# Pointwise order and undominated copulas with prescribed sections along arbitrary tracks

Matjaž Omladič[a], Damjan Škulj[b], Aljaž Zalar[a,c,d]

[a] *University of Ljubljana, Faculty of Mathematics and Physics, Jadranska 19, SI-1000 Ljubljana, Slovenia*
[b] *University of Ljubljana, Faculty of Social Sciences, Kardeljeva ploščad 5, SI-1000 Ljubljana, Slovenia*
[c] *University of Ljubljana, Faculty of Computer and Information Science, Večna pot 113, SI-1000 Ljubljana, Slovenia*
[d] *Institute of Mathematics, Physics and Mechanics, Jadranska 19, SI-1000 Ljubljana, Slovenia*

**Abstract**

The first two authors introduced a canonical family of copulas with a prescribed track section and showed that every undominated copula belongs to this family. For the identity track they also proved the converse, using an incomparability theorem for distinct canonical copulas. The corresponding order structure for a genuinely non-diagonal track remained open because the prescribed section may meet one coordinate bound without meeting the other. We solve this problem for arbitrary continuous strictly increasing tracks. The parameter interval is decomposed into a nondegeneracy set, two unilateral-contact sets, and the simultaneous-contact set. For two eligible parameters $\psi_1, \psi_2$, we prove that $C_{\psi_1} \leq C_{\psi_2}$ if and only if their derivatives agree almost everywhere on the nondegeneracy set, while $\psi_1' \leq \psi_2'$ on the contact set where $\delta(t) = t < \varphi(t)$ and $\psi_1' \geq \psi_2'$ on the contact set where $\delta(t) = \varphi(t) < t$. The order comparison imposes no additional derivative condition on the simultaneous-contact set. This yields a complete characterization of undominated copulas for arbitrary tracks: a canonical copula $C_\psi$ is undominated exactly when $\psi' = 1$ almost everywhere on the first unilateral-contact set and $\psi' = 0$ almost everywhere on the second. We also explicitly construct the unique undominated copula above each canonical copula. The identity-track theorem and the null-unilateral-contact case follow

*Email addresses:* matjaz@omladic.net (Matjaž Omladič), damjan.skulj@fdv.uni-lj.si (Damjan Škulj), aljaz.zalar@fri.uni-lj.si (Aljaž Zalar)

as immediate corollaries, while examples with positive-measure unilateral contact illustrate the new order regime.

*Keywords:* bivariate copula, track section, undominated copula, maximal copula, pointwise order

**2020 Mathematics Subject Classification.** Primary 62H05; Secondary 60E05.

## 1. Introduction

Let $\varphi : [0,1] \to [0,1]$ be continuous and strictly increasing, with $\varphi(0) = 0$ and $\varphi(1) = 1$, and let

$$B_\varphi = \{(x, \varphi(x)) : x \in [0,1]\}$$

be its track. If $C$ is a copula, the function

$$\delta(x) = C(x, \varphi(x))$$

is the track section of $C$ on $B_\varphi$. Track sections, also called curvilinear sections, have been studied from the viewpoints of existence and curve-based asymmetry [2, 1]. Sharp bounds and pointwise suprema were investigated in [11, 8, 9]. Curved splicing and curvilinear patchwork constructions were developed in [4, 10], while recent work has also considered the existence and construction of absolutely continuous copulas with prescribed curvilinear sections [7, 12]. For diagonal splicing and related identity-track constructions, see [5].

The first two authors [6] proposed an order-theoretic notion of maximality for copulas with a prescribed section. In the class

$$\mathcal{C}(B_\varphi, \delta) := \{C : C(x, \varphi(x)) = \delta(x),\ x \in [0,1]\},$$

a copula is called *undominated* if there is no $D \in \mathcal{C}(B_\varphi, \delta)$ such that $D \neq C$ and $C \leq D$ pointwise. Their construction associates to each eligible one-dimensional function $\psi$ a canonical copula $C_\psi$. The construction itself is valid for arbitrary tracks, and every undominated copula is canonical. The converse, however, was established in [6, Theorem 5.6 and Corollary 5.7] only when $\varphi = \mathrm{id}$.

For the identity track, the proof relies on a reflection argument. If two canonical copulas differ, then their difference has opposite signs at a pair $(u, v)$ and $(v, u)$. On a non-diagonal track, the natural reflected pair is

$$(u, \varphi(v)), \qquad (v, \varphi(u)),$$

which is exchanged by the involution

$$(x, y) \longmapsto (\varphi^{-1}(y), \varphi(x)).$$

The obstruction is that the prescribed section can touch exactly one of the coordinate bounds, $t$ or $\varphi(t)$. At such a unilateral contact, one of the two reflected evaluations may be clipped by the Fréchet–Hoeffding upper bound, so the identity-track sign reversal need not occur. Indeed, unilateral contact of positive measure can produce comparable canonical copulas.

The purpose of the present paper is to determine exactly what replaces incomparability in the arbitrary-track case. We split the parameter interval into four sets. On the open nondegeneracy set, where $\delta$ lies strictly below both coordinate bounds, pointwise domination forces two eligible functions to have the same derivative. On the set where $\delta(t) = t < \varphi(t)$, domination is equivalent to one inequality between their derivatives; on the set where $\delta(t) = \varphi(t) < t$, the inequality is reversed. Simultaneous contacts $\delta(t) = t = \varphi(t)$ impose no condition because they act as separators on which all copulas in the prescribed-section class agree across the track.

This leads to a complete pointwise-order theorem for the canonical family, followed by a complete characterization of undominated copulas for every track. In particular, an eligible parameter $\psi$ produces an undominated copula exactly when its derivative takes the extreme value 1 on the first unilateral-contact set and the extreme value 0 on the second. Moreover, every canonical copula admits a canonical "maximalization" obtained by changing only those derivative values; the resulting copula is the unique undominated canonical copula above the original one.

The identity-track theorem follows because unilateral contacts are impossible. Thus the derivative criterion both explains the earlier incomparability phenomenon and describes precisely what happens beyond it.

The remainder of the paper is organized as follows. Section 2 recalls the canonical construction and introduces the decomposition into the nondegeneracy, unilateral-contact, and simultaneous-contact sets. Section 3 develops the local comparison formulas and the separator property needed in the sequel. Section 4 establishes the pointwise-order theorem and derives the corresponding equality and incomparability criteria. Section 5 constructs the maximalization of an arbitrary eligible parameter, while Section 6 uses it to characterize all undominated copulas and prove the uniqueness of the undominated copula above each canonical copula. Section 7 presents examples and special cases, and Section 8 concludes the paper and indicates directions for future work.

## 2. Canonical copulas and the contact decomposition

We use the notation and canonical construction of [6]. A track section $\delta$ corresponding to $\varphi$ satisfies, in particular,

$$\delta(t) \le t \wedge \varphi(t), \qquad t \in [0,1], \tag{1}$$

and is continuous and increasing.

Throughout, increasing means nondecreasing, $\lambda$ denotes Lebesgue measure on $(0,1)$, and $\mathbf{1}_A$ denotes the indicator function of a measurable set $A \subseteq (0,1)$. We write

$$M(x,y) := x \wedge y = \min\{x,y\}, \qquad (x,y) \in [0,1]^2,$$

for the Fréchet–Hoeffding upper bound.

**Definition 2.1** (Eligible parameter)**.** A function $\psi : [0,1] \to \mathbb{R}$ is *eligible for* $(\varphi,\delta)$ if $\psi(0) = 0$ and

$$\psi, \qquad \varphi - \delta + \psi, \qquad \delta - \psi, \qquad \mathrm{id} - \psi \tag{2}$$

are increasing on $[0,1]$. The set of all eligible functions is denoted by $\mathcal{E}(\varphi,\delta)$.

For $\psi \in \mathcal{E}(\varphi,\delta)$, the associated canonical copula is given by [6, Corollary 3.7]

$$C_\psi(x,y) = \min\left\{x,\, y,\, \psi(x) - \psi(\varphi^{-1}(y)) + \delta(\varphi^{-1}(y))\right\}. \tag{3}$$

By [6, Theorem 3.4], $C_\psi \in \mathcal{C}(B_\varphi,\delta)$. Conversely, the canonical envelope construction of [6, Proposition 3.3] associates with every $C \in \mathcal{C}(B_\varphi,\delta)$ an eligible function $\widehat{\psi}$ such that

$$C \le C_{\widehat{\psi}}.$$

Consequently, by [6, Corollary 3.5], every undominated copula is of the form (3).

Eligibility gives the regularity needed throughout the paper.

**Lemma 2.2.** *If $\psi \in \mathcal{E}(\varphi,\delta)$, then*

$$0 \le \psi(y) - \psi(x) \le y - x \qquad (0 \le x \le y \le 1).$$

*Hence $\psi$ is 1-Lipschitz and absolutely continuous, and*

$$0 \le \psi'(t) \le 1 \qquad \textit{for almost every } t \in (0,1). \tag{3}$$

*Proof.* The lower bound follows from the monotonicity of $\psi$. The monotonicity of $\mathrm{id} - \psi$ gives

$$(y - \psi(y)) - (x - \psi(x)) \geq 0,$$

which is equivalent to the upper bound. □

We now separate the four possible contact regimes. Define

$$G := \{t \in (0,1) : \delta(t) < t \text{ and } \delta(t) < \varphi(t)\}, \tag{4}$$
$$U_+ := \{t \in (0,1) : \delta(t) = t < \varphi(t)\}, \tag{5}$$
$$U_- := \{t \in (0,1) : \delta(t) = \varphi(t) < t\}, \tag{6}$$
$$Z := \{t \in (0,1) : \delta(t) = t = \varphi(t)\}. \tag{7}$$

All four sets are Borel measurable, $G$ is open, and the sets are pairwise disjoint. Moreover,

$$(0,1) = G \dot\cup U_+ \dot\cup U_- \dot\cup Z. \tag{8}$$

The sets $U_+$ and $U_-$ are the two unilateral-contact sets. When $\varphi = \mathrm{id}$, both are empty.

## 3. Local comparison tools and simultaneous-contact points

The proof of the order theorem relies on three elementary tools for comparing canonical copulas locally and for handling points at which both coordinate bounds are attained simultaneously.

Simultaneous-contact points have a particularly strong consequence: they force every copula with the prescribed track section to attain the upper Fréchet–Hoeffding bound across the corresponding regions. Thus, such points act as separators, making the relevant cross-region values independent of the particular copula.

**Lemma 3.1** (Separator property)**.** *Let $C \in \mathcal{C}(B_\varphi, \delta)$ and let $s \in Z$. If $0 \leq r \leq s \leq q \leq 1$, then*

$$C(r,q) = C(q,r) = r. \tag{9}$$

*Consequently, if $x, t \in [0,1]$ and a point of $Z$ lies between $x$ and $t$, then*

$$C(x, \varphi(t)) = x \wedge \varphi(t), \tag{10}$$

*independently of the choice of $C \in \mathcal{C}(B_\varphi, \delta)$.*

*Proof.* Since $s \in Z$,

$$C(s,s) = C(s,\varphi(s)) = \delta(s) = s.$$

The 1-Lipschitz property gives

$$C(r,s) \geq C(s,s) - (s-r) = r,$$

while the Fréchet–Hoeffding upper bound gives $C(r,s) \leq r$. Thus $C(r,s) = r$. Monotonicity in the second coordinate yields

$$r = C(r,s) \leq C(r,q) \leq C(r,1) = r,$$

and hence $C(r,q) = r$. The identity $C(q,r) = r$ follows by the same argument with the coordinates interchanged.

If $x = t$, then the hypothesis implies $t \in Z$, and hence

$$C(x,\varphi(t)) = C(t,\varphi(t)) = \delta(t) = t = x \wedge \varphi(t).$$

We may therefore assume that $x \neq t$.

Suppose first that $x < t$ and $s \in Z$ satisfies $x \leq s \leq t$. Then

$$x \leq s = \varphi(s) \leq \varphi(t),$$

so (9) gives $C(x,\varphi(t)) = x$. If $t < x$, then

$$\varphi(t) \leq \varphi(s) = s \leq x,$$

and (9) gives $C(x,\varphi(t)) = \varphi(t)$. □

At a point of the nondegeneracy set $G$, none of the inequalities defining the canonical copulas is an equality; each is separated from equality by a positive amount. By continuity, the inequalities remain strict in a sufficiently small neighborhood of the point. Consequently, no boundary case arises at nearby crossed track points, and their values can be written directly in terms of the increments of the parameter functions.

**Lemma 3.2** (Nondegenerate local crossing)**.** *Let $\psi_1, \psi_2 \in \mathcal{E}(\varphi,\delta)$ and $s \in G$. There are an open interval $I \ni s$ and a number $\alpha > 0$ such that, whenever $u, v \in I$ and $|u - v| < \alpha$, for $i = 1, 2$,*

$$C_{\psi_i}(u,\varphi(v)) = \psi_i(u) - \psi_i(v) + \delta(v), \tag{11}$$
$$C_{\psi_i}(v,\varphi(u)) = \psi_i(v) - \psi_i(u) + \delta(u). \tag{12}$$

*Proof.* Put

$$K_i(p,q) := \psi_i(p) - \psi_i(q) + \delta(q).$$

At $(s,s)$,

$$K_i(s,s) = \delta(s) < \min\{s, \varphi(s)\}.$$

All functions involved are continuous. Hence, after shrinking a neighborhood of $s$ if necessary, there is $\alpha > 0$ such that

$$K_i(p,q) < \min\{p, \varphi(q)\}$$

for $p, q$ in that neighborhood with $|p-q| < \alpha$, simultaneously for $i = 1, 2$. The third entry in the minimum (3) is then active at both track-transposed points, giving (11)–(12). □

At a unilateral-contact point, the relevant one-sided variation of $\psi$ can be recovered from a single nearby value of the canonical copula. The following identities give the corresponding evaluation formulas.

**Lemma 3.3** (One-sided formulas at unilateral contacts)**.** *Let $\psi \in \mathcal{E}(\varphi, \delta)$.*

1. *If $s \in U_+$ and*

$$0 < h < \min\{1 - s, \varphi(s) - s\},$$

*then*

$$C_\psi(s + h, \varphi(s)) = s + \psi(s + h) - \psi(s). \tag{13}$$

2. *If $s \in U_-$ and*

$$0 < h < \min\{s, s - \varphi(s)\},$$

*then*

$$C_\psi(s - h, \varphi(s)) = \varphi(s) + \psi(s - h) - \psi(s). \tag{14}$$

*Proof.* If $s \in U_+$, then $\delta(s) = s < \varphi(s)$. By Lemma 2.2,

$$s \le s + \psi(s + h) - \psi(s) \le s + h < \varphi(s).$$

Thus the third entry of (3) is no larger than either coordinate bound at $(s + h, \varphi(s))$, which proves (13).

If $s \in U_-$, then $\delta(s) = \varphi(s) < s$. Again by Lemma 2.2,

$$\varphi(s) - h \le \varphi(s) + \psi(s - h) - \psi(s) \le \varphi(s) < s - h.$$

Hence the third entry is active at $(s - h, \varphi(s))$, proving (14). □

## 4. The pointwise order theorem

We can now describe the order relation between any two canonical copulas entirely in terms of almost-everywhere derivatives of their eligible parameters.

**Theorem 4.1** (Criterion for pointwise order)**.** *Let $\psi_1, \psi_2 \in \mathcal{E}(\varphi, \delta)$, and let $C_{\psi_1}$ and $C_{\psi_2}$ be the corresponding canonical copulas. Then*

$$C_{\psi_1} \leq C_{\psi_2} \qquad \text{pointwise on } [0,1]^2 \tag{15}$$

*if and only if*

$$\psi_1' = \psi_2' \quad \text{a.e. on } G, \tag{16}$$
$$\psi_1' \leq \psi_2' \quad \text{a.e. on } U_+, \tag{17}$$
$$\psi_1' \geq \psi_2' \quad \text{a.e. on } U_-. \tag{18}$$

*The order comparison imposes no additional condition on $Z$.*

*Proof.* Set $d := \psi_1 - \psi_2$. Since $\psi_1$ and $\psi_2$ are Lipschitz by Lemma 2.2, the function $d$ is absolutely continuous.

*Necessity.* Assume that $C_{\psi_1} \leq C_{\psi_2}$.

Let $s \in G$. By Lemma 3.2, for sufficiently close $u < v$ in a neighborhood of $s$,

$$C_{\psi_1}(u, \varphi(v)) - C_{\psi_2}(u, \varphi(v)) = d(u) - d(v),$$
$$C_{\psi_1}(v, \varphi(u)) - C_{\psi_2}(v, \varphi(u)) = d(v) - d(u).$$

Both left-hand sides are nonpositive, and hence $d(u) = d(v)$. Thus $d$ is locally constant on $G$, so $d' = 0$ almost everywhere on $G$. This proves (16).

Let $s \in U_+$. For every sufficiently small $h > 0$, Lemma 3.3 and the order assumption give

$$d(s+h) - d(s) = C_{\psi_1}(s+h, \varphi(s)) - C_{\psi_2}(s+h, \varphi(s)) \leq 0.$$

At every point $s \in U_+$ where $d'$ exists, division by $h$ followed by the limit $h \downarrow 0$ yields $d'(s) \leq 0$. Since $d$ is absolutely continuous, it is differentiable almost everywhere by [3, Theorem 3.35]. This proves (17).

Similarly, if $s \in U_-$, then for every sufficiently small $h > 0$, Lemma 3.3 gives

$$d(s-h) - d(s) = C_{\psi_1}(s-h, \varphi(s)) - C_{\psi_2}(s-h, \varphi(s)) \leq 0.$$

Equivalently, $(d(s) - d(s-h))/h \geq 0$. Passing to the limit at points of differentiability gives $d'(s) \geq 0$ almost everywhere on $U_-$, proving (18).

*Sufficiency.* Assume (16)–(18). Fix an arbitrary point $(x, y) \in [0,1]^2$. Since $\varphi$ is a bijection of $[0,1]$, there is a unique $t \in [0,1]$ such that $y = \varphi(t)$. It therefore suffices to show that $C_{\psi_1}(x, \varphi(t)) \leq C_{\psi_2}(x, \varphi(t))$. If $x \in \{0,1\}$ or $t \in \{0,1\}$, the two copulas agree by the copula boundary conditions. Hence we may assume that $x, t \in (0,1)$.

If a point of $Z$ lies between $x$ and $t$, then Lemma 3.1 gives $C_{\psi_1}(x, \varphi(t)) = C_{\psi_2}(x, \varphi(t))$. We may therefore assume that the interval between $x$ and $t$ does not meet $Z$.

For $i = 1, 2$, set $K_i(x,t) := \psi_i(x) - \psi_i(t) + \delta(t)$. If $d(x) \leq d(t)$, then $K_1(x,t) \leq K_2(x,t)$. The canonical formula, which is monotone in its third entry, therefore gives $C_{\psi_1}(x, \varphi(t)) \leq C_{\psi_2}(x, \varphi(t))$. It remains to consider the case $d(x) > d(t)$.

Suppose first that $t < x$. By absolute continuity,

$$0 < d(x) - d(t) = \int_t^x d'(s)\, ds.$$

Since $(t,x) \cap Z = \varnothing$, the partition (8) and the derivative conditions imply that, if $\lambda((t,x) \cap U_-) = 0$, then $d' \leq 0$ almost everywhere on $(t,x)$. This would contradict the preceding integral identity. Hence $\lambda((t,x) \cap U_-) > 0$; in particular, we may choose $s \in (t,x) \cap U_-$. Then $\delta(s) = \varphi(s) < s$, and hence $\varphi(t) < \varphi(s) < s < x$.

The function $\varphi - \delta + \psi_2$ is increasing and equals $\psi_2$ at $s$. Consequently, $\varphi(t) - \delta(t) + \psi_2(t) \leq \psi_2(s) \leq \psi_2(x)$, which is equivalent to $K_2(x,t) \geq \varphi(t)$. Since $\varphi(t) < x$, the canonical formula gives $C_{\psi_2}(x, \varphi(t)) = \varphi(t)$. The Fréchet–Hoeffding upper bound then yields $C_{\psi_1}(x, \varphi(t)) \leq \varphi(t) = C_{\psi_2}(x, \varphi(t))$.

Finally, suppose that $x < t$. By absolute continuity,

$$d(t) - d(x) = \int_x^t d'(s)\, ds < 0.$$

Since $(x,t) \cap Z = \varnothing$, the partition (8) and the derivative conditions imply that, if $\lambda((x,t) \cap U_+) = 0$, then $d' \geq 0$ almost everywhere on $(x,t)$. This contradicts the preceding integral identity. Hence $\lambda((x,t) \cap U_+) > 0$; in particular, we may choose $s \in (x,t) \cap U_+$. Then $\delta(s) = s < \varphi(s)$, and therefore $x < s < \varphi(s) < \varphi(t)$.

Since $\delta - \psi_2$ and $\mathrm{id} - \psi_2$ are increasing, we have

$$\delta(t) - \psi_2(t) \geq s - \psi_2(s) \geq x - \psi_2(x).$$

Equivalently, $K_2(x,t) \geq x$, and hence $C_{\psi_2}(x,\varphi(t)) = x$. Another application of the Fréchet–Hoeffding upper bound gives $C_{\psi_1}(x,\varphi(t)) \leq x = C_{\psi_2}(x,\varphi(t))$.

This completes the proof. □

The equality relation follows at once.

**Corollary 4.2** (Equality criterion)**.** *For $\psi_1, \psi_2 \in \mathcal{E}(\varphi,\delta)$,*

$$C_{\psi_1} = C_{\psi_2}$$

*if and only if*

$$\psi_1' = \psi_2' \qquad \text{a.e. on } (0,1) \setminus Z. \tag{19}$$

*Equivalently, if $C_{\psi_1} = C_{\psi_2}$, then $\psi_1 - \psi_2$ is constant on each interval contained in $(0,1) \setminus Z$; any variation in this difference can occur only on $Z$.*

*Proof.* Equality is equivalent to both $C_{\psi_1} \leq C_{\psi_2}$ and $C_{\psi_2} \leq C_{\psi_1}$. Applying Theorem 4.1 in both directions gives equality of the derivatives on $G$, $U_+$ and $U_-$. The converse is immediate from the same theorem. □

**Corollary 4.3** (Null unilateral contact)**.** *If $\lambda(U_+ \cup U_-) = 0$, then any two distinct canonical copulas are pointwise incomparable.*

*Proof.* If $C_{\psi_1} \leq C_{\psi_2}$, Theorem 4.1 gives equality of the derivatives on $G$. The unilateral-contact sets are null, so the derivatives agree almost everywhere on $(0,1) \setminus Z$. By Corollary 4.2, the copulas are equal. □

In particular, when $\varphi = \mathrm{id}$, $U_+ = U_- = \varnothing$, and Corollary 4.3 recovers the incomparability conclusion of [6, Theorem 5.6].

## 5. Maximalization of an eligible parameter

We next show that the derivative order in Theorem 4.1 has an explicit maximal element above every canonical copula. Two standard facts about monotone functions will be used.

**Lemma 5.1.** *Let $F\colon [0,1] \to \mathbb{R}$ be increasing, and let $r \in L^1(0,1)$ satisfy*

$$0 \leq r(t) \leq F'(t) \qquad \text{for almost every } t \in (0,1).$$

*Then the function*

$$x \longmapsto F(x) - \int_0^x r(t)\,dt$$

*is increasing.*

*Proof.* An increasing function is differentiable almost everywhere, its derivative is integrable, and

$$F(y) - F(x) \geq \int_x^y F'(t)\,dt, \qquad 0 \leq x < y \leq 1;$$

see [3, Section 3.5]. Therefore,

$$\begin{aligned}\left(F(y) - \int_0^y r(t)\,dt\right) - \left(F(x) - \int_0^x r(t)\,dt\right) &= F(y) - F(x) - \int_x^y r(t)\,dt \\ &\geq \int_x^y \big(F'(t) - r(t)\big)\,dt \geq 0.\end{aligned}$$

Hence the stated function is increasing. □

**Lemma 5.2** (Derivatives on a coincidence set)**.** *Let $f, g\colon [0,1] \to \mathbb{R}$ be functions of bounded variation, and set*

$$E := \{t \in [0,1] : f(t) = g(t)\}.$$

*Then $f'(t) = g'(t)$ for almost every $t \in E$.*

*Proof.* Functions of bounded variation are differentiable almost everywhere, and almost every point of the measurable set $E$ is a density point of $E$; see [3, Sections 3.4–3.5]. Put $h := f - g$. At almost every $t \in E$, the function $h$ is differentiable at $t$ and $t$ is a density point of $E$. We may therefore choose a sequence

$$t_n \in E \setminus \{t\}, \qquad t_n \to t.$$

Since $h(t_n) = h(t) = 0$, we have

$$\frac{h(t_n) - h(t)}{t_n - t} = 0$$

for every $n$. Passing to the limit gives $h'(t) = 0$, and hence $f'(t) = g'(t)$. □

Let $\psi \in \mathcal{E}(\varphi, \delta)$. Define

$$H_+(x) := \int_0^x (1 - \psi'(t))\,\mathbf{1}_{U_+}(t)\,dt, \tag{20}$$

$$H_-(x) := \int_0^x \psi'(t)\,\mathbf{1}_{U_-}(t)\,dt, \tag{21}$$

and set

$$\psi^\star := \psi + H_+ - H_-. \tag{22}$$

**Theorem 5.3** (Canonical maximalization)**.** *For every $\psi \in \mathcal{E}(\varphi, \delta)$, the function $\psi^\star$ defined by* (22) *is eligible. Moreover,*

$$(\psi^\star)'(t) = \begin{cases} 1, & \text{for a.e. } t \in U_+, \\ 0, & \text{for a.e. } t \in U_-, \\ \psi'(t), & \text{for a.e. } t \in G \cup Z, \end{cases} \tag{23}$$

*and*

$$C_\psi \le C_{\psi^\star}. \tag{24}$$

*Proof.* By Lemma 2.2, $0 \le \psi' \le 1$ almost everywhere. Hence $H_+$ and $H_-$ are increasing and absolutely continuous, and direct differentiation of (22) gives (23); in particular,

$$0 \le (\psi^\star)' \le 1$$

almost everywhere. Thus $\psi^\star$ and $\mathrm{id} - \psi^\star$ are increasing.

Put

$$P := \varphi - \delta + \psi, \qquad Q := \delta - \psi.$$

Both $P$ and $Q$ are increasing by eligibility. On $U_-$ we have $\delta = \varphi$, hence $P = \psi$. By Lemma 5.2,

$$P' = \psi' \qquad \text{a.e. on } U_-.$$

Since $P' \ge 0$ almost everywhere, it follows that

$$0 \le H_-' = \psi' \mathbf{1}_{U_-} \le P' \qquad \text{almost everywhere.}$$

Lemma 5.1 therefore shows that $P - H_-$ is increasing. Adding the increasing function $H_+$ shows that

$$\varphi - \delta + \psi^\star = P + H_+ - H_-$$

is increasing.

On $U_+$ we have $\delta = \mathrm{id}$, hence

$$Q = \mathrm{id} - \psi.$$

Since $Q' \ge 0$ almost everywhere, we have

$$0 \le H_+' = (1 - \psi')\mathbf{1}_{U_+} \le Q' \qquad \text{almost everywhere.}$$

Therefore $Q - H_+$ is increasing by Lemma 5.1; adding $H_-$ gives the monotonicity of

$$\delta - \psi^\star = Q - H_+ + H_-.$$

Thus all four functions in (2) are increasing, and $\psi^\star$ is eligible.

Finally, let $d = \psi - \psi^\star = -H_+ + H_-$. Then

$$d' = 0 \text{ on } G, \qquad d' \le 0 \text{ on } U_+, \qquad d' \ge 0 \text{ on } U_-$$

almost everywhere. The order criterion Theorem 4.1 yields (24). □

## 6. Complete characterization of undominated copulas

The maximalization theorem turns the order criterion into an exact characterization of maximal elements.

**Theorem 6.1** (Undominated canonical copulas)**.** *Let $\psi \in \mathcal{E}(\varphi, \delta)$. Then $C_\psi$ is undominated in $\mathcal{C}(B_\varphi, \delta)$ if and only if*

$$\psi'(t) = 1 \qquad \text{for a.e. } t \in U_+, \tag{25}$$

$$\psi'(t) = 0 \qquad \text{for a.e. } t \in U_-. \tag{26}$$

*Proof.* Assume first that (25)–(26) hold, and suppose

$$C_\psi \le D$$

for some $D \in \mathcal{C}(B_\varphi, \delta)$. By the construction in [6, Proposition 3.3 and the proof of Corollary 3.5], there exists an eligible $\eta \in \mathcal{E}(\varphi, \delta)$ such that

$$D \le C_\eta.$$

Hence $C_\psi \le C_\eta$. Applying Theorem 4.1 to $d = \psi - \eta$ gives $d' = 0$ on $G$, $d' \le 0$ on $U_+$ and $d' \ge 0$ on $U_-$ almost everywhere.

On $U_+$, (25) and Lemma 2.2 imply

$$d' = 1 - \eta' \ge 0.$$

Thus $d' = 0$ almost everywhere on $U_+$. On $U_-$, (26) gives

$$d' = -\eta' \le 0,$$

so again $d' = 0$ almost everywhere. Therefore $\psi' = \eta'$ almost everywhere on $(0,1) \setminus Z$, and Corollary 4.2 yields

$$C_\psi = C_\eta.$$

The inequalities $C_\psi \le D \le C_\eta$ force $D = C_\psi$. Thus $C_\psi$ is undominated.

Conversely, suppose that at least one of (25)–(26) fails. Then the corresponding correction in (20) or (21) is nonzero on a set of positive measure, so

$$(\psi^\star)' \neq \psi'$$

on a positive-measure subset of $(0,1) \setminus Z$. By Corollary 4.2,

$$C_\psi \neq C_{\psi^\star}.$$

Together with (24), this shows that $C_\psi$ is dominated by a different copula in the same prescribed-section class. □

Combining this with the canonical representation of every undominated copula gives the arbitrary-track analogue of [6, Corollary 5.7].

**Corollary 6.2** (All undominated copulas)**.** *A copula $C \in \mathcal{C}(B_\varphi, \delta)$ is undominated if and only if*

$$C = C_\psi$$

*for some eligible $\psi$ satisfying* (25)–(26)*.*

*Proof.* Every undominated copula is canonical by the general construction of [6]. The additional derivative conditions are then necessary and sufficient by Theorem 6.1. □

The maximalization has the following uniqueness property in the full prescribed-section class.

**Corollary 6.3** (Unique undominated copula above $C_\psi$)**.** *For every eligible $\psi$, the copula $C_{\psi^\star}$ is undominated and is the unique $D \in \mathcal{C}(B_\varphi, \delta)$ satisfying $C_\psi \leq D$ and such that $D$ is undominated.*

*Proof.* By Theorem 5.3, the function $\psi^\star$ is eligible, $C_\psi \leq C_{\psi^\star}$, and

$$(\psi^\star)' = 1 \quad \text{a.e. on } U_+, \qquad (\psi^\star)' = 0 \quad \text{a.e. on } U_-.$$

It follows from Theorem 6.1 that $C_{\psi^\star}$ is undominated.

To prove uniqueness, let $D \in \mathcal{C}(B_\varphi, \delta)$ be an undominated copula such that $C_\psi \leq D$. By Corollary 6.2, there is an eligible $\eta$ such that $D = C_\eta$. Applying Theorem 4.1 to $C_\psi \leq C_\eta$ gives

$$\eta' = \psi' \quad \text{a.e. on } G, \qquad \eta' \geq \psi' \quad \text{a.e. on } U_+, \qquad \eta' \leq \psi' \quad \text{a.e. on } U_-.$$

Since $C_\eta$ is undominated, Theorem 6.1 gives

$$\eta' = 1 \quad \text{a.e. on } U_+, \qquad \eta' = 0 \quad \text{a.e. on } U_-.$$

Comparing these relations with (23), we obtain

$$\eta' = (\psi^\star)' \qquad \text{a.e. on } (0,1) \setminus Z.$$

Corollary 4.2 therefore yields $D = C_\eta = C_{\psi^\star}$. This proves uniqueness. □

*Remark* 6.4 (Order components). The order theorem implies that if two canonical copulas $C_{\psi_1}$ and $C_{\psi_2}$ are comparable, then

$$\psi_1' = \psi_2' \qquad \text{a.e. on } G.$$

Thus, the derivative profile on $G$ is constant on every chain in the canonical family. Within a fixed profile on $G$, the pointwise order is the ordinary derivative order on $U_+$ and the reversed derivative order on $U_-$. The maximalization (22) preserves the derivative on $G$, sets it equal to 1 on $U_+$, and sets it equal to 0 on $U_-$, almost everywhere. Consequently, comparable canonical copulas have the same unique undominated copula above them.

## 7. Examples and special cases

### *7.1. Identity and null unilateral contact*

If $\varphi = \mathrm{id}$, then neither $U_+$ nor $U_-$ can occur. Hence every eligible canonical copula automatically satisfies the conditions of Theorem 6.1. We recover the main identity-track conclusion of [6]: all canonical copulas are undominated, and distinct ones are incomparable.

More generally, if $\lambda(U_+ \cup U_-) = 0$, the derivative saturation conditions are vacuous. Thus every canonical copula is undominated, recovering Corollary 4.3.

### *7.2. A positive-measure $U_+$ example*

The following example exhibits the fundamental obstruction to a general incomparability theorem. Our results provide a complete characterization of the resulting order structure.

**Example 7.1.** Let

$$\varphi(t) = \sqrt{t}, \qquad \delta(t) = t, \qquad 0 \le t \le 1.$$

Then

$$U_+ = (0,1), \qquad G = U_- = Z = \varnothing.$$

For $c \in [0, \frac{1}{2}]$, define

$$\psi_c(t) = (1-c)t.$$

The four functions in (2) are

$$(1-c)t, \qquad \sqrt{t}-ct, \qquad ct, \qquad ct,$$

and they are increasing, so $\psi_c$ is eligible. Since $\varphi^{-1}(y) = y^2$,

$$C_c(x,y) = \min\{x, y, (1-c)x + cy^2\}. \tag{27}$$

By Theorem 4.1,

$$c_1 \geq c_2 \quad \Longrightarrow \quad C_{c_1} \leq C_{c_2}.$$

In particular,

$$C_c \leq C_0 = M,$$

where, as above, $M(x,y) = x \wedge y$ denotes the Fréchet–Hoeffding upper bound. The undominatedness criterion requires $\psi_c' = 1$ almost everywhere on $U_+$, hence $c = 0$. Thus $M$ is the unique undominated canonical copula in this family. For example,

$$C_{1/2}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{3}{8} < \frac{1}{2} = C_0\left(\frac{1}{2}, \frac{1}{2}\right).$$

*7.3. The dual positive-measure $U_-$ example*

**Example 7.2.** Let

$$\varphi(t) = t^2, \qquad \delta(t) = t^2.$$

Then

$$U_- = (0,1), \qquad G = U_+ = Z = \varnothing.$$

For $c \in [0, \frac{1}{2}]$, put

$$\psi_c(t) = ct^2.$$

The four eligibility functions reduce to

$$ct^2, \qquad ct^2, \qquad (1-c)t^2, \qquad t - ct^2,$$

which are increasing. Since $\varphi^{-1}(y) = \sqrt{y}$,

$$C_c(x,y) = \min\{x, y, cx^2 + (1-c)y\}.$$

Theorem 4.1 gives

$$c_1 \geq c_2 \quad \Longrightarrow \quad C_{c_1} \leq C_{c_2},$$

because the order is reversed in the derivative on $U_-$. The undominatedness condition is $\psi_c' = 0$ almost everywhere, again selecting $c = 0$ and hence $M$.

*7.4. A track with both kinds of unilateral contact*

**Example 7.3.** Let $0 < a < 1$ and define

$$\varphi_a(t) = t + \frac{a}{2\pi}\sin(2\pi t), \qquad \delta_a(t) = \min\{t, \varphi_a(t)\}.$$

Since

$$\varphi_a'(t) = 1 + a\cos(2\pi t) > 0,$$

$\varphi_a$ is a track function, and $\delta_a$ is the track section of the Fréchet–Hoeffding upper bound $M$. In this case

$$U_+ = (0, \tfrac{1}{2}), \qquad U_- = (\tfrac{1}{2}, 1), \qquad Z = \{\tfrac{1}{2}\}, \qquad G = \varnothing.$$

Thus an undominated eligible parameter must satisfy

$$\psi' = 1 \text{ a.e. on } (0, \tfrac{1}{2}), \qquad \psi' = 0 \text{ a.e. on } (\tfrac{1}{2}, 1).$$

Since $Z = \{\frac{1}{2}\}$ is null and $\psi$ is absolutely continuous with $\psi(0) = 0$, these derivative conditions determine $\psi$ uniquely:

$$\psi(t) = t \wedge \frac{1}{2}.$$

The corresponding canonical copula is $M$. This example shows how the two unilateral-contact orientations are combined by the derivative criterion.

## 8. Conclusion

For a prescribed section on an arbitrary track, the order of canonical copulas is completely determined by where the section lies relative to the two coordinate bounds. On the nondegeneracy set $G$, domination is possible only when the eligible parameters have the same derivative. On $U_+$ the derivative is ordered in the usual direction, while on $U_-$ it is ordered in the reverse direction. Simultaneous contacts form separator sets and do not affect the represented copula.

This yields a complete arbitrary-track characterization of undominated copulas. The derivative of an undominated parameter must be maximal, namely 1, on $U_+$ and minimal, namely 0, on $U_-$. No additional restriction is imposed on the admissible derivative profile in $G$ beyond eligibility itself. The explicit transformation $\psi \mapsto \psi^\star$ provides a canonical maximalization and identifies the unique undominated canonical copula above any given canonical copula.

The identity-track theorem and the null-contact incomparability result are therefore special cases of a more general order decomposition. Positive-measure unilateral contact is not an exceptional pathology but a genuine ordered regime: it is precisely where nontrivial domination inside the canonical family can occur.

An interesting direction for future work is to investigate whether this order decomposition extends to higher-dimensional copulas. In dimensions $d \geq 3$, a prescribed section may interact with several coordinate bounds simultaneously, potentially producing a richer family of unilateral and multiple-contact regimes. Identifying suitable higher-dimensional canonical parameters, separator sets, and saturation conditions could lead to an analogue of the present characterization of undominated copulas.


## Funding

Damjan Škulj was supported by the Slovenian Research and Innovation Agency (ARIS) research core funding No. P5-0168; Matjaž Omladič by ARIS research core funding No. P1-0448; and Aljaž Zalar by ARIS research core funding No. P1-0288 and grants No. J1-50002, J1-60011, and J1-70017.


## Data availability

No data were used for the research described in this article.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work, the authors used OpenAI's GPT-5.6 Pro to explore possible proof strategies, organize intermediate LaTeX drafts, and check elementary estimates. The authors subsequently reviewed and verified all mathematical arguments and take full responsibility for the content of the manuscript.

## References


[1] B. De Baets, H. De Meyer, T. Jwaid, On the degree of asymmetry of a quasi-copula with respect to a curve, *Fuzzy Sets Syst.* 354 (2019) 84–103.

[2] J. Fernández-Sánchez, M. Úbeda-Flores, Copulas with given track and opposite track sections: solution to a problem on diagonals, *Fuzzy Sets Syst.* 308 (2017) 133–137.

[3] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, 2nd ed., John Wiley & Sons, New York, 1999.

[4] T. Jwaid, H. De Meyer, A. Haj Ismail, B. De Baets, Curved splicing of copulas, *Inf. Sci.* 556 (2021) 95–110.

[5] R.B. Nelsen, J.J. Quesada-Molina, J.A. Rodríguez-Lallena, M. Úbeda-Flores, On the construction of copulas and quasi-copulas with given diagonal sections, *Insur. Math. Econ.* 42 (2008) 473–483.

[6] M. Omladič, D. Škulj, A complete characterization of maximal copulas with given track section, *Fuzzy Sets Syst.* 541 (2026) 109949.

[7] Y. Ouyang, Q. Cao, H.-P. Zhang, Absolutely continuous copulas with a given curvilinear section, *Fuzzy Sets Syst.* 530 (2026) 109764.

[8] Y. Ouyang, Y. Sun, H.-P. Zhang, On an upper bound of the set of copulas with a given curvilinear section, *Fuzzy Sets Syst.* 500 (2025) 109199.

[9] Y. Ouyang, Y. Sun, H.-P. Zhang, On the pointwise supremum of the set of copulas with a given curvilinear section, *Fuzzy Sets Syst.* 518 (2025) 109515.

[10] J. Xie, B. Wu, W. Zou, C. Jiang, Curvilinear patchwork constructions of (quasi-)copulas with given curvilinear sections, *Fuzzy Sets Syst.* 473 (2023) 108720.

[11] W. Zou, L. Sun, J. Xie, Best-possible bounds on the sets of copulas and quasi-copulas with given curvilinear sections, *Fuzzy Sets Syst.* 441 (2022) 335–365.

[12] W. Zou, J. Xie, Construction methods for absolutely continuous copulas with specified curvilinear sections, *Commun. Stat. Theory Methods* (2026), published online, doi:10.1080/03610926.2026.2698249.