb{C}$-algebras, meaning that $\mathfrak{m}_{B'} J = 0$. For every deformation $\mathcal{X}_B$ of X over B, there is an obstruction class

$$\mathrm{ob}(\mathcal{X}_B, B') \in T^2_X \otimes_{\mathbb{C}} J$$

whose vanishing is equivalent to the existence of a lifting of $\mathcal{X}_B$ to B'. If this obstruction vanishes, the set of isomorphism classes of liftings is a torsor under $T^1_X \otimes_{\mathbb{C}} J$.

We say that X is *unobstructed* if its deformation functor Def_X is formally smooth, that is, if the natural map $\mathrm{Def}_X(B') \longrightarrow \mathrm{Def}_X(B)$ is surjective for every small extension $B' \to B$.

We now specialise to $X = X_P$. The torus action on X_P induces $\widetilde{M}$-gradings

$$T^i_{X_P} = \bigoplus_{r \in \widetilde{M}} T^i_{X_P}(-r), \qquad i = 1, 2.$$

Moreover, the obstruction theory is torus-equivariant. Consequently, obstruction classes decompose into their $\widetilde{M}$-homogeneous components, and the equations governing every finite-dimensional graded deformation problem may be chosen $\widetilde{M}$-homogeneous.

For a graded deformation, a base parameter of degree r has Kodaira–Spencer class in $T^1_{X_P}(-r)$.

In the following two subsections, we recall the combinatorial descriptions of the homogeneous components of $T^1_{X_P}$ and $T^2_{X_P}$. These descriptions will allow us to determine the possible tangent parameter degrees and the degrees in which obstructions can occur.

2.3. **The tangent space.** Let σ be as in (2.1). For $r \in \widetilde{M}$ and $j = 1, \dots, p$, recall from [AS98, Alt00] the subsets

$$K^r_j := \big\{ m \in S_P \;\big|\; \langle a^j, m \rangle < \langle a^j, r \rangle \big\}, \qquad S_P = \sigma^\vee \cap \widetilde{M}.$$

More generally, for every nonzero face $\tau \preceq \sigma$, define

$$K^r_\tau := \bigcap_{a^j \in \tau} K^r_j,$$

and, for the zero face, put $K^r_0 := S_P$. Thus, if $\gamma \preceq \tau$, then $K^r_\tau \subseteq K^r_\gamma$.

The complex $\operatorname{span}(K^r)_\bullet$ of free abelian groups is defined by

$$\operatorname{span}(K^r)_{-k} := \bigoplus_{\substack{\tau \preceq \sigma \\ \dim \tau = k}} \operatorname{span}_{\mathbb{Z}}(K^r_\tau),$$

with differentials induced by the inclusions of faces and the usual incidence signs. We write

$$\operatorname{span}(K^r)^*_\bullet := \operatorname{Hom}_{\mathbb{Z}}\big(\operatorname{span}(K^r)_\bullet, \mathbb{Z}\big).$$

Since X_P is Gorenstein, and hence Gorenstein in codimension two, we have

$$T^k_{X_P}(-r) \cong H^k\left(\operatorname{span}(K^r)^*_\bullet \otimes_{\mathbb{Z}} \mathbb{C}\right), \qquad k = 1, 2; \tag{2.3}$$

see [AS98, Propositions 5.3 and 5.4] and [Alt00, Theorem 2.1]. Here $T^k_{X_P}(-r)$ denotes the homogeneous component of degree $-r \in \widetilde{M}$. The torus action induces the decomposition

$$T^k_{X_P} = \bigoplus_{r \in \widetilde{M}} T^k_{X_P}(-r).$$

We next recall the more explicit description of $T^1_{X_P}$ obtained in [Alt00]. For $r \in \widetilde{M}$, write

$$\varphi_r(v) := \langle r, (v, 1) \rangle, \qquad v \in N_{\mathbb{R}},$$

and consider the cross-section

$$Q(r) := \sigma \cap \left\{ x \in \widetilde{N}_{\mathbb{R}} \;\middle|\; \langle r, x \rangle = 1 \right\}.$$

Let $\mathcal{E}_c(r)$ denote the set of compact edges of $Q(r)$. For every $E \in \mathcal{E}_c(r)$, choose an orientation and denote its direction vector by d_E. For every compact two-dimensional face $\varepsilon \subset Q(r)$, choose incidence signs $\delta_\varepsilon(E) \in \{0, \pm 1\}$ such that

$$\sum_{E \subset \varepsilon} \delta_\varepsilon(E) d_E = 0.$$

Define

$$V(r) := \left\{ (t_E)_{E \in \mathcal{E}_c(r)} \;\middle|\; \sum_{E \subset \varepsilon} \delta_\varepsilon(E) t_E d_E = 0 \text{ for every compact two-face } \varepsilon \right\}.$$

If $E \in \mathcal{E}_c(r)$ is induced by an edge $[v^i, v^j] \subset P$, let $\ell(E)$ denote the lattice length of this edge. Define

$$V'(r) := \left\{ (t_E)_E \in V(r) \;\middle|\; t_E \neq 0 \Longrightarrow 1 \leq \varphi_r(v^i) = \varphi_r(v^j) \leq \ell(E) \right\}.$$

We write

$$V(r)_{\mathbb{C}} := V(r) \otimes_{\mathbb{R}} \mathbb{C}, \qquad V'(r)_{\mathbb{C}} := V'(r) \otimes_{\mathbb{R}} \mathbb{C}.$$

Theorem 2.3 ([Alt00, Theorem 4.1])**.** *The following statements hold.*

(1) Suppose that $\varphi_r(v) \leq 1$ for every vertex v of P. Then

$$T^1_{X_P}(-r) \cong V(r)_{\mathbb{C}} / \mathbb{C} \cdot \mathbf{1},$$

where $\mathbf{1} = (1, \ldots, 1)$ is the homothety direction.

(2) Suppose that $\max_{v\in P}\varphi_r(v)\geq 2$. *Then*

$$T^1_{X_P}(-r)\cong V'(r)_{\mathbb{C}}.$$

In particular, an edge coordinate can be nonzero only on an edge $E=[v^i,v^j]$ *satisfying*

$$\varphi_r(v^i)=\varphi_r(v^j)=q_E,\qquad 1\leq q_E\leq \ell(E).$$

2.4. **The obstruction space.** For every vertex v^j of P, one has

$$\operatorname{span}_{\mathbb{C}}(K^r_j)=\begin{cases}\widetilde{M}_{\mathbb{C}}, & \langle a^j,r\rangle\geq 2,\\ (a^j)^{\perp}_{\mathbb{C}}, & \langle a^j,r\rangle = 1,\\ 0, & \langle a^j,r\rangle\leq 0.\end{cases}\tag{2.4}$$

For a face $F\preceq P$, write

$$\widehat{F}:=\operatorname{cone}\{(v,1)\mid v\in F\}\preceq\sigma\tag{2.5}$$

and set $K^r_F:=K^r_{\widehat{F}}$. In particular, if $F=\{v^j\}$, then $K^r_F=K^r_j$.

By (2.3), the vector space $T^2_{X_P}(-r)$ is the middle cohomology of the complex

$$\bigoplus_{j=1}^{p}\big(\operatorname{span}_{\mathbb{C}}K^r_j\big)^*\xrightarrow{g_1}\bigoplus_{E\in\mathcal{E}(P)}\big(\operatorname{span}_{\mathbb{C}}K^r_E\big)^*\xrightarrow{g_2}\bigoplus_{\varepsilon\in\mathcal{F}_2(P)}\big(\operatorname{span}_{\mathbb{C}}K^r_\varepsilon\big)^*,\tag{2.6}$$

where $\mathcal{E}(P)$ and $\mathcal{F}_2(P)$ denote the sets of edges and two-dimensional faces of P, respectively.

Choose an orientation of every edge of P. For an oriented edge $E=[v^i,v^j]$, write $d_E:=v^j-v^i$. The first map in (2.6) is given by

$$g_1\big((\varphi_i)_i\big)_E=(\varphi_j-\varphi_i)|_{\operatorname{span}_{\mathbb{C}}K^r_E}.$$

For every two-dimensional face $\varepsilon\subset P$, choose an orientation of its boundary and incidence signs $\delta^\varepsilon_E\in\{0,\pm1\}$ such that

$$\sum_{E\subset\varepsilon}\delta^\varepsilon_E d_E=0.$$

The second map in (2.6) is given by

$$g_2\big((\varphi_E)_E\big)_\varepsilon=\sum_{E\subset\varepsilon}\delta^\varepsilon_E\,\varphi_E|_{\operatorname{span}_{\mathbb{C}}K^r_\varepsilon}.$$

2.5. **Affine terminal Gorenstein toric singularities.** An *affine lattice automorphism* of $N_{\mathbb{R}}$ is a map

$$\varphi\colon N_{\mathbb{R}}\longrightarrow N_{\mathbb{R}},\qquad x\longmapsto A(x)+b,$$

where

$$A\in\mathrm{GL}(N)\qquad\text{and}\qquad b\in N.$$

Such a map is also called an *affine unimodular transformation*.

Definition 2.4. Let $P,P'\subset N_{\mathbb{R}}$ be lattice polytopes. We say that P and P' are *lattice equivalent*, or *unimodularly equivalent*, if there exists an affine lattice automorphism

$$\varphi(x)=A(x)+b,\qquad A\in\mathrm{GL}(N),\quad b\in N,$$

such that $\varphi(P)=P'$.

Definition 2.5. A lattice polytope $P \subset N_{\mathbb{R}}$ is called *empty* if $P \cap N = \operatorname{Vert}(P)$, that is, if its only lattice points are its vertices. An affine toric Gorenstein variety X_P is *terminal*, if P is empty.

Note that terminal X_P has only terminal singularities, see e.g. [CLS11, Proposition 11.4.12]. The following vanishing result will be useful.

Theorem 2.6. *Let P be empty. For $r \in \widetilde{M}$, write*

$$\varphi_r(v) := \langle r, (v,1) \rangle, \qquad F_r := P \cap (\varphi_r = 1).$$

Then the following hold:

(1) If there exists a vertex v of P such that $\varphi_r(v) > 1$, then

$$T^k_{X_P}(-r) = 0, \qquad k = 1, 2.$$

(2) Suppose that $\varphi_r \leq 1$ on P. If every ℓ-dimensional face of F_r is a simplex, then

$$T^k_{X_P}(-r) = 0 \qquad \text{for } k < \ell, \quad k = 1, 2.$$

(3) If $\dim F_r \leq 2$, then $T^2_{X_P}(-r) = 0$.

Proof. This follows from [AvS00, Proposition 4.1, Theorem 6.6]. □

3. Two-parameter deformations associated with compatible deformation pairs

We retain the notation of Subsection 2.1 and follow the sign and grading conventions of [Fil26, Sections 2 and 3]. For convenience, we recall below the conventions used in the construction. We construct a simultaneous two-parameter deformation associated with two deformation pairs having possibly different Minkowski summands, generalizing the one-parameter construction of [Fil26, Section 3]. This result will be crucial in proving the unobstructedness of affine terminal Gorenstein toric varieties; see Theorem 5.3.

We use the convention $\mathbb{N} = \{0, 1, 2, \ldots\}$. For multi-indices $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^2$, the notation $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ means componentwise inequality. For a lattice polytope $Q \subset N_{\mathbb{R}}$, our support function is

$$\eta_Q(c) := -\min_{v \in Q} \langle c, v \rangle, \qquad c \in M_{\mathbb{R}}.$$

Thus η_Q is positively homogeneous and subadditive. We say that a polytope A is a Minkowski summand of a polytope B if $B = A + C$ for some polytope C. All semigroup algebras and polynomial presentations below are $\widetilde{M}$-graded, with $\deg(\chi^s) = s$.

We begin with a finite generating set $S_P = \operatorname{cone}(s_1, \ldots, s_r, R^*)$, where each $s_j = (c_j, \eta_P(c_j))$ is a boundary element. The set is not required to be minimal and will be enlarged below, after the two deformation pairs have been fixed. Put $\mathbf{x} = (x_1, \ldots, x_r)$, and let

$$\Theta_P \colon \mathbb{C}[\mathbf{x}, u] \longrightarrow \mathbb{C}[S_P]$$

be the surjective homomorphism determined by

$$\Theta_P(x_j) = \chi^{s_j}, \qquad \Theta_P(u) = \chi^{R^*}.$$

Setting $I_P := \ker(\Theta_P)$, we have

$$X_P = \operatorname{Spec} \mathbb{C}[S_P] \cong \operatorname{Spec} \frac{\mathbb{C}[\mathbf{x}, u]}{I_P}.$$

For $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$, set

$$s_{\mathbf{k}} := \sum_{j=1}^{r} k_j s_j, \qquad \mathbf{x}^{\mathbf{k}} := \prod_{j=1}^{r} x_j^{k_j},$$

and, for a lattice polytope $Q \subset N_{\mathbb{R}}$, define

$$\eta_Q(\mathbf{k}) := \sum_{j=1}^{r} \eta_Q(k_j c_j) - \eta_Q\left(\sum_{j=1}^{r} k_j c_j\right).$$

Every $s \in S_P$ has a unique decomposition

$$s = \partial_P(s) + n_P(s) R^*, \qquad \partial_P(s) - R^* \notin S_P, \qquad n_P(s) \in \mathbb{N}.$$

For every element $b \in \partial_P(S_P)$, choose an exponent vector

$$\partial(b) \in \mathbb{N}^r \qquad \text{such that} \qquad s_{\partial(b)} = b.$$

After fixing the two deformation pairs, we shall replace these choices by representatives adapted simultaneously to their two support functions. For $s \in S_P$, define $\partial(s) := \partial\big(\partial_P(s)\big)$ and put

$$\chi_P^s := \mathbf{x}^{\partial(s)} u^{n_P(s)}.$$

For $s = s_{\mathbf{k}}$, write

$$\partial(\mathbf{k}) := \partial(s_{\mathbf{k}}).$$

By positive homogeneity of η_P, the preceding definition gives

$$\eta_P(\mathbf{k}) = \sum_{j=1}^{r} k_j \eta_P(c_j) - \eta_P\left(\sum_{j=1}^{r} k_j c_j\right) = n_P(s_{\mathbf{k}}).$$

Consequently,

$$s_{\mathbf{k}} = s_{\partial(\mathbf{k})} + \eta_P(\mathbf{k}) R^*, \qquad \chi_P^{s_{\mathbf{k}}} = \mathbf{x}^{\partial(\mathbf{k})} u^{\eta_P(\mathbf{k})}.$$

The following elementary presentation lemma will allow us to enlarge the semigroup generating set without changing the form of the equations or of their relations.

Lemma 3.1. *For any finite set of boundary generators as above and any choices of boundary representatives $\partial(b)$, the binomials*

$$f_{\mathbf{k}} := \mathbf{x}^{\mathbf{k}} - \chi_P^{s_{\mathbf{k}}}, \qquad \mathbf{k} \in \mathbb{N}^r,$$

generate I_P. Let $\mathcal{P} := \mathbb{C}[\mathbf{x}, u]$ and let

$$\mathcal{F} := \bigoplus_{\mathbf{k} \in \mathbb{N}^r} \mathcal{P} e_{\mathbf{k}}$$

be the free $\mathcal{P}$-module with basis $\{e_{\mathbf{k}} \mid \mathbf{k} \in \mathbb{N}^r\}$. Let

$$\psi \colon \mathcal{F} \longrightarrow I_P$$

be the unique $\mathcal{P}$-linear homomorphism determined by

$$\psi(e_{\mathbf{k}}) = f_{\mathbf{k}}.$$

Then ψ is surjective, and its kernel is generated as a $\mathcal{P}$-module by the elements

$$\rho_{\mathbf{a},\mathbf{k}} := e_{\mathbf{a}+\mathbf{k}} - \mathbf{x}^{\mathbf{a}} e_{\mathbf{k}} - u^{\eta_P(\mathbf{k})} e_{\partial(\mathbf{k})+\mathbf{a}}, \qquad \mathbf{a}, \mathbf{k} \in \mathbb{N}^r.$$

Equivalently, the first syzygy module of the family $\{f_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^r}$ *is generated by the* $\rho_{\mathbf{a},\mathbf{k}}$.

Proof. For the corresponding statement for the Hilbert-basis presentation, see [ACF22a, Lemmas 5.3 and 5.6]. The same argument applies to any finite set of boundary generators and any fixed choice of boundary representatives. □

Consequently, the module of relations among the equations $f_{\mathbf{k}}$ is generated by

$$r_{\mathbf{a},\mathbf{k}} := f_{\mathbf{a}+\mathbf{k}} - \mathbf{x}^{\mathbf{a}} f_{\mathbf{k}} - u^{\eta_P(\mathbf{k})} f_{\partial(\mathbf{k})+\mathbf{a}} = 0, \qquad \mathbf{a},\mathbf{k}\in\mathbb{N}^r. \tag{3.1}$$

Example 3.2. Continue with the polytope of Example 2.2. Recall that $H_P = \{s_1,\dots,s_8\}$, whereas $R^* = (0,0,0,1) = s_2 + s_7 = s_3 + s_8$. Although R^* is decomposable and hence does not belong to the Hilbert basis, we adjoin it as a redundant semigroup generator and denote the corresponding variable by u.

Let $\mathbf{e}_1,\dots,\mathbf{e}_8$ denote the standard basis of $\mathbb{N}^8$, and write $f_{ij} := f_{\mathbf{e}_i+\mathbf{e}_j}$. The following additive relations hold in S_P:

$$\begin{aligned} &s_1+s_5 = s_2+R^*, && s_1+s_6 = s_3+R^*, && s_1+s_7 = s_3+s_4,\\ &s_1+s_8 = s_2+s_4, && s_2+s_6 = s_3+s_5, && s_2+s_7 = R^*,\\ &s_3+s_8 = R^*, && s_4+s_5 = s_8+R^*, && s_4+s_6 = s_7+R^*,\\ &s_5+s_7 = s_6+s_8. \end{aligned}$$

Choose the boundary representatives in accordance with the right-hand sides of these relations. The corresponding binomials are

$$\begin{aligned} &f_{15} = x_1x_5 - x_2u, && f_{16} = x_1x_6 - x_3u, && f_{17} = x_1x_7 - x_3x_4,\\ &f_{18} = x_1x_8 - x_2x_4, && f_{26} = x_2x_6 - x_3x_5, && f_{27} = x_2x_7 - u,\\ &f_{38} = x_3x_8 - u, && f_{45} = x_4x_5 - x_8u, && f_{46} = x_4x_6 - x_7u,\\ &f_{57} = x_5x_7 - x_6x_8. \end{aligned}$$

For instance, $s_1+s_5 = s_2+R^*$ gives $\partial(\mathbf{e}_1+\mathbf{e}_5) = \mathbf{e}_2$ and $\eta_P(\mathbf{e}_1+\mathbf{e}_5) = 1$, hence $f_{15} = x_1x_5 - x_2u$. Similarly, $s_1+s_7 = s_3+s_4$ gives $\partial(\mathbf{e}_1+\mathbf{e}_7) = \mathbf{e}_3+\mathbf{e}_4$ and $\eta_P(\mathbf{e}_1+\mathbf{e}_7) = 0$, hence $f_{17} = x_1x_7 - x_3x_4$.

Let

$$J := (f_{15}, f_{16}, f_{17}, f_{18}, f_{26}, f_{27}, f_{38}, f_{45}, f_{46}, f_{57}) \subseteq \mathbb{C}[x_1,\dots,x_8,u].$$

Since every displayed binomial comes from an additive relation in S_P, we have $J \subseteq I_P$. We verify the reverse inclusion using a monomial parametrization.

Consider the unimodular lattice automorphism $U\colon \widetilde{M} \to \widetilde{M}$ defined by $U(a,b,c,d) = (d, a+d, b+d, c+d)$. Its inverse is $U^{-1}(\alpha,\beta,\gamma,\delta) = (\beta-\alpha, \gamma-\alpha, \delta-\alpha, \alpha)$, so $U \in \mathrm{GL}_4(\mathbb{Z})$. Moreover, the vectors $U(s_1),\dots,U(s_8),U(R^*)$ have nonnegative coordinates.

Let w,p,q,z be auxiliary variables corresponding, in this order, to the four coordinates of $U(\widetilde{M})$. Thus $(\alpha,\beta,\gamma,\delta)$ is represented by the monomial $w^\alpha p^\beta q^\gamma z^\delta$. The transformed generators determine a monomial homomorphism $\Phi\colon \mathbb{C}[x_1,\dots,x_8,u] \to \mathbb{C}[w,p,q,z]$ given by

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	u
$\Phi(\,\cdot\,)$	p	q	z	wp^2	wq^2z	wqz^2	wpz	wpq	$wpqz.$

For example, $U(s_1) = (0,1,0,0)$ and $U(s_4) = (1,2,0,0)$, which explains the assignments $\Phi(x_1) = p$ and $\Phi(x_4) = wp^2$.

Since U is unimodular, it preserves all additive relations among the semigroup generators. Hence $\operatorname{im}(\Phi) \cong \mathbb{C}[S_P]$ and $\ker(\Phi) = I_P$. A Gröbner-basis computation in Magma gives

$$\begin{aligned}\mathcal{G} = \big\{&x_2x_6x_8 - x_5u,\ x_1x_7x_8 - x_4u,\ x_2x_4 - x_1x_8,\ x_3x_4 - x_1x_7,\\ &x_1x_5 - x_2u,\ x_3x_5 - x_2x_6,\ x_4x_5 - x_8u,\ x_1x_6 - x_3u,\\ &x_4x_6 - x_7u,\ x_2x_7 - u,\ x_5x_7 - x_6x_8,\ x_3x_8 - u\big\}\end{aligned}$$

for I_P. Ten elements of $\mathcal{G}$ are the generators of J, up to multiplication by -1. The remaining two cubic binomials are redundant: indeed, $x_2x_6x_8 - x_5u = x_8f_{26} + x_5f_{38}$ and $x_1x_7x_8 - x_4u = x_8f_{17} + x_4f_{38}$. Thus $\mathcal{G} \subseteq J$, and consequently $I_P \subseteq J$. Together with $J \subseteq I_P$, this proves $I_P = J$. Therefore

$$X_P \cong \operatorname{Spec} \frac{\mathbb{C}[x_1, \dots, x_8, u]}{(f_{15}, f_{16}, f_{17}, f_{18}, f_{26}, f_{27}, f_{38}, f_{45}, f_{46}, f_{57})}.$$

We finally illustrate the syzygy statement of Lemma 3.1. Take $\mathbf{k} = \mathbf{e}_1 + \mathbf{e}_5$ and $\mathbf{a} = \mathbf{e}_7$. As observed above, $\partial(\mathbf{k}) = \mathbf{e}_2$ and $\eta_P(\mathbf{k}) = 1$. The corresponding standard syzygy is

$$\rho_{\mathbf{e}_7,\mathbf{e}_1+\mathbf{e}_5} = e_{\mathbf{e}_1+\mathbf{e}_5+\mathbf{e}_7} - x_7e_{\mathbf{e}_1+\mathbf{e}_5} - ue_{\mathbf{e}_2+\mathbf{e}_7} \in \mathcal{F}.$$

Applying the $\mathcal{P}$-linear homomorphism $\psi\colon \mathcal{F} \to I_P$ gives

$$\psi\left(\rho_{\mathbf{e}_7,\mathbf{e}_1+\mathbf{e}_5}\right) = f_{157} - x_7f_{15} - uf_{27},$$

where $f_{157} := f_{\mathbf{e}_1+\mathbf{e}_5+\mathbf{e}_7} = x_1x_5x_7 - u^2$, since $s_1 + s_5 + s_7 = 2R^*$. Indeed,

$$f_{157} - x_7f_{15} - uf_{27} = (x_1x_5x_7 - u^2) - x_7(x_1x_5 - x_2u) - u(x_2x_7 - u) = 0.$$

Thus $\rho_{\mathbf{e}_7,\mathbf{e}_1+\mathbf{e}_5} \in \ker(\psi)$, as asserted by Lemma 3.1.

We now recall the notation used in the definition of a deformation pair. For $m \in \widetilde{M}$, let

$$\varphi_m\colon N_{\mathbb{R}} \longrightarrow \mathbb{R}, \qquad \varphi_m(v) := \langle m, (v,1)\rangle = \langle \pi_M(m), v\rangle + \pi_{\mathbb{Z}}(m).$$

Definition 3.3 ([Fil26])**.** Let $m \in \widetilde{M}$, and let $Q \subset \ker\big(\pi_M(m)\big)$ be a lattice polytope. The pair (m, Q) is called a *deformation pair of* P if, for every $i \in \mathbb{N}$ such that the section

$$P_i(m) := P \cap \{\varphi_m = i\}$$

is nonempty, there exists a polytope R_i such that

$$P_i(m) = iQ + R_i.$$

Here $iQ = Q + \ldots + Q$ denotes the i-fold Minkowski sum of Q for $i \geq 1$, and $0Q := \{0\}$. Equivalently, the condition says that iQ is a Minkowski summand of every nonempty integral-level section $P_i(m)$.

A deformation pair (m, Q) determines a one-parameter deformation of X_P [Fil26, Corollary 3.9]. We denote its deformation parameter by $t_{(m,Q)}$ and assign it degree $\deg t_{(m,Q)} = m$. The deformation-pair condition implies the following regularity property:

$$s_{\mathbf{k}} - jm \in \sigma^\vee \qquad \text{whenever } 0 \leq j \leq \eta_Q(\mathbf{k}). \tag{3.2}$$

Indeed, this follows by evaluating on the generators $(v, 1)$ of σ and using the Minkowski decomposition $P_i(m) = iQ + R_i$.

Let

$$\mathcal{D} = \big\{(m_1, Q_1), (m_2, Q_2)\big\}$$

be two deformation pairs of P. We assume that they are *mutually compatible*, meaning that

$$Q_1, Q_2 \subset \ker \pi_M(m_1) \cap \ker \pi_M(m_2). \tag{3.3}$$

We also assume that

$$\varphi_{m_1}(v)\varphi_{m_2}(v) = 0 \qquad \text{for every vertex } v \text{ of } P. \tag{3.4}$$

We now make the boundary choices above simultaneously compatible with Q_1 and Q_2. The following lemma is the reason for allowing a nonminimal semigroup generating set.

Lemma 3.4. *After adjoining finitely many boundary elements to $s_1, \dots, s_r$, every boundary element $b \in \partial_P(S_P)$ admits a representative $\partial(b)$ such that*

$$\eta_{Q_i}(\partial(b)) = 0, \qquad i = 1, 2. \tag{3.5}$$

The enlarged elements together with R^* still generate S_P, and the presentation and syzygy statement of Lemma 3.1 applies to this enlarged presentation.

Proof. Let Σ be a finite common rational polyhedral refinement of the normal fans of P, Q_1, and Q_2. Equivalently, the three support functions $\eta_P,\ \eta_{Q_1},\ \eta_{Q_2}$ are linear on every cone $\tau \in \Sigma$. For such a cone, set

$$B_\tau := \big\{(c, \eta_P(c)) \mid c \in \tau \cap M\big\} \subset \partial_P(S_P).$$

Since η_P is linear on τ, the projection to M identifies B_τ with the affine semigroup $\tau \cap M$. Therefore B_τ has a finite Hilbert basis. Adjoin the Hilbert bases of B_τ for all cones $\tau \in \Sigma$. This adds only finitely many boundary generators.

Let $b = (c, \eta_P(c)) \in \partial_P(S_P)$. Choose a cone $\tau \in \Sigma$ containing c, express b as a sum of elements of the Hilbert basis of B_τ, and use this expression to define $\partial(b)$. All generators occurring in this representative have first coordinates in the same cone τ. Since each η_{Q_i} is linear on τ,

$$\eta_{Q_i}(\partial(b)) = \sum_j k_j \eta_{Q_i}(c_j) - \eta_{Q_i}\left(\sum_j k_j c_j\right) = 0, \qquad i = 1, 2.$$

Finally, every $s \in S_P$ has the form $s = (c, \eta_P(c)) + nR^*$, $n \in \mathbb{N}$, so the enlarged boundary elements together with R^* generate S_P. Lemma 3.1 concludes the proof. ☐

From now on, we use this enlarged generating set and these adapted representatives, retaining the notation $s_j, x_j, \mathbf{x}$, and ∂. The enlargement only introduces redundant toric coordinates and does not change X_P; such coordinates may be eliminated in explicit smaller presentations.

The adapted vanishing also gives, for every $\mathbf{a}, \mathbf{k} \in \mathbb{N}^r$,

$$\eta_{Q_i}(\mathbf{a} + \mathbf{k}) = \eta_{Q_i}(\mathbf{k}) + \eta_{Q_i}(\partial(\mathbf{k}) + \mathbf{a}), \qquad i = 1, 2. \tag{3.6}$$

Indeed, the projections to M of $s_{\mathbf{k}}$ and $s_{\partial(\mathbf{k})}$ agree, and (3.5) applies to $\partial(\mathbf{k})$.

Write

$$t_i := t_{(m_i, Q_i)}, \qquad \mathbf{t} := (t_1, t_2), \qquad \deg(t_i) = m_i.$$

For $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, set

$$\mathbf{t}^{\boldsymbol{\alpha}} := t_1^{\alpha_1} t_2^{\alpha_2}, \qquad m_{\boldsymbol{\alpha}} := \alpha_1 m_1 + \alpha_2 m_2.$$

For $\mathbf{k} \in \mathbb{N}^r$, write

$$A_i(\mathbf{k}) := \eta_{Q_i}(\mathbf{k}), \qquad \mathbf{A}(\mathbf{k}) := \big(A_1(\mathbf{k}), A_2(\mathbf{k})\big),$$

and

$$\binom{\mathbf{A}(\mathbf{k})}{\boldsymbol{\alpha}} := \binom{A_1(\mathbf{k})}{\alpha_1}\binom{A_2(\mathbf{k})}{\alpha_2}.$$

We use the convention that this coefficient is zero if $\alpha_i > A_i(\mathbf{k})$ for some i.

For every $\mathbf{k} \in \mathbb{N}^r$, define

$$F_{\mathbf{k}}(\mathbf{x}, u, t_1, t_2) := \mathbf{x}^{\mathbf{k}} - \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^2 \\ \alpha_i \le A_i(\mathbf{k}),\, i=1,2}} \binom{\mathbf{A}(\mathbf{k})}{\boldsymbol{\alpha}} \mathbf{t}^{\boldsymbol{\alpha}} \chi_P^{s_{\mathbf{k}}-m_{\boldsymbol{\alpha}}}. \tag{3.7}$$

The characters occurring in (3.7) are well-defined. Indeed, let v be a vertex of P. By (3.4), one of $\varphi_{m_1}(v)$ and $\varphi_{m_2}(v)$ is zero. If, for example, $\varphi_{m_2}(v) = 0$, then $s_{\mathbf{k}} - \alpha_1 m_1 \in \sigma^\vee$ by (3.2), and hence

$$\langle s_{\mathbf{k}} - m_{\boldsymbol{\alpha}}, (v, 1)\rangle = \langle s_{\mathbf{k}} - \alpha_1 m_1, (v, 1)\rangle - \alpha_2 \varphi_{m_2}(v) \geq 0.$$

The other case is identical. Thus $s_{\mathbf{k}} - m_{\boldsymbol{\alpha}} \in \sigma^\vee$.

For every admissible $\boldsymbol{\alpha}$, set

$$\mathbf{k}_{\boldsymbol{\alpha}} := \partial\big(s_{\mathbf{k}} - m_{\boldsymbol{\alpha}}\big) \in \mathbb{N}^r$$

and $\nu_{\mathbf{k},\boldsymbol{\alpha}} := n_P\big(s_{\mathbf{k}} - m_{\boldsymbol{\alpha}}\big)$. Thus

$$s_{\mathbf{k}} - m_{\boldsymbol{\alpha}} = s_{\mathbf{k}_{\boldsymbol{\alpha}}} + \nu_{\mathbf{k},\boldsymbol{\alpha}} R^*. \tag{3.8}$$

For $\mathbf{a}, \mathbf{k} \in \mathbb{N}^r$, define

$$R_{\mathbf{a},\mathbf{k}} \quad := \quad F_{\mathbf{a}+\mathbf{k}} \;-\; \mathbf{x}^{\mathbf{a}} F_{\mathbf{k}} \;-\; u^{\eta_P(\mathbf{k})} F_{\partial(\mathbf{k})+\mathbf{a}} \;-\; \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^2\setminus\{\mathbf{0}\} \\ \alpha_i \le A_i(\mathbf{k}),\, i=1,2}} u^{\nu_{\mathbf{k},\boldsymbol{\alpha}}} \binom{\mathbf{A}(\mathbf{k})}{\boldsymbol{\alpha}} \mathbf{t}^{\boldsymbol{\alpha}} F_{\mathbf{k}_{\boldsymbol{\alpha}}+\mathbf{a}}. \tag{3.9}$$

Lemma 3.5. *For every* $\mathbf{a} \in \mathbb{N}^r$*,*

$$\eta_{Q_i}(\mathbf{k}_{\boldsymbol{\alpha}} + \mathbf{a}) = \eta_{Q_i}(\partial(\mathbf{k}) + \mathbf{a}), \qquad i = 1, 2.$$

Proof. Fix $i \in \{1, 2\}$, and write

$$c_{\mathbf{b}} := \sum_{j=1}^{r} b_j c_j \qquad \text{for } \mathbf{b} \in \mathbb{N}^r.$$

Since $\partial(\mathbf{k})$ and $\mathbf{k}_{\boldsymbol{\alpha}}$ are the chosen adapted representatives of boundary elements, Lemma 3.4 gives

$$\eta_{Q_i}(\partial(\mathbf{k})) = \eta_{Q_i}(\mathbf{k}_{\boldsymbol{\alpha}}) = 0.$$

Moreover, projecting the boundary decomposition to M yields $c_{\mathbf{k}_{\boldsymbol{\alpha}}} = c_{\partial(\mathbf{k})} - \pi_M(m_{\boldsymbol{\alpha}})$. By mutual compatibility,

$$Q_i \subset \ker \pi_M(m_1) \cap \ker \pi_M(m_2),$$

and hence $\pi_M(m_{\boldsymbol{\alpha}})$ vanishes identically on Q_i. Therefore $\eta_{Q_i}(c_{\mathbf{k}_{\boldsymbol{\alpha}}}) = \eta_{Q_i}(c_{\partial(\mathbf{k})})$ and

$$\eta_{Q_i}(c_{\mathbf{a}} + c_{\mathbf{k}_{\boldsymbol{\alpha}}}) = \eta_{Q_i}(c_{\mathbf{a}} + c_{\partial(\mathbf{k})}).$$

Using the definition of $\eta_{Q_i}(\,\cdot\,)$ and the two vanishing equalities above, we conclude that

$$\eta_{Q_i}(\mathbf{k}_{\boldsymbol{\alpha}} + \mathbf{a}) = \eta_{Q_i}(\partial(\mathbf{k}) + \mathbf{a}),$$

as required. ∎

Proposition 3.6. *For every $\mathbf{a}, \mathbf{k} \in \mathbb{N}^r$, the expression $R_{\mathbf{a},\mathbf{k}}$ is a linear relation among the equations $F_{\mathbf{b}}$.*

Proof. Set

$$A_i := \eta_{Q_i}(\mathbf{k}), \qquad B_i := \eta_{Q_i}(\partial(\mathbf{k}) + \mathbf{a}), \qquad i = 1, 2.$$

By (3.6), $\eta_{Q_i}(\mathbf{a} + \mathbf{k}) = A_i + B_i$. Moreover, Lemma 3.5 implies

$$\eta_{Q_i}(\mathbf{k}_{\boldsymbol{\alpha}} + \mathbf{a}) = B_i, \qquad i = 1, 2.$$

Expanding (3.9) and using (3.8), all terms of degree $\mathbf{t}^{\boldsymbol{\beta}}$ are multiples of the same character. Their common coefficient is

$$-\binom{A_1 + B_1}{\beta_1}\binom{A_2 + B_2}{\beta_2} + \sum_{\boldsymbol{\alpha} \leq \boldsymbol{\beta}} \binom{A_1}{\alpha_1}\binom{A_2}{\alpha_2}\binom{B_1}{\beta_1 - \alpha_1}\binom{B_2}{\beta_2 - \alpha_2},$$

which is zero by the Vandermonde's convolution applied twice [GKP94, Equation (5.27), pp. 170]. Hence $R_{\mathbf{a},\mathbf{k}} = 0$. □

Corollary 3.7. *The equations $F_{\mathbf{k}}$ define a formal two-parameter deformation over $\mathbb{C}[[t_1, t_2]]$ of X_P. The restriction to the t_i-axis is the one-parameter deformation associated with (m_i, Q_i), for $i = 1, 2$.*

Proof. Modulo (t_1, t_2), the equations $F_{\mathbf{k}}$ reduce to $f_{\mathbf{k}}$, and the relations $R_{\mathbf{a},\mathbf{k}}$ reduce to $r_{\mathbf{a},\mathbf{k}}$. By Lemma 3.1, the latter generate all relations among the equations of the central fibre, while Proposition 3.6 provides their lifts. By the equational flatness criterion [Fil26, Lemma 3.5], the equations therefore define a formal flat deformation over $\mathbb{C}[[t_1, t_2]]$. □

For every facet $F \subset P$, let $s_F \in \widetilde{M}$ denote the primitive generator of the ray of $\sigma^\vee$ corresponding to F. Equivalently, s_F is the primitive inward support element satisfying

$$\langle s_F, (v, 1)\rangle = 0 \qquad \text{for every } v \in F.$$

Corollary 3.8. *Let P be an empty three-dimensional lattice polytope, and let $F_1, F_2 \subset P$ be square facets satisfying $s_{F_1} + s_{F_2} = R^*$. For $i = 1, 2$, let $(R^* - s_{F_i}, Q_i)$ be a deformation pair of P. Then the corresponding one-parameter deformations extend to a formal two-parameter deformation over $\mathbb{C}[[t_1, t_2]]$.*

Proof. The relation $s_{F_1} + s_{F_2} = R^*$ gives $R^* - s_{F_1} = s_{F_2}$, $R^* - s_{F_2} = s_{F_1}$, and

$$(R^* - s_{F_1}) + (R^* - s_{F_2}) = R^*.$$

Since $s_{F_1}, s_{F_2} \in \sigma^\vee$, for every vertex v of P, the values $\varphi_{R^*-s_{F_1}}(v)$ and $\varphi_{R^*-s_{F_2}}(v)$ are nonnegative integers whose sum is $\varphi_{R^*}(v) = 1$. Hence $\varphi_{R^*-s_{F_1}}(v)\varphi_{R^*-s_{F_2}}(v) = 0$.

Moreover, since $\pi_M(R^*) = 0$, we have $\pi_M(R^* - s_{F_1}) = -\pi_M(R^* - s_{F_2})$, and therefore

$$\ker \pi_M(R^* - s_{F_1}) = \ker \pi_M(R^* - s_{F_2}).$$

Together with $Q_i \subset \ker \pi_M(R^* - s_{F_i})$ for $i = 1, 2$, this shows that the two deformation pairs satisfy the compatibility conditions. The result now follows from Corollary 3.7. □

The following example makes the preceding construction explicit for the eight-vertex polytope of Example 2.2. Its two pairs of opposite square facets satisfy the hypotheses of Corollary 3.8, and we write down the resulting two-parameter deformations explicitly.

Example 3.9. Let P be the polytope of Example 2.2, retain the notation $F_{y,0}, F_{y,1}, F_{z,0}, F_{z,1}$ for its four square facets, and use the coordinate-ring presentation of Example 3.2. Recall that their primitive support elements are $s_{F_{y,0}} = s_2$, $s_{F_{y,1}} = s_7$, $s_{F_{z,0}} = s_3$, and $s_{F_{z,1}} = s_8$.

In particular,

$$s_{F_{y,0}} + s_{F_{y,1}} = s_2 + s_7 = R^*, \qquad s_{F_{z,0}} + s_{F_{z,1}} = s_3 + s_8 = R^*.$$

Let e_1, e_2, e_3 be the standard basis of $N = \mathbb{Z}^3$, and set $L := [0, e_1]$. The decompositions

$$F_{z,0} = [0, e_1] + [0, e_2], \qquad F_{z,1} = e_3 + [0, e_1] + [0, e_1 + e_2],$$
$$F_{y,0} = [0, e_1] + [0, e_3], \qquad F_{y,1} = e_2 + [0, e_1] + [0, e_1 + e_3].$$

show that L is a Minkowski summand of each square facet. For $m_F := R^* - s_F$, the only nontrivial Minkowski-summand condition in Definition 3.3 is the one for the level-one section

$$P \cap (\varphi_{m_F} = 1) = F;$$

the condition at level zero is automatic because $0L = \{0\}$. Consequently, (m_F, L) is a deformation pair for each of the four square facets. The two relations above and Corollary 3.8 therefore give one two-parameter deformation for each pair of opposite square facets.

For $c = (c_x, c_y, c_z) \in M$, one has $\eta_L(c) = \max\{0, -c_x\}$. Consequently, for $\mathbf{k} = (k_1, \ldots, k_8)$,

$$\eta_L(\mathbf{k}) = \min\{k_1 + k_4, k_5 + k_6\}. \tag{3.10}$$

Among the ten generators f_{ij} of the toric ideal displayed in Example 3.2, the equality $\eta_L(\mathbf{e}_i + \mathbf{e}_j) = 1$ holds precisely for f_{15}, f_{16}, f_{45}, and f_{46}.

It is zero for the remaining six generators, which therefore remain unchanged in both families.

The pair $F_{z,0}, F_{z,1}$. The corresponding deformation degrees are

$$m_{z,0} := R^* - s_3 = s_8, \qquad m_{z,1} := R^* - s_8 = s_3.$$

Let τ_0, τ_1 be the associated parameters, with $\deg(\tau_0) = s_8$ and $\deg(\tau_1) = s_3$. Formula (3.7) gives

$$F^z_{15} = x_1x_5 - x_2u - \tau_0x_2x_3 - \tau_1x_2x_8 - \tau_0\tau_1x_2,$$
$$F^z_{16} = x_1x_6 - x_3u - \tau_0x_3^2 - \tau_1u - \tau_0\tau_1x_3,$$
$$F^z_{45} = x_4x_5 - x_8u - \tau_0u - \tau_1x_8^2 - \tau_0\tau_1x_8,$$
$$F^z_{46} = x_4x_6 - x_7u - \tau_0x_3x_7 - \tau_1x_7x_8 - \tau_0\tau_1x_7.$$

The pair $F_{y,0}, F_{y,1}$. The corresponding deformation degrees are

$$m_{y,0} := R^* - s_2 = s_7, \qquad m_{y,1} := R^* - s_7 = s_2.$$

Let λ_0, λ_1 be the associated parameters, with $\deg(\lambda_0) = s_7$ and $\deg(\lambda_1) = s_2$. In this case, formula (3.7) gives

$$F^y_{15} = x_1x_5 - x_2u - \lambda_0x_2^2 - \lambda_1u - \lambda_0\lambda_1x_2,$$
$$F^y_{16} = x_1x_6 - x_3u - \lambda_0x_2x_3 - \lambda_1x_3x_7 - \lambda_0\lambda_1x_3,$$
$$F^y_{45} = x_4x_5 - x_8u - \lambda_0x_2x_8 - \lambda_1x_7x_8 - \lambda_0\lambda_1x_8,$$
$$F^y_{46} = x_4x_6 - x_7u - \lambda_0u - \lambda_1x_7^2 - \lambda_0\lambda_1x_7.$$

Setting the two parameters equal to zero recovers the defining equations of X_P in each case.

4. Affine terminal Gorenstein toric varieties

In this section, we study the tangent and obstruction spaces of affine terminal Gorenstein toric varieties. We give a combinatorial description of the nonzero homogeneous components of T^1_X and T^2_X, and show that $T^2_X = 0$ when $\dim X \le 3$. In particular, affine terminal Gorenstein toric varieties of dimension at most three are unobstructed.

The following two lemmas are well known; we include brief proofs for convenience.

Lemma 4.1. *Let $P \subset N_{\mathbb{R}}$ be an empty lattice polytope of dimension d, that is, $P \cap N = \mathrm{Vert}(P)$. Then $\#\,\mathrm{Vert}(P) \le 2^d$.*

Proof. After replacing the ambient lattice by $\mathrm{span}_{\mathbb{R}}(P-P)\cap N$, we may assume that $N \cong \mathbb{Z}^d$ and that P is full-dimensional. There are exactly 2^d congruence classes in $N/2N \cong (\mathbb{Z}/2\mathbb{Z})^d$. If P had more than 2^d vertices, then two distinct vertices v, w would be congruent modulo $2N$. Hence $\frac{v+w}{2} \in N$. Since this point lies in the relative interior of the segment $[v,w] \subset P$, it is a non-vertex lattice point of P, which contradicts the assumption that P is empty. ☐

Lemma 4.2. *Let $P \subset N_{\mathbb{R}}$, where $N \simeq \mathbb{Z}^2$, be an empty two-dimensional lattice polytope. Then P is lattice equivalent to either the standard triangle*

$$\Delta_2 := \mathrm{conv}\{(0,0),(1,0),(0,1)\}$$

or the unit square

$$\square := \mathrm{conv}\{(0,0),(1,0),(0,1),(1,1)\}.$$

Proof. By Lemma 4.1, P has either three or four vertices. Since P is empty, every edge is primitive and P has no interior lattice points. Thus Pick's formula [BR15, Theorem 2.8] gives $\mathrm{Area}(P) = \frac{r}{2} - 1$, where r is the number of vertices of P.

If $r = 3$, then $\mathrm{Area}(P) = \frac{1}{2}$. Hence the normalized area of P is one, so P is a unimodular triangle and therefore lattice equivalent to Δ_2.

Suppose that $r = 4$, and write the vertices as v_0, v_1, v_2, v_3 in cyclic order. Then $\mathrm{Area}(P) = 1$. Each diagonal divides P into two lattice triangles. Since every nondegenerate lattice triangle has area at least $\frac{1}{2}$, all four triangles arising from the two diagonals have area $\frac{1}{2}$ and are therefore unimodular.

After a lattice equivalence, we may assume that $v_0 = 0$, $v_1 = e_1$, and $v_2 = e_2$. Writing $v_3 = (a,b)$, the unimodularity and cyclic ordering of $\mathrm{conv}\{v_0, v_2, v_3\}$ and $\mathrm{conv}\{v_0, v_1, v_3\}$ give $-a = 1$ and $b = 1$. Thus $v_3 = e_2 - e_1$, and P is lattice equivalent to the unit square. ☐

Lemma 4.3. *Let X be an affine terminal Gorenstein toric variety. If $\dim X \le 3$, then $T^2_X = 0$ and thus X is unobstructed.*

Proof. If $\dim X \le 2$, then $X \cong \mathbb{A}^1$ or $X \cong \mathbb{A}^2$. Hence X is smooth and $T^2_X = 0$. Suppose now that $\dim X = 3$. Then the associated Gorenstein polytope P is two-dimensional and, by terminality, empty. By Definition 2.5 and Lemma 4.2, P is lattice equivalent either to the standard triangle Δ_2 or to the unit square $\square$. If P is lattice equivalent to Δ_2, then the primitive generators of the cone over P form a lattice basis. Hence the cone is unimodular and $X \cong \mathbb{A}^3$; thus $T^2_X = 0$. If P is lattice equivalent to the unit square, then the Hilbert basis of $\sigma^\vee \cap \widetilde{M}$ consists of four elements u_1, u_2, u_3, u_4, with the unique primitive relation $u_1 + u_4 = u_2 + u_3$. Consequently,

$$X \cong \mathrm{Spec}\,\frac{\mathbb{C}[x_1,x_2,x_3,x_4]}{(x_1x_4 - x_2x_3)},$$

which is the three-dimensional ordinary double point. In particular, X is a hypersurface and $T^2_X = 0$. So X is unobstructed. $\square$

If X is a four-dimensional affine terminal Gorenstein toric variety, then the obstruction space T^2_X need not vanish, as the following example shows. Thus, unlike in dimensions at most three, unobstructedness cannot be deduced simply from the vanishing of T^2_X; one has to show that the actual obstruction equations vanish.

Example 4.4. Let P be the empty three-dimensional lattice polytope from Example 2.2, and let X_P be its associated four-dimensional affine toric variety. Retain the notation $\mathcal{E}(P)$ for its fourteen edges and $F_{y,0}, F_{y,1}, F_{z,0}, F_{z,1}$ for its four square facets. The remaining four facets are triangles. We compute $T^2_{X_P}(-R^*)$.

Recall that $a^i = (v^i, 1)$. Since $\langle R^*, a^i\rangle = 1$ for every vertex v^i, the complex computing $T^2_{X_P}(-R^*)$ is

$$\bigoplus_{i=1}^{8} \big((a^i)^\perp\big)^* \xrightarrow{g_1} \bigoplus_{E=[v^i,v^j]\in\mathcal{E}(P)} \big((a^i,a^j)^\perp\big)^* \xrightarrow{g_2} \bigoplus_{F\in\mathcal{F}_2(P)} \big(\widehat{F}^\perp\big)^*. \tag{4.1}$$

Each vertex summand has dimension three, each edge summand has dimension two, and each facet summand has dimension one. Thus the dimensions of the three terms are 24, 28, and 8, respectively.

We first determine $V(R^*)$. Recall that the closure relation for a two-dimensional face F is equal to $\sum_{E\subset F} \delta^F_E t_E d_E = 0$. It expresses the condition that the scaled oriented edge vectors $t_E d_E$ form a closed polygon. For a triangular facet, it forces all three edge parameters to be equal, whereas for a square facet it identifies the parameters on opposite edges. For a triangular facet, the closure relation forces its three edge parameters to be equal. The four triangular facets therefore give $t_{13} = t_{15} = t_{35} = t_{37} = t_{57}$ and $t_{24} = t_{26} = t_{28} = t_{48} = t_{68}$.

For each square facet, the closure relation identifies the parameters on opposite edges:

$$\begin{aligned} F_{y,0}:&\quad t_{12} = t_{56}, &\quad t_{15} = t_{26},\\ F_{z,0}:&\quad t_{12} = t_{34}, &\quad t_{13} = t_{24},\\ F_{y,1}:&\quad t_{34} = t_{78}, &\quad t_{37} = t_{48},\\ F_{z,1}:&\quad t_{56} = t_{78}, &\quad t_{57} = t_{68}. \end{aligned}$$

Combining these relations, we obtain

$$V(R^*) = \left\{ (t_E)_{E\in\mathcal{E}(P)} \;\middle|\; \begin{array}{l} t_{12} = t_{34} = t_{56} = t_{78} = \beta, \\ t_E = \alpha \text{ for every other edge } E \end{array} \right\}.$$

Thus $\dim_{\mathbb{C}} V(R^*)_{\mathbb{C}} = 2$. The homothety direction $\mathbf{1} = (1)_{E\in\mathcal{E}(P)}$ is given by $\alpha = \beta$. Consequently, Theorem 2.3 gives $T^1_{X_P}(-R^*) \cong V(R^*)_{\mathbb{C}}/\mathbb{C}\mathbf{1} \cong \mathbb{C}$. The nontrivial class is represented by the Minkowski decomposition $P = [0, e_1] + \operatorname{conv}\{0, e_2, e_3, e_1 + e_2 + e_3\}$.

Since P is full-dimensional, the restriction map from global functionals to the direct sum of the vertex summands is injective and has four-dimensional image. Moreover, $T^1_{X_P}(-R^*) \cong \ker(g_1)/\operatorname{im}(\widetilde{M}^*_{\mathbb{C}})$. It follows that $\dim_{\mathbb{C}} \ker(g_1) = 4 + 1 = 5$ and $\operatorname{rank}(g_1) = 24 - 5 = 19$.

The map g_2 is surjective. Indeed, suppose that an element of the dual of the last term lies in $\ker(g_2^*)$. If two facets F and F' meet along an edge E, then $\widehat{F}^\perp$ and $\widehat{F'}^\perp$ are distinct one-dimensional subspaces of the two-dimensional space $\widehat{E}^\perp$. The E-component of its image under g_2^* can therefore vanish only if its coefficients at both F and F' are zero. Hence g_2^* is injective, so g_2 is surjective. Consequently, $\operatorname{rank}(g_2) = 8$ and $\dim_{\mathbb{C}} \ker(g_2) = 28 - 8 = 20$.

Finally,
$$\dim_{\mathbb{C}} T^2_{X_P}(-R^*) = \dim_{\mathbb{C}} \ker(g_2) - \operatorname{rank}(g_1) = 20 - 19 = 1.$$
Therefore $T^2_{X_P}(-R^*) \cong \mathbb{C}$.

4.1. **The tangent space.**

Proposition 4.5. *Let $P \subset N_{\mathbb{R}}$ be an empty three-dimensional lattice polytope, and let X_P be the associated affine Gorenstein toric variety. For $r \in \widetilde{M}$, set*
$$\varphi_r(v) := \langle r, (v, 1)\rangle, \qquad F_r := P \cap \{\varphi_r = 1\}.$$
For every facet $F \subset P$, let $s_F \in \widetilde{M}$ denote the primitive generator of the corresponding ray of $\sigma^\vee$, and let $\mathcal{S}(P)$ denote the set of square facets of P.

Apart from the Gorenstein-degree component $T^1_{X_P}(-R^*)$, the nonzero homogeneous components of $T^1_{X_P}$ are precisely those associated with square facets. More explicitly, for $r \neq R^*$,
$$T^1_{X_P}(-r) \neq 0 \quad\iff\quad r = R^* - ns_F$$
for some $F \in \mathcal{S}(P)$ and $n \geq 1$. In this case, $F_r = F$ and
$$T^1_{X_P}\big(-(R^* - ns_F)\big) \cong \mathbb{C}.$$
Moreover, $F_{R^*} = P$, and every r for which $T^1_{X_P}(-r) \neq 0$ satisfies $\varphi_r \leq 1$ on P. Consequently,
$$T^1_{X_P} = T^1_{X_P}(-R^*) \oplus \bigoplus_{F \in \mathcal{S}(P)} \bigoplus_{n \geq 1} T^1_{X_P}\big(-(R^* - ns_F)\big).$$

Proof. Since P is empty, every two-dimensional face of P is lattice equivalent either to the standard triangle or to the unit square, by Lemma 4.2. Theorem 2.6 therefore shows that $T^1_{X_P}(-r) \neq 0$ only if $\varphi_r \leq 1$ on P, and that in this case either $F_r = P$ or F_r is a square facet.

If $F_r = P$, then $\varphi_r = 1$ on the full-dimensional polytope P. Hence $\varphi_r = \varphi_{R^*}$, and therefore $r = R^*$. Conversely, $\varphi_{R^*} = 1$ on P, so $F_{R^*} = P$.

Suppose now that $r \neq R^*$ and $T^1_{X_P}(-r) \neq 0$. Then $F := F_r$ is a square facet. Since both φ_r and φ_{R^*} equal 1 on F, the element $R^* - r$ vanishes on F. Moreover, $\varphi_r \leq 1$ on P, so $R^* - r$ is nonnegative on P. It therefore belongs to the ray of $\sigma^\vee$ corresponding to F. Since s_F is the primitive lattice generator of this ray, $R^* - r = ns_F$ for some $n \geq 1$, or equivalently $r = R^* - ns_F$.

Conversely, let F be a square facet and set $r = R^* - ns_F$ for some $n \geq 1$. Since $\varphi_{s_F} \geq 0$ on P and vanishes precisely on F, we have $\varphi_r = 1 - n\varphi_{s_F} \leq 1$ on P, with equality precisely on F. Thus $F_r = F$.

For a unit square, $V(r)_{\mathbb{C}} \cong \mathbb{C}^2$, with one coordinate for each pair of opposite edges. Theorem 2.3 therefore gives
$$T^1_{X_P}(-r) \cong V(r)_{\mathbb{C}}/\mathbb{C}\mathbf{1} \cong \mathbb{C}.$$
This proves the classification of the nonzero homogeneous components, and the asserted direct-sum decomposition follows. □

The following example applies Proposition 4.5 to the eight-vertex polytope and records all nonzero homogeneous components of its tangent space.

Example 4.6. Continue with the polytope P of Example 2.2. Retain the notation $F_{y,0}, F_{y,1}, F_{z,0}, F_{z,1}$ for its four square facets. Their primitive support elements are $s_{F_{y,0}} = s_2$, $s_{F_{y,1}} = s_7$, $s_{F_{z,0}} = s_3$, and $s_{F_{z,1}} = s_8$.

By Proposition 4.5, for every $n \geq 1$ and $i \in \{2,3,7,8\}$, we have $T^1_{X_P}(-(R^* - ns_i)) \cong \mathbb{C}$. Moreover, Example 4.4 showed that $T^1_{X_P}(-R^*) \cong \mathbb{C}$. Hence

$$T^1_{X_P} = T^1_{X_P}(-R^*) \oplus \bigoplus_{i\in\{2,3,7,8\}} \bigoplus_{n\geq 1} T^1_{X_P}\big(-(R^* - ns_i)\big),$$

and every homogeneous component appearing in this decomposition is one-dimensional.

4.2. **The obstruction space.**

Proposition 4.7. *Let $P \subset N_{\mathbb{R}}$, where $N \simeq \mathbb{Z}^3$, be an empty three-dimensional lattice polytope, and let X_P be the associated four-dimensional affine Gorenstein toric variety. Then $T^2_{X_P}(-r) = 0$ for every $r \in \widetilde{M}$ with $r \neq R^*$.*

Proof. If there exists a vertex v of P such that $\varphi_r(v) > 1$, then the assertion follows from Theorem 2.6(1). Suppose therefore that $\varphi_r \leq 1$ on P. Since $r \neq R^*$, the section $F_r = P \cap (\varphi_r = 1)$ is a proper face of P, and hence $\dim F_r \leq 2$. The assertion now follows from Theorem 2.6(3). □

5. UNOBSTRUCTEDNESS OF AFFINE TERMINAL GORENSTEIN TORIC FOURFOLDS

In this section, we prove that every affine terminal Gorenstein toric fourfold is unobstructed.

Throughout this section, let $P \subset N_{\mathbb{R}}$ be an empty three-dimensional lattice polytope and let X_P be the associated affine toric fourfold. We denote by $\mathcal{S}(P)$ the set of square facets of P. For $F \in \mathcal{S}(P)$, let $s_F \in \widetilde{M}$ be the primitive generator of the ray of $\sigma^\vee$ corresponding to F, and set $m_F := R^* - s_F$.

By Proposition 4.7, the only homogeneous component of $T^2_{X_P}$ which can be nonzero is $T^2_{X_P}(-R^*)$. On the other hand, Proposition 4.5 gives $T^1_{X_P}(-m_F) \cong \mathbb{C}$ for every $F \in \mathcal{S}(P)$. Choose a tangent parameter t_F corresponding to this component and assign it degree $\deg(t_F) = m_F$.

A monomial

$$\prod_{F\in\mathcal{S}(P)} t_F^{n_F}, \qquad n_F \in \mathbb{N},$$

can contribute to a nonzero obstruction only if its degree is R^*, that is, only if

$$\sum_{F\in\mathcal{S}(P)} n_F m_F = R^*.$$

The following lemma classifies all such relations. It shows that the only possibility involves two parallel square facets, each occurring with coefficient one.

Lemma 5.1. *With the notation above, suppose that*

$$\sum_{F\in\mathcal{S}(P)} n_F (R^* - s_F) = R^*$$

for some $n_F \in \mathbb{N}$. Then there exist precisely two square facets $F_1, F_2 \in \mathcal{S}(P)$ such that $n_{F_1} = n_{F_2} = 1$, while $n_F = 0$ for every $F \notin \{F_1, F_2\}$. Moreover, F_1 and F_2 are parallel.

Proof. We first show that, for every square facet F and every vertex $v \in P \setminus F$,

$$\langle s_F, (v,1)\rangle = 1. \tag{5.1}$$

After an affine lattice change of coordinates, we may assume that

$$F = \operatorname{conv}\{(0,0,0),(1,0,0),(0,1,0),(1,1,0)\} \tag{5.2}$$

and that $\langle s_F, (x,y,z,1)\rangle = z$. Let $v = (a,b,h)$ be a vertex of P not contained in F. Then $h \geq 1$. Suppose that $h \geq 2$. The section of the pyramid $\operatorname{conv}(F,v)$ at height 1 is

$$\operatorname{conv}(F,v) \cap \{z=1\} = \left(\frac{a}{h}, \frac{b}{h}, 1\right) + \left(1 - \frac{1}{h}\right)([0,1]^2 \times \{0\}).$$

Indeed, every point of $\operatorname{conv}(F,v)$ can be written as $(1-\lambda)w + \lambda v$, where $w = (x,y,0) \in F$ and $0 \leq \lambda \leq 1$. Its z-coordinate is λh, so a point lies in the section $z = 1$ precisely when $\lambda = 1/h$. Consequently, the points of this section are exactly

$$\left(\frac{a}{h} + \left(1 - \frac{1}{h}\right)x,\, \frac{b}{h} + \left(1 - \frac{1}{h}\right)y,\, 1\right), \qquad (x,y) \in [0,1]^2,$$

which verifies (5.2).

We next verify that this section contains a lattice point. Write $a = dh + r$, where $0 \leq r < h$. If $r = 0$, then $\lceil a/h \rceil = a/h$. If $r > 0$, then $\lceil a/h \rceil - a/h = (h-r)/h \leq (h-1)/h = 1 - 1/h$. Thus $\lceil a/h \rceil \in [a/h, a/h + 1 - 1/h]$, and the same argument applies to b. Hence

$$q := \left(\left\lceil \frac{a}{h} \right\rceil, \left\lceil \frac{b}{h} \right\rceil, 1\right)$$

belongs to $\operatorname{conv}(F,v) \cap \{z=1\}$ and is a lattice point.

Since $0 < 1 < h$, the point q is neither a vertex of F nor the vertex v. Moreover, it is a nontrivial convex combination of v and a point of F, so it is not a vertex of P. This contradicts the assumption that P is empty. Therefore $h = 1$, proving (5.1).

Consequently, for every vertex v of P,

$$\langle R^* - s_F, (v,1)\rangle = \begin{cases} 1, & v \in F, \\ 0, & v \notin F. \end{cases}$$

Pairing

$$\sum_{F \in \mathcal{S}(P)} n_F (R^* - s_F) = R^*$$

with $(v,1)$, for an arbitrary vertex v of P, gives

$$\sum_{\substack{F \in \mathcal{S}(P) \\ v \in F}} n_F = 1. \tag{5.3}$$

Set $\mathcal{S}' := \{F \in \mathcal{S}(P) \mid n_F > 0\}$. Equation (5.3) shows that $n_F = 1$ for every $F \in \mathcal{S}'$, that distinct facets in $\mathcal{S}'$ are vertex-disjoint, and that every vertex of P belongs to exactly one such facet. Since every facet in $\mathcal{S}'$ has four vertices and P has at most eight vertices by Lemma 4.1, we have $|\mathcal{S}'| \leq 2$. Moreover, $|\mathcal{S}'| \neq 1$, since otherwise all vertices of P would lie in a single two-dimensional facet. Hence

$$\mathcal{S}' = \{F_1, F_2\}, \qquad n_{F_1} = n_{F_2} = 1,$$

and all the remaining coefficients vanish.

The original relation becomes

$$(R^* - s_{F_1}) + (R^* - s_{F_2}) = R^*,$$

or equivalently $s_{F_1} + s_{F_2} = R^*$. Applying the projection $\pi_M \colon \widetilde{M} \longrightarrow M$ gives

$$\pi_M(s_{F_1}) + \pi_M(s_{F_2}) = 0.$$

Thus the primitive normal vectors of F_1 and F_2 are opposite, and hence F_1 and F_2 are parallel. □

Example 5.2. Let P be the polytope of Example 2.2. For its four square facets, the corresponding primitive tangent degrees are

$$m_{y,0} = R^* - s_{F_{y,0}} = R^* - s_2 = s_7, \qquad m_{y,1} = R^* - s_{F_{y,1}} = R^* - s_7 = s_2,$$
$$m_{z,0} = R^* - s_{F_{z,0}} = R^* - s_3 = s_8, \qquad m_{z,1} = R^* - s_{F_{z,1}} = R^* - s_8 = s_3.$$

Consequently, the two pairs of opposite square facets give the relations

$$m_{y,0} + m_{y,1} = s_7 + s_2 = R^* \qquad \text{and} \qquad m_{z,0} + m_{z,1} = s_8 + s_3 = R^*.$$

In Example 3.9, we constructed a two-parameter deformation associated with each of these two relations. Thus this polytope admits two distinct two-parameter deformation families arising from its two pairs of opposite square facets.

Theorem 5.3. *Let X_P be an affine terminal Gorenstein toric variety with $\dim X_P \leq 4$. Then X_P is unobstructed.*

Proof. The assertion for $\dim X_P \leq 3$ follows from Lemma 4.3. We therefore assume that $\dim X_P = 4$, so that P is an empty three-dimensional lattice polytope.

By Proposition 4.5, for each square facet $F \in \mathcal{S}(P)$, the corresponding nonzero tangent components occur in degrees $-(R^* - ns_F)$, with $n \geq 1$. We call the component corresponding to $n = 1$ the primitive tangent component associated with F.

We first restrict to the finite-dimensional primitive tangent subspace

$$V_{\text{prim}} := T^1_{X_P}(-R^*) \oplus \bigoplus_{F \in \mathcal{S}(P)} T^1_{X_P}\big(-(R^* - s_F)\big).$$

Thus V_{prim} consists of the Gorenstein-degree component together with the primitive tangent component associated with each square facet of P.

For every $F \in \mathcal{S}(P)$, put $m_F := R^* - s_F$. By (5.1),

$$\langle m_F, (v,1) \rangle = \begin{cases} 1, & v \in F, \\ 0, & v \notin F. \end{cases}$$

Consequently, $m_F \in \sigma^\vee \setminus \{0\}$.

By Lemma 4.2, the facet F is lattice equivalent to the unit square. Hence there exist a vertex $p \in F$ and primitive lattice vectors $e_1, e_2 \in N$ such that

$$F = p + [0, e_1] + [0, e_2].$$

Since e_1 and e_2 are parallel to F, they are annihilated by $\pi_M(s_F)$, and therefore also by $\pi_M(m_F) = -\pi_M(s_F)$. Thus $Q_F := [0, e_1]$ is a primitive lattice segment contained in $\ker \pi_M(m_F)$, parallel to an edge of F, and a Minkowski summand of F. Then (m_F, Q_F) is a deformation pair. Let t_F denote the parameter of the corresponding one-parameter deformation and set

$$\xi_F := \mathrm{KS}\left(\frac{\partial}{\partial t_F}\right) \in T^1_{X_P}(-m_F).$$

By restricting (3.7) to the t_F-axis, its first-order part is

$$F_{\mathbf{k}} = f_{\mathbf{k}} - \eta_{Q_F}(\mathbf{k}) t_F \chi^{s_{\mathbf{k}} - m_F} + O(t_F^2).$$

Hence ξ_F is the Minkowski-summand class determined by Q_F. Under the identification of Theorem 2.3, this class is nonzero because Q_F is a nontrivial Minkowski summand of the unit square F. Since $T^1_{X_P}(-m_F)$ is one-dimensional, ξ_F is a generator.

Let $d := \dim_{\mathbb{C}} T^1_{X_P}(-R^*)$ and choose a basis $\zeta_1, \ldots, \zeta_d$ of $T^1_{X_P}(-R^*)$. Introduce variables $z_1, \ldots, z_d$ of degree R^*, and put

$$R_{\mathrm{prim}} := \mathbb{C}\left[\left[z_1, \ldots, z_d, (t_F)_{F \in \mathcal{S}(P)}\right]\right],$$

where

$$\deg(z_i) = R^*, \qquad \deg(t_F) = m_F = R^* - s_F.$$

Let

$$\mathfrak{m}_{\mathrm{prim}} := \big(z_1, \ldots, z_d, (t_F)_{F \in \mathcal{S}(P)}\big)$$

and, for $k \geq 1$, set

$$R^{(k)}_{\mathrm{prim}} := R_{\mathrm{prim}} / \mathfrak{m}^{k+1}_{\mathrm{prim}}.$$

Since $T^1_{X_P}$ may be infinite-dimensional, no complete local noetherian base can carry independent parameters for all its tangent directions. We therefore begin with the finite-dimensional subspace V_{prim} and construct a formal deformation in all orders.

More precisely, for each $k \geq 1$, we construct inductively a graded k-th order deformation

$$\mathcal{X}^{(k)}_{\mathrm{prim}} \longrightarrow \operatorname{Spec} R^{(k)}_{\mathrm{prim}}$$

of X_P. These deformations are compatible in the sense that the pullback of $\mathcal{X}^{(k+1)}_{\mathrm{prim}}$ to $\operatorname{Spec} R^{(k)}_{\mathrm{prim}}$ is isomorphic to $\mathcal{X}^{(k)}_{\mathrm{prim}}$. Their Kodaira–Spencer maps identify the coordinate tangent vectors with

$$\zeta_1, \ldots, \zeta_d, \qquad (\xi_F)_{F \in \mathcal{S}(P)}.$$

The total degree-zero element

$$\sum_{i=1}^{d} \zeta_i \otimes z_i + \sum_{F \in \mathcal{S}(P)} \xi_F \otimes t_F \in T^1_{X_P} \otimes_{\mathbb{C}} \mathfrak{m}_{\mathrm{prim}} / \mathfrak{m}^2_{\mathrm{prim}}$$

determines a graded first-order deformation $\mathcal{X}^{(1)}_{\mathrm{prim}} \longrightarrow \operatorname{Spec} R^{(1)}_{\mathrm{prim}}$.

Suppose inductively that $\mathcal{X}^{(k)}_{\mathrm{prim}}$ has been constructed. The obstruction to lifting it to $R^{(k+1)}_{\mathrm{prim}}$ is a sum of terms

$$\omega_{\boldsymbol{\alpha}, \mathbf{a}} \otimes z_1^{\alpha_1} \cdots z_d^{\alpha_d} \prod_{F \in \mathcal{S}(P)} t_F^{a_F},$$

where

$$\omega_{\boldsymbol{\alpha},\mathbf{a}} \in T^2_{X_P}\left(-\left(|\boldsymbol{\alpha}|R^* + \sum_{F\in\mathcal{S}(P)} a_F m_F\right)\right)$$

and

$$|\boldsymbol{\alpha}| + \sum_F a_F = k+1.$$

By Proposition 4.7, this coefficient can be nonzero only if

$$|\boldsymbol{\alpha}|R^* + \sum_{F\in\mathcal{S}(P)} a_F m_F = R^*. \tag{5.4}$$

Suppose first that $|\boldsymbol{\alpha}| \geq 1$. Equation (5.4) is equivalent to

$$\big(|\boldsymbol{\alpha}| - 1\big)R^* + \sum_{F\in\mathcal{S}(P)} a_F m_F = 0.$$

All the terms in this sum belong to the strongly convex cone $\sigma^\vee$. Hence every term is zero. Since $R^* \neq 0$ and $m_F \neq 0$, we obtain $|\boldsymbol{\alpha}| = 1$ and $a_F = 0$ for every F. This gives a monomial of ordinary order one, whereas an obstruction to lifting $R^{(k)}_{\text{prim}}$ to $R^{(k+1)}_{\text{prim}}$ has ordinary order $k+1 \geq 2$. Consequently, no obstruction term can involve one of the variables z_i.

It remains to consider monomials of the form

$$\prod_{F\in\mathcal{S}(P)} t_F^{a_F}.$$

Their degree is R^* precisely when

$$\sum_{F\in\mathcal{S}(P)} a_F(R^* - s_F) = R^*.$$

By Lemma 5.1, this can occur only when there are two parallel square facets F_1, F_2 such that

$$a_{F_1} = a_{F_2} = 1, \qquad a_F = 0 \quad \text{for } F \notin \{F_1, F_2\},$$

and $s_{F_1} + s_{F_2} = R^*$. Thus the only possible obstruction monomial is $t_{F_1}t_{F_2}$. In particular, such an obstruction can occur only in the passage from first to second order.

Let $\omega_{F_1,F_2} \in T^2_{X_P}(-R^*)$ denote the coefficient of this monomial in the second-order obstruction. Put

$$B := \mathbb{C}[[u_1, u_2]], \qquad \mathfrak{n} := (u_1, u_2), \qquad B^{(k)} := B/\mathfrak{n}^{k+1}.$$

Consider the graded homomorphism $R^{(1)}_{\text{prim}} \longrightarrow B^{(1)}$ given by

$$t_{F_1} \longmapsto u_1, \qquad t_{F_2} \longmapsto u_2,$$

and by sending all the remaining variables to zero.

By Corollary 3.8, there is a formal two-parameter deformation over B whose Kodaira–Spencer classes in the u_1- and u_2-directions are ξ_{F_1} and ξ_{F_2}, respectively. Its first-order truncation therefore represents the pullback of $\mathcal{X}^{(1)}_{\text{prim}}$, while its second-order truncation provides a lifting of that first-order deformation to $B^{(2)}$.

By functoriality of obstruction classes, the pullback of the second-order obstruction is consequently zero. On the other hand, this pullback is

$$\omega_{F_1,F_2} \otimes u_1u_2 \in T^2_{X_P}(-R^*) \otimes_{\mathbb{C}} \mathfrak{n}^2/\mathfrak{n}^3.$$

Since

$$u_1u_2 \neq 0 \qquad \text{in} \qquad \mathfrak{n}^2/\mathfrak{n}^3,$$

it follows that $\omega_{F_1,F_2} = 0$.

Equivalently, after choosing compatible parameter coordinates, the comparison homomorphism has the form

$$t_{F_i} \longmapsto u_i + O(\mathfrak{n}^2), \qquad i = 1, 2.$$

Therefore

$$t_{F_1}t_{F_2} \longmapsto u_1u_2 \pmod{\mathfrak{n}^3},$$

so a higher-order change of parameter cannot alter the coefficient of the mixed quadratic term.

We have proved that every second-order obstruction vanishes. For orders at least three, the degree calculation above shows that there is no monomial whose coefficient could lie in a nonzero homogeneous component of $T^2_{X_P}$. We may therefore construct inductively compatible graded liftings

$$\mathcal{X}^{(k+1)}_{\text{prim}} \longrightarrow \operatorname{Spec} R^{(k+1)}_{\text{prim}}$$

for all $k \geq 1$. For a complete local ring $(R, \mathfrak{m})$, we write $\operatorname{Spf} R$ for its formal spectrum, namely the affine formal scheme determined by the infinitesimal thickenings $\operatorname{Spec}(R/\mathfrak{m}^{k+1})$, $k \geq 0$. Passing to the inverse limit gives a formal deformation $\mathcal{X}_{\text{prim}} \longrightarrow \operatorname{Spf} R_{\text{prim}}$ whose Kodaira–Spencer map is an isomorphism onto

$$V_{\text{prim}} = T^1_{X_P}(-R^*) \oplus \bigoplus_{F \in \mathcal{S}(P)} T^1_{X_P}\big(-(R^* - s_F)\big).$$

We next incorporate the remaining homogeneous tangent directions. For every square facet F, let $x_F \in \mathbb{C}[S_P]$ denote the character χ^{s_F}. For $n \geq 1$, put $r_{F,n} := R^* - ns_F$. By (5.1),

$$\langle r_{F,n}, (v, 1)\rangle = \begin{cases} 1, & v \in F, \\ 1 - n, & v \notin F. \end{cases}$$

It follows directly from the definition of the sets K^r_τ that $K^{r_{F,n}}_\tau = K^{r_{F,1}}_\tau$ for every face $\tau \preceq \sigma$. Indeed, at a vertex of F the upper bound is one for every n, while at a vertex outside F it is nonpositive, so the corresponding K-set is empty.

By [AS98, Theorem 3.3], multiplication by x_F^{n-1} is induced on the complexes computing $T^1_{X_P}$ by the inclusions $K^{r_{F,n}}_\tau \subseteq K^{r_{F,1}}_\tau$. Since these inclusions are equalities, multiplication induces an isomorphism

$$x_F^{n-1} \cdot - \colon T^1_{X_P}\big(-(R^* - s_F)\big) \xrightarrow{\sim} T^1_{X_P}\big(-(R^* - ns_F)\big).$$

Consequently,

$$T^1_{X_P}\big(-(R^* - ns_F)\big) = \mathbb{C}\, x_F^{n-1}\xi_F. \tag{5.5}$$

We shall use the following flatness lemma for the coordinate-dependent substitution below.

Lemma 5.4. *Let $(B, \mathfrak{m}_B)$ and $(C, \mathfrak{m}_C)$ be complete local Noetherian $\mathbb{C}$-algebras with residue field $\mathbb{C}$, let $\mathcal{P}_0$ be a polynomial $\mathbb{C}$-algebra, and put*

$$\mathcal{P}_B := B \widehat{\otimes}_{\mathbb{C}} \mathcal{P}_0, \qquad \mathcal{P}_C := C \widehat{\otimes}_{\mathbb{C}} \mathcal{P}_0.$$

Suppose that

$$\mathcal{A}_B = \mathcal{P}_B/(G_1, \ldots, G_e)$$

is B-flat. Then one can choose finitely many syzygies

$$V_1, \ldots, V_b \in \mathcal{P}_B^{\oplus e}, \qquad (G_1, \ldots, G_e)V_j = 0,$$

whose reductions modulo $\mathfrak{m}_B$ generate all syzygies among the reduced equations $g_1, \ldots, g_e$.

Let $\Phi \colon \mathcal{P}_B \to \mathcal{P}_C$ be a continuous $\mathbb{C}$-algebra homomorphism such that

$$\Phi(\mathfrak{m}_B \mathcal{P}_B) \subseteq \mathfrak{m}_C \mathcal{P}_C$$

and whose reduction modulo the maximal ideals is the identity on $\mathcal{P}_0$. Put

$$G_i' := \Phi(G_i), \qquad V_j' := \Phi(V_j).$$

If $(G_1', \ldots, G_e')V_j' = 0$ for every j, then

$$\mathcal{P}_C/(G_1', \ldots, G_e')$$

is C-flat.

Proof. Set $I_B := (G_1, \ldots, G_e) \subset P_B$, and let $K_B := \ker(P_B^{\oplus e} \to I_B)$, where the i-th basis vector maps to G_i. By construction, P_B is flat over B, while $\mathcal{A}_B = P_B/I_B$ is flat over B by assumption. The exact sequence

$$0 \longrightarrow I_B \longrightarrow P_B \longrightarrow \mathcal{A}_B \longrightarrow 0$$

therefore implies that I_B is flat over B.

Tensoring this sequence and

$$0 \longrightarrow K_B \longrightarrow P_B^{\oplus e} \longrightarrow I_B \longrightarrow 0$$

with $B/\mathfrak{m}_B \cong \mathbb{C}$, and using the flatness of $\mathcal{A}_B$ and I_B, gives

$$K_B/\mathfrak{m}_B K_B \cong \ker\bigl(P_0^{\oplus e} \longrightarrow I_0\bigr), \qquad I_0 := (g_1, \ldots, g_e).$$

Since P_0 is noetherian, the module on the right is finitely generated. Choose generators and lift them to syzygies $V_1, \ldots, V_b \in K_B$. Their reductions modulo $\mathfrak{m}_B$ generate all syzygies among $g_1, \ldots, g_e$. If the data are graded, the generators and their lifts may be chosen homogeneous.

Put $\mathcal{A}_C := P_C/(G_1', \ldots, G_e')$. Since $G_i' = \Phi(G_i)$ and $V_j' = \Phi(V_j)$, we have $(G_1', \ldots, G_e')V_j' = 0$. Moreover, because Φ reduces to the identity on P_0, the reductions of the G_i' and V_j' are g_i and $V_j \bmod \mathfrak{m}_B$, respectively.

For $q \geq 0$, put $C_q := C/\mathfrak{m}_C^{q+1}$ and $P_q := P_C/\mathfrak{m}_C^{q+1} P_C$, and let $I_q \subset P_q$ be the ideal generated by the images of $G_1', \ldots, G_e'$. We claim that $A_q := P_q/I_q$ is flat over C_q.

Let $h \in I_q \cap \mathfrak{m}_C P_q$, and write $h = (G_1', \ldots, G_e')c$ for some $c \in P_q^{\oplus e}$. Reducing modulo $\mathfrak{m}_C$, we obtain the syzygy $(g_1, \ldots, g_e)\overline{c} = 0$. Hence $\overline{c} = \sum_{j=1}^{b} \overline{d}_j \, \overline{V}_j$ for suitable $\overline{d}_j \in P_0$. After choosing lifts $d_j \in P_q$, we have $c - \sum_j d_j V_j' \in \mathfrak{m}_C P_q^{\oplus e}$. Since $(G_1', \ldots, G_e')V_j' = 0$, it follows that $h \in \mathfrak{m}_C I_q$. Therefore

$$I_q \cap \mathfrak{m}_C P_q = \mathfrak{m}_C I_q.$$

Since $A_q/\mathfrak{m}_C A_q \cong P_0/I_0$ is a $\mathbb{C}$-vector space, the local criterion for flatness over the Artinian local ring C_q shows that A_q is flat over C_q. This is the presentation form of the equational flatness criterion; see, e.g., [Fil26, Lemma 3.5].

Finally, P_C is noetherian and $\mathfrak{m}_C$-adically complete. Hence the finitely generated ideal $(G'_1, \dots, G'_e)$ is closed, $\mathcal{A}_C$ is $\mathfrak{m}_C$-adically complete, and

$$\mathcal{A}_C \cong \varprojlim_q A_q.$$

The transition maps $A_{q+1} \to A_q$ are surjective, and every A_q is flat over C_q. The inverse-limit criterion for flatness over a noetherian ring therefore implies that $\mathcal{A}_C$ is flat over C. $\square$

We now perform the substitution at the level of equations and relations. Let

$$\mathcal{P}_0 := \mathbb{C}[\mathbf{x}, u], \qquad A_0 := \mathbb{C}[S_P] = \mathcal{P}_0/I_P,$$

and put

$$\mathcal{P}_{\text{prim}} := R_{\text{prim}} \widehat{\otimes}_{\mathbb{C}} \mathcal{P}_0.$$

After lifting the ambient coordinate functions to the primitive family, write

$$\mathcal{A}_{\text{prim}} = \mathcal{P}_{\text{prim}}/I_{\text{prim}}.$$

The ring $\mathcal{P}_{\text{prim}}$ is Noetherian, so we may choose finite homogeneous generators

$$G_1, \dots, G_e$$

of I_{prim}. Since $\mathcal{A}_{\text{prim}}$ is flat over R_{prim}, their reductions $g_1, \dots, g_e$ generate I_P. Choose finitely many homogeneous syzygies among the G_i as in Lemma 5.4, and let v_{prim} be the matrix whose columns are these syzygies. Let u_{prim} be the row matrix with components G_i, and let u_0, v_0 denote their reductions modulo $\mathfrak{m}_{\text{prim}}$. We then have a homogeneous complex

$$\mathcal{P}_{\text{prim}}^{\oplus b} \xrightarrow{v_{\text{prim}}} \mathcal{P}_{\text{prim}}^{\oplus e} \xrightarrow{u_{\text{prim}}} \mathcal{P}_{\text{prim}} \longrightarrow \mathcal{A}_{\text{prim}} \longrightarrow 0, \tag{5.6}$$

with $u_{\text{prim}} \circ v_{\text{prim}} = 0$, whose reduction is the finite homogeneous presentation

$$\mathcal{P}_0^{\oplus b} \xrightarrow{v_0} \mathcal{P}_0^{\oplus e} \xrightarrow{u_0} \mathcal{P}_0 \longrightarrow A_0 \longrightarrow 0. \tag{5.7}$$

In particular, the columns of v_0 generate all relations among the central equations g_i.

For every $F \in \mathcal{S}(P)$, choose a homogeneous polynomial, still denoted by x_F, in $\mathcal{P}_0$ whose image in A_0 is χ^{s_F}. For $N \geq 1$, introduce parameters

$$t_{F,n}, \qquad F \in \mathcal{S}(P), \quad 1 \leq n \leq N,$$

of degree $\deg(t_{F,n}) = R^* - ns_F$, and put

$$R_N := \mathbb{C}\left[\left[z_1, \dots, z_d, (t_{F,n})_{\substack{F \in \mathcal{S}(P) \\ 1 \leq n \leq N}}\right]\right].$$

Set

$$\widehat{t}_F^{(N)} := \sum_{n=1}^{N} x_F^{n-1} t_{F,n}.$$

Every summand is homogeneous of degree

$$\deg\bigl(x_F^{n-1} t_{F,n}\bigr) = (n-1)s_F + (R^* - ns_F) = R^* - s_F = \deg(t_F).$$

Let $\mathcal{P}_N := R_N \widehat{\otimes}_{\mathbb{C}} \mathcal{P}_0$. There is a continuous homogeneous $\mathbb{C}$-algebra homomorphism

$$\Phi_N : \mathcal{P}_{\mathrm{prim}} \longrightarrow \mathcal{P}_N$$

which is the identity on $\mathcal{P}_0$ and on the variables z_i, and satisfies $\Phi_N(t_F) = \widehat{t}_F^{(N)}$. Notice that Φ_N is not asserted to be a base change homomorphism: the coefficients x_F^{n-1} belong to the ambient coordinate ring.

Apply Φ_N to both matrices in (5.6), and set

$$u_N := \Phi_N(u_{\mathrm{prim}}), \qquad v_N := \Phi_N(v_{\mathrm{prim}}).$$

Then

$$u_N \circ v_N = \Phi_N(u_{\mathrm{prim}} \circ v_{\mathrm{prim}}) = 0.$$

Moreover, modulo the maximal ideal of R_N, the resulting complex reduces to (5.7). Thus

$$\mathcal{P}_N^{\oplus b} \xrightarrow{v_N} \mathcal{P}_N^{\oplus e} \xrightarrow{u_N} \mathcal{P}_N \longrightarrow \mathcal{A}_N \longrightarrow 0, \qquad \mathcal{A}_N := \mathcal{P}_N / \operatorname{im}(u_N),$$

is a complex which is exact except possibly at $\mathcal{P}_N^{\oplus e}$, and whose reduction over the closed point is exact.

The homomorphism Φ_N preserves the parameter ideals and induces the identity on the closed fibre. The equality $u_N \circ v_N = 0$ shows that all substituted syzygies still vanish, and their reductions generate all syzygies among the central equations. Lemma 5.4 therefore implies that $\mathcal{A}_N$ is flat over R_N. Hence it defines a formal deformation $\mathcal{X}_N \longrightarrow \operatorname{Spf} R_N$ of X_P. Writing $G_i^{(N)} := \Phi_N(G_i)$, we have

$$\left.\frac{\partial G_i^{(N)}}{\partial t_{F,n}}\right|_{\mathbf{z}=\mathbf{t}=0} = x_F^{n-1} \left.\frac{\partial G_i}{\partial t_F}\right|_{\mathbf{z}=\mathbf{t}=0}.$$

By the $\mathbb{C}[S_P]$-module structure on $T^1_{X_P}$, this gives

$$\mathrm{KS}\left(\frac{\partial}{\partial t_{F,n}}\right) = x_F^{n-1} \xi_F.$$

By (5.5), this is a generator of $T^1_{X_P}\big(-(R^* - n s_F)\big)$. The z_i-directions map to the chosen basis $\zeta_1, \dots, \zeta_d$. Consequently, the Kodaira–Spencer map of $\mathcal{X}_N$ is an isomorphism onto

$$V_N := T^1_{X_P}(-R^*) \oplus \bigoplus_{F \in \mathcal{S}(P)} \bigoplus_{1 \le n \le N} T^1_{X_P}\big(-(R^* - n s_F)\big).$$

The families $\mathcal{X}_N$ are compatible as N varies: $\mathcal{X}_{N+1}$ restricts to $\mathcal{X}_N$ after setting $t_{F,N+1} = 0$ for every F. Furthermore,

$$T^1_{X_P} = \bigcup_{N \ge 1} V_N.$$

We claim that every deformation of X_P over a local Artinian $\mathbb{C}$-algebra B is induced from $\mathcal{X}_N$ for some N.

We prove this by induction on the length of B. The assertion is clear for $B = \mathbb{C}$. Choose a small extension

$$0 \longrightarrow J \longrightarrow B \longrightarrow \overline{B} \longrightarrow 0$$

and let $\mathcal{Y}$ be a deformation of X_P over B. By induction, its restriction $\overline{\mathcal{Y}} := \mathcal{Y} \otimes_B \overline{B}$ is induced from $\mathcal{X}_N$ by a homomorphism $\varphi_N : R_N \longrightarrow \overline{B}$ for some N.

Since R_N is a formal power-series ring in finitely many variables, φ_N lifts to a homomorphism

$$\widetilde{\varphi}_N : R_N \longrightarrow B.$$

Let $\mathcal{Y}_0$ be the corresponding pullback of $\mathcal{X}_N$. Then $\mathcal{Y}$ and $\mathcal{Y}_0$ are two liftings of $\overline{\mathcal{Y}}$ across the same small extension. Their difference is represented by an element $\delta \in T^1_{X_P} \otimes_{\mathbb{C}} J$.

Since $T^1_{X_P}$ is the algebraic direct sum of its homogeneous components, δ has finite homogeneous support. Hence there exists $N' \geq N$ such that $\delta \in V_{N'} \otimes_{\mathbb{C}} J$. Regard $\widetilde{\varphi}_N$ as a homomorphism $R_{N'} \longrightarrow B$ by sending the additional parameters to zero. Since the Kodaira–Spencer map of $\mathcal{X}_{N'}$ is an isomorphism onto $V_{N'}$, there exists

$$\lambda \in \mathrm{Hom}_{\mathbb{C}}\left(\mathfrak{m}_{R_{N'}}/\mathfrak{m}^2_{R_{N'}}, J\right)$$

whose image under the Kodaira–Spencer map is δ.

Because the extension is small, one has $\mathfrak{m}_B J = 0$. We may therefore modify the parameter homomorphism by setting

$$\widetilde{\varphi}'_{N'}(q) = \widetilde{\varphi}_{N'}(q) + \lambda(\overline{q})$$

for each parameter q, where $\overline{q}$ denotes its class in $\mathfrak{m}_{R_{N'}}/\mathfrak{m}^2_{R_{N'}}$. This again defines a local homomorphism

$$\widetilde{\varphi}'_{N'} : R_{N'} \longrightarrow B$$

lifting the original homomorphism to $\overline{B}$. The corresponding change of the pullback deformation is exactly δ. Hence the pullback defined by $\widetilde{\varphi}'_{N'}$ is isomorphic to $\mathcal{Y}$, proving the claim.

Finally, let $B' \longrightarrow B$ be a small extension of local Artinian $\mathbb{C}$-algebras and let $\mathcal{Y}$ be a deformation of X_P over B. By the claim, $\mathcal{Y}$ is induced from $\mathcal{X}_N$, for some N, by a homomorphism

$$R_N \longrightarrow B.$$

Since R_N is a formal power-series ring in finitely many variables, this homomorphism lifts to a homomorphism $R_N \longrightarrow B'$. Pulling back $\mathcal{X}_N$ gives a lifting of $\mathcal{Y}$ to B'. Therefore

$$\mathrm{Def}_{X_P}(B') \longrightarrow \mathrm{Def}_{X_P}(B)$$

is surjective for every small extension $B' \to B$. Thus Def_{X_P} is formally smooth, and X_P is unobstructed. □

REFERENCES

[Alt97] K. Altmann: *The versal deformation of an isolated, toric Gorenstein singularity*, Invent. Math. **128** (1997) 443–479. 3

[Alt00] K. Altmann: *One parameter families containing three-dimensional toric Gorenstein singularities*, Explicit birational geometry of 3-folds, London Math. Soc. Lecture Note Ser., vol. **281**, Cambridge Univ. Press, Cambridge (2000), 21–50. 5, 6

[ACF22a] K. Altmann, A. Constantinescu, and M. Filip, *Versality in toric geometry*, J. Algebra **609** (2022), 1–43. 10

[AS98] K. Altmann, A. B. Sletsjøe: *André-Quillen cohomology of monoid algebras*, J. Alg. **210** (1998), 708–718. 1, 5, 6, 24

[AvS00] K. Altmann and D. van Straten, *The polyhedral Hodge number $h^{2,1}$ and vanishing of obstructions*, Tohoku Math. J. **52** (2000), 579–602. 1, 8

[BR15] M. Beck and S. Robins, *Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra*, 2nd ed., Undergraduate Texts in Mathematics, Springer, New York, 2015. 16

[CLS11] D. Cox, J. Little, H. Schenk: *Toric varieties*, Graduate Studies in Mathematics **124** (2011). 8

[Fil26] M. Filip: *Laurent polynomials and deformations of non-isolated Gorenstein toric singularities*, Adv. Math. **487** (2026). 2, 8, 11, 14, 26

[FL24] R. Friedman and R. Laza, *Higher Du Bois and higher rational singularities*, Duke Math. J. **173** (2024), no. 10, 1839–1881. 1

[FL25] R. Friedman and R. Laza, *Deformations of singular Fano and Calabi–Yau varieties*, J. Differential Geom. **131** (2025), no. 1, 65–131. 1

[GKP94] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd ed., Addison–Wesley, Reading, MA, 1994. 14

[Kaw92] Y. Kawamata, *Unobstructed deformations: a remark on a paper of Z. Ran*, J. Algebraic Geom. **1** (1992), no. 2, 183–190. 1

[Nam94] Y. Namikawa, *On deformations of Calabi–Yau 3-folds with terminal singularities*, Topology **33** (1994), no. 3, 429–446. 1

[Ran92a] Z. Ran, *Deformations of Calabi–Yau Kleinfolds*, in *Essays on Mirror Manifolds*, International Press, Hong Kong, 1992, 451–457. 1

[Ran92b] Z. Ran, *Deformations of manifolds with torsion or negative canonical bundle*, J. Algebraic Geom. **1** (1992), no. 2, 279–291. 1

[San14] T. Sano, *Unobstructedness of deformations of weak Fano manifolds*, Int. Math. Res. Not. IMRN (2014), no. 18, 5124–5133. 1

[Ser06] E. Sernesi, *Deformations of Algebraic Schemes*, Grundlehren der mathematischen Wissenschaften **334**, Springer-Verlag, Berlin, 2006. 5

[Tia92] G. Tian, *Smoothing 3-folds with trivial canonical bundle and ordinary double points*, in *Essays on Mirror Manifolds*, International Press, Hong Kong, 1992, 458–479. 1

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