EXTREME MASS DISTRIBUTIONS FOR k-INCREASING QUASI-COPULAS

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Abstract. The recent paper [2] introduced a hierarchy of classes of special d-quasi-copulas satisfying the k-increasing property for some $k \leq d$. These classes lie strictly between all quasi-copulas and all copulas, extending the supermodularity property from the bivariate setting to arbitrary dimension. In our previous work, we studied the extreme values of the mass distributions associated with multidimensional quasi-copulas. In the present paper, we address the maximal-volume problem within each of the aforementioned subclasses. By formulating and solving suitably simplified primal and dual linear programs, we derive the exact maximal negative and positive masses together with the corresponding extremal boxes.

1. Introduction

The seminal paper [3] by Arias-García, Mesiar, and De Baets (2020) presents a survey of the most significant results and open problems in quasi-copula theory. The fifth problem on their list—also referred to as *Hitchhiker's problem #5*—asks for the maximal negative and maximal positive mass *over all boxes and over all quasi-copulas*, and for the characterization of the boxes attaining these extrema. Here, *maximal negative* refers to the negative value with the largest absolute magnitude.

This problem has now been solved in full generality using a linear programming approach in our recent work [12]. Special cases had previously been addressed in [4, 5, 11, 13]. It turns out that the maximal volumes grow exponentially, and the extremal values can be computed using a simple algorithm determining the smallest positive integer satisfying a certain inequality.

In [7, 8], the authors studied properties of *2-increasing* binary aggregation operators and provided a method for constructing copulas from them. In dimensions $d > 2$, beyond the class of 2-increasing functions, there exists a hierarchy of intermediate classes satisfying the k-increasing property for $k = 2, \ldots, d$. These classes were introduced in [2, Section 5] (see also [3, Section 6.3]), extending the volume-based characterization of quasi-copulas to k-dimensional slices of the d-dimensional box. In this way one obtains special subclasses of quasi-copulas: they are strictly contained within the class of all quasi-copulas (corresponding to the 1-increasing property) and contain all copulas (corresponding to the d-increasing property).

The motivation for the present paper is to study the maximal volume problem described above within these intermediate classes of quasi-copulas, and thereby to quantify how restrictive the k-increasing property becomes in terms of maximal volumes. Our results may also serve as indicators of the "distance" between the classes of quasi-copulas (resp. copulas) and the class of k-increasing quasi-copulas. As noted in [2, Section 6], certain properties of bivariate copulas do not extend to higher dimensions d, but they do hold for some of the intermediate classes associated with small k, suggesting that quasi-copulas are more closely related to these. On the other hand, classes with larger k have been used to study copulas, illustrating the close connection between the two settings.

In the bivariate case, the 2-increasing property coincides with *supermodularity* (see, e.g., [10, Definition 2.1]). Consequently, the notion of k-increasingness can be viewed as a natural generalization of supermodularity to higher dimensions. Very recently, a related problem to the main problem of this paper, was studied by Anzilli and Durante [1], who derived bounds on the average F-volume of closed

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rectangles in $[0,1]^2$ for a supermodular aggregation function F. These average-volume bounds offer an additional perspective for understanding such aggregation functions and, in particular, copulas.

The paper is organized as follows. In Section 2 we present our main results (Theorems 1 and 2), together with a numerical analysis of the extremal values (Tables 1 and 2). Section 3 formulates the maximal-volume problems as linear programs (Proposition 1), simplifies them using a symmetrization argument (Corollary 1), introduces new sets of variables (Proposition 3), and derives the dual linear programs (Proposition 4). In Section 4 we prove Theorems 1 and 2 by solving these dual programs exactly. Finally, in Section 5 we summarize the paper and give some directions for further work.

2. Statements of the main theorems

In this section we introduce the notation and state our main results, i.e., Theorem 1 solves the maximal negative volume problem for k-increasing quasi-copulas, while Theorem 2 solves the maximal positive volume problem. Concrete numerical solutions examples are also presented in the form of tables. (See Tables 1 and 2 for the minimal and the maximal volume question, respectively.)

Let $\mathcal{D} \subseteq [0,1]^d$ be a non-empty set and $d \in \mathbb{N}$, $d \geq 2$. We say that a function $F : \mathcal{D} \to [0,1]$:

- is *d-increasing* if for any d-box $\mathcal{B} := \prod_{i=1}^d [a_i, b_i] \subseteq \mathcal{D}$ it holds that
$$V_F(\mathcal{B}) := \sum_{z \in \operatorname{Vert}(\mathcal{B})} (-1)^{S(z)}\, F(z) \ \geq 0,$$
where $\operatorname{Vert}(\mathcal{B})$ stands for the vertices of $\mathcal{B}$ and $S(z) = \operatorname{card}\{\, j \in \{1,2,\ldots,d\} \colon z_j = a_j \,\}$. The quantity $V_F(\mathcal{B})$ is called the *F-volume* of $\mathcal{B}$. Here $\operatorname{card} A$ denotes the cardinality of the set A.
- satisfies the *boundary condition* if for $\underline{u} := (u_1, \ldots, u_d) \in \mathcal{D}$ the following hold:
  - (a) If $\underline{u} = (u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_d)$ for some i, then $F(\underline{u}) = 0$.
  - (b) If $\underline{u} = (1, \ldots, 1, u_i, 1, \ldots, 1)$ for some i, then $F(\underline{u}) = u_i$.
- satisfies the *monotonicity condition* if it is nondecreasing in every variable, i.e., for each $i = 1, \ldots, d$ and each pair of d-tuples
$$\underline{u} := (u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_d) \in \mathcal{D},$$
$$\underline{\widetilde{u}} := (u_1, \ldots, u_{i-1}, \widetilde{u}_i, u_{i+1}, \ldots, u_d) \in \mathcal{D},$$
such that $u_i \leq \widetilde{u}_i$, it follows that $F(\underline{u}) \leq F(\underline{\widetilde{u}})$.
- satisfies the *Lipschitz condition* if for given d–tuples $(u_1, \ldots, u_d)$ and $(v_1, \ldots, v_d)$ in $\mathcal{D}$, it holds that
$$|F(u_1, \ldots, u_d) - F(v_1, \ldots, v_d)| \leq \sum_{i=1}^d |u_i - v_i|.$$

If $\mathcal{D} = [0,1]^d$ and F satisfies the boundary, the monotonicity and the Lipschitz conditions, then F is called a *d-variate quasi-copula* (or *d-quasi-copula*). We will omit the dimension d when it is clear from the context and write quasi-copula for short.

Let $\mathcal{B} := \prod_{j=1}^d [x_j, y_j]$, where $0 \leq x_j \leq y_j \leq 1$ for each j. Next we recall the definition of a k-dimensional section of a function $F : \mathcal{B} \to [0,1]$ (see [2, Definition 5]). For any $\underline{a} := (a_1, \ldots, a_d) \in \mathcal{B}$ and a set of indices $A \subseteq \{1, 2, \ldots, d\}$ with $0 < \operatorname{card} A = k \leq d$, the *$k$-dimensional slice* (or *k-slice*) $\mathcal{B}_{\underline{a},A}$ of $\mathcal{B}$ with fixed values given by $\underline{a}$ in the positions determined by A, is defined by
$$\mathcal{B}_{\underline{a},A} := \{(z_1, \ldots, z_d) \colon z_j = a_j \text{ if } j \notin A \text{ and } z_j \in [x_j, y_j] \text{ if } j \in A.\}$$
In particular, if $\underline{a}$ is a vertex of $\mathcal{B}$, then $\mathcal{B}_{\underline{a},A}$ is called a *k-dimensional face* (or *k-face*) of $\mathcal{B}$. A *k-dimensional section* (or *k-section*) of F with fixed values given by $\underline{a}$ in the positions determined by A, is the function

$$F_{\underline{a},A} : B_{\underline{a},A} \to [0,1], \tag{2.1}$$

defined by
$$F_{\underline{a},A}(\underline{z}) := F(\underline{w}), \quad \text{where} \quad w_j = \begin{cases} z_j, & \text{if } j \in A, \\ a_j, & \text{if } j \notin A. \end{cases}$$

We say that a function $F : \mathcal{B} \to [0,1]$ is *k-dimensionally-increasing* (or *k-increasing*), with $k \in \{1,\dots,d\}$, if any of its k-sections is k-increasing.

The solution to the maximal negative volume problem, i.e., the largest in absolute value among negative ones, over the set of k-increasing d-quasi-copulas, is the following.

Theorem 1. *Assume the notation above. Let $d,k \in \mathbb{N}$ and $2 \le k \le d$. The maximal negative volume $V_Q(\mathcal{B})$ of some d-box $\mathcal{B}$ over all k-increasing d-quasi-copulas Q is equal to*

$$w_{k,d,-} := \min\left(0, \min_{i=k,\dots,d} (-1)^{d-i}\binom{d-k}{i-k}\left(\gamma_i^{(k)}\right)^{-1}\right),$$

where $\gamma_i^{(j)}$ are defined recursively by

$$\gamma_i^{(j)} = \begin{cases} d+1-i & \text{for } j=2 \text{ and } i=2,\dots,d,\\ \gamma_i^{(j-1)} & \text{for } j=3,\dots,k \text{ and } i=2,\dots,j-2,\\ \sum_{\ell=i}^{d}\gamma_\ell^{(j-1)} & \text{for } j=3,\dots,k \text{ and } i=j-1,\dots,d.\end{cases} \tag{2.2}$$

Let $i_0 \in \{k,\dots,d\}$ be such that

$$w_{k,d,-} = (-1)^{d-i_0}\binom{d-k}{i_0-k}\left(\gamma_{i_0}^{(k)}\right)^{-1}.$$

One of the realizations of Q and $\mathcal{B}$ such that $V_Q(\mathcal{B}) = w_{k,d,-}$ has the following properties:

$$\mathcal{B} = [a,1]^d, \qquad a = \frac{\alpha_{i_0}^{(k)}}{\gamma_{i_0}^{(k)}}, \qquad Q(z) = q_{d-S(z)}, \quad z \in \mathrm{Vert}(\mathcal{B}), \tag{2.3}$$

where q_i are defined as follows

$$q_i = \begin{cases} 0, & \text{for } i=0,\dots,i_0-1,\\ \left(\gamma_{i_0}^{(k)}\right)^{-1}, & \text{for } i=i_0,\\ \sum_{j=1}^{k}\binom{k}{j}(-1)^{j+1}q_{i-j}, & \text{for } i=i_0+1,\dots,d,\end{cases} \tag{2.4}$$

and $\alpha_i^{(j)}$ are defined recursively by

$$\alpha_i^{(j)} = \begin{cases} 1 & \text{for } j=2 \text{ and } i=2,\dots,d,\\ \alpha_i^{(j-1)} & \text{for } j=3,\dots,k \text{ and } i=2,\dots,j-2,\\ \sum_{\ell=i}^{d}\alpha_\ell^{(j-1)} & \text{for } j=3,\dots,k \text{ and } i=j-1,\dots,d.\end{cases} \tag{2.5}$$

This realization of Q on $\{a,1\}^d$ indeed extends to a quasi-copula $Q : [0,1]^d \to [0,1]$ by Proposition 1 below.

The solution to the maximal positive volume question over the set of k-increasing d-quasi-copulas, is the following.

Theorem 2. *Assume the notation above. Let $d,k \in \mathbb{N}$ and $2 \le k \le d$. The maximal positive volume $V_Q(\mathcal{B})$ of some d-box $\mathcal{B}$ over all k-increasing d-quasi-copulas Q is equal to*

$$w_{k,d,+} := \max_{i=k,\dots,d} (-1)^{d-i}\binom{d-k}{i-k}(\gamma_i^{(k)})^{-1},$$

where $\gamma_i^{(j)}$ are defined recursively by (2.2).

Let $i_0 \in \{k,\dots,d\}$ be such that

$$w_{k,d,+} = (-1)^{d-i_0}\binom{d-k}{i_0-k}(\gamma_{i_0}^{(k)})^{-1}.$$

One of the realizations of $\mathcal{B}$ and Q is as in (2.3).

In Table 1 and 2[1] we give maximal negative and positive volumes of the d-boxes over all k-increasing d-quasi-copulas for dimensions up to 15. The cases $k \ge 2$ are computed according to Theorems 1 and 2 above, while the case $k = 1$ was studied in our previous work [12, Theorems 1 and 2].

[1]The numerical analysis in arithmetic over $\mathbb{Q}$ was performed using the software tool *Mathematica* [14]. The source code is available at `https://github.com/ZalarA/Quasi-copulas-k-increasing-extreme-volumes`.

TABLE 1. Minimal values of $V_Q(\mathcal{B})$ over all d-variate k-increasing quasi-copulas Q and all d-boxes $\mathcal{B} \subseteq [0,1]^d$.

$k\backslash d$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	$-\frac{1}{3}$	$-\frac{4}{5}$	$-\frac{9}{7}$	$-\frac{32}{13}$	$-\frac{75}{16}$	$-\frac{19}{2}$	$-\frac{55}{3}$	-37	$-\frac{209}{3}$	$-\frac{251}{2}$	$-\frac{791}{3}$	$-\frac{1583}{3}$	$-\frac{3002}{3}$	$-\frac{7435}{4}$
2	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2	$-\frac{5}{2}$	-5	$-\frac{35}{4}$	-14	-21	-42	-77	-132	$-\frac{429}{2}$
3		0	$-\frac{1}{3}$	$-\frac{2}{3}$	-1	$-\frac{4}{3}$	$-\frac{5}{3}$	-2	$-\frac{7}{2}$	$-\frac{28}{5}$	$-\frac{42}{5}$	-12	-22	$-\frac{264}{7}$
4			0	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{3}{4}$	-1	$-\frac{5}{4}$	$-\frac{3}{2}$	$-\frac{7}{4}$	$-\frac{14}{5}$	$-\frac{21}{5}$	-6	$-\frac{33}{4}$
5				0	$-\frac{1}{5}$	$-\frac{2}{5}$	$-\frac{3}{5}$	$-\frac{4}{5}$	-1	$-\frac{6}{5}$	$-\frac{7}{5}$	$-\frac{8}{5}$	$-\frac{12}{5}$	$-\frac{24}{7}$
6					0	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{5}{6}$	-1	$-\frac{7}{6}$	$-\frac{4}{3}$	$-\frac{3}{2}$
7						0	$-\frac{1}{7}$	$-\frac{2}{7}$	$-\frac{3}{7}$	$-\frac{4}{7}$	$-\frac{5}{7}$	$-\frac{6}{7}$	-1	$-\frac{8}{7}$
8							0	$-\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{3}{8}$	$-\frac{1}{2}$	$-\frac{5}{8}$	$-\frac{3}{4}$	$-\frac{7}{8}$
9								0	$-\frac{1}{9}$	$-\frac{2}{9}$	$-\frac{1}{3}$	$-\frac{4}{9}$	$-\frac{5}{9}$	$-\frac{2}{3}$
10									0	$-\frac{1}{10}$	$-\frac{1}{5}$	$-\frac{3}{10}$	$-\frac{2}{5}$	$-\frac{1}{2}$
11										0	$-\frac{1}{11}$	$-\frac{2}{11}$	$-\frac{3}{11}$	$-\frac{4}{11}$
12											0	$-\frac{1}{12}$	$-\frac{1}{6}$	$-\frac{1}{4}$
13												0	$-\frac{1}{13}$	$-\frac{2}{13}$
14													0	$-\frac{1}{14}$
15														0

Table 2. Maximal values of $V_Q(\mathcal{B})$ over all d-variate k-increasing quasi-copulas Q and all d-boxes $\mathcal{B} \subseteq [0,1]^d$.

$k\backslash d$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	2	$\frac{7}{2}$	$\frac{11}{2}$	$\frac{31}{3}$	19	$\frac{71}{2}$	$\frac{211}{3}$	$\frac{421}{3}$	$\frac{793}{3}$	$\frac{1915}{4}$	$\frac{3004}{3}$	$\frac{6007}{3}$
2	1	1	1	1	2	$\frac{10}{3}$	5	7	14	$\frac{126}{5}$	42	66	132	$\frac{1716}{7}$
3		1	1	1	1	1	$\frac{5}{3}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{14}{3}$	$\frac{42}{5}$	14	22	33
4			1	1	1	1	1	1	$\frac{3}{2}$	$\frac{21}{10}$	$\frac{14}{5}$	$\frac{18}{5}$	6	$\frac{66}{7}$
5				1	1	1	1	1	1	1	$\frac{7}{5}$	$\frac{28}{15}$	$\frac{12}{5}$	3
6					1	1	1	1	1	1	1	1	$\frac{4}{3}$	$\frac{12}{7}$
7						1	1	1	1	1	1	1	1	1
8							1	1	1	1	1	1	1	1
9								1	1	1	1	1	1	1
10									1	1	1	1	1	1
11										1	1	1	1	1
12											1	1	1	1
13												1	1	1
14													1	1
15														1

3. Towards the proof of Theorems 1 and 2

In this section, we formulate the maximal volume problems as linear programs (see Proposition 1). We then simplify these programs in two steps: first, by applying the symmetrization trick (see Corollary 1), and second, by introducing new sets of variables (see Proposition 3). We note that the symmetrization trick played a crucial role in solving the maximal volume problems for quasi-copulas in our previous work [12], whereas the introduction of new sets of variables is the main novelty of the present paper, as it substantially reduces the complexity of the resulting linear programs. Finally, we derive the corresponding dual linear programs (see Proposition 4), which will be used in the next section to prove Theorems 1 and 2.

Fix $d \in \mathbb{N}$. For multi-indices

$$\mathbb{I} = (\mathbb{I}_1, \ldots, \mathbb{I}_d) \in \{0,1\}^d \quad \text{and} \quad \mathbb{J} = (\mathbb{J}, \ldots, \mathbb{J}_d) \in \{0,1\}^d$$

let

$$\mathbb{J} - \mathbb{I} = (\mathbb{J} - \mathbb{I}_1, \ldots, \mathbb{J}_d - \mathbb{I}_d) \in \{-1,0,1\}^d$$

stand for their usual coordinate-wise difference. Let $\mathbb{E}^{(\ell)}$ stand for the multi-index with the only non-zero coordinate the ℓ–th one, which is equal to 1. For each $\ell = 1, \ldots, d$ we define a relation on $\{0,1\}^d$ by

$$\mathbb{I} \prec_\ell \mathbb{J} \quad \Leftrightarrow \quad \mathbb{J} - \mathbb{I} = \mathbb{E}^{(\ell)}.$$

We write

$$\mathbb{I} \prec \mathbb{J} \quad \Leftrightarrow \quad \mathbb{I} \prec_\ell \mathbb{J} \quad \text{for some } \ell \in \{1,2,\ldots,d\}.$$

For a point $\underline{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we define the functions

$$G_d : \mathbb{R}^d \to \mathbb{R}, \quad G_d(\underline{x}) := \sum_{i=1}^{d} x_i - d + 1,$$

$$H_d : \mathbb{R}^d \to \mathbb{R}, \quad H_d(\underline{x}) := \min\{x_1, x_2, \ldots, x_d\}.$$

Let Q be a quasi-copula and $\mathcal{B} = \prod_{i=1}^d [a_i, b_i] \subseteq [0,1]^d$ a d-box with $a_i \le b_i$ for each i. We will use multi-indices of the form $\mathbb{I} := (\mathbb{I}_1, \mathbb{I}_2 \dots \mathbb{I}_d) \in \{0,1\}^d$ to index 2^d vertices of $\mathcal{B}$. Let $\|\mathbb{I}\|_1 := \sum_{j=1}^d \mathbb{I}_j$ be the 1-norm of $\mathbb{I}$. We write

$$x_{\mathbb{I}} := ((x_{\mathbb{I}})_1, \dots, (x_{\mathbb{I}})_d)$$

to denote the vertex with coordinates

$$(x_{\mathbb{I}})_k = \begin{cases} a_k, & \text{if } \mathbb{I}_k = 0, \\ b_k, & \text{if } \mathbb{I}_k = 1. \end{cases} \tag{3.1}$$

Let us denote the value of Q in the point $x_{\mathbb{I}}$ by

$$q_{\mathbb{I}} := Q(x_{\mathbb{I}}).$$

We write $\operatorname{sign}(\mathbb{I}) := (-1)^{d-\|\mathbb{I}\|_1}$. In the notation above, the Q-volume of $\mathcal{B}$ is equal to

$$V_Q(\mathcal{B}) = \sum_{\mathbb{I} \in \{0,1\}^d} \operatorname{sign}(\mathbb{I}) \cdot q_{\mathbb{I}}.$$

Let F be a face of the unit d-cube $[0,1]^d$. We denote with

$$\mathbb{I}(F) := \{\mathbb{I} \colon \mathbb{I} \in \operatorname{Vert}(F)\} \subseteq \{0,1\}^d$$

the set of all multi-indices, corresponding to the vertices of the face F. Let

$$\mathbb{I}_{\max}(F) := (\max_{\mathbb{I} \in \mathbb{I}(F)} \mathbb{I}_1, \max_{\mathbb{I} \in \mathbb{I}(F)} \mathbb{I}_2, \dots, \max_{\mathbb{I} \in \mathbb{I}(F)} \mathbb{I}_d)$$

be the vertex of F such that each of its coordinates is the largest among all vertices. We define

$$\operatorname{sign}_F(\mathbb{I}) := (-1)^{\|\mathbb{I}_{\max}(F) - \mathbb{I}\|_1}.$$

We will prove Theorems 1 and 2 in several steps using linear programming as the main tool.

Proposition 1. *For any $k \in \{1, 2, \dots, d\}$ define the following linear program*

$$\begin{aligned} \min_{\substack{a_1,\dots,a_d,\\ b_1,\dots,b_d,\\ q_{\mathbb{I}} \text{ for } \mathbb{I} \in \{0,1\}^d}} \quad & \sum_{\mathbb{I} \in \{0,1\}^d} \operatorname{sign}(\mathbb{I}) q_{\mathbb{I}}, \\ \text{subject to} \quad & 0 \le a_i < b_i \le 1 \quad \text{for all } i = 1, 2, \dots, d, \\ & q_{\mathbb{J}} - q_{\mathbb{I}} \le b_\ell - a_\ell \quad \text{for all } \ell = 1, 2, \dots, d \text{ and all } \mathbb{I} \prec_\ell \mathbb{J}, \\ & 0 \le \sum_{\mathbb{J} \in \mathbb{I}(F)} \operatorname{sign}_F(\mathbb{J}) \cdot q_{\mathbb{J}} \quad \text{for all } j\text{-dimensional faces } F \text{ of } [0,1]^d \\ & \qquad\qquad\qquad \text{and for all } j = 1, \dots, k, \\ & \max\{0, G_d(x_{\mathbb{I}})\} \le q_{\mathbb{I}} \le H_d(x_{\mathbb{I}}) \quad \text{for all } \mathbb{I} \in \{0,1\}^d. \end{aligned} \tag{3.2}$$

Let $\mathbb{I}^{(1)}, \dots, \mathbb{I}^{(2^d)}$ be some order of all multi-indices $\mathbb{I} \in \{0,1\}^d$. If there exists an optimal solution

$$(a_1^*, \dots, a_d^*, b_1^*, \dots, b_d^*, q_{\mathbb{I}^{(1)}}, \dots, q_{\mathbb{I}^{(2^d)}})$$

to (3.2)*, which satisfies*

$$b_1^* = \dots = b_d^* = 1, \tag{3.3}$$

then the optimal value of (3.2) *is the maximal negative volume of some box over all k-increasing d-quasi-copulas.*

Moreover, if there exists an optimal solution to (3.2) *where* min *is replaced with* max*, which satisfies* (3.3)*, then the optimal value of* (3.2) *is the maximal positive volume of some box over all k-increasing d-quasi-copulas.*

In the proof of Proposition 1 we will use the fact that the k-increasing property of a multilinear function on a d-box follows from the k-increasing property on all k-faces of the box (see Lemma 1 below).

Lemma 1. *Let F be a multilinear function on a d-box $\mathcal{B} = \prod_{i=1}^d [x_i, y_i] \subseteq [0,1]^d$. The following statements are equivalent:*

(1) F is k-increasing.

(2) F is k-increasing on each k-face of $\mathcal{B}$.

Proof. The nontrivial implication is (2) $\Rightarrow$ (1). We will use induction on ℓ to show that F is k-increasing on each ℓ-dimensional face of $\mathcal{B}$ for $\ell = k, k+1, \ldots, d$. For $\ell = d$ we get that F is k-increasing on the whole box $\mathcal{B}$, proving (1). The base of induction is $\ell = k$ and holds by the assumption of (2). Assume now that F is k-increasing on each ℓ-face for some ℓ with $\ell \le d-1$ and prove that it is k-increasing on each $(\ell+1)$-face. Let $\mathcal{B}_{\underline{u},A}$ be an arbitrary $(\ell+1)$-face of $\mathcal{B}$, where $\underline{u} \in \operatorname{Vert}\mathcal{B}$ and $A \subseteq \{1, 2, \ldots, d\}$ with $\operatorname{card} A = \ell + 1$. Let $\underline{z} \in \mathcal{B}_{\underline{u},A}$ and $(\mathcal{B}_{\underline{u},A})_{\underline{z},B}$ be a ℓ-section of $\mathcal{B}_{\underline{u},A}$, where $\underline{z} \in \mathcal{B}_{\underline{u},A}$ and $B \subset A$ with $\operatorname{card} B = \ell$. Let $\{i_0\} := A \setminus B$. Without loss of generality we can assume that $\underline{u}_{i_0} = x_{u_0}$. Then $z_{i_0} = \lambda x_{i_0} + (1-\lambda) y_{i_0}$ for some $\lambda \in [0,1]$ and thus the l section $(\mathcal{B}_{\underline{u},A})_{\underline{z},B}$ can be written as a convex combination of two parallel l sub-faces $\mathcal{B}_{\underline{u},B}$ and $\mathcal{B}_{\underline{v},B}$

$$(\mathcal{B}_{\underline{u},A})_{\underline{z},B} = \lambda \mathcal{B}_{\underline{u},B} + (1-\lambda)\mathcal{B}_{\underline{v},B}, \tag{3.4}$$

where $\underline{v}$ is a neighbor vertex of $\underline{u}$ that only differs in coordinate i_0

$$v_j = \begin{cases} u_j & j \neq i_0, \\ y_{i_0}, & j = i_0. \end{cases}$$

By the induction hypothesis, F is k-increasing on $\mathcal{B}_{\underline{u},B}$ and $\mathcal{B}_{\underline{v},B}$. B by multilinearity of F and by (3.4), F is k-increasing on $\mathcal{B}_{\underline{a},A}$. This proves the induction step and concludes the proof of the lemma. $\square$

Now we can prove Proposition 1.

Proof of Proposition 1. The linear program (3.2) is obtained by extending the linear program for general quasi-copulas introduced in [12, Proposition 2] with additional constraints enforcing the k-increasing property. By [2, Lemma 2], any k-increasing quasi-copula is also j-increasing for every $j \le k$. Nevertheless, in the linear program it is necessary to impose the j-increasing constraints explicitly for all $j \le k$. Otherwise, the resulting quasi-copula would satisfy the k-increasing property only on the box $\prod_{i=1}^{k}[a_i, b_i]$, which is insufficient to invoke [2, Lemma 2].

It remains to verify that the solution of the linear program (3.2) can be extended to a k-increasing quasi-copula.

Let points $x_{\mathbb{I}} \in \prod_{i=1}^{d}\{a_i, 1\} =: \mathcal{D}$ be defined by (3.1) and real numbers $q_{\mathbb{I}}$ for all $\mathbb{I} \in \{0,1\}^d$, satisfy conditions (3.2). Let $\mathcal{L}_i$ denote the $(d-1)$–dimensional faces of $[0,1]^d$ containing $(0, \ldots, 0)$, i.e.,

$$\mathcal{L}_i = \{(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_d) \colon x_i \in [0,1]\}.$$

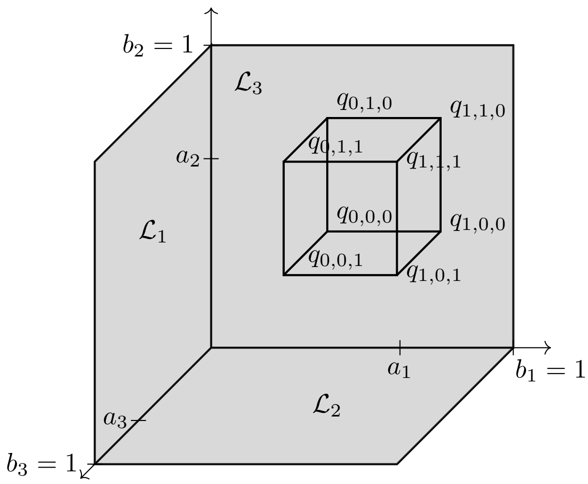

FIGURE 1. To construct a k-increasing 3-quasi-copula Q given the values $q_{\mathbb{I}}$ at $x_{\mathbb{I}} \in \mathcal{D} := \prod_{i=1}^{3}\{a_i, 1\}$, we first define it to be 0 on all 2-dimensional faces $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and prove that the extension indeed meets the requirements of a k-increasing quasi-copula.

Let

$$\mathcal{D}^{(\mathrm{ext})} := \mathcal{D} \cup \left(\cup_{i=1}^{d} \mathcal{L}_i \right)$$

and define

$$Q : \mathcal{D}^{(\text{ext})} \to \mathbb{R}$$

by

$$Q(x_1,\ldots,x_d) = \begin{cases} q_{\mathbb{I}}, & \text{if } (x_1,\ldots,x_d) = x_{\mathbb{I}} \text{ for some } \mathbb{I} \in \{0,1\}^d, \\ 0, & \text{if } (x_1,\ldots,x_d) \in \mathcal{L}_i \text{ for some } i \in \{1,\ldots,d\}. \end{cases}$$

We subdivide the box $[0,1]^d$ into 2^d smaller d-boxes

$$\mathcal{B}_{\mathbb{I}} = \prod_{j=1}^{d} \delta_j(\mathbb{I}) \tag{3.5}$$

for $\mathbb{I} = (\mathbb{I}_1,\ldots,\mathbb{I}_d) \in \{0,1\}^d$, where

$$\delta_j(\mathbb{I}) = \begin{cases} [0,a_j], & \text{if } \mathbb{I}_j = 0, \\ [a_j,1], & \text{if } \mathbb{I}_j = 1. \end{cases}$$

In particular,

$$\mathcal{B}_{(0,0\ldots0)} = \prod_{k=1}^{d}[0,a_k], \quad \mathcal{B}_{(1,0\ldots0)} = [a_1,1] \times \prod_{k=2}^{d}[0,a_k], \ldots, \quad \mathcal{B}_{(1,1\ldots1)} = \prod_{k=1}^{d}[a_k,1].$$

Note that the Q-volume $V_Q(\mathcal{B}_{\mathbb{I}})$ of each box $\mathcal{B}_{\mathbb{I}}$ is determined by the value of Q on the vertices of $\mathcal{B}_{\mathbb{I}}$. For each $\mathbb{I} \in \{0,1\}^d$ we define a constant function

$$\rho_{\mathbb{I}} : \mathcal{B}_{\mathbb{I}} \to \mathbb{R}, \quad \rho_{\mathbb{I}} := \frac{V_Q(\mathcal{B}_{\mathbb{I}})}{\prod_{j=1}^{d} V(\delta_j(\mathbb{I}))},$$

where $V([a,b]) = |b-a|$ is the length of the interval $[a,b]$. Let us define a piecewise constant function

$$\rho : [0,1]^d \to \mathbb{R}, \quad \rho(\underline{x}) := \begin{cases} \rho_{\mathbb{I}}, & \text{if } \underline{x} \in \operatorname{int}(\mathcal{B}_{\mathbb{I}}) \text{ for some } \mathbb{I} \in \{0,1\}^d, \\ 0, & \text{otherwise}, \end{cases}$$

where $\operatorname{int}(A)$ stands for the topological interior of the set A in the usual Euclidean topology. We will prove that a function $Q : [0,1]^d \to \mathbb{R}$, defined by

$$Q(x_1,x_2,\ldots x_d) = \int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_d} \rho(x_1,x_2,\ldots x_d)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_d, \tag{3.6}$$

is a k–increasing quasi-copula satisfying the statement of the proposition.

By the same proof as in [4, Theorem 2.1], it follows that Q is a quasi-copula. It remains to prove that it is k-increasing. By Lemma 3.4, it suffices to prove that Q is k-increasing on each k-face of each box $\mathcal{B}_{\mathbb{I}}$ for $\mathbb{I} \in \{0,1\}^d$. By construction, this is true for $\mathbb{I}_{\text{main}} = (1,1,\ldots,1)$. Let us now consider $\mathbb{I} \in \{0,1\}^d \setminus \mathbb{I}_{\text{main}}$. If the face has constant 0 on some component, then Q is constantly equal to 0 on the whole face and Q is k-increasing. Otherwise every constant component is equal to a_j or 1. Further on, we can assume that all non-constant components are $[a_j,1]$ and not $[0,a_j]$, because otherwise we can further project the section on the j-th component to $\{a_j\}$, hence obtaining a lower dimensional face of $\mathcal{B}_{\mathbb{I}_{\text{main}}}$. However, this reduction give us a k-dimensional face of $\mathcal{B}_{\mathbb{I}_{\text{main}}}$, whence it is k-increasing by construction. □

By the following proposition it suffices to consider symmetric solutions to the linear program (3.2).

Proposition 2. *Assume the notation from Proposition 1. There exists an optimal solution to the linear program* (3.2) *of the form*

$$\big(\underbrace{a^*,\ldots,a^*}_{d},\underbrace{b^*,\ldots,b^*}_{d}, q^*_{\|\mathbb{I}^{(1)}\|_1},\ldots,q^*_{\|\mathbb{I}^{(2^d)}\|_1}\big) \tag{3.7}$$

for some $a,b,q_1,\ldots,q_d \in [0,1]$.

Analogously, replacing min *with* max *in* (3.2) *above, the same statement holds.*

Proof. The proof is the same as for [4, Proposition 3], but we include it for the sake of completeness. Let S_d be the set of all permutations of a d-element set $\{1,\ldots,d\}$. For $\Phi \in S_d$ and $\mathbb{I}^{(j)} := (\mathbb{I}_1^{(j)},\ldots,\mathbb{I}_d^{(j)}) \in \{0,1\}^d$, let $\Phi(\mathbb{I}^{(j)}) := \big(\mathbb{I}^{(j)}_{\Phi(1)},\ldots,\mathbb{I}^{(j)}_{\Phi(d)}\big)$. For every optimal solution $\big(a_1^*,\ldots,a_d^*,b_1^*,\ldots,b_d^*,q^*_{\mathbb{I}^{(1)}},\ldots,q^*_{\mathbb{I}^{(2^d)}}\big)$ to (3.2), $\big(a_1^*,\ldots,a_d^*,b_1^*,\ldots,b_d^*,q^*_{\mathbb{I}^{(1)}},\ldots,q^*_{\mathbb{I}^{(2^d)}}\big)$ is also an optimal solution to (3.2), whence an optimal solution to (3.2) of the form (3.7) is equal to $\frac{1}{d!}\sum_{\Phi\in S_d}\big(a^*_{\Phi(1)},\ldots,a^*_{\Phi(d)},b^*_{\Phi(1)},\ldots,b^*_{\Phi(d)},q^*_{\Phi(\mathbb{I}^{(1)})},\ldots,q^*_{\Phi(\mathbb{I}^{(2^d)})}\big)$. □

An immediate corollary to Proposition 2 is the following.

Corollary 1. *The optimal value of the linear program* (3.2) *is equal to the optimal value of the linear program*

$$
\begin{aligned}
\min_{a,b,q_0,q_1,\dots,q_d} \quad & \sum_{i=0}^{d}(-1)^{d-i}\binom{d}{i}q_i,\\
\text{subject to} \quad & 0\le a<b\le 1,\\
& q_i-q_{i-1}\le b-a \quad \text{for } i=1,2,\dots,d,\\
& 0\le \sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}q_{\ell-i} \quad \text{for all } j\in 1,2,\dots,k \text{ and all } \ell=j,j+1,\dots,d,\\
& \max\{0,(d-i)a+ib-d+1\}\le q_i\le a \quad \text{for } i=0,1,\dots,d-1,\\
& \max\{0,db-d+1\}\le q_d\le b.
\end{aligned}
\tag{3.8}
$$

Analogously, replacing min *with* max *in* (3.8) *above, the same statement holds.*

It turns out that some of the constraints in (3.8) are redundant, while introducing new variables

$$
\begin{aligned}
\delta_i^{(1)} &:= q_i-q_{i-1} \quad \text{for } i=1,2,\dots,d,\\
\delta_i^{(j)} &:= \delta_i^{(j-1)}-\delta_{i-1}^{(j-1)} \quad \text{for } j\in 2,3,\dots,k \text{ and } i=j,j+1,\dots,d.
\end{aligned}
\tag{3.9}
$$

further decreases the number of constraints. Let us also define the numbers $\beta_i^{(j)}$ recursively by

$$
\beta_i^{(j)}=\begin{cases} d-i & \text{for } j=2 \text{ and } i=2,3,\dots,d-1,\\ \beta_i^{(j-1)} & \text{for } j=3,4,\dots,k \text{ and } i=2,3,\dots,j-2,\\ \sum_{\ell=i}^{d-1}\beta_\ell^{(j-1)} & \text{for } j=3,4,\dots,k \text{ and } i=j-1,j,\dots,d-1, \end{cases}. \tag{3.10}
$$

Lemma 2. *Let $\alpha_i^{(j)}$, $\beta_i^{(j)}$ and $\gamma_i^{(j)}$ be as in* (2.2), (2.5) *and* (3.10)*. For $j=2,3,\dots,k$ and $i=2,3,\dots,d$, we have*

$$
\alpha_i^{(j)}+\beta_i^{(j)}=\gamma_i^{(j)}. \tag{3.11}
$$

Proof. For $j=2$, we have $\alpha_i^{(2)}+\beta_i^{(2)}=d+1-i=\gamma_i^{(2)}$ for $i=2,3,\dots,d$, which proves (3.11). For $j>2$, (3.11) follows inductively using that (3.11) holds for smaller values of j. ☐

Proposition 3. *Let $\alpha_i^{(j)}$, $\beta_i^{(j)}$ and $\gamma_i^{(j)}$ be as in* (2.2), (2.5) *and* (3.10)*. The optimal value of the linear program* (3.8) *is equal to the optimal value of the linear program*

$$
\begin{aligned}
\min_{\substack{a,b,q_0,\\ \delta_1^{(1)},\delta_2^{(2)},\dots,\delta_{k-1}^{(k-1)},\\ \delta_k^{(k)},\delta_{k+1}^{(k)},\dots,\delta_d^{(k)}}} \quad & \sum_{j=k}^{d}(-1)^{d+j}\binom{d-k}{j-k}\delta_j^{(k)},\\
\text{subject to} \quad & b\le 1,\\
& a-b+\delta_1^{(1)}+\sum_{i=2}^{k-1}\alpha_i^{(k)}\delta_i^{(i)}+\sum_{i=k}^{d}\alpha_i^{(k)}\delta_i^{(k)}\le 0,\\
& -a+q_0+(d-1)\delta_1^{(1)}+\sum_{i=2}^{k-1}\beta_i^{(k)}\delta_i^{(i)}+\sum_{i=k}^{d-1}\beta_i^{(k)}\delta_i^{(k)}\le 0,\\
& db-q_0-d\delta_1^{(1)}-\sum_{i=2}^{k-1}\gamma_i^{(k)}\delta_i^{(i)}-\sum_{i=k}^{d}\gamma_i^{(k)}\delta_i^{(k)}\le d-1,\\
& a\ge 0,\ b\ge 0,\ q_0\ge 0,\ \delta_i^{(i)}\ge 0 \ \text{for } i=1,2,\dots,k-1,\\
& \delta_i^{(k)}\ge 0 \ \text{for } i=k,k+1,\dots,d.
\end{aligned}
\tag{3.12}
$$

Analogously, replacing min *with* max *in* (3.12) *above, the same statement holds.*

Proof. We will use induction on k to prove Proposition 3.

We start by $k = 2$. By [12, Proposition 5], (3.8) is equivalent to

$$
\begin{aligned}
\min_{a,b,q_0,\delta_1^{(1)},\ldots,\delta_d^{(1)}} \quad & \sum_{j=1}^{d}(-1)^{d+j}\binom{d-1}{j-1}\delta_j^{(1)}, \\
\text{subject to} \quad & b \le 1, \\
& \delta_i^{(1)} \le b-a \quad \text{for } i=1,2,\ldots,d, \\
& 0 \le q_i - 2q_{i-1} + q_{i-2} \quad \text{for } i=2,3,\ldots,d, \\
& q_0 + \sum_{i=1}^{d-1}\delta_i^{(1)} \le a \\
& db - d + 1 \le q_0 + \sum_{i=1}^{d}\delta_i^{(1)} \\
& a \ge 0,\ b \ge 0,\ q_0 \ge 0,\ \delta_i^{(1)} \ge 0 \quad \text{for } i=1,2,\ldots,d.
\end{aligned}
\tag{3.13}
$$

Since $\delta_i^{(1)}, \delta_i^{(2)}$ are as in (3.9), we can replace the inequalities

$$0 \le q_i - 2q_{i-1} + q_{i-2} \quad \text{for } i = 2,3,\ldots,d$$

with the inequalities

$$0 \le \delta_i^{(2)} \quad \text{for } i = 2,3,\ldots,d. \tag{3.14}$$

Using (3.14), the constraints $\delta_i^{(1)} \ge 0$ and $\delta_i^{(1)} \le b-a$ for $i = 1,2,\ldots,d$ in (3.13) can be replaced by the constraints $\delta_1^{(1)} \ge 0$ and $\delta_d^{(1)} \le b-a$. Since $\delta_j^{(1)} = \delta_j^{(2)} + \delta_{j-1}^{(1)}$ for $j = 2,3,\ldots,d$, it follows inductively that

$$\delta_j^{(1)} = \sum_{l=2}^{j}\delta_l^{(2)} + \delta_1^{(1)} \quad \text{for } j = 2,3,\ldots,d. \tag{3.15}$$

Using (3.15) in the constraints of (3.13), it is easy to see that we get the constraints of (3.8) with $\alpha_i^{(2)}, \beta_i^{(2)}, \gamma_i^{(2)}$. It remains to show that the objective function of (3.8) becomes the objective function of (3.8) after substitutions (3.15). Indeed,

$$
\begin{aligned}
\sum_{j=1}^{d}(-1)^{d-j}\binom{d-1}{j-1}\delta_j^{(1)} &= \sum_{j=1}^{d}(-1)^{d-j}\binom{d-1}{j-1}\Big(\delta_1^{(1)} + \sum_{\ell=2}^{j}\delta_\ell^{(2)}\Big) \\
&= \delta_1^{(1)}\underbrace{\sum_{j=1}^{d}(-1)^{d-j}\binom{d-1}{j-1}}_{0} + \sum_{\ell=2}^{d}\delta_\ell^{(2)}\underbrace{\Big(\sum_{k=\ell}^{d}(-1)^{d-k}\binom{d-1}{k-1}\Big)}_{(-1)^{d+\ell}\binom{d-2}{\ell-2}},
\end{aligned}
$$

where in the last equality we used that for $0 \le n \le r$ we have

$$\sum_{k=n}^{r}(-1)^k\binom{r}{k} \underbrace{=}_{\sum_{k=0}^{r}(-1)^k\binom{r}{k}=0} -\sum_{k=0}^{n-1}(-1)^k\binom{r}{k} = (-1)^n\binom{r-1}{n-1}. \tag{3.16}$$

Assume know that the proposition holds for some k, $2 \le k \le d-1$ and prove it for $k+1$. By the induction hypothesis the optimal value of the linear program (3.8) is equal to the optimal value of the linear program

$$
\begin{aligned}
\min_{\substack{a,b,q_0,\\ \delta_1^{(1)},\delta_2^{(2)},\ldots,\delta_{k-1}^{(k-1)},\\ \delta_k^{(k)},\delta_{k+1}^{(k)},\ldots,\delta_d^{(k)}}} \quad & \sum_{j=k}^{d}(-1)^{d+j}\binom{d-k}{j-k}\delta_j^{(k)},\\
\text{subject to} \quad & b\leq 1,\\
& a-b+\delta_1^{(1)}+\sum_{i=2}^{k-1}\alpha_i^{(k)}\delta_i^{(i)}+\sum_{i=k}^{d}\alpha_i^{(k)}\delta_i^{(k)}\leq 0,\\
& 0\leq \sum_{i=0}^{k+1}(-1)^{k+1-i}\binom{k+1}{i}q_{\ell-i} \quad \text{for all } \ell=k+1,k+2,\ldots,d,\\
& -a+q_0+(d-1)\delta_1^{(1)}+\sum_{i=2}^{k-1}\beta_i^{(k)}\delta_i^{(i)}+\sum_{i=k}^{d-1}\beta_i^{(k)}\delta_i^{(k)}\leq 0,\\
& db-q_0-d\delta_1^{(1)}-\sum_{i=2}^{k-1}\gamma_i^{(k)}\delta_i^{(i)}-\sum_{i=k}^{d}\gamma_i^{(k)}\delta_i^{(k)}\leq d-1,\\
& a\geq 0,\ b\geq 0,\ q_0\geq 0,\ \delta_i^{(i)}\geq 0 \ \text{ for } i=1,2,\ldots,k-1,\\
& \delta_i^{(k)}\geq 0 \ \text{ for } i=k,k+1,\ldots,d.
\end{aligned}
\tag{3.17}
$$

Since $\delta_i^{(j)}$ are as in (3.9), we can replace the inequalities

$$
0\leq \sum_{i=0}^{k+1}(-1)^{k+1-i}\binom{k+1}{i}q_{\ell-i} \quad \text{for all } \ell=k+1,k+2,\ldots,d,
$$

with the inequalities

$$
0\leq \delta_\ell^{(k+1)} \quad \text{for } \ell=k+1,k+2,\ldots,d.
$$

Since $\delta_j^{(k)}=\delta_j^{(k+1)}+\delta_{j-1}^{(k)}$ for $j=k+1,k+2,\ldots,d$, it follows inductively that

$$
\delta_j^{(k)}=\sum_{\ell=k+1}^{j}\delta_\ell^{(k+1)}+\delta_k^{(k)} \quad \text{for } j=k+1,k+2,\ldots,d. \tag{3.18}
$$

Using (3.18) in the constraints of (3.17), it is easy to see that the constraints become as in (3.12), where k is replaced by $k+1$. It remains to show that the objective function of (3.17) becomes the objective function of (3.12) after substitutions (3.18). Indeed,

$$
\begin{aligned}
\sum_{j=k}^{d}(-1)^{d+j}\binom{d-k}{j-k}\delta_j^{(k)} &= \sum_{j=k}^{d}(-1)^{d+j}\binom{d-k}{j-k}\left(\delta_k^{(k)}+\sum_{\ell=k+1}^{j}\delta_\ell^{(k+1)}\right)\\
&= \delta_k^{(k)}\underbrace{\sum_{j=k}^{d}(-1)^{d+j}\binom{d-k}{j-k}}_{0}+\sum_{\ell=k+1}^{d}\delta_\ell^{(k+1)}\underbrace{\left(\sum_{j=\ell}^{d}(-1)^{d+j}\binom{d-k}{j-k}\right)}_{(-1)^{d+\ell}\binom{d-k-1}{\ell-k-1}},
\end{aligned}
$$

where in the last equality we used (3.16). □

Using duality, the following proposition holds.

Proposition 4. *Let* $\alpha_i^{(j)}$, $\beta_i^{(j)}$ *and* $\gamma_i^{(j)}$ *be as in* (2.2), (2.5) *and* (3.10). *The optimal value of the linear program* (3.12) *is equal to the optimal value of the linear program*

$$
\begin{aligned}
\max_{y_1,y_2,y_3,y_4} \quad & -y_1-(d-1)y_4,\\
\text{subject to} \quad & y_1-y_2+dy_4\geq 0,\\
& y_2-y_3\geq 0,\\
& y_3-y_4\geq 0,\\
& y_2+(d-1)y_3-dy_4\geq 0,\\
& \alpha_i^{(k)}y_2+\beta_i^{(k)}y_3-\gamma_i^{(k)}y_4\geq 0 \quad \text{for } i=2,3,\ldots,k-1,\\
& \alpha_i^{(k)}y_2+\beta_i^{(k)}y_3-\gamma_i^{(k)}y_4\geq (-1)^{d+1+i}\binom{d-k}{i-k} \quad \text{for } i=k,k+1,\ldots,d,\\
& y_1\geq 0,\ y_2\geq 0,\ y_3\geq 0,\ y_4\geq 0.
\end{aligned}
\tag{3.19}
$$

Analogously, the optimal value of the linear program (3.12) *with the objective function*

$$
\max_{\substack{a,b,q_0,\\ \delta_1^{(1)},\delta_2^{(2)},\ldots,\delta_{k-1}^{(k-1)},\\ \delta_k^{(k)},\delta_{k+1}^{(k)},\ldots,\delta_d^{(k)}}} \sum_{j=k}^{d}(-1)^{d+j}\binom{d-k}{j-k}\delta_j^{(k)},
$$

is equal to the optimal value of the linear program

$$
\begin{aligned}
\min_{y_1,y_2,y_3,y_4} \quad & y_1+(d-1)y_4,\\
\text{subject to} \quad & y_1-y_2+dy_4\geq 0,\\
& y_2-y_3\geq 0,\\
& y_3-y_4\geq 0,\\
& y_2+(d-1)y_3-dy_4\geq 0,\\
& \alpha_i^{(k)}y_2+\beta_i^{(k)}y_3-\gamma_i^{(k)}y_4\geq 0 \quad \text{for } i=2,3,\ldots,k-1,\\
& \alpha_i^{(k)}y_2+\beta_i^{(k)}y_3-\gamma_i^{(k)}y_4\geq (-1)^{d+i}\binom{d-k}{i-k} \quad \text{for } i=k,k+1,\ldots,d,\\
& y_1\geq 0,\ y_2\geq 0,\ y_3\geq 0,\ y_4\geq 0.
\end{aligned}
\tag{3.20}
$$

Proof. Note that (3.19) is the dual linear program to (3.12), whence the statement of Proposition 4 follows by the strong duality. □

4. Proofs of Theorems 1 and 2

Finally we can prove Theorem 1.

Proof of Theorem 1. By the results above to prove Theorem 1, it suffices to find the optimal value of (3.19). First we prove the following claim.

Claim 1. Their exists an optimal solution $(y_1^*, y_2^*, y_3^*, y_4^*)$ to the linear program (3.19) which satisfies $y_2^* = y_3^*$.

Proof of Claim 1. Let $(y_1^*, y_2^*, y_3^*, y_4^*)$ be an optimal solution to (3.19). If $y_2^* > y_3^*$, then replacing y_3^* with y_2^* it is clear that $(y_1^*, y_2^*, y_2^*, y_4^*)$ satisfies all constraints and the value of the objective function remains unchanged. ■

Using Claim 1 and (3.11), the linear program (3.19) simplifies to:

$$\begin{aligned}
&\max_{y_2,y_4} && -y_1-(d-1)y_4,\\
&\text{subject to} && y_1-y_2+dy_4\geq 0,\\
&&& y_2-y_4\geq 0,\\
&&& \gamma_i^{(k)}(y_2-y_4)\geq(-1)^{d+1-i}\binom{d-k}{i-k}\quad\text{for } i=k,k+1,\ldots,d,\\
&&& y_1\geq 0,\ y_2\geq 0,\ y_4\geq 0.
\end{aligned}\tag{4.1}$$

Claim 2. There is an optimal solution (y_1^*, y_2^*, y_4^*) to the linear program (4.1) such that

$$y_2^* = y_1^* + dy_4^*.$$

Proof of Claim 2. Let (y_1^*, y_2^*, y_4^*) be an optimal solution to (4.1). If $y_2^* < y_1^* + dy_4^*$, then we can replace y_2^* with $y_1^* + dy_4^*$, still satisfying all constraints of (4.1) and not changing its objective function. This proves Claim 2. ■

Using Claim 2, (4.1) simplifies to

$$\begin{aligned}
&\max_{y_1,y_4} && -y_1-(d-1)y_4,\\
&\text{subject to} && \gamma_i^{(k)}(y_1+(d-1)y_4)\geq(-1)^{d+1-i}\binom{d-k}{i-k}\quad\text{for } i=k,k+1,\ldots,d,\\
&&& y_1\geq 0,\ y_4\geq 0.
\end{aligned}\tag{4.2}$$

Clearly, the optimal solution satisfies

$$y_1^* + (d-1)y_4^* = \max\big(0, \max_{i\in k,\ldots,d}(-1)^{d+1-i}\binom{d-k}{i-k}(\gamma_i^{(k)})^{-1}\big),\tag{4.3}$$

which proves the first part of Theorem 1.

It remains to prove the part about the realization of one optimal solution. Define

$$q_0^* = (\delta_1^{(1)})^* = (\delta_2^{(2)})^* = \ldots = (\delta_{k-1}^{(k-1)})^* = (\delta_k^{(k)})^* = \ldots = (\delta_{i_0-1}^{(k)})^* = (\delta_{i_0+1}^{(k)})^* = \ldots = (\delta_d^{(k)})^* = 0.$$

Since there is a solution to (3.19) such that $y_i^* \neq 0$ for $i = 1, 2, 3, 4$ (This is due to the fact that any pair $(y_1, y_4) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ such that $y_1 + (d-1)y_4 = w_{k,d,-}$, is an optimal solution.), it follows by complementary slackness that in the optimal solution to (3.12) we have four equalities. From the optimal value of the objective function of (3.12) we conclude $(\delta_{i_0}^{(k)})^* = (\gamma_{i_0}^{(k)})^{-1}$. From the four equalities stated above we conclude (2.3) using also that $\delta_j^{(i)} = \sum_{\ell=0}^{i}\binom{i}{\ell}(-1)^\ell q_{j-\ell}$. □

Proof of Theorem 2. The proof is analogous to the proof of Theorem 1. □

5. Concluding remarks and future research

In this paper, we addressed the problem of determining the extreme values of the mass distribution associated with a multidimensional k-increasing d-quasi-copula for $k = 2, 3, \ldots, d$, building on the linear programming approach that solved the case $k = 1$ (see [5] for $d = 3$, [13] for $d = 4$, [4] for $d \leq 17$, and [12] for the general case). We derived closed formulas for the extreme volumes as well as an explicit realization of one corresponding quasi-copula. Besides the symmetrization trick from [12], the main novelty enabling us to solve the linear programs was the introduction of new sets of variables, which significantly reduced their complexity. We also provided a numerical analysis of the results for dimensions $d \leq 15$.

Finally, we outline several directions for future research.

In the bivariate case, the 2-increasing property coincides with the supermodularity of a function F. Very recently, Anzilli and Durante [1] derived bounds on the average F-volume of closed rectangles in $[0, 1]^2$ for a supermodular aggregation function F. In line with these results, it would be interesting to study the behavior of extreme volumes for supermodular quasi-copulas in higher dimensions, as well as for ultramodular and modular quasi-copulas investigated in [10].

It is clear that every extreme point (in the Krein–Milman sense) of the convex set of k-increasing quasi-copulas must be a maximal-volume one. Our construction of the extremal maximal-volume solution is symmetric—due to the symmetrization step—and is therefore not extreme. To better understand the geometry of the convex set of k-increasing quasi-copulas, it would be natural to characterize their extreme points. For semilinear copulas, such a characterization was obtained in [6].

References

[1] L. Anzilli, F. Durante, The average rectangular volume induced by supermodular aggregation functions, *J. Math. Anal. Appl.* 555 (2026) 21 pp.

[2] J.J. Arias-García, R. Mesiar and B. De Baets, The unwalked path between quasi-copulas and copulas: Stepping stones in higher dimensions, *International Journal of Approximate Reasoning*, 80 (2017) 89–99.

[3] J.J. Arias-García, R. Mesiar and B. De Baets, A hitchhiker's guide to quasi-copulas, *Fuzzy Sets and Systems* 393 (2020), 1–28.

[4] M. Belšak, M. Omladič, M. Vuk, A. Zalar, Extreme values of the mass distribution associated with d-quasi-copulas via linear programming, *Fuzzy Sets and Systems*, 517 (2025) 17 pp.

[5] B. De Baets, H. De Meyer and M. Úbeda-Flores, Extremes of the mass distribution associated with a trivariate quasi-copula, *Comptes Rendus Mathematique*, 344 (2007), 587–590.

[6] F. Durante, J. Fernández-Sánche and M. Úbeda-Flores Extreme semilinear copulas, *Fuzzy Sets and Systems*, 428 (2020) 121–137.

[7] F. Durante, R. Mesiar, P.L. Papini, C. Sempi, 2-increasing binary aggregation operators, *Inf. Sci.* 177 (1) (2007), 111-–129.

[8] F. Durante, S. Saminger-Platz, P. Sarkoci, On representations of 2-increasing binary aggregation functions, *Inf. Sci.* 178 (23) (2009), 914–921.

[9] F. Durante and C. Sempi, *Principles of Copula Theory*, CRC Press, 2015.

[10] E.P. Klement, M. Manzi, R. Mesiar, Ultramodular aggregation functions, *Inf. Sci.* **18** (2011), 4101–4111.

[11] R.B. Nelsen, J.J. Quesada-Molina, J.A. Rodríguez-Lallena and M. Úbeda-Flores, Some New Properties of Quasi-Copulas, *Distributions With Given Marginals and Statistical Modelling*, Springer Netherlands, (2002), 187–194.

[12] M. Omladič, M. Vuk, A. Zalar, Extreme mass distributions for quasi-copulas, arXiv preprint `https://arxiv.org/abs/2506.20672`.

[13] M. Úbeda-Flores, Extreme values of the mass distribution associated with a tetravariate quasi-copula, *Fuzzy Sets and Systems* 473 (2023).

[14] Wolfram Research, Inc., Mathematica, Version 14.1, Wolfram Research, Inc., Champaign, IL, 2024.

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