A LINEAR PROGRAMMING APPROACH TO FINITE COPULA INTERPOLATION

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Abstract. We consider exact and interval-valued interpolation of multivariate copulas at arbitrary finite configurations in the unit cube. We first characterize when the prescribed data are compatible with a quasi-copula and derive explicit extremal quasi-copula envelopes. The copula interpolation problem is thereby reduced to a copula sandwich problem between these envelopes. Restricting them to the canonical rectangular grid generated by the data coordinates yields a finite linear feasibility system whose elementary-volume constraints are encoded by a cell–vertex incidence matrix. Farkas' lemma then identifies the criterion of Omladič and Stopar with an explicit certificate of infeasibility. Although this criterion is indexed by infinitely many formal cell combinations, we show that it suffices to test finitely many vertices of a sign-polytopal subdivision. We also extend the construction to k-increasing discrete quasi-copulas for $2 \leq k \leq d$.

1. Introduction

Let $E = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subseteq [0,1]^d$ be an arbitrary finite set. The finite copula interpolation problem asks whether there exists a d-copula C such that

$$C(\mathbf{x}_j) = q_j, \qquad j = 1, \ldots, n, \tag{1.1}$$

for prescribed values $q_1, \ldots, q_n \in [0,1]$. More generally, given intervals $[a_j, b_j] \subseteq [0,1]$, $j = 1, \ldots, n$, the question is whether there exists a d-copula satisfying

$$a_j \leq C(\mathbf{x}_j) \leq b_j, \qquad j = 1, \ldots, n. \tag{1.2}$$

No product-grid assumption is imposed on E: the interpolation points may form an arbitrary finite configuration in the unit cube.

Copula interpolation under finitely many pointwise constraints has previously been studied in several forms. Best-possible bounds and existence results for copulas with a prescribed value at one point are closely related to the corresponding one-point quasi-copula bounds; see [10] and the references therein. The existence and construction of multivariate copulas attaining prescribed quasi-copula values at two arbitrary points were established in [6]. In the trivariate case, a linear-programming approach was used in [2] to treat prescribed values at several points: existence is guaranteed for two arbitrary points, whereas it may fail for three points. These results motivate the general finite interpolation problem considered here, in which both the number and the coordinates of the interpolation points are arbitrary.

The interval-valued problem (1.2) is naturally related to the more general problem of finding a function between prescribed lower and upper bounds. The existence of a k-increasing multivariate function satisfying $A \leq C \leq B$ has been studied in the framework of coherence and avoidance of sure loss for standardized functions and semicopulas; see [5] and the references therein. For copulas and quasi-copulas, related discrete sandwich criteria were obtained by Omladič and Stopar in the bivariate setting [7] and subsequently in arbitrary dimension [8].

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Our starting point is the compatibility of finite data with the class of quasi-copulas. For exact data, the Fréchet–Hoeffding bounds and the directed Lipschitz inequalities give necessary compatibility conditions. We show that these conditions are also sufficient and construct explicitly the smallest and the greatest quasi-copulas interpolating the prescribed values. For interval-valued data, represented by lower and upper functions $A, B\colon E \to [0,1]$ satisfying $A \le B$, we obtain analogous extremal lower and upper quasi-copula envelopes, denoted by $\underline{Q}_A$ and $\overline{Q}_B$. The original finite interpolation constraints are then equivalent to the global sandwich condition $\underline{Q}_A \le Q \le \overline{Q}_B$. Consequently, the interval-valued copula interpolation problem is equivalent to the existence of a copula between these two quasi-copula envelopes.

The finite set E need not itself have a rectangular structure. Nevertheless, its coordinates generate a canonical rectangular grid $\mathcal{D}_E$. For each coordinate, we collect 0, 1, and all coordinate values occurring among the points of E, and then take the Cartesian product of the resulting finite coordinate sets. Every point of E is a node of the grid $\mathcal{D}_E$. The generated grid is therefore an auxiliary finite-dimensional representation of the original interpolation problem, rather than an additional assumption on the geometry of the data. Restricting the extremal envelopes to the generated grid gives discrete lower and upper bounds $A_E := \underline{Q}_A|_{\mathcal{D}_E}$ and $B_E := \overline{Q}_B|_{\mathcal{D}_E}$. The existence of a copula satisfying the original interval constraints is thereby transformed into a finite discrete copula sandwich problem. More precisely, we seek a discrete copula C_E on $\mathcal{D}_E$ such that $A_E \le C_E \le B_E$.

We formulate this discrete sandwich problem in terms of values assigned to the grid nodes. The elementary-cell volumes are encoded by a cell–vertex incidence matrix H_E. If $\mathbf{c} = (C_E(\mathbf{z}_1), \dots, C_E(\mathbf{z}_N))^\intercal$ is the vector of node values, then $H_E\mathbf{c} \ge \mathbf{0}$ expresses the nonnegativity of all elementary-cell volumes. The sandwich problem is therefore equivalent to the finite feasibility system $H_E\mathbf{c} \ge \mathbf{0}$, $\mathbf{a}_E \le \mathbf{c} \le \mathbf{b}_E$, where $\mathbf{a}_E$ and $\mathbf{b}_E$ are the node-value vectors of A_E and B_E, respectively.

Applying Farkas' lemma to this system gives a new interpretation of the $L^{(A,B)}$-criterion introduced by Omladič and Stopar [7, 8]. A nonnegative multiplier vector assigns weights to the elementary cells and determines a formal nonnegative combination R, whose node-multiplicity vector is obtained by applying $H_E^\intercal$ to the multiplier vector. The support function of the interval box $[\mathbf{a}_E, \mathbf{b}_E]$ in this direction is precisely $L^{(A_E,B_E)}(R)$. We prove that the discrete copula sandwich problem is feasible if and only if $L^{(A_E,B_E)}(R) \ge 0$ for every formal nonnegative integer combination R of elementary cells. Equivalently, a combination with $L^{(A_E,B_E)}(R) < 0$ is an explicit certificate of infeasibility. The canonical-grid construction therefore extends the discrete sandwich criterion from bounds on a prescribed rectangular grid to data on an arbitrary finite set.

Although the $L^{(A_E,B_E)}$-criterion is initially formulated through infinitely many formal combinations, we show that it reduces to finitely many inequalities. The associated dual functional Φ_E is continuous, positively homogeneous, and piecewise linear. Positive homogeneity allows us to normalize every nonzero nonnegative Farkas multiplier to the standard simplex. The signs of the coordinates of the corresponding node-multiplicity vector subdivide this simplex into finitely many sign polytopes. On each sign polytope, Φ_E is linear. It is therefore sufficient to verify its nonnegativity at the vertices of all nonempty sign polytopes.

Finally, we extend this construction from d-increasing copulas to k-increasing discrete quasi-copulas, where $k \in \{2, \dots, d\}$. Elementary d-dimensional cells are replaced by elementary k-dimensional coordinate boxes, whose volumes are encoded by a k-cell–vertex incidence matrix $H_E^{(k)}$. The same Farkas argument yields a corresponding $L_k^{(A_E,B_E)}$-criterion and a finite reduction to the vertices of sign polytopes. For $k = d$, this construction reduces to the discrete copula sandwich criterion.

Organization of the paper. Section 2 fixes the notation and recalls the relevant continuous and discrete notions. Section 3 establishes the finite-data compatibility conditions and the extremal quasi-copula envelopes. Section 4 constructs the canonical grid, proves the finite copula interpolation criterion, and illustrates its primal and dual certificates. Section 5 extends the argument to k-increasing discrete quasi-copulas.

2. Preliminaries

Throughout the paper, $d \geq 2$, inequalities between vectors are understood coordinatewise, and $[d] = \{1, \ldots, d\}$.

2.1. **Copulas and quasi-copulas.** For

$$\mathbf{x} = (x_1, \ldots, x_d), \qquad \mathbf{y} = (y_1, \ldots, y_d) \in [0,1]^d$$

satisfying $\mathbf{x} \leq \mathbf{y}$, a d-box is a set of the form

$$B = [\mathbf{x}, \mathbf{y}] := \prod_{\ell=1}^{d} [x_\ell, y_\ell].$$

The set of vertices of B is

$$\operatorname{ver}(B) := \prod_{\ell=1}^{d} \{x_\ell, y_\ell\}.$$

The box B is called *nondegenerate* if $x_\ell < y_\ell$ for every $\ell \in [d]$, and degenerate otherwise.

Let F be a real-valued function whose domain contains $\operatorname{ver}(B)$. If B is nondegenerate, the F-volume of B is defined by

$$V_F(B) := \sum_{\mathbf{z} \in \operatorname{ver}(B)} (-1)^{\#\{\ell \in [d] \colon z_\ell = x_\ell\}} F(\mathbf{z}). \tag{2.1}$$

If B is degenerate, we put $V_F(B) := 0$. The function F is called *d-increasing* if

$$V_F(B) \geq 0$$

for every d-box $B \subseteq [0,1]^d$.

A function $C \colon [0,1]^d \to [0,1]$ is called a *d-copula*, or simply a *copula*, if it satisfies the following conditions:

(1) C is *grounded*, i.e., $C(u_1, \ldots, u_d) = 0$ whenever $u_\ell = 0$ for some $\ell \in [d]$;
(2) C has *uniform one-dimensional marginals*, i.e., $C(1, \ldots, 1, u_\ell, 1, \ldots, 1) = u_\ell$ for every $u_\ell \in [0,1]$ and every $\ell \in [d]$;
(3) C is d-increasing.

A function $Q \colon [0,1]^d \to [0,1]$ is called a *d-quasi-copula*, or simply a *quasi-copula*, if it satisfies the following conditions:

(1) Q is grounded;
(2) Q has uniform one-dimensional margins;
(3) Q is increasing in each variable;
(4) Q is 1-*Lipschitz*, i.e.,

$$|Q(\mathbf{u}) - Q(\mathbf{v})| \leq \sum_{\ell=1}^{d} |u_\ell - v_\ell|$$

for all $\mathbf{u}, \mathbf{v} \in [0,1]^d$.

Here and throughout the paper, increasing means nondecreasing. Every d-copula is a d-quasi-copula, but the converse does not hold in general. For standard background, see [9, 3]; for the lattice structure of multivariate copulas and quasi-copulas, see [4].

Every d-quasi-copula Q satisfies the Fréchet–Hoeffding bounds [3, Section 3.3]; see also [1]:

$$W_d(\mathbf{u}) \leq Q(\mathbf{u}) \leq M_d(\mathbf{u}), \qquad \mathbf{u} \in [0,1]^d,$$

where

$$W_d(\mathbf{u}) := \max\left\{0, \sum_{\ell=1}^{d} u_\ell - d + 1\right\}, \qquad M_d(\mathbf{u}) := \min_{1\le\ell\le d} u_\ell. \tag{2.2}$$

2.2. **k-increasing functions.** We will also use lower-dimensional coordinate boxes. Let $J \subseteq [d]$ be a nonempty set of indices. A $|J|$-dimensional coordinate box is a set of the form

$$B_J = \prod_{\ell=1}^{d} I_\ell,$$

where

$$I_\ell = \begin{cases} [x_\ell, y_\ell], & \ell \in J, \\ \{q_\ell\}, & \ell \notin J, \end{cases}$$

for some $x_\ell < y_\ell$ when $\ell \in J$ and some fixed $q_\ell \in [0,1]$ when $\ell \notin J$. The coordinates indexed by J are called *active*, and the dimension of B_J is $|J|$.

The set of vertices of B_J is

$$\operatorname{ver}(B_J) = \left\{\mathbf{z} \in [0,1]^d : z_\ell \in \{x_\ell, y_\ell\} \text{ for } \ell \in J, \quad z_\ell = q_\ell \text{ for } \ell \notin J\right\}.$$

For a real-valued function F whose domain contains $\operatorname{ver}(B_J)$, the F-volume of B_J is defined by

$$V_F(B_J) := \sum_{\mathbf{z}\in\operatorname{ver}(B_J)} (-1)^{\#\{\ell\in J: z_\ell = x_\ell\}} F(\mathbf{z}). \tag{2.3}$$

For $k \in [d]$, the function F is called *k-increasing* if

$$V_F(B_J) \ge 0$$

for every k-dimensional coordinate box B_J with $|J| = k$.

2.3. **Discrete copulas and quasi-copulas.** Let

$$\mathcal{D} = D_1 \times \cdots \times D_d \subseteq [0,1]^d$$

be a finite rectangular grid, where $0, 1 \in D_\ell$ for every $\ell \in [d]$. A function $Q\colon \mathcal{D} \to [0,1]$ is called a *discrete d-quasi-copula* if it satisfies the following conditions:

1. Q is grounded, i.e., $Q(\mathbf{u}) = 0$ whenever some coordinate of $\mathbf{u}$ is zero;
2. Q has uniform one-dimensional margins, i.e., $Q(1,\dots,1,u_\ell,1,\dots,1) = u_\ell$ for every $u_\ell \in D_\ell$ and every $\ell \in [d]$;
3. Q is increasing in each variable on $\mathcal{D}$;
4. Q satisfies
$$|Q(\mathbf{u}) - Q(\mathbf{v})| \le \sum_{\ell=1}^{d} |u_\ell - v_\ell|$$
for all $\mathbf{u}, \mathbf{v} \in \mathcal{D}$.

Equivalently, the last two conditions may be checked along coordinate edges: whenever $\mathbf{u}, \mathbf{v} \in \mathcal{D}$ differ only in coordinate ℓ and $u_\ell \le v_\ell$, one has

$$0 \le Q(\mathbf{v}) - Q(\mathbf{u}) \le v_\ell - u_\ell.$$

For a coordinate box B_J whose vertices belong to $\mathcal{D}$, its Q-volume is defined by (2.3). A discrete quasi-copula Q is called k-increasing if

$$V_Q(B_J) \ge 0$$

for every k-dimensional coordinate box B_J with $|J| = k$ and $\operatorname{ver}(B_J) \subseteq \mathcal{D}$. A *discrete d-copula* is a d-increasing discrete d-quasi-copula.

Proposition 2.1 (Extension of a discrete copula). *Let $\mathcal{D} = D_1 \times \cdots \times D_d \subseteq [0,1]^d$ be a finite rectangular grid such that $0, 1 \in D_\ell$ for every $\ell \in [d]$. Every discrete d-copula $C_{\mathcal{D}} \colon \mathcal{D} \to [0,1]$ admits a d-copula extension $C \colon [0,1]^d \to [0,1]$ satisfying $C|_{\mathcal{D}} = C_{\mathcal{D}}$.*

Proof. A discrete d-copula on $\mathcal{D}$ is a d-subcopula whose domain is the product set $D_1 \times \cdots \times D_d$. The conclusion follows from the extension theorem for subcopulas; see [3, Section 2.3]. □

3. Finite interpolation data and quasi-copula envelopes

We first characterize finite exact data that admit a quasi-copula interpolant and construct the least and greatest such interpolants. We then derive analogous envelopes for interval-valued data. These envelopes turn the original pointwise constraints into the global sandwich bounds used in Section 4.

For $\mathbf{u}, \mathbf{v} \in [0,1]^d$, define the directed distance from $\mathbf{v}$ to $\mathbf{u}$ by

$$d_+(\mathbf{u}, \mathbf{v}) := \sum_{\ell=1}^{d} (u_\ell - v_\ell)_+, \qquad t_+ := \max\{t, 0\}. \tag{3.1}$$

Recall the Fréchet–Hoeffding bounds W_d and M_d from (2.2).

3.1. **Exact data.** Let $q_j \in [0,1]$ be prescribed at $\mathbf{x}_j$. We call the exact data $(\mathbf{x}_j, q_j)_{j=1}^n$ *quasi-copula compatible* if

$$W_d(\mathbf{x}_j) \le q_j \le M_d(\mathbf{x}_j), \qquad j = 1, \ldots, n, \tag{3.2}$$

$$q_i - q_j \le d_+(\mathbf{x}_i, \mathbf{x}_j), \qquad i, j = 1, \ldots, n. \tag{3.3}$$

Define

$$\underline{Q}(\mathbf{u}) := \max\left\{ W_d(\mathbf{u}), \max_{1 \le i \le n} \big[q_i - d_+(\mathbf{x}_i, \mathbf{u})\big] \right\}, \tag{3.4}$$

$$\overline{Q}(\mathbf{u}) := \min\left\{ M_d(\mathbf{u}), \min_{1 \le i \le n} \big[q_i + d_+(\mathbf{u}, \mathbf{x}_i)\big] \right\}. \tag{3.5}$$

The one-point versions of the bounds below are due to Rodríguez-Lallena and Úbeda-Flores [10, Theorem 3.2]; see also [6, Theorem 2.1]. The finite-point result follows by combining those bounds with the fact that finite pointwise maxima and minima of d-quasi-copulas remain d-quasi-copulas. We include the details because the extremal envelopes are used below.

Proposition 3.1 (Finite-point quasi-copula interpolation). *The exact data admit a quasi-copula interpolant if and only if (3.2)–(3.3) hold. In that case, $\underline{Q}$ and $\overline{Q}$ are quasi-copulas,*

$$\underline{Q}(\mathbf{x}_j) = q_j = \overline{Q}(\mathbf{x}_j), \qquad j = 1, \ldots, n,$$

and every quasi-copula Q satisfying $Q(\mathbf{x}_j) = q_j$ obeys

$$\underline{Q} \le Q \le \overline{Q}.$$

Proof. Assume first that a quasi-copula Q interpolating the prescribed data exists. Every d-quasi-copula satisfies the Fréchet–Hoeffding inequalities

$$W_d(\mathbf{u}) \le Q(\mathbf{u}) \le M_d(\mathbf{u}), \qquad \mathbf{u} \in [0,1]^d.$$

Evaluating these inequalities at $\mathbf{u} = \mathbf{x}_j$ gives (3.2).

Moreover, coordinatewise monotonicity and the Lipschitz property of a quasi-copula imply the directed inequality

$$Q(\mathbf{u}) - Q(\mathbf{v}) \le \sum_{\ell=1}^{d} (u_\ell - v_\ell)_+ = d_+(\mathbf{u}, \mathbf{v}), \qquad \mathbf{u}, \mathbf{v} \in [0,1]^d. \tag{3.6}$$

Indeed, let

$$\mathbf{w} := \mathbf{u} \vee \mathbf{v} = \big(\max\{u_1, v_1\}, \ldots, \max\{u_d, v_d\}\big).$$

coordinatewise monotonicity and the Lipschitz condition give

$$Q(\mathbf{u}) - Q(\mathbf{v}) \le Q(\mathbf{w}) - Q(\mathbf{v}) \le \sum_{\ell=1}^{d} (w_\ell - v_\ell) = \sum_{\ell=1}^{d} (u_\ell - v_\ell)_+.$$

Applying (3.6) to $\mathbf{u} = \mathbf{x}_i$ and $\mathbf{v} = \mathbf{x}_j$ yields (3.3).

Conversely, assume that (3.2)–(3.3) hold. For every $i \in \{1, \dots, n\}$, define

$$Q_i^-(\mathbf{u}) := \max\{W_d(\mathbf{u}), q_i - d_+(\mathbf{x}_i, \mathbf{u})\},$$
$$Q_i^+(\mathbf{u}) := \min\{M_d(\mathbf{u}), q_i + d_+(\mathbf{u}, \mathbf{x}_i)\}.$$

Since $W_d(\mathbf{x}_i) \le q_i \le M_d(\mathbf{x}_i)$, the one-point quasi-copula bounds [10, Theorem 3.2] imply that Q_i^- and Q_i^+ are quasi-copulas, that $Q_i^-(\mathbf{x}_i) = q_i = Q_i^+(\mathbf{x}_i)$, and that every quasi-copula Q satisfying $Q(\mathbf{x}_i) = q_i$ fulfills $Q_i^- \le Q \le Q_i^+$.

Finite pointwise maxima and minima of d-quasi-copulas are again d-quasi-copulas. Indeed, groundedness, uniform one-dimensional margins, coordinatewise monotonicity, and the common 1-Lipschitz bound are preserved under both operations. Consequently,

$$\underline{Q} = \max_{1\le i\le n} Q_i^-, \qquad \overline{Q} = \min_{1\le i\le n} Q_i^+,$$

are quasi-copulas. These are precisely the functions defined in (3.4) and (3.5).

It remains to verify that they interpolate all prescribed values. For $i, j \in \{1, \dots, n\}$, condition (3.3) gives $q_i - d_+(\mathbf{x}_i, \mathbf{x}_j) \le q_j$. Together with $W_d(\mathbf{x}_j) \le q_j$, this yields $Q_i^-(\mathbf{x}_j) \le q_j$. For $i = j$, equality holds: $Q_j^-(\mathbf{x}_j) = q_j$. Therefore,

$$\underline{Q}(\mathbf{x}_j) = \max_{1\le i\le n} Q_i^-(\mathbf{x}_j) = q_j.$$

Similarly, applying (3.3) to the ordered pair (j, i) gives $q_j - q_i \le d_+(\mathbf{x}_j, \mathbf{x}_i)$, or equivalently, $q_j \le q_i + d_+(\mathbf{x}_j, \mathbf{x}_i)$. Since $q_j \le M_d(\mathbf{x}_j)$, we obtain $q_j \le Q_i^+(\mathbf{x}_j)$. Again, equality holds for $i = j$, and hence

$$\overline{Q}(\mathbf{x}_j) = \min_{1\le i\le n} Q_i^+(\mathbf{x}_j) = q_j.$$

Finally, let Q be any quasi-copula interpolating the prescribed data. Applying the one-point extremal bounds at every $(\mathbf{x}_i, q_i)$ gives $Q_i^- \le Q \le Q_i^+$, $i = 1, \dots, n$. Taking the pointwise maximum of the lower bounds and the pointwise minimum of the upper bounds yields

$$\underline{Q} = \max_{1\le i\le n} Q_i^- \le Q \le \min_{1\le i\le n} Q_i^+ = \overline{Q}.$$

Thus, $\underline{Q}$ and $\overline{Q}$ are, respectively, the least and greatest quasi-copulas interpolating the prescribed data in the pointwise order. □

3.2. **Interval-valued data.** Let $0 \le a_j \le b_j \le 1$. We call $(\mathbf{x}_j; [a_j, b_j])_{j=1}^n$ *quasi-copula compatible* if

$$a_i \le M_d(\mathbf{x}_i), \qquad i = 1, \dots, n, \tag{3.7}$$
$$W_d(\mathbf{x}_j) \le b_j, \qquad j = 1, \dots, n, \tag{3.8}$$
$$a_i - b_j \le d_+(\mathbf{x}_i, \mathbf{x}_j), \qquad i, j = 1, \dots, n. \tag{3.9}$$

Define the interval envelopes

$$\underline{Q}_A(\mathbf{u}) = \max\left\{W_d(\mathbf{u}), \max_{1\le i\le n}\big[a_i - d_+(\mathbf{x}_i, \mathbf{u})\big]\right\}, \tag{3.10}$$

$$\overline{Q}_B(\mathbf{u}) = \min\left\{M_d(\mathbf{u}), \min_{1\le j\le n}\big[b_j + d_+(\mathbf{u}, \mathbf{x}_j)\big]\right\}. \tag{3.11}$$

Proposition 3.2 (Extremal interval envelopes)**.** *The interval data admit a quasi-copula Q satisfying*

$$a_j \le Q(\mathbf{x}_j) \le b_j, \qquad j = 1, \dots, n,$$

if and only if (3.7)–(3.9) *hold. In that case,* $\underline{Q}_A$ *and* $\overline{Q}_B$ *are quasi-copulas,* $\underline{Q}_A \leq \overline{Q}_B$*, and a quasi-copula* Q *satisfies the interval constraints if and only if*

$$\underline{Q}_A \leq Q \leq \overline{Q}_B. \tag{3.12}$$

Proof. If Q satisfies the interval constraints, the Fréchet–Hoeffding and directed Lipschitz inequalities give (3.7)–(3.9). They also give, for every $\mathbf{u}$,

$$Q(\mathbf{u}) \geq a_i - d_+(\mathbf{x}_i, \mathbf{u}) \qquad \text{and} \qquad Q(\mathbf{u}) \leq b_j + d_+(\mathbf{u}, \mathbf{x}_j),$$

so (3.12) follows.

Conversely, suppose that the compatibility conditions hold. Put

$$\widetilde{a}_i := \max\{a_i, W_d(\mathbf{x}_i)\}, \qquad \widetilde{b}_i := \min\{b_i, M_d(\mathbf{x}_i)\}.$$

Then

$$W_d(\mathbf{x}_i) \leq \widetilde{a}_i \leq M_d(\mathbf{x}_i), \qquad W_d(\mathbf{x}_i) \leq \widetilde{b}_i \leq M_d(\mathbf{x}_i).$$

If $a_i < W_d(\mathbf{x}_i)$, the directed Lipschitz inequality for W_d gives

$$a_i - d_+(\mathbf{x}_i, \mathbf{u}) \leq W_d(\mathbf{x}_i) - d_+(\mathbf{x}_i, \mathbf{u}) \leq W_d(\mathbf{u}),$$

so both the original and clipped terms in (3.10) are redundant. Similarly, if $b_i > M_d(\mathbf{x}_i)$, then

$$M_d(\mathbf{u}) \leq M_d(\mathbf{x}_i) + d_+(\mathbf{u}, \mathbf{x}_i) \leq b_i + d_+(\mathbf{u}, \mathbf{x}_i),$$

so both the original and clipped terms in (3.11) are redundant. Thus the envelopes are unchanged if a_i, b_i are replaced by $\widetilde{a}_i, \widetilde{b}_i$, respectively. The one-point bounds [10, Theorem 3.2] therefore show that every function entering the two finite extrema is a quasi-copula. Since finite pointwise maxima and minima preserve the quasi-copula properties, $\underline{Q}_A$ and $\overline{Q}_B$ are quasi-copulas.

It remains to verify that $\underline{Q}_A \leq \overline{Q}_B$. Fix $\mathbf{u} \in [0,1]^d$. First, $W_d(\mathbf{u}) \leq M_d(\mathbf{u})$. Moreover, the directed Lipschitz inequalities for the Fréchet–Hoeffding bounds give

$$W_d(\mathbf{u}) \leq W_d(\mathbf{x}_j) + d_+(\mathbf{u}, \mathbf{x}_j) \leq b_j + d_+(\mathbf{u}, \mathbf{x}_j)$$

for every j, and

$$a_i - d_+(\mathbf{x}_i, \mathbf{u}) \leq M_d(\mathbf{x}_i) - d_+(\mathbf{x}_i, \mathbf{u}) \leq M_d(\mathbf{u})$$

for every i. Finally, by (3.9) and the directed triangle inequality,

$$a_i - d_+(\mathbf{x}_i, \mathbf{u}) \leq b_j + d_+(\mathbf{x}_i, \mathbf{x}_j) - d_+(\mathbf{x}_i, \mathbf{u}) \leq b_j + d_+(\mathbf{u}, \mathbf{x}_j)$$

for every i, j. Thus every term in the maximum defining $\underline{Q}_A(\mathbf{u})$ is bounded above by every term in the minimum defining $\overline{Q}_B(\mathbf{u})$. Hence $\underline{Q}_A(\mathbf{u}) \leq \overline{Q}_B(\mathbf{u})$. Since $\mathbf{u}$ was arbitrary, this proves $\underline{Q}_A \leq \overline{Q}_B$.

At every interpolation point $\mathbf{x}_j$, the terms with index j in (3.10) and (3.11) give $\underline{Q}_A(\mathbf{x}_j) \geq a_j$ and $\overline{Q}_B(\mathbf{x}_j) \leq b_j$. Consequently, every quasi-copula Q satisfying $\underline{Q}_A \leq Q \leq \overline{Q}_B$ also satisfies $a_j \leq Q(\mathbf{x}_j) \leq b_j$, $j = 1, \ldots, n$. In particular, both $\underline{Q}_A$ and $\overline{Q}_B$ satisfy the prescribed interval constraints. □

Remark 3.3. Under the compatibility assumptions of Proposition 3.2, the finite interval interpolation problem is feasible if and only if there exists a d-copula C such that

$$\underline{Q}_A \leq C \leq \overline{Q}_B.$$

For exact data, the analogous bounds are $\underline{Q}$ and $\overline{Q}$, which coincide at every interpolation point.

4. Copula sandwich problems and Farkas duality

We reformulate finite copula interpolation as a linear feasibility problem in the node values of a canonical rectangular grid. Linear-programming formulations for prescribed copula values were used in the trivariate setting in [2]; discrete sandwich problems and the associated $L^{(A,B)}$-criterion were studied in [7, 8]. Here a cell–vertex incidence matrix encodes all elementary-cell volumes. Farkas' lemma identifies a negative L-value with an infeasibility certificate, while a sign-polytopal subdivision reduces the resulting infinite family of inequalities to a finite vertex test. The final examples give complementary primal and dual certificates.

Let

$$E = \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subseteq [0,1]^d, \qquad \mathbf{x}_j = (x_{1j}, \dots, x_{dj}).$$

The finite set E need not itself form a rectangular grid. We therefore begin by constructing the smallest rectangular grid containing it. For each coordinate $\ell \in \{1, \dots, d\}$, define

$$D_\ell(E) = \{0, 1, x_{\ell 1}, \dots, x_{\ell n}\}, \tag{4.1}$$

where repeated values are omitted. Enumerate the resulting set in increasing order as

$$0 = t_{\ell,0} < t_{\ell,1} < \cdots < t_{\ell,n_\ell} = 1.$$

The rectangular grid generated by E is

$$\mathcal{D}_E = D_1(E) \times \cdots \times D_d(E). \tag{4.2}$$

By construction, $E \subseteq \mathcal{D}_E$, even though E itself need not have a product structure.

The elementary cells of $\mathcal{D}_E$ are indexed by

$$\mathcal{I}_E = \prod_{\ell=1}^{d} \{1, \dots, n_\ell\}.$$

For each multi-index $\mathbf{i} = (i_1, \dots, i_d) \in \mathcal{I}_E$, let

$$B_{\mathbf{i}} = \prod_{\ell=1}^{d} [t_{\ell, i_\ell - 1}, t_{\ell, i_\ell}]. \tag{4.3}$$

This formulation connects the $L^{(A,B)}$-criterion of Omladič and Stopar [7, Proposition 16] and its multivariate extension [8, Proposition 14] to standard linear-programming duality.

Enumerate the nodes of $\mathcal{D}_E$ as

$$\mathcal{D}_E = \{\mathbf{z}_1, \dots, \mathbf{z}_N\}.$$

For $\mathbf{i} \in \mathcal{I}_E$, write

$$B_{\mathbf{i}} = [\mathbf{x}^{\mathbf{i}}, \mathbf{y}^{\mathbf{i}}] = \prod_{\ell=1}^{d} [t_{\ell, i_\ell - 1}, t_{\ell, i_\ell}], \qquad \text{where} \quad x^{\mathbf{i}}_\ell = t_{\ell, i_\ell - 1},\ y^{\mathbf{i}}_\ell = t_{\ell, i_\ell}.$$

Define the vertex-multiplicity function of $B_{\mathbf{i}}$ by

$$m_{\mathbf{i}}(\mathbf{z}) = \begin{cases} (-1)^{\#\{\ell : z_\ell = x^{\mathbf{i}}_\ell\}}, & \mathbf{z} \in \operatorname{ver}(B_{\mathbf{i}}), \\ 0, & \mathbf{z} \notin \operatorname{ver}(B_{\mathbf{i}}). \end{cases} \tag{4.4}$$

The corresponding cell–vertex incidence matrix is

$$H_E = \big[m_{\mathbf{i}}(\mathbf{z}_s)\big]_{\mathbf{i} \in \mathcal{I}_E,\ 1 \le s \le N} \in \{-1, 0, 1\}^{|\mathcal{I}_E| \times N}. \tag{4.5}$$

If

$$\mathbf{c} = \big(c(\mathbf{z}_1), \dots, c(\mathbf{z}_N)\big)^{\mathsf{T}}$$

is a vector of values assigned to the grid nodes, then

$$(H_E \mathbf{c})_{\mathbf{i}} = \sum_{\mathbf{z} \in \operatorname{ver}(B_{\mathbf{i}})} m_{\mathbf{i}}(\mathbf{z}) c(\mathbf{z})$$

is the volume assigned to the elementary cell $B_{\mathbf{i}}$. Consequently, $H_E\mathbf{c} \geq 0$ is equivalent to requiring that every elementary cell have nonnegative volume. Since every box with vertices in $\mathcal{D}_E$ is a union of elementary cells, finite additivity implies that every such box also has nonnegative volume.

Example 4.1 (A bivariate grid generated by two incomparable points)**.** Let

$$E = \left\{ \left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{2}{3}, \frac{1}{3}\right) \right\} \subseteq (0,1)^2.$$

Note that the two points are incomparable in the coordinatewise order. They are also nonsymmetric in the sense that E is not invariant under interchange of the two coordinates.

The coordinate sets generated by E are

$$D_1(E) = \left\{0, \frac{1}{4}, \frac{2}{3}, 1\right\}, \qquad D_2(E) = \left\{0, \frac{1}{3}, \frac{3}{4}, 1\right\}.$$

Hence

$$\mathcal{D}_E = \left\{0, \frac{1}{4}, \frac{2}{3}, 1\right\} \times \left\{0, \frac{1}{3}, \frac{3}{4}, 1\right\}.$$

Order the sixteen grid nodes lexicographically:

$$\begin{aligned}
&\mathbf{z}_1 = (0,0), && \mathbf{z}_2 = \left(0, \frac{1}{3}\right), && \mathbf{z}_3 = \left(0, \frac{3}{4}\right), && \mathbf{z}_4 = (0,1),\\
&\mathbf{z}_5 = \left(\frac{1}{4}, 0\right), && \mathbf{z}_6 = \left(\frac{1}{4}, \frac{1}{3}\right), && \mathbf{z}_7 = \left(\frac{1}{4}, \frac{3}{4}\right), && \mathbf{z}_8 = \left(\frac{1}{4}, 1\right),\\
&\mathbf{z}_9 = \left(\frac{2}{3}, 0\right), && \mathbf{z}_{10} = \left(\frac{2}{3}, \frac{1}{3}\right), && \mathbf{z}_{11} = \left(\frac{2}{3}, \frac{3}{4}\right), && \mathbf{z}_{12} = \left(\frac{2}{3}, 1\right),\\
&\mathbf{z}_{13} = (1,0), && \mathbf{z}_{14} = \left(1, \frac{1}{3}\right), && \mathbf{z}_{15} = \left(1, \frac{3}{4}\right), && \mathbf{z}_{16} = (1,1).
\end{aligned}$$

For $i, j \in \{1,2,3\}$, let

$$B_{ij} = [t_{1,i-1}, t_{1,i}] \times [t_{2,j-1}, t_{2,j}],$$

where

$$(t_{1,0}, t_{1,1}, t_{1,2}, t_{1,3}) = \left(0, \frac{1}{4}, \frac{2}{3}, 1\right) \qquad \text{and} \qquad (t_{2,0}, t_{2,1}, t_{2,2}, t_{2,3}) = \left(0, \frac{1}{3}, \frac{3}{4}, 1\right).$$

Thus, the nine elementary cells are ordered as

$$B_{11}, B_{12}, B_{13}, B_{21}, B_{22}, B_{23}, B_{31}, B_{32}, B_{33}.$$

With this ordering, the cell–vertex incidence matrix is

$$H_E = \begin{pmatrix}
1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1
\end{pmatrix}.$$

For the node-value vector

$$\mathbf{c} = \big(c(\mathbf{z}_1), \ldots, c(\mathbf{z}_{16})\big)^{\mathsf{T}},$$

we therefore have

$$H_E\mathbf{c} = \begin{pmatrix} V_c(B_{11}) \\ V_c(B_{12}) \\ V_c(B_{13}) \\ V_c(B_{21}) \\ V_c(B_{22}) \\ V_c(B_{23}) \\ V_c(B_{31}) \\ V_c(B_{32}) \\ V_c(B_{33}) \end{pmatrix}.$$

Consequently,

$$H_E\mathbf{c} \geq \mathbf{0}_9$$

is precisely the requirement that all nine elementary cells have nonnegative c-volume. △

We next consider formal nonnegative integer combinations of elementary cells. Let

$$\mathcal{R}_E = \left\{ \bigsqcup_{\mathbf{i}\in\mathcal{I}_E} \lambda_{\mathbf{i}} B_{\mathbf{i}} : \boldsymbol{\lambda} = (\lambda_{\mathbf{i}})_{\mathbf{i}\in\mathcal{I}_E} \in \mathbb{N}_0^{\mathcal{I}_E} \right\}. \tag{4.6}$$

Here, $\bigsqcup$ denotes a formal multiset union: the coefficient $\lambda_{\mathbf{i}}$ records the multiplicity with which the elementary cell $B_{\mathbf{i}}$ occurs. Thus, the elements of $\mathcal{R}_E$ need not be geometric boxes or disjoint unions of cells.

For

$$R \equiv R_\lambda := \bigsqcup_{\mathbf{i}\in\mathcal{I}_E} \lambda_{\mathbf{i}} B_{\mathbf{i}} \in \mathcal{R}_E, \tag{4.7}$$

define its node-multiplicity function by

$$m_R(\mathbf{z}) = \sum_{\mathbf{i}\in\mathcal{I}_E} \lambda_{\mathbf{i}} m_{\mathbf{i}}(\mathbf{z}), \qquad \mathbf{z} \in \mathcal{D}_E. \tag{4.8}$$

The corresponding multiplicity vector is

$$\mathbf{m}_R = \big(m_R(\mathbf{z}_1), \ldots, m_R(\mathbf{z}_N)\big)^{\mathsf{T}}.$$

After fixing an ordering of $\mathcal{I}_E$ and regarding $\boldsymbol{\lambda}$ as a column vector in $\mathbb{N}_0^{|\mathcal{I}_E|}$, we have

$$\mathbf{m}_R = H_E^{\mathsf{T}} \boldsymbol{\lambda}. \tag{4.9}$$

For functions $A, B\colon \mathcal{D}_E \to \mathbb{R}$ satisfying $A \leq B$, define

$$L^{(A,B)}(R) = \sum_{\substack{\mathbf{z}\in\mathcal{D}_E \\ m_R(\mathbf{z})>0}} m_R(\mathbf{z})B(\mathbf{z}) + \sum_{\substack{\mathbf{z}\in\mathcal{D}_E \\ m_R(\mathbf{z})<0}} m_R(\mathbf{z})A(\mathbf{z}). \tag{4.10}$$

Proposition 4.2 (Support-function identity)**.** *Let $R \in \mathcal{R}_E$, and let*

$$\mathbf{a} = (A(\mathbf{z}_s))_{s=1}^N, \qquad \mathbf{b} = (B(\mathbf{z}_s))_{s=1}^N.$$

Then

$$L^{(A,B)}(R) = \max_{\mathbf{a}\leq\mathbf{c}\leq\mathbf{b}} \mathbf{m}_R^{\mathsf{T}} \mathbf{c}. \tag{4.11}$$

Proof. For a positive coefficient the maximum is attained at the upper bound, and for a negative coefficient it is attained at the lower bound. □

We use the following inequality form of Farkas' lemma; see [11, Section 7.3].

Theorem 4.3 (Farkas' lemma)**.** *Let $G \in \mathbb{R}^{p\times q}$ and $\mathbf{h} \in \mathbb{R}^p$. Exactly one of the following systems is feasible:*

(1) *$G\mathbf{x} \leq \mathbf{h}$, with $\mathbf{x} \in \mathbb{R}^q$;*
(2) *$\mathbf{y} \geq 0$, $G^{\mathsf{T}}\mathbf{y} = 0$, and $\mathbf{h}^{\mathsf{T}}\mathbf{y} < 0$, with $\mathbf{y} \in \mathbb{R}^p$.*

Let $A, B\colon E \to [0,1]$ satisfy $A \le B$, and suppose that the corresponding interval-valued data are quasi-copula compatible. Let $\underline{Q}_A$ and $\overline{Q}_B$ be the extremal interval envelopes from (3.10)–(3.11), and set

$$A_E := \underline{Q}_A|_{\mathcal{D}_E}, \qquad B_E := \overline{Q}_B|_{\mathcal{D}_E}. \tag{4.12}$$

Then $A_E, B_E\colon \mathcal{D}_E \to [0,1]$ are discrete quasi-copulas and $A_E \le B_E$. Put $M := |\mathcal{I}_E|$. Let

$$\mathbf{a}_E = \big(A_E(\mathbf{z}_1), \dots, A_E(\mathbf{z}_N)\big)^{\mathsf{T}}, \qquad \mathbf{b}_E = \big(B_E(\mathbf{z}_1), \dots, B_E(\mathbf{z}_N)\big)^{\mathsf{T}}.$$

For $\boldsymbol{\lambda} \in \mathbb{R}^M_{\ge 0}$, define

$$\begin{aligned}\Phi_E(\boldsymbol{\lambda}) &:= \sum_{(H_E^{\mathsf{T}}\boldsymbol{\lambda})_s > 0} (H_E^{\mathsf{T}}\boldsymbol{\lambda})_s B_E(\mathbf{z}_s) + \sum_{(H_E^{\mathsf{T}}\boldsymbol{\lambda})_s < 0} (H_E^{\mathsf{T}}\boldsymbol{\lambda})_s A_E(\mathbf{z}_s) \\ &= \sum_{(H_E^{\mathsf{T}}\boldsymbol{\lambda})_s > 0} (H_E^{\mathsf{T}}\boldsymbol{\lambda})_s (\mathbf{b}_E)_s + \sum_{(H_E^{\mathsf{T}}\boldsymbol{\lambda})_s < 0} (H_E^{\mathsf{T}}\boldsymbol{\lambda})_s (\mathbf{a}_E)_s.\end{aligned} \tag{4.13}$$

If $\boldsymbol{\lambda} \in \mathbb{N}_0^M$ and $R_{\boldsymbol{\lambda}}$ is as in (4.7), then

$$\Phi_E(\boldsymbol{\lambda}) = L^{(A_E, B_E)}(R_{\boldsymbol{\lambda}}). \tag{4.14}$$

For each $S \subseteq \{1, \dots, N\}$, define the sign polytope

$$\mathcal{P}_S := \left\{\boldsymbol{\lambda} \in \mathbb{R}^M_{\ge 0} : \mathbf{1}^{\mathsf{T}}\boldsymbol{\lambda} = 1,\ (H_E^{\mathsf{T}}\boldsymbol{\lambda})_s \ge 0 \text{ for } s \in S,\ (H_E^{\mathsf{T}}\boldsymbol{\lambda})_s \le 0 \text{ for } s \notin S\right\}. \tag{4.15}$$

Each nonempty $\mathcal{P}_S$ is a polytope and hence has finitely many vertices; see, for example, [13]. Its set of vertices is denoted by $\operatorname{ver}(\mathcal{P}_S)$.

The following theorem gives the promised criterion for the original finite interpolation problem.

Theorem 4.4 (Finite copula interpolation criterion)**.** *Let $A, B\colon E \to [0,1]$ be as above, and let*

$$A_E = \underline{Q}_A|_{\mathcal{D}_E}, \qquad B_E = \overline{Q}_B|_{\mathcal{D}_E}.$$

Then the following statements are equivalent:

(1) *There exists a d-copula C on $[0,1]^d$ such that*
$$A \le C|_E \le B.$$
(2) *There exists a discrete d-copula C_E on $\mathcal{D}_E$ such that*
$$A_E \le C_E \le B_E.$$
(3) *For every $R \in \mathcal{R}_E$,*
$$L^{(A_E,B_E)}(R) \ge 0.$$
(4) *For every $S \subseteq \{1, \dots, N\}$ such that $\mathcal{P}_S \ne \varnothing$, and every $\boldsymbol{\lambda} \in \operatorname{ver}(\mathcal{P}_S)$, one has*
$$\Phi_E(\boldsymbol{\lambda}) \ge 0.$$

In particular, finite interval-valued copula interpolation can be decided by checking finitely many inequalities. Exact interpolation is the special case $A = B$.

Proof. If (1) holds, then $C_E := C|_{\mathcal{D}_E}$ is a discrete d-copula. The extremal-envelope property in Proposition 3.2 gives $A_E \le C_E \le B_E$, so (2) holds. Conversely, if (2) holds, then Proposition 2.1 provides a d-copula extension C of C_E. Because $E \subseteq \mathcal{D}_E$ and the envelope bounds enforce the original intervals at the points of E, we have $A \le C|_E \le B$. Thus (1) and (2) are equivalent.

Let

$$\mathbf{c} = \big(C_E(\mathbf{z}_1), \dots, C_E(\mathbf{z}_N)\big)^{\mathsf{T}}$$

be the node-value vector of C_E. Since A_E and B_E are discrete quasi-copulas, they agree at every node where the copula boundary conditions prescribe a fixed value. Moreover, nonnegative elementary-cell volumes imply the lower-order monotonicity and Lipschitz conditions; see Lemma 5.1 below. Hence (2) is equivalent to the feasibility of

$$H_E\mathbf{c} \geq \mathbf{0}_M, \qquad \mathbf{a}_E \leq \mathbf{c} \leq \mathbf{b}_E, \tag{4.16}$$

where vector inequalities are understood componentwise.

We next prove that (2) implies (3). Suppose that $\mathbf{c}$ satisfies (4.16), and let $R = \bigsqcup_{\mathbf{i}\in\mathcal{I}_E} \lambda_{\mathbf{i}} B_{\mathbf{i}} \in \mathcal{R}_E$. After fixing an ordering of $\mathcal{I}_E$, write $\boldsymbol{\lambda} = (\lambda_{\mathbf{i}})_{\mathbf{i}\in\mathcal{I}_E} \in \mathbb{N}_0^M$. The multiplicity vector of R is

$$\mathbf{m}_R = H_E^{\mathsf{T}}\boldsymbol{\lambda}.$$

It follows that

$$\mathbf{m}_R^{\mathsf{T}}\mathbf{c} = \boldsymbol{\lambda}^{\mathsf{T}} H_E \mathbf{c} \geq 0.$$

Moreover, since $\mathbf{a}_E \leq \mathbf{c} \leq \mathbf{b}_E$, the support-function identity in Proposition 4.2 yields

$$\mathbf{m}_R^{\mathsf{T}}\mathbf{c} \leq \max_{\mathbf{a}_E\leq\mathbf{x}\leq\mathbf{b}_E} \mathbf{m}_R^{\mathsf{T}}\mathbf{x} = L^{(A_E,B_E)}(R).$$

Consequently, $L^{(A_E,B_E)}(R) \geq 0$ for every $R \in \mathcal{R}_E$. This proves that (2) implies (3).

We prove the converse by contraposition. Suppose that (4.16) is infeasible. Write this system as

$$G\mathbf{c} \leq \mathbf{h},$$

where

$$G = \begin{bmatrix} -H_E \\ I_N \\ -I_N \end{bmatrix} \in \mathbb{R}^{(M+2N)\times N}, \qquad \mathbf{h} = \begin{bmatrix} \mathbf{0}_M \\ \mathbf{b}_E \\ -\mathbf{a}_E \end{bmatrix} \in \mathbb{R}^{M+2N}. \tag{4.17}$$

By Farkas' lemma, in the form stated in Theorem 4.3, there exist $\boldsymbol{\lambda} \in \mathbb{R}_{\geq 0}^M$ and $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{R}_{\geq 0}^N$ such that

$$-H_E^{\mathsf{T}}\boldsymbol{\lambda} + \boldsymbol{\mu} - \boldsymbol{\nu} = \mathbf{0}_N \tag{4.18}$$

and

$$\boldsymbol{\mu}^{\mathsf{T}}\mathbf{b}_E - \boldsymbol{\nu}^{\mathsf{T}}\mathbf{a}_E < 0. \tag{4.19}$$

Set $\mathbf{m} = H_E^{\mathsf{T}}\boldsymbol{\lambda}$. By (4.18),

$$\mathbf{m} = \boldsymbol{\mu} - \boldsymbol{\nu}.$$

For each $s \in \{1, \ldots, N\}$, if $m_s \geq 0$, then $\mu_s = m_s + \nu_s$, and hence

$$\mu_s B_E(\mathbf{z}_s) - \nu_s A_E(\mathbf{z}_s) = m_s B_E(\mathbf{z}_s) + \nu_s\big(B_E(\mathbf{z}_s) - A_E(\mathbf{z}_s)\big) \geq m_s B_E(\mathbf{z}_s).$$

If $m_s < 0$, then $\nu_s = \mu_s - m_s$, and therefore

$$\mu_s B_E(\mathbf{z}_s) - \nu_s A_E(\mathbf{z}_s) = m_s A_E(\mathbf{z}_s) + \mu_s\big(B_E(\mathbf{z}_s) - A_E(\mathbf{z}_s)\big) \geq m_s A_E(\mathbf{z}_s).$$

Summing over s and using (4.19), we obtain

$$\Phi_E(\boldsymbol{\lambda}) < 0. \tag{4.20}$$

The function $\Phi_E\colon \mathbb{R}_{\geq 0}^M \to \mathbb{R}$ is continuous and piecewise linear. It is also positively homogeneous of degree one:

$$\Phi_E(t\boldsymbol{\lambda}) = t\Phi_E(\boldsymbol{\lambda}) \qquad \text{for all } \boldsymbol{\lambda} \in \mathbb{R}_{\geq 0}^M \text{ and } t \geq 0. \tag{4.21}$$

Indeed, $H_E^{\mathsf{T}}(t\boldsymbol{\lambda}) = tH_E^{\mathsf{T}}\boldsymbol{\lambda}$, and multiplication by $t \geq 0$ preserves the coordinate signs. Hence every term in Φ_E is multiplied by t.

Since the inequality in (4.20) is strict, there exists a nonnegative rational vector $\boldsymbol{\lambda}' \in \mathbb{Q}_{\geq 0}^M$ sufficiently close to $\boldsymbol{\lambda}$ such that $\Phi_E(\boldsymbol{\lambda}') < 0$. Choose a positive integer r such that $\widehat{\boldsymbol{\lambda}} := r\boldsymbol{\lambda}' \in \mathbb{N}_0^M$. By positive homogeneity, $\Phi_E(\widehat{\boldsymbol{\lambda}}) = r\Phi_E(\boldsymbol{\lambda}') < 0$. The vector $\widehat{\boldsymbol{\lambda}}$ defines the formal combination $R_{\widehat{\boldsymbol{\lambda}}} = \bigsqcup_{\mathbf{i}\in\mathcal{I}_E} \widehat{\lambda}_{\mathbf{i}} B_{\mathbf{i}} \in \mathcal{R}_E$. By (4.14),

$$L^{(A_E,B_E)}(R_{\widehat{\boldsymbol{\lambda}}}) = \Phi_E(\widehat{\boldsymbol{\lambda}}) < 0.$$

Thus, if (2) fails, then (3) fails. This proves the equivalence of (2) and (3).

It remains to prove the equivalence of (3) and (4). We first observe that (3) is equivalent to

$$\Phi_E(\boldsymbol{\lambda}) \geq 0 \qquad \text{for every } \boldsymbol{\lambda} \in \mathbb{R}^M_{\geq 0}. \tag{4.22}$$

Indeed, by (4.14), condition (3) gives this inequality for every $\boldsymbol{\lambda} \in \mathbb{N}_0^M$. Positive homogeneity then extends it to every nonnegative rational vector, and continuity extends it to every vector in $\mathbb{R}^M_{\geq 0}$. The converse follows by restricting (4.22) to $\mathbb{N}_0^M$ and applying (4.14).

Since Φ_E is positively homogeneous, it is enough to check its nonnegativity on the standard simplex

$$\Delta_M = \left\{ \boldsymbol{\lambda} \in \mathbb{R}^M_{\geq 0} : \mathbf{1}^\mathsf{T}\boldsymbol{\lambda} = 1 \right\}.$$

Indeed, if $\boldsymbol{\lambda} \in \mathbb{R}^M_{\geq 0} \setminus \{\mathbf{0}_M\}$, then $\widehat{\boldsymbol{\lambda}} = \frac{\boldsymbol{\lambda}}{\mathbf{1}^\mathsf{T}\boldsymbol{\lambda}} \in \Delta_M$ and $\Phi_E(\boldsymbol{\lambda}) = \big(\mathbf{1}^\mathsf{T}\boldsymbol{\lambda}\big)\Phi_E(\widehat{\boldsymbol{\lambda}})$. Moreover, $\Phi_E(\mathbf{0}_M) = 0$.

The sign polytopes $\mathcal{P}_S$, $S \subseteq \{1, \dots, N\}$, cover Δ_M. Indeed, for each $\boldsymbol{\lambda} \in \Delta_M$, one may take

$$S(\boldsymbol{\lambda}) = \left\{ s \in \{1, \dots, N\} : (H_E^\mathsf{T}\boldsymbol{\lambda})_s \geq 0 \right\},$$

in which case $\boldsymbol{\lambda} \in \mathcal{P}_{S(\boldsymbol{\lambda})}$.

Fix $S \subseteq \{1, \dots, N\}$ such that $\mathcal{P}_S \neq \varnothing$. On $\mathcal{P}_S$, the function Φ_E is given by the linear expression

$$\Phi_E(\boldsymbol{\lambda}) = \sum_{s \in S} (H_E^\mathsf{T}\boldsymbol{\lambda})_s B_E(\mathbf{z}_s) + \sum_{s \notin S} (H_E^\mathsf{T}\boldsymbol{\lambda})_s A_E(\mathbf{z}_s). \tag{4.23}$$

If a coordinate of $H_E^\mathsf{T}\boldsymbol{\lambda}$ vanishes, its contribution is zero, so its assignment to either sign class is immaterial.

Since Φ_E is linear on $\mathcal{P}_S$, it is nonnegative on $\mathcal{P}_S$ if and only if it is nonnegative at every vertex of $\mathcal{P}_S$. Therefore,

$$\Phi_E(\boldsymbol{\lambda}) \geq 0 \quad \text{for every } \boldsymbol{\lambda} \in \mathcal{P}_S \quad \Longleftrightarrow \quad \Phi_E(\boldsymbol{\lambda}) \geq 0 \quad \text{for every } \boldsymbol{\lambda} \in \operatorname{ver}(\mathcal{P}_S).$$

Since the sign polytopes $\mathcal{P}_S$ cover Δ_M, condition (4.22) is equivalent to condition (4). This proves the equivalence of (3) and (4).

Finally, there are only finitely many subsets $S \subseteq \{1, \dots, N\}$, and each nonempty polytope $\mathcal{P}_S$ has finitely many vertices. Hence condition (4) consists of finitely many inequalities. $\square$

Example 4.5 (The complexity of the sign-polytope reduction)**.** We now describe the vertices occurring in the sign-polytope reduction for the grid from Example 4.1.

Write

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}^\mathsf{T} \in \Delta_9,$$

where

$$\Delta_9 = \left\{ \boldsymbol{\lambda} \in \mathbb{R}^9_{\geq 0} : \sum_{i=1}^{3} \sum_{j=1}^{3} \lambda_{ij} = 1 \right\}.$$

The coordinates correspond, respectively, to the elementary cells

$$B_{11}, B_{12}, B_{13}, B_{21}, B_{22}, B_{23}, B_{31}, B_{32}, B_{33}.$$

Using the incidence matrix from Example 4.1, we obtain

$$H_E^{\mathsf{T}}\boldsymbol{\lambda} = \begin{pmatrix} \lambda_{11} \\ -\lambda_{11}+\lambda_{12} \\ -\lambda_{12}+\lambda_{13} \\ -\lambda_{13} \\ -\lambda_{11}+\lambda_{21} \\ \lambda_{11}-\lambda_{12}-\lambda_{21}+\lambda_{22} \\ \lambda_{12}-\lambda_{13}-\lambda_{22}+\lambda_{23} \\ \lambda_{13}-\lambda_{23} \\ -\lambda_{21}+\lambda_{31} \\ \lambda_{21}-\lambda_{22}-\lambda_{31}+\lambda_{32} \\ \lambda_{22}-\lambda_{23}-\lambda_{32}+\lambda_{33} \\ \lambda_{23}-\lambda_{33} \\ -\lambda_{31} \\ \lambda_{31}-\lambda_{32} \\ \lambda_{32}-\lambda_{33} \\ \lambda_{33} \end{pmatrix}. \tag{4.24}$$

Thus, the sign-polytopal subdivision of Δ_9 is induced by the sixteen coordinate hyperplanes of $H_E^{\mathsf{T}}\boldsymbol{\lambda}$, together with the nine facets

$$\lambda_{ij} = 0, \qquad i,j \in \{1,2,3\}.$$

A direct exact enumeration[1] gives

$$\left| \bigcup_{\substack{S\subseteq\{1,\dots,16\} \\ \mathcal{P}_S\neq\varnothing}} \operatorname{ver}(\mathcal{P}_S) \right| = 159, \tag{4.25}$$

where repetitions are omitted.

Enumerate the multiplier coordinates as

$$(\lambda_1,\dots,\lambda_9) = (\lambda_{11},\lambda_{12},\lambda_{13},\lambda_{21},\lambda_{22},\lambda_{23},\lambda_{31},\lambda_{32},\lambda_{33}),$$

and let $\mathbf{e}_1,\dots,\mathbf{e}_9$ denote the standard basis of $\mathbb{R}^9$. For every nonempty $I\subseteq\{1,\dots,9\}$, put

$$\mathbf{v}_I := \frac{1}{|I|}\sum_{r\in I}\mathbf{e}_r.$$

Of the 159 distinct vertices, 147 are uniform-support vertices of the form $\mathbf{v}_I$. If $\mathcal{I}_r$ denotes the collection of their supports of size r, the enumeration gives

r	1	2	3	4	5	6	7	8	9	total
$\lvert\mathcal{I}_r\rvert$	9	12	22	20	32	24	18	9	1	147.

The complete support lists are included with the verification code cited above. Write

$$\mathcal{V}_{\mathrm{unif}} := \left\{ \mathbf{v}_I : I \in \bigcup_{r=1}^{9}\mathcal{I}_r \right\}.$$

There are twelve additional vertices that are not uniform on their supports. Four of them have six nonzero coordinates:

$$\mathbf{w}_1 = \frac{1}{7}(0,1,0,1,2,1,0,1,1)^{\mathsf{T}}, \qquad \mathbf{w}_2 = \frac{1}{7}(0,1,0,1,2,1,1,1,0)^{\mathsf{T}},$$
$$\mathbf{w}_3 = \frac{1}{7}(0,1,1,1,2,1,0,1,0)^{\mathsf{T}}, \qquad \mathbf{w}_4 = \frac{1}{7}(1,1,0,1,2,1,0,1,0)^{\mathsf{T}}.$$

[1]The computation used Wolfram Mathematica [12]. Verification code is available at `https://github.com/ZalarA/Copula-Interpolation`.

The remaining eight exceptional vertices have eight nonzero coordinates:

$$\mathbf{w}_5 = \frac{1}{9}(0,1,1,1,2,1,1,1,1)^\mathsf{T}, \qquad \mathbf{w}_6 = \frac{1}{9}(1,1,0,1,2,1,1,1,1)^\mathsf{T},$$
$$\mathbf{w}_7 = \frac{1}{9}(1,1,1,1,0,1,1,1,2)^\mathsf{T}, \qquad \mathbf{w}_8 = \frac{1}{9}(1,1,1,1,0,1,2,1,1)^\mathsf{T},$$
$$\mathbf{w}_9 = \frac{1}{9}(1,1,1,1,2,1,0,1,1)^\mathsf{T}, \qquad \mathbf{w}_{10} = \frac{1}{9}(1,1,1,1,2,1,1,1,0)^\mathsf{T},$$
$$\mathbf{w}_{11} = \frac{1}{9}(1,1,2,1,0,1,1,1,1)^\mathsf{T}, \qquad \mathbf{w}_{12} = \frac{1}{9}(2,1,1,1,0,1,1,1,1)^\mathsf{T}.$$

Consequently, the complete set of distinct sign-polytope vertices is

$$\boxed{\mathcal{V}_{\text{sign}} = \bigcup_{\substack{S\subseteq\{1,\dots,16\}\\ \mathcal{P}_S\neq\varnothing}} \operatorname{ver}(\mathcal{P}_S) = \mathcal{V}_{\text{unif}} \cup \{\mathbf{w}_1,\dots,\mathbf{w}_{12}\}.}$$

In particular, $|\mathcal{V}_{\text{sign}}| = 147 + 12 = 159$.

Example 4.6 (An explicit test of the discrete copula sandwich criterion)**.** We exhibit lower and upper extremal quasi-copula envelopes such that neither envelope is a discrete copula, although a discrete copula exists between them.

Let
$$E = \left\{0, \frac{1}{4}, \frac{2}{3}, 1\right\} \times \left\{0, \frac{1}{3}, \frac{3}{4}, 1\right\}.$$
Thus, E is the entire grid from Example 4.1, and the grid generated by E is $\mathcal{D}_E = E$. In particular, the incidence matrix H_E and the set of 159 sign-polytope vertices from Examples 4.1 and 4.5 remain unchanged.

Using the node ordering from Example 4.1, define $A, B\colon E \to [0,1]$ by their node-value vectors

$$\mathbf{a} = \big(A(\mathbf{z}_1),\dots,A(\mathbf{z}_{16})\big)^\mathsf{T} = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{6} & \tfrac{1}{4} & 0 & \tfrac{1}{3} & \tfrac{5}{12} & \tfrac{2}{3} & 0 & \tfrac{1}{3} & \tfrac{3}{4} & 1\end{pmatrix}^\mathsf{T}$$

and

$$\mathbf{b} = \big(B(\mathbf{z}_1),\dots,B(\mathbf{z}_{16})\big)^\mathsf{T} = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{4} & \tfrac{1}{4} & 0 & \tfrac{1}{3} & \tfrac{1}{2} & \tfrac{2}{3} & 0 & \tfrac{1}{3} & \tfrac{3}{4} & 1\end{pmatrix}^\mathsf{T}.$$

Clearly,
$$A \le B \qquad \text{on } E.$$

Both A and B satisfy the discrete quasi-copula boundary conditions. Moreover, their increments along every elementary horizontal or vertical edge are nonnegative and do not exceed the length of the corresponding edge. Therefore, A and B are discrete quasi-copulas.

Since the data are prescribed at every node of the generated grid, the extremal envelopes reproduce the prescribed functions on $\mathcal{D}_E$. Consequently,

$$A_E = \underline{Q}_A|_{\mathcal{D}_E} = A, \qquad B_E = \overline{Q}_B|_{\mathcal{D}_E} = B.$$

Thus, their node-value vectors are

$$\mathbf{a}_E = \mathbf{a}, \qquad \mathbf{b}_E = \mathbf{b}.$$

Neither of them is a discrete copula. Indeed,

$$\big(H_E\mathbf{a}_E\big)^\mathsf{T} = \begin{pmatrix}0 & \tfrac{1}{6} & \tfrac{1}{12} & \tfrac{1}{3} & -\tfrac{1}{12} & \tfrac{1}{6} & 0 & \tfrac{1}{3} & 0\end{pmatrix}$$

and

$$\big(H_E\mathbf{b}_E\big)^\mathsf{T} = \begin{pmatrix}0 & \tfrac{1}{4} & 0 & \tfrac{1}{3} & -\tfrac{1}{12} & \tfrac{1}{6} & 0 & \tfrac{1}{4} & \tfrac{1}{12}\end{pmatrix}.$$

In both cases, the volume of the central elementary cell

$$B_{22} = \left[\frac{1}{4}, \frac{2}{3}\right] \times \left[\frac{1}{3}, \frac{3}{4}\right]$$

is negative:

$$V_{A_E}(B_{22}) = V_{B_E}(B_{22}) = -\frac{1}{12}.$$

Thus, neither A_E nor B_E is 2-increasing.

We now use the sign-polytope reduction from Example 4.5. Retain the notation

$$\mathcal{V}_{\text{sign}} = \mathcal{V}_{\text{unif}} \cup \{\mathbf{w}_1, \dots, \mathbf{w}_{12}\}$$

for the complete set of its 159 distinct vertices. For every $\boldsymbol{\lambda} \in \mathcal{V}_{\text{sign}}$, evaluate

$$\Phi_E(\boldsymbol{\lambda}) = \sum_{\left(H_E^\mathsf{T}\boldsymbol{\lambda}\right)_s > 0} \left(H_E^\mathsf{T}\boldsymbol{\lambda}\right)_s B_E(\mathbf{z}_s) + \sum_{\left(H_E^\mathsf{T}\boldsymbol{\lambda}\right)_s < 0} \left(H_E^\mathsf{T}\boldsymbol{\lambda}\right)_s A_E(\mathbf{z}_s).$$

An exact calculation gives

$$\min_{\boldsymbol{\lambda} \in \mathcal{V}_{\text{sign}}} \Phi_E(\boldsymbol{\lambda}) = 0.$$

The minimum is attained precisely at $\mathbf{e}_1$, $\mathbf{e}_5$, $\mathbf{e}_7$. In particular,

$$\Phi_E(\boldsymbol{\lambda}) \geq 0 \qquad \text{for every } \boldsymbol{\lambda} \in \mathcal{V}_{\text{sign}}.$$

It follows from Theorem 4.4 that there exists a discrete copula C_E satisfying

$$A_E \leq C_E \leq B_E.$$

Thus, the sign-polytope inequalities provide a finite dual certificate for the feasibility of the sandwich problem.

We may also recover a copula between the bounds directly by solving a linear feasibility problem. Let

$$\mathbf{c} = \left(c(\mathbf{z}_1), \dots, c(\mathbf{z}_{16})\right)^\mathsf{T} \in \mathbb{R}^{16}$$

be the unknown vector of grid-node values. Consider the linear program

$$\begin{aligned} \min_{\mathbf{c} \in \mathbb{R}^{16}} \quad & 0 \\ \text{subject to} \quad & H_E \mathbf{c} \geq \mathbf{0}_9, \\ & \mathbf{a}_E \leq \mathbf{c} \leq \mathbf{b}_E. \end{aligned} \tag{4.26}$$

The objective function is identically zero, so (4.26) is simply a linear feasibility problem. The inequality

$$H_E \mathbf{c} \geq \mathbf{0}_9$$

requires all nine elementary-cell volumes to be nonnegative. The box constraints require the node values to lie between A_E and B_E. No separate copula boundary constraints are needed because A_E and B_E agree at every boundary node and satisfy the copula boundary conditions there. Thus,

$$\mathbf{a}_E \leq \mathbf{c} \leq \mathbf{b}_E$$

already forces $\mathbf{c}$ to satisfy those boundary conditions.

Solving (4.26) gives, for example, the feasible solution

$$\mathbf{c}_E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{6} & \tfrac{1}{4} & 0 & \tfrac{1}{3} & \tfrac{1}{2} & \tfrac{2}{3} & 0 & \tfrac{1}{3} & \tfrac{3}{4} & 1 \end{pmatrix}^\mathsf{T}.$$

It satisfies

$$\mathbf{a}_E \leq \mathbf{c}_E \leq \mathbf{b}_E.$$

Moreover,

$$\left(H_E \mathbf{c}_E\right)^\mathsf{T} = \begin{pmatrix} 0 & \tfrac{1}{6} & \tfrac{1}{12} & \tfrac{1}{3} & 0 & \tfrac{1}{12} & 0 & \tfrac{1}{4} & \tfrac{1}{12} \end{pmatrix} \geq \mathbf{0}_9^\mathsf{T}.$$

Hence all elementary-cell volumes are nonnegative. Their sum is

$$0 + \frac{1}{6} + \frac{1}{12} + \frac{1}{3} + 0 + \frac{1}{12} + 0 + \frac{1}{4} + \frac{1}{12} = 1.$$

Since $\mathbf{c}_E$ also satisfies the copula boundary conditions, it is the node-value vector of a discrete copula C_E satisfying

$$A_E \leq C_E \leq B_E.$$

This example illustrates both sides of the primal–dual description: the nonnegativity of Φ_E at the 159 sign-polytope vertices provides a finite dual certificate of feasibility, while $\mathbf{c}_E$ provides an explicit primal witness.

5. A sandwich criterion for k-increasing discrete quasi-copulas

We extend the incidence-matrix argument from Section 4 to k-increasing discrete quasi-copulas, where $2 \le k \le d$. Elementary d-cells are replaced by elementary k-dimensional coordinate boxes. The existence of a k-increasing multivariate function between prescribed bounds was studied in [5] through coherence and avoidance of sure loss. Here we specialize the problem to finite data, derive the corresponding Farkas certificate, and obtain a finite sign-polytope test. For $k = d$, the construction reduces to Theorem 4.4.

Recall that

$$E = \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subseteq [0,1]^d, \qquad \mathbf{x}_j = (x_{1j}, \dots, x_{dj}),$$

is the finite set on which the lower and upper bounds are prescribed. For each $\ell \in \{1, \dots, d\}$, the corresponding coordinate set is

$$D_\ell(E) = \{0, 1, x_{\ell 1}, \dots, x_{\ell n}\} = \{t_{\ell,0}, \dots, t_{\ell,n_\ell}\},$$

where repeated values are omitted and

$$0 = t_{\ell,0} < t_{\ell,1} < \cdots < t_{\ell,n_\ell} = 1.$$

The rectangular grid generated by E is

$$\mathcal{D}_E = D_1(E) \times \cdots \times D_d(E).$$

For the remainder of this section, fix $k \in \{2, \dots, d\}$. The incidence matrix below records the volumes of all elementary k-dimensional coordinate boxes in $\mathcal{D}_E$.

For every $J \subseteq \{1, \dots, d\}$ with $|J| = k$, define

$$\mathcal{I}_{E,J} = \prod_{\ell \in J} \{1, \dots, n_\ell\}, \qquad \mathcal{Q}_{E,J} = \prod_{\ell \notin J} D_\ell(E).$$

An elementary k-box of $\mathcal{D}_E$ is a set of the form

$$B_{J,\mathbf{i},\mathbf{q}} = \prod_{\ell=1}^{d} I_\ell, \tag{5.1}$$

where

$$I_\ell = \begin{cases} [t_{\ell,i_\ell-1}, t_{\ell,i_\ell}], & \ell \in J, \\ \{q_\ell\}, & \ell \notin J, \end{cases}$$

for some

$$\mathbf{i} \in \mathcal{I}_{E,J}, \qquad \mathbf{q} \in \mathcal{Q}_{E,J}.$$

Thus, the coordinates indexed by J are active, whereas the remaining coordinates are fixed.

Let

$$\mathcal{I}_E^{(k)} = \{(J, \mathbf{i}, \mathbf{q}) : J \subseteq \{1, \dots, d\},\ |J| = k,\ \mathbf{i} \in \mathcal{I}_{E,J},\ \mathbf{q} \in \mathcal{Q}_{E,J}\}, \tag{5.2}$$

and put

$$M_k = |\mathcal{I}_E^{(k)}|.$$

For $\alpha = (J, \mathbf{i}, \mathbf{q}) \in \mathcal{I}_E^{(k)}$, write

$$B_\alpha = B_{J,\mathbf{i},\mathbf{q}}.$$

Its set of vertices consists of the 2^k nodes obtained by choosing, for each $\ell \in J$, one of the two endpoints $t_{\ell,i_\ell-1}$ and t_{ℓ,i_ℓ}, while keeping $z_\ell = q_\ell$ for $\ell \notin J$.

Define the vertex-multiplicity function of B_α by

$$m_\alpha(\mathbf{z}) = \begin{cases} (-1)^{\#\{\ell \in J : z_\ell = t_{\ell,i_\ell-1}\}}, & \mathbf{z} \in \mathrm{ver}(B_\alpha), \\ 0, & \mathbf{z} \notin \mathrm{ver}(B_\alpha). \end{cases} \tag{5.3}$$

After fixing an ordering of $\mathcal{I}_E^{(k)}$, define the k-cell–vertex incidence matrix by

$$H_E^{(k)} = \big[m_\alpha(\mathbf{z}_s)\big]_{\alpha\in\mathcal{I}_E^{(k)},\ 1\le s\le N} \in \{-1,0,1\}^{M_k\times N}. \tag{5.4}$$

For a node-value vector

$$\mathbf{c} = \big(c(\mathbf{z}_1),\dots,c(\mathbf{z}_N)\big)^{\mathsf{T}},$$

the component

$$\big(H_E^{(k)}\mathbf{c}\big)_\alpha$$

is the k-dimensional C-volume of B_α. Hence

$$H_E^{(k)}\mathbf{c} \ge \mathbf{0}_{M_k} \tag{5.5}$$

is equivalent to the nonnegativity of the C-volume of every elementary k-box. By finite additivity, this is equivalent to k-increasingness on $\mathcal{D}_E$.

Lemma 5.1. *Let*

$$\mathcal{D} = D_1\times\cdots\times D_d \subseteq [0,1]^d$$

be a finite rectangular grid such that $0,1\in D_\ell$ for every $\ell\in[d]$. Let $F\colon\mathcal{D}\to[0,1]$ be grounded and have uniform one-dimensional margins. If F is k-increasing for some $k\in\{2,\dots,d\}$, then F is r-increasing for every $r\in\{1,\dots,k\}$ and, in particular, F is a discrete quasi-copula.

Proof. We first show that r-increasingness implies $(r-1)$-increasingness for every $r\in\{2,\dots,d\}$.

Let B_J be an $(r-1)$-dimensional coordinate box with $|J|=r-1$. If one of its fixed coordinates is zero, then every vertex of B_J lies in a grounded face. Hence $V_F(B_J)=0$.

Suppose, therefore, that all fixed coordinates of B_J are positive. Choose $s\notin J$, and denote the fixed s-th coordinate of B_J by $q_s>0$. Let $\widetilde{B}$ be the r-dimensional coordinate box obtained from B_J by replacing the fixed coordinate $\{q_s\}$ with the interval $[0,q_s]$. By the definition of the alternating volume,

$$V_F(\widetilde{B}) = V_F(B_J) - V_F(B_J^{(s=0)}),$$

where $B_J^{(s=0)}$ denotes the copy of B_J contained in the face whose s-th coordinate is zero. Groundedness gives $V_F(B_J^{(s=0)})=0$. Consequently,

$$V_F(B_J) = V_F(\widetilde{B}) \ge 0.$$

Thus r-increasingness implies $(r-1)$-increasingness. Iterating this argument shows that F is r-increasing for every $r\in\{1,\dots,k\}$.

In particular, F is 1-increasing and hence increasing in each variable. It remains to verify the discrete Lipschitz condition. Let $\mathbf{u},\mathbf{v}\in\mathcal{D}$ differ only in coordinate i, with $u_i\le v_i$. Since F is increasing,

$$F(\mathbf{v})-F(\mathbf{u})\ge 0.$$

Moreover, F is 2-increasing. For every $j\ne i$, 2-increasingness implies that the increment of F over $[u_i,v_i]$ is nondecreasing in the j-th coordinate. Indeed, if $u_j\le u_j'$, then applying 2-increasingness to the coordinate box with active coordinates i,j, intervals $[u_i,v_i]$, $[u_j,u_j']$, and all remaining coordinates fixed gives

$$\begin{aligned}F(u_1,\dots,v_i,\dots,u_j,\dots,u_d) &- F(u_1,\dots,u_i,\dots,u_j,\dots,u_d)\\ &\le F(u_1,\dots,v_i,\dots,u_j',\dots,u_d) - F(u_1,\dots,u_i,\dots,u_j',\dots,u_d).\end{aligned}$$

Raising all coordinates other than i successively to 1, we obtain

$$F(\mathbf{v})-F(\mathbf{u}) \le F(1,\dots,1,v_i,1,\dots,1) - F(1,\dots,1,u_i,1,\dots,1).$$

By the uniform one-dimensional marginal condition, the right-hand side is v_i-u_i. Therefore,

$$0\le F(\mathbf{v})-F(\mathbf{u})\le v_i-u_i.$$

Thus F satisfies the discrete quasi-copula edge conditions. Hence

$$|F(\mathbf{u}) - F(\mathbf{v})| \le \sum_{\ell=1}^{d} |u_\ell - v_\ell|$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{D}$, and F is a discrete quasi-copula. □

Remark 5.2. The assumption $k \ge 2$ in Lemma 5.1 is essential. For $k = 1$, groundedness, uniform one-dimensional margins, and 1-increasingness imply that the function is coordinatewise nondecreasing, but they do not imply the Lipschitz condition. Consequently, a grounded and standardized 1-increasing function need not be a discrete quasi-copula.

For example, let

$$D = \left\{0, \frac{2}{5}, \frac{1}{2}, 1\right\}, \qquad \mathcal{D} = D^2,$$

and define $F\colon \mathcal{D} \to [0,1]$ by the usual groundedness and marginal conditions,

$$F(x,0) = F(0,y) = 0, \qquad F(x,1) = x, \qquad F(1,y) = y,$$

and by

$F(x,y)$	$x = \frac{2}{5}$	$x = \frac{1}{2}$
$y = \frac{2}{5}$	0	$\frac{2}{5}$
$y = \frac{1}{2}$	0	$\frac{2}{5}$

at the remaining interior nodes. Then F is grounded, has uniform one-dimensional margins, and is increasing in each coordinate. However,

$$F\left(\frac{1}{2}, \frac{2}{5}\right) - F\left(\frac{2}{5}, \frac{2}{5}\right) = \frac{2}{5} > \frac{1}{10} = \frac{1}{2} - \frac{2}{5}.$$

Thus F fails the discrete Lipschitz condition and is not a discrete quasi-copula. Hence 1-increasingness alone is not sufficient; the implication in Lemma 5.1 relies on the resulting 2-increasingness when $k \ge 2$.

We next introduce formal nonnegative integer combinations of elementary k-boxes:

$$\mathcal{R}_E^{(k)} = \left\{ \bigsqcup_{\alpha \in \mathcal{I}_E^{(k)}} \lambda_\alpha B_\alpha : \boldsymbol{\lambda} = (\lambda_\alpha)_{\alpha \in \mathcal{I}_E^{(k)}} \in \mathbb{N}_0^{M_k} \right\}. \tag{5.6}$$

As before, $\bigsqcup$ denotes a formal multiset union.

For

$$R = \bigsqcup_{\alpha \in \mathcal{I}_E^{(k)}} \lambda_\alpha B_\alpha \in \mathcal{R}_E^{(k)},$$

define

$$m_R^{(k)}(\mathbf{z}) = \sum_{\alpha \in \mathcal{I}_E^{(k)}} \lambda_\alpha m_\alpha(\mathbf{z}), \qquad \mathbf{z} \in \mathcal{D}_E. \tag{5.7}$$

The corresponding multiplicity vector is

$$\mathbf{m}_R^{(k)} = \left(m_R^{(k)}(\mathbf{z}_1), \dots, m_R^{(k)}(\mathbf{z}_N)\right)^\mathsf{T} = \left(H_E^{(k)}\right)^\mathsf{T} \boldsymbol{\lambda}. \tag{5.8}$$

For functions $A, B\colon \mathcal{D}_E \to \mathbb{R}$ satisfying $A \le B$, define

$$L_k^{(A,B)}(R) = \sum_{\substack{\mathbf{z} \in \mathcal{D}_E \\ m_R^{(k)}(\mathbf{z}) > 0}} m_R^{(k)}(\mathbf{z}) B(\mathbf{z}) + \sum_{\substack{\mathbf{z} \in \mathcal{D}_E \\ m_R^{(k)}(\mathbf{z}) < 0}} m_R^{(k)}(\mathbf{z}) A(\mathbf{z}). \tag{5.9}$$

The proof of the following identity is identical to that of Proposition 4.2.

Proposition 5.3 (k-dimensional support-function identity). *Let $R \in \mathcal{R}_E^{(k)}$, and let*

$$\mathbf{a} = \big(A(\mathbf{z}_1), \dots, A(\mathbf{z}_N)\big)^\mathsf{T}, \qquad \mathbf{b} = \big(B(\mathbf{z}_1), \dots, B(\mathbf{z}_N)\big)^\mathsf{T}.$$

Then

$$L_k^{(A,B)}(R) = \max_{\mathbf{a} \le \mathbf{c} \le \mathbf{b}} \big(\mathbf{m}_R^{(k)}\big)^\mathsf{T} \mathbf{c}. \tag{5.10}$$

Proof. The maximum is attained coordinatewise: at b_s when $m_R^{(k)}(\mathbf{z}_s) > 0$, and at a_s when $m_R^{(k)}(\mathbf{z}_s) < 0$. This is the same argument as in the proof of Proposition 4.2. □

Let $A, B \colon E \to [0,1]$ satisfy $A \le B$, and suppose that the corresponding interval-valued data are quasi-copula compatible. Let $\underline{Q}_A$ and $\overline{Q}_B$ be the extremal envelopes from (3.10)–(3.11), and set

$$A_E := \underline{Q}_A|_{\mathcal{D}_E}, \qquad B_E := \overline{Q}_B|_{\mathcal{D}_E}. \tag{5.11}$$

Then A_E and B_E are discrete quasi-copulas satisfying $A_E \le B_E$. The pointwise argument from Proposition 3.2, which uses only monotonicity and the Lipschitz inequalities at the relevant points, also applies on the grid. Hence every discrete quasi-copula Q_E on $\mathcal{D}_E$ satisfies

$$A \le Q_E|_E \le B \iff A_E \le Q_E \le B_E. \tag{5.12}$$

In particular, this equivalence holds for every k-increasing discrete quasi-copula on $\mathcal{D}_E$.

Set

$$\mathbf{a}_E = \big(A_E(\mathbf{z}_1), \dots, A_E(\mathbf{z}_N)\big)^\mathsf{T}, \qquad \mathbf{b}_E = \big(B_E(\mathbf{z}_1), \dots, B_E(\mathbf{z}_N)\big)^\mathsf{T}.$$

For $\boldsymbol{\lambda} \in \mathbb{R}_{\ge 0}^{M_k}$, define

$$\Phi_E^{(k)}(\boldsymbol{\lambda}) := \sum_{\left((H_E^{(k)})^\mathsf{T}\boldsymbol{\lambda}\right)_s > 0} \big((H_E^{(k)})^\mathsf{T}\boldsymbol{\lambda}\big)_s B_E(\mathbf{z}_s) + \sum_{\left((H_E^{(k)})^\mathsf{T}\boldsymbol{\lambda}\right)_s < 0} \big((H_E^{(k)})^\mathsf{T}\boldsymbol{\lambda}\big)_s A_E(\mathbf{z}_s). \tag{5.13}$$

If $\boldsymbol{\lambda} \in \mathbb{N}_0^{M_k}$, then

$$\Phi_E^{(k)}(\boldsymbol{\lambda}) = L_k^{(A_E,B_E)}\big(R_{\boldsymbol{\lambda}}\big), \tag{5.14}$$

where $R_{\boldsymbol{\lambda}} = \bigsqcup_{\alpha \in \mathcal{I}_E^{(k)}} \lambda_\alpha B_\alpha$.

For each $S \subseteq \{1, \dots, N\}$, define the sign polytope

(5.15)
$$\mathcal{P}_S^{(k)} = \Big\{ \boldsymbol{\lambda} \in \mathbb{R}_{\ge 0}^{M_k} : \mathbf{1}^\mathsf{T}\boldsymbol{\lambda} = 1,\ \big((H_E^{(k)})^\mathsf{T}\boldsymbol{\lambda}\big)_s \ge 0 \text{ for } s \in S,\ \big((H_E^{(k)})^\mathsf{T}\boldsymbol{\lambda}\big)_s \le 0 \text{ for } s \notin S \Big\}.$$

Theorem 5.4 (k-increasing discrete sandwich criterion). *Fix $k \in \{2, \dots, d\}$. Let $A, B \colon E \to [0,1]$ be as above, and let*

$$A_E = \underline{Q}_A|_{\mathcal{D}_E}, \qquad B_E = \overline{Q}_B|_{\mathcal{D}_E}.$$

Then the following statements are equivalent:

1. *There exists a k-increasing discrete quasi-copula C_E on $\mathcal{D}_E$ such that*
$$A_E \le C_E \le B_E.$$
Equivalently, there exists such a quasi-copula whose restriction to E satisfies $A \le C_E|_E \le B$.
2. *For every $R \in \mathcal{R}_E^{(k)}$,*
$$L_k^{(A_E,B_E)}(R) \ge 0.$$
3. *For every $S \subseteq \{1, \dots, N\}$ such that $\mathcal{P}_S^{(k)} \ne \varnothing$, and every $\boldsymbol{\lambda} \in \operatorname{ver}(\mathcal{P}_S^{(k)})$, one has*
$$\Phi_E^{(k)}(\boldsymbol{\lambda}) \ge 0.$$

In particular, the existence of a k-increasing discrete quasi-copula between A_E and B_E can be decided by checking finitely many inequalities.

Proof. The proof follows the proof of Theorem 4.4, with

$$H_E, \quad M, \quad \mathcal{R}_E, \quad \mathbf{m}_R, \quad L^{(A_E,B_E)}, \quad \Phi_E, \quad \mathcal{P}_S$$

replaced, respectively, by

$$H_E^{(k)}, \quad M_k, \quad \mathcal{R}_E^{(k)}, \quad \mathbf{m}_R^{(k)}, \quad L_k^{(A_E,B_E)}, \quad \Phi_E^{(k)}, \quad \mathcal{P}_S^{(k)}.$$

Indeed, by (5.12), assertion (1) is equivalent to the existence of a k-increasing discrete quasi-copula C_E on $\mathcal{D}_E$ satisfying $A_E \le C_E \le B_E$. Let

$$\mathbf{c} = \big(C_E(\mathbf{z}_1), \dots, C_E(\mathbf{z}_N)\big)^\top$$

be its node-value vector. Since A_E and B_E are discrete quasi-copulas, they agree at all nodes at which the quasi-copula boundary conditions prescribe a fixed value. Consequently, the inequalities

$$\mathbf{a}_E \le \mathbf{c} \le \mathbf{b}_E$$

force the corresponding function C_E to be grounded and to have uniform one-dimensional margins.

Moreover,

$$H_E^{(k)}\mathbf{c} \ge \mathbf{0}_{M_k}$$

is equivalent to k-increasingness on $\mathcal{D}_E$, because its components are the volumes of the elementary k-boxes and every k-dimensional coordinate box with vertices in $\mathcal{D}_E$ is a finite union of elementary k-boxes. Since $k \ge 2$, Lemma 5.1 implies that the corresponding function C_E is a discrete quasi-copula.

Thus, assertion (1) is equivalent to the feasibility of

$$H_E^{(k)}\mathbf{c} \ge \mathbf{0}_{M_k}, \qquad \mathbf{a}_E \le \mathbf{c} \le \mathbf{b}_E. \tag{5.16}$$

If $\mathbf{c}$ satisfies (5.16), then for every $R \in \mathcal{R}_E^{(k)}$, with coefficient vector $\boldsymbol{\lambda} \in \mathbb{N}_0^{M_k}$,

$$\big(\mathbf{m}_R^{(k)}\big)^\top \mathbf{c} = \boldsymbol{\lambda}^\top H_E^{(k)}\mathbf{c} \ge 0.$$

Together with Proposition 5.3, this gives

$$L_k^{(A_E,B_E)}(R) \ge 0.$$

Conversely, if (5.16) is infeasible, apply Farkas' lemma in the form stated in Theorem 4.3 to

$$G_k = \begin{bmatrix} -H_E^{(k)} \\ I_N \\ -I_N \end{bmatrix}, \qquad \mathbf{h}_k = \begin{bmatrix} \mathbf{0}_{M_k} \\ \mathbf{b}_E \\ -\mathbf{a}_E \end{bmatrix}.$$

Exactly as in the proof of Theorem 4.4, the resulting Farkas multiplier yields

$$\boldsymbol{\lambda} \in \mathbb{R}_{\ge 0}^{M_k} \qquad \text{such that} \qquad \Phi_E^{(k)}(\boldsymbol{\lambda}) < 0.$$

The function $\Phi_E^{(k)}$ is continuous, piecewise linear, and positively homogeneous of degree one. Approximating $\boldsymbol{\lambda}$ by a nonnegative rational vector and clearing denominators therefore produces $\widehat{\boldsymbol{\lambda}} \in \mathbb{N}_0^{M_k}$ such that

$$L_k^{(A_E,B_E)}\big(R_{\widehat{\boldsymbol{\lambda}}}\big) = \Phi_E^{(k)}(\widehat{\boldsymbol{\lambda}}) < 0.$$

This proves the equivalence of (1) and (2).

Finally, positive homogeneity reduces the nonnegativity of $\Phi_E^{(k)}$ on $\mathbb{R}_{\ge 0}^{M_k}$ to its nonnegativity on the standard simplex

$$\Big\{\boldsymbol{\lambda} \in \mathbb{R}_{\ge 0}^{M_k} : \mathbf{1}^\top \boldsymbol{\lambda} = 1\Big\}.$$

The sign polytopes $\mathcal{P}_S^{(k)}$ cover this simplex, and $\Phi_E^{(k)}$ is linear on each of them. It is therefore nonnegative on every $\mathcal{P}_S^{(k)}$ if and only if it is nonnegative at every vertex of every nonempty $\mathcal{P}_S^{(k)}$. This proves the equivalence of (2) and (3).

Since there are finitely many sign polytopes and each has finitely many vertices, only finitely many inequalities need to be checked. □

Remark 5.5. For $k = d$, the only active coordinate set is $J = \{1, \ldots, d\}$. Hence $H_E^{(d)} = H_E$, $\mathcal{R}_E^{(d)} = \mathcal{R}_E$, and Theorem 5.4 reduces to Theorem 4.4.

6. Conclusion and outlook

We established a finite criterion for exact and interval-valued copula interpolation on an arbitrary finite set $E \subseteq [0,1]^d$. The first step is an explicit pair of extremal quasi-copula envelopes. The coordinates of E then generate a canonical rectangular grid on which the interpolation problem becomes

$$H_E \mathbf{c} \geq \mathbf{0}, \qquad \mathbf{a}_E \leq \mathbf{c} \leq \mathbf{b}_E.$$

Farkas' lemma identifies the $L^{(A_E,B_E)}$-criterion with a dual infeasibility certificate for this system. Positive homogeneity and piecewise linearity further reduce the infinite family of formal-cell conditions to a finite test at sign-polytope vertices.

The same construction applies to k-increasing discrete quasi-copulas after replacing elementary d-cells by elementary k-dimensional coordinate boxes. For $k = d$, it recovers the copula criterion.

The reduction is finite but may still be large: both the canonical grid and the collection of sign-polytope vertices can grow rapidly with the dimension and the number of distinct data coordinates. A natural next step is therefore to compare the node-value formulation with the alternative cell-mass formulation, exploit redundancy and symmetry in the incidence matrices, and develop algorithms that generate only the dual certificates relevant to a given data set.

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