THICK COMPRESSIONS OF THIN ALGEBRAS

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Abstract. Let $\mathcal{A}$ be the algebra generated by a nonderogatory matrix in $M_{n^2}(\mathbb{F})$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. We prove that there exist matrices $V \in M_{n^2,n}(\mathbb{F})$ and $W \in M_{n,n^2}(\mathbb{F})$ such that $WV = I_n$ and $W\mathcal{A}V = M_n(\mathbb{F})$. Over $\mathbb{C}$, this is equivalent to the existence of a rank n orthogonal projection P for which $P\mathcal{A}P|_{\operatorname{Im} P} = M_n(\mathbb{C})$. As a consequence, the diagonal algebra in $M_{n^2}(\mathbb{C})$ has a quantum n-clique. The corresponding optimal result for the complex diagonal algebra was obtained independently recently by Weaver. Over $\mathbb{R}$, ordinary orthogonal compressions are generally too restrictive, and the corresponding geometric result is formulated using general rank-n idempotents, possibly oblique. We also give an elementary two-sided construction for the diagonal algebra over both fields and determine the dimensions of the other three block spaces associated with a maximal orthogonal compression of a commutative self-adjoint algebra.

1. Introduction

This paper concerns large compressions of "thin" matrix algebras. Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$. Throughout, A^* denotes the conjugate transpose of a matrix A; over $\mathbb{R}$, this is simply the transpose A^T. Let $A \in M_N(\mathbb{F})$ be *nonderogatory*, i.e., A has a cyclic vector: there exists $x \in \mathbb{F}^N$ such that $x, Ax, \dots, A^{N-1}x$ is a basis of $\mathbb{F}^N$. Equivalently, the minimal polynomial of A coincides with its characteristic polynomial. The algebra generated by A is then

$$\mathcal{A} = \operatorname{span}_{\mathbb{F}}\{I, A, \dots, A^{N-1}\},$$

and hence has dimension N, compared with the dimension N^2 of the full matrix algebra.

We ask whether such a thin algebra can nevertheless have a "thick" corner. The extremal case is $N = n^2$, where dimension counting permits an at most $n \times n$ compression to fill $M_n(\mathbb{F})$. Our principal result gives an affirmative answer: if $\mathcal{A}$

2020 *Mathematics Subject Classification.* Primary 15A30; Secondary 47L05, 81P45.

Key words and phrases. Matrix algebra, compression, nonderogatory matrix, operator system, quantum clique, diagonal algebra, idempotent.

M. Omladič, K. Šivic and A. Zalar acknowledge financial support from the Slovenian Research and Innovation Agency through research core funding Nos. P1-0448, P1-0222, P1-0288, respectively. K. Šivic and A. Zalar also acknowledge grants Nos. J1-50002, J1-60011, and J1-70017 from the Slovenian Research and Innovation Agency.

is generated by a nonderogatory matrix in $M_{n^2}(\mathbb{F})$, then there exist matrices

$$V \in M_{n^2,n}(\mathbb{F}), \qquad W \in M_{n,n^2}(\mathbb{F}),$$

such that

$$WV = I_n \qquad \text{and} \qquad W\mathcal{A}V = M_n(\mathbb{F}).$$

Equivalently, after a suitable change of basis, the upper-left $n \times n$ blocks of the elements of $\mathcal{A}$ run through all of $M_n(\mathbb{F})$.

As a concrete new example, let $\mathcal{T}_{n^2}$ denote the algebra of upper triangular $n^2 \times n^2$ Toeplitz matrices. Since $\mathcal{T}_{n^2}$ is generated by a nilpotent Jordan block, our theorem shows that it admits a full $n \times n$ compression. This complements the diagonal-algebra example, for which the corresponding result was already known.

Matrix compression problems arise in the theory of higher-rank numerical ranges and quantum error correction; see [2]. The diagonal case also has an interpretation in quantum tomography. Indeed, if $\mathcal{D}_{n^2}$ denotes the algebra of all diagonal matrices in $M_{n^2}(\mathbb{C})$, then an isometry $V \in M_{n^2,n}(\mathbb{C})$ satisfying

$$V^*\mathcal{D}_{n^2}V = M_n(\mathbb{C})$$

determines n^2 rank-one positive operators that form a minimal informationally complete measurement; compare [4].

Over $\mathbb{C}$, the two-sided compression can be replaced by an orthogonal compression: there exists a rank n orthogonal projection P such that

$$P\mathcal{A}P|_{\operatorname{Im} P} = M_n(\mathbb{C}).$$

This connects the problem with quantum graph theory. If $\mathcal{V} \subseteq M_N(\mathbb{C})$ is an operator system, a rank n projection P is called a *quantum n-clique* when

$$P\mathcal{V}P = PM_N(\mathbb{C})P,$$

or equivalently when $\dim(P\mathcal{V}P) = n^2$; see [3, 10, 11].

Weaver proved that the diagonal operator system $\mathcal{D}_N$ has a quantum n-clique whenever $N \geq n^2 + n - 1$; see [11, Proposition 2.2]. More recently, he obtained the sharp bound by proving that $\mathcal{D}_{n^2}$ has a quantum n-clique; see [13, Theorem 2.7]. Thus the complex diagonal case of our theorem gives the same optimal conclusion by a different method. Our principal result applies more generally to every algebra generated by a nonderogatory matrix, including algebras generated by nonnormal or nondiagonalizable matrices, and also gives a real two-sided analogue. Its proof proceeds from an explicit compression of a nilpotent Jordan block to arbitrary nonderogatory matrices by means of companion matrices.

Weaver's related quantum Turán problem is studied in [12]. For recent work on quantum Ramsey numbers and quantum Turán parameters, see also [1]. Kennedy, Kolomatski, and Spivak established an infinite-dimensional extension: every operator system on an infinite-dimensional Hilbert space has either an infinite quantum clique, an infinite quantum anticlique, or an infinite quantum obstruction [7]. More recently, Mijares extended this result to suitably chosen families of infinite-rank projections [8].

The real case requires a different geometric formulation. If $\mathcal{A}$ consists of real symmetric matrices and P is an orthogonal projection, then every element of $P\mathcal{A}P$ is symmetric on $\operatorname{Im} P$, so such a compression cannot equal $M_n(\mathbb{R})$. Instead, there exists a rank n idempotent $E \in M_{n^2}(\mathbb{R})$ such that

$$E\mathcal{A}E|_{\operatorname{Im} E} = M_n(\mathbb{R}).$$

Thus the complex and real results coincide in the formulation $W\mathcal{A}V = M_n(\mathbb{F})$ with $WV = I_n$, but their geometric formulations differ: orthogonal projections suffice over $\mathbb{C}$, whereas idempotents are generally necessary over $\mathbb{R}$.

An important special case is that of maximal abelian self-adjoint algebras, or *masas*. Every masa in $M_N(\mathbb{C})$ is unitarily equivalent to the diagonal algebra $\mathcal{D}_N$: its elements are commuting normal matrices and hence are simultaneously unitarily diagonalizable, while maximality gives the full diagonal algebra. For the general theory of simultaneous triangularization and diagonalization of matrix families, see [9]; see also [5, Section 2.5]. Since $\mathcal{D}_N$ is generated by a diagonal matrix with distinct entries, our main theorem implies that every masa in $M_{n^2}(\mathbb{C})$ has a quantum n-clique. In the diagonal case, this conclusion also follows from [13, Theorem 2.7]. The analogous real statement holds for the real diagonal algebra, and more generally for its orthogonal conjugates, with an idempotent in place of an orthogonal projection.

We also examine the remaining blocks determined by a thick orthogonal compression. Let $\mathcal{A} \subseteq M_{n^2}(\mathbb{F})$ be a commutative self-adjoint algebra of dimension n^2. If P is a rank n orthogonal projection satisfying

$$\dim(P\mathcal{A}P) = n^2$$

and $Q = I - P$, then

$$\dim(Q\mathcal{A}Q) = n^2 \qquad \text{and} \qquad \dim(P\mathcal{A}Q) = \dim(Q\mathcal{A}P) = n^2 - 1.$$

Thus maximal thickness of one diagonal block determines the dimensions of all three remaining block spaces.

2. Equivalent formulations of thick compressions

Before passing to algebras generated by nonderogatory matrices, we record several equivalent formulations of the existence of a thick $n \times n$ compression. We first establish the equivalence between the similarity and two-sided formulations over both $\mathbb{C}$ and $\mathbb{R}$. We then treat the geometric formulations separately. Over $\mathbb{C}$, a two-sided compression may be replaced by an orthogonal compression, represented either by a unitary matrix, an isometry, or an orthogonal projection. Over $\mathbb{R}$, orthogonal compressions are generally too restrictive, and the appropriate geometric formulation uses an idempotent.

Throughout, $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. For an $\mathbb{F}$-vector space $\mathcal{V}$, let $\mathrm{End}_{\mathbb{F}}(\mathcal{V})$ denote the algebra of all $\mathbb{F}$-linear maps from $\mathcal{V}$ to itself. If $\dim_{\mathbb{F}} \mathcal{V} = n$, then a choice of basis identifies $\mathrm{End}_{\mathbb{F}}(\mathcal{V})$ with $M_n(\mathbb{F})$.

Theorem 2.1. *Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$, and let $\mathcal{A}$ be an algebra in $M_{n^2}(\mathbb{F})$. Let P_0 denote the coordinate projection onto the first n coordinates. Then the following conditions are equivalent.*

(i) There exists an invertible matrix $T \in M_{n^2}(\mathbb{F})$ such that

$$P_0(T^{-1}\mathcal{A}T)P_0\big|_{\operatorname{Im} P_0} = M_n(\mathbb{F}).$$

Equivalently, the upper-left $n \times n$ block of $T^{-1}AT$, as A ranges over $\mathcal{A}$, runs through all of $M_n(\mathbb{F})$.

(ii) There exist matrices $V \in M_{n^2,n}(\mathbb{F})$, $W \in M_{n,n^2}(\mathbb{F})$, such that $WV = I_n$ and

$$W\mathcal{A}V = M_n(\mathbb{F}).$$

Proof. Suppose first that *(i)* holds. Define $V = TP_0|_{\operatorname{Im} P_0}$ and $W = P_0T^{-1}$. Then

$$WV = P_0T^{-1}TP_0|_{\operatorname{Im} P_0} = I_n.$$

Moreover, for every $A \in \mathcal{A}$,

$$WAV = P_0T^{-1}ATP_0|_{\operatorname{Im} P_0}.$$

Thus $W\mathcal{A}V$ is precisely the coordinate $n \times n$ compression of $T^{-1}\mathcal{A}T$, which proves *(ii)*. Conversely, suppose *(ii)* holds. In particular, the columns of V are linearly independent and

$$\operatorname{Im} V \cap \ker W = \{0\}.$$

Since $\dim \operatorname{Im} V = n$ and $\dim \ker W = n^2 - n$, it follows that $\mathbb{F}^{n^2} = \operatorname{Im} V \oplus \ker W$. Choose a basis of $\mathbb{F}^{n^2}$ whose first n vectors are the columns of V and whose remaining vectors form a basis of $\ker W$, and let T be the matrix whose columns are these basis vectors. Then T is invertible and $TP_0|_{\operatorname{Im} P_0} = V$. We identify $\operatorname{Im} P_0$ with $\mathbb{F}^n$. To see that $P_0T^{-1} = W$ under this identification, let $y \in \mathbb{F}^{n^2}$. There are unique vectors $x \in \mathbb{F}^n$ and $z \in \ker W$ such that $y = Vx + z$. By the definition of T, the first n coordinates of $T^{-1}y$ are x, and hence $P_0T^{-1}y = x$. On the other hand,

$$Wy = W(Vx + z) = WVx + Wz = x.$$

Thus $P_0T^{-1}y = Wy$ for every $y \in \mathbb{F}^{n^2}$, proving that $P_0T^{-1} = W$. Therefore, for every $A \in \mathcal{A}$,

$$P_0T^{-1}ATP_0|_{\operatorname{Im} P_0} = WAV,$$

which proves *(i)*. □

Theorem 2.2. *Let $\mathcal{A}$ be an algebra in $M_{n^2}(\mathbb{C})$. Then the equivalent conditions of Theorem 2.1 are also equivalent to each of the following conditions.*

(i) There exists a unitary matrix $U \in M_{n^2}(\mathbb{C})$ such that

$$P_0 U^* \mathcal{A} U P_0|_{\operatorname{Im} P_0} = M_n(\mathbb{C}).$$

Equivalently, the upper-left $n \times n$ block of U^*AU, as A ranges over $\mathcal{A}$, runs through all of $M_n(\mathbb{C})$.

(ii) There exists an isometry $V \in M_{n^2,n}(\mathbb{C})$ such that

$$V^* \mathcal{A} V = M_n(\mathbb{C}).$$

(iii) There exists a rank n orthogonal projection $P \in M_{n^2}(\mathbb{C})$ such that

$$P \mathcal{A} P|_{\operatorname{Im} P} = \operatorname{End}_{\mathbb{C}}(\operatorname{Im} P).$$

Proof. Condition *(i)* clearly implies condition *(i)* of Theorem 2.1, by taking $T = U$. We prove the converse: the similarity formulation of Theorem 2.1 implies the unitary coordinate-compression formulation. Suppose that

$$P_0 (T^{-1} \mathcal{A} T) P_0|_{\operatorname{Im} P_0} = M_n(\mathbb{C}),$$

where $T \in M_{n^2}(\mathbb{C})$ is invertible. Write the polar decomposition of T as $T = RU$, where R is positive definite and U is unitary. Then $T^{-1} = U^* R^{-1}$, and hence

$$P_0 (U^* R^{-1} \mathcal{A} R U) P_0|_{\operatorname{Im} P_0} = M_n(\mathbb{C}).$$

Choose $A_1, \dots, A_{n^2} \in \mathcal{A}$ such that the matrices

$$P_0 (U^* R^{-1} A_j R U) P_0|_{\operatorname{Im} P_0}, \qquad j = 1, \dots, n^2,$$

form a basis of $M_n(\mathbb{C})$. If these matrices are written as columns of an $n^2 \times n^2$ matrix, its determinant is nonzero. Let

$$R = Y \operatorname{Diag}(r_1, \dots, r_{n^2}) Y^*$$

be a spectral decomposition of R, where Y is unitary and $r_i > 0$. Keeping Y, U, and the matrices $A_1, \dots, A_{n^2}$ fixed, regard the eigenvalues as free nonzero complex parameters and set

$$R(z) = Y \operatorname{Diag}(z_1, \dots, z_{n^2}) Y^*.$$

The determinant obtained from the matrices

$$P_0 \big(U^* R(z)^{-1} A_j R(z) U \big) P_0|_{\operatorname{Im} P_0}, \qquad j = 1, \dots, n^2, \tag{2.1}$$

is a rational function of $z_1, \dots, z_{n^2}$. It is not identically zero, since it is nonzero at the original point $(z_1, \dots, z_{n^2}) = (r_1, \dots, r_{n^2})$. Therefore we may choose $z_1, \dots, z_{n^2}$ on the complex unit circle so that this determinant remains nonzero. Indeed, after clearing denominators we obtain a nonzero Laurent polynomial in $z_1, \dots, z_{n^2}$. Such a Laurent polynomial cannot vanish on the whole torus $\mathbb{T}^{n^2}$; otherwise its restriction $(t_1, \dots, t_{n^2}) \mapsto q(e^{it_1}, \dots, e^{it_{n^2}})$ would be the zero trigonometric polynomial, forcing all its Fourier coefficients to be zero. For this choice of z_i's, the matrix $R(z)$ is unitary. Hence $R(z)U$ is unitary. Moreover, since the above determinant is still nonzero, the matrices (2.1) still form a basis of $M_n(\mathbb{C})$. But, because $R(z)$ is unitary,

$$U^* R(z)^{-1} A_j R(z) U = (R(z)U)^* A_j (R(z)U).$$

Consequently

$$P_0\big((R(z)U)^* \mathcal{A} (R(z)U)\big) P_0|_{\operatorname{Im} P_0} = M_n(\mathbb{C}).$$

So the invertible-similarity formulation implies the unitary coordinate-compression formulation over $\mathbb{C}$.

It remains to relate condition *(i)* to conditions *(ii)* and *(iii)*. The equivalence of conditions *(i)* and *(ii)* is immediate: the first n columns of a unitary matrix form an isometry, and every isometry from $\mathbb{C}^n$ into $\mathbb{C}^{n^2}$ can be extended to a unitary matrix. Finally, conditions *(i)* and *(iii)* are equivalent because every rank n orthogonal projection has the form $P = UP_0U^*$ for some unitary matrix $U \in M_{n^2}(\mathbb{C})$. □

Theorem 2.3. *Let $\mathcal{A}$ be an algebra in $M_{n^2}(\mathbb{R})$. Then the equivalent conditions of Theorem 2.1 are also equivalent to each of the following conditions.*

(i) There exist an orthogonal matrix $U \in O(n^2)$ and a matrix $X \in M_{n,n^2-n}(\mathbb{R})$ such that, for the rank n idempotent

$$E_0 = \begin{pmatrix} I_n & X \\ 0 & 0 \end{pmatrix},$$

one has

$$E_0 U^T \mathcal{A} U E_0\big|_{\operatorname{Im} E_0} = M_n(\mathbb{R}).$$

Equivalently, the upper-left $n \times n$ block of $E_0 U^T A U E_0$, as A ranges over $\mathcal{A}$, runs through all of $M_n(\mathbb{R})$. Here and below, $\operatorname{Im} E_0$ is identified with $\mathbb{R}^n$ through the first n coordinates.

(ii) There exists a rank n idempotent $E \in M_{n^2}(\mathbb{R})$ such that

$$E \mathcal{A} E\big|_{\operatorname{Im} E} = \operatorname{End}_{\mathbb{R}}(\operatorname{Im} E).$$

Proof. Suppose first that condition *(ii)* of Theorem 2.1 holds, so that $WV = I_n$. Take a QR decomposition $V = V_0 R$, where $V_0^T V_0 = I_n$ and $R \in GL_n(\mathbb{R})$. Put $W_0 = RW$. Then

$$W_0 V_0 = I_n, \qquad W_0 \mathcal{A} V_0 = M_n(\mathbb{R}).$$

Extend the columns of V_0 to an orthonormal basis and write

$$U = \begin{pmatrix} V_0 & V_\perp \end{pmatrix} \in O(n^2).$$

Then

$$W_0 U = \begin{pmatrix} W_0 V_0 & W_0 V_\perp \end{pmatrix} = \begin{pmatrix} I_n & X \end{pmatrix},$$

where $X = W_0 V_\perp$. Thus

$$W_0 = \begin{pmatrix} I_n & X \end{pmatrix} U^T.$$

Define

$$E_0 = \begin{pmatrix} I_n & X \\ 0 & 0 \end{pmatrix}.$$

Then $E_0^2 = E_0$. If

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

then the upper-left block of $E_0 M E_0$ is

$$M_{11} + X M_{21} = \begin{pmatrix} I_n & X \end{pmatrix} M \begin{pmatrix} I_n \\ 0 \end{pmatrix}.$$

Applying this to $M = U^T A U$, with $A \in \mathcal{A}$, gives

$$\begin{pmatrix} I_n & X \end{pmatrix} U^T A U \begin{pmatrix} I_n \\ 0 \end{pmatrix} = W_0 A V_0.$$

Since $W_0 \mathcal{A} V_0 = M_n(\mathbb{R})$, we obtain

$$E_0 U^T \mathcal{A} U E_0\big|_{\operatorname{Im} E_0} = M_n(\mathbb{R}).$$

This proves the first real idempotent formulation.

Conversely, suppose there exist $U \in O(n^2)$ and

$$E_0 = \begin{pmatrix} I_n & X \\ 0 & 0 \end{pmatrix}$$

such that

$$E_0 U^T \mathcal{A} U E_0\big|_{\operatorname{Im} E_0} = M_n(\mathbb{R}).$$

Write $U = \begin{pmatrix} V & V_\perp \end{pmatrix}$, where V is the matrix consisting of the first n columns of U, and define $W = \begin{pmatrix} I_n & X \end{pmatrix} U^T$. Then $V^T V = I_n$, $WV = I_n$. Moreover, the compression $E_0\, U^T \mathcal{A}\, U\, E_0\big|_{\operatorname{Im} E_0}$ is represented by $W\mathcal{A}V$. Thus condition *(ii)* of Theorem 2.1 holds.

Finally, we relate this to arbitrary rank n idempotents. If the coordinate idempotent formulation holds, then $E = U E_0 U^T$ is a rank n idempotent and

$$E\mathcal{A}E\big|_{\operatorname{Im} E} = \operatorname{End}_{\mathbb{R}}(\operatorname{Im} E). \tag{2.2}$$

Conversely, suppose $E \in M_{n^2}(\mathbb{R})$ is a rank n idempotent such that (2.2) holds. Choose a basis of $\operatorname{Im} E$ and let $V \in M_{n^2,n}(\mathbb{R})$ be the matrix whose columns are this basis. Let $W \in M_{n,n^2}(\mathbb{R})$ be the coordinate map associated with the decomposition

$$\mathbb{R}^{n^2} = \operatorname{Im} E \oplus \ker E.$$

Then $E = VW$, $WV = I_n$, and the induced compression of $E\mathcal{A}E$ on $\operatorname{Im} E$ is represented by $W\mathcal{A}V$. Hence $W\mathcal{A}V = M_n(\mathbb{R})$, which is condition *(ii).* This completes the proof. $\square$

Remark 2.4. For both $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$, the invertible-similarity formulation and the two-sided formulation $W\mathcal{A}V$ are the same. The difference between the complex and real cases lies in the orthogonal formulation. Over $\mathbb{C}$ the invertible-similarity formulation can be replaced by a unitary compression. Over $\mathbb{R}$, this is generally impossible: if $\mathcal{A}$ is the real diagonal algebra and $V^TV = I_n$, then $V^T\mathcal{A}V$ consists only of symmetric matrices. Thus the real orthogonal substitute is the idempotent formulation above.

3. Algebras generated by nonderogatory matrices

In this section we prove that algebras generated by nonderogatory matrices satisfy the equivalent conditions of Theorem 2.1. We use the two-sided formulation, since it applies uniformly over both $\mathbb{C}$ and $\mathbb{R}$. The proof is based on an explicit nilpotent Jordan model whose compressed powers form a basis of $M_n(\mathbb{F})$, followed by an approximation argument using companion matrices. As a consequence of Theorem 3.1, we obtain Theorem 3.2 for complex masas and for real diagonal algebras and their orthogonal conjugates.

We shall use the following standard facts about nonderogatory matrices. If

$$p(\lambda) = \lambda^N + a_{N-1}\lambda^{N-1} + \cdots + a_1\lambda + a_0$$

is a monic polynomial over a field $\mathbb{F}$, its companion matrix is

$$C(p) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{N-1} \end{pmatrix}.$$

A matrix $A \in M_N(\mathbb{F})$ is *nonderogatory* if and only if its minimal polynomial equals its characteristic polynomial; equivalently, its rational canonical form consists of a single companion block. Thus a nonderogatory matrix is similar over $\mathbb{F}$ to the companion matrix of its characteristic polynomial. These are standard consequences of the rational canonical form theorem; see, for example, [5, 6].

Theorem 3.1. *Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$, and let $\mathcal{A}$ be the algebra generated by a nonderogatory matrix in $M_{n^2}(\mathbb{F})$. Then there exist matrices $V \in M_{n^2,n}(\mathbb{F})$, $W \in M_{n,n^2}(\mathbb{F})$, such that $WV = I_n$ and $W\mathcal{A}V = M_n(\mathbb{F})$.*

Proof. Put $N = n^2$. We divide the proof into four steps. First we prove the result for the nilpotent Jordan block. Then we compute the resulting compressed powers explicitly. Next we approximate an arbitrary nonderogatory matrix by companion matrices close to the nilpotent one. Finally, we transport the compression back by similarity.

Step 1: The nilpotent model. We first treat the nilpotent Jordan block of size $N = n^2$. Let J be the backward shift on $\mathbb{F}^n$, and let S be the forward shift. Thus

$$Je_i = e_{i-1}, \qquad Se_i = e_{i+1},$$

with the convention that $e_0 = e_{n+1} = 0$. Consider the following $N \times N$ block matrix, with blocks of size $n \times n$:

$$X = \begin{pmatrix} 0 & 0 & \cdots & 0 & S \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{pmatrix}.$$

This matrix is permutation-similar to the nilpotent Jordan block of size N. Hence it is enough to prove the desired two-sided compression statement for the algebra generated by X. Indeed, starting with the first vector in the first block, the action of X moves successively through the first vectors of all blocks and then to the second vector of the first block; continuing in this way, one obtains a single Jordan chain of length N.

We claim that the algebra generated by the nilpotent Jordan block admits the desired two-sided compression. Since the nilpotent Jordan block is permutation-similar to X, it suffices to prove this for the algebra generated by X. Define

$$V_0 = \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad W_0 = \begin{pmatrix} I & J & \cdots & J^{n-1} \end{pmatrix}.$$

Then clearly $W_0V_0 = I_n$.

Step 2: Computing the compressed powers. We now compute the matrices

$$W_0X^mV_0, \qquad 0 \leq m \leq N-1.$$

Write

$$m = qn + r, \qquad 0 \leq q \leq n-1, \qquad 0 \leq r \leq n-1.$$

The matrix X shifts the nonzero block of V_0 down by one block row. After one full cycle through the n block rows, the nonzero block returns to the first block row and acquires a factor of S. Therefore $X^{qn+r}V_0$ has exactly one nonzero block, namely S^q, in the $(r+1)$-st block row. Multiplying by W_0 we get

$$W_0X^{qn+r}V_0 = J^rS^q.$$

Thus the matrices $W_0X^mV_0$, $0 \leq m \leq N-1$, are precisely J^rS^q, $0 \leq r, q \leq n-1$, in some order.

We next recall why these matrices form a basis of $M_n(\mathbb{F})$. For $a = b$, the matrices

$$J^0S^0,\ J^1S^1,\ \ldots,\ J^{n-1}S^{n-1}$$

are diagonal matrices with consecutive initial strings of ones and then zeros. Their consecutive differences give the standard diagonal matrix units. If $a - b = d > 0$, then $J^a S^b$ is supported on the d-th upper diagonal, and varying b gives initial segments whose consecutive differences give all matrix units on that diagonal. Similarly, if $b - a = d > 0$, the matrices are supported on the d-th lower diagonal and give all matrix units there. Hence

$$\{J^a S^b : 0 \le a, b \le n-1\}$$

is a basis of $M_n(\mathbb{F})$. Therefore

$$\operatorname{span}_{\mathbb{F}}\{W_0 X^m V_0 : 0 \le m \le N-1\} = M_n(\mathbb{F}).$$

Equivalently, for the algebra $\mathcal{J}$ generated by the nilpotent Jordan block of size N,

$$W_0 \mathcal{J} V_0 = M_n(\mathbb{F}).$$

Step 3: Passing from the nilpotent model to a nonderogatory matrix. We now pass to an arbitrary nonderogatory matrix. Let $A \in M_N(\mathbb{F})$ be nonderogatory and let

$$\mathcal{A} = \operatorname{span}_{\mathbb{F}}\{I, A, A^2, \dots, A^{N-1}\}$$

be the algebra generated by A. Since A is nonderogatory, so is tA for every nonzero scalar $t \in \mathbb{F}$. Indeed, if

$$x, Ax, \dots, A^{N-1}x$$

is a basis, then

$$x, (tA)x, \dots, (tA)^{N-1}x$$

is obtained from it by multiplying the m-th vector by the nonzero scalar t^m, and is therefore again a basis. Thus tA is similar over $\mathbb{F}$ to the companion matrix C_t of its characteristic polynomial.

We compare C_t with the nilpotent model X. If

$$p_A(\lambda) = \lambda^N + a_{N-1}\lambda^{N-1} + \cdots + a_1\lambda + a_0,$$

then

$$p_{tA}(\lambda) = \lambda^N + t a_{N-1}\lambda^{N-1} + t^2 a_{N-2}\lambda^{N-2} + \cdots + t^N a_0.$$

Hence every non-leading coefficient tends to zero as $t \to 0$. It follows that the companion matrix C_t converges, as $t \to 0$, to the nilpotent companion matrix C_0. The nilpotent companion matrix C_0 is permutation-similar to the nilpotent model X constructed above. Thus there exists a permutation matrix Π such that

$$C_0 = \Pi X \Pi^{-1}.$$

Consequently,

$$C_t \longrightarrow \Pi X \Pi^{-1} \qquad \text{as } t \to 0.$$

To compare the compressed powers of C_t with those of X, define

$$\widetilde{V}_0 = \Pi V_0, \qquad \widetilde{W}_0 = W_0 \Pi^{-1}.$$

Then
$$\widetilde{W}_0\widetilde{V}_0 = W_0\Pi^{-1}\Pi V_0 = W_0V_0 = I_n.$$
Moreover, for each fixed $m = 0, 1, \ldots, N-1$,
$$\widetilde{W}_0 C_t^m \widetilde{V}_0 \longrightarrow W_0 X^m V_0 \qquad \text{as } t \to 0.$$
Indeed,
$$\widetilde{W}_0 C_t^m \widetilde{V}_0 = W_0\Pi^{-1}C_t^m \Pi V_0 \longrightarrow W_0\Pi^{-1}(\Pi X^m \Pi^{-1})\Pi V_0 = W_0 X^m V_0.$$
Since the matrices
$$W_0X^mV_0, \qquad 0 \le m \le N-1,$$
form a basis of $M_n(\mathbb{F})$, linear independence persists for all sufficiently small nonzero t. Thus, for such a t,
$$\operatorname{span}_{\mathbb{F}}\{\widetilde{W}_0 C_t^m \widetilde{V}_0 : 0 \le m \le N-1\} = M_n(\mathbb{F}).$$

Step 4: Transporting the compression back by similarity. We now transfer the compression from the companion matrix C_t back to the original matrix A. Choose a sufficiently small nonzero t. Since tA is similar over $\mathbb{F}$ to C_t, there exists $R \in GL_N(\mathbb{F})$ such that
$$tA = RC_tR^{-1}.$$
Define
$$V = R\widetilde{V}_0, \qquad W = \widetilde{W}_0 R^{-1}.$$
Then
$$WV = \widetilde{W}_0 R^{-1} R \widetilde{V}_0 = \widetilde{W}_0\widetilde{V}_0 = I_n.$$
Moreover, for $0 \le m \le N-1$,
$$W(tA)^m V = \widetilde{W}_0 R^{-1}(tA)^m R\widetilde{V}_0 = \widetilde{W}_0 C_t^m \widetilde{V}_0.$$
Hence
$$\operatorname{span}_{\mathbb{F}}\{W(tA)^mV : 0 \le m \le N-1\} = M_n(\mathbb{F}).$$
Since $t \neq 0$, the powers of tA span the same algebra as the powers of A. Therefore $W\mathcal{A}V = M_n(\mathbb{F})$, which proves the theorem. $\square$

We next record the diagonal, or masa, case as a consequence of the nonderogatory result. Over $\mathbb{C}$, the resulting optimal quantum-clique statement for the diagonal algebra was independently proved by Weaver using Parseval frames; see [13, Theorem 2.7]. Our main theorem places this phenomenon in the broader setting of arbitrary nonderogatory generators and also provides a real two-sided version.

Theorem 3.2. *Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$. If $\mathbb{F} = \mathbb{C}$, let $\mathcal{A}$ be a masa in $M_{n^2}(\mathbb{C})$. If $\mathbb{F} = \mathbb{R}$, let $\mathcal{A}$ be orthogonally equivalent to the algebra of all real diagonal matrices in $M_{n^2}(\mathbb{R})$. Then there exist matrices $V \in M_{n^2,n}(\mathbb{F})$, $W \in M_{n,n^2}(\mathbb{F})$, such that $WV = I_n$ and $W\mathcal{A}V = M_n(\mathbb{F})$.*

Proof. Let $\mathcal{D} \subseteq M_{n^2}(\mathbb{F})$ denote the algebra of all diagonal matrices over $\mathbb{F}$. Choose distinct scalars $\lambda_1, \dots, \lambda_{n^2} \in \mathbb{F}$ and set $D = \operatorname{diag}(\lambda_1, \dots, \lambda_{n^2})$. Then D is non-derogatory, since its minimal polynomial equals its characteristic polynomial. Moreover, the algebra generated by D is precisely $\mathcal{D}$. By Theorem 3.1, the conclusion of Theorem 3.2 follows for $\mathcal{D}$. Now let $\mathcal{A}$ be an algebra as in the statement. If $\mathbb{F} = \mathbb{C}$, then $\mathcal{A}$ is unitarily equivalent to $\mathcal{D}$; if $\mathbb{F} = \mathbb{R}$, then by assumption $\mathcal{A}$ is orthogonally equivalent to $\mathcal{D}$. Transporting V and W through the corresponding unitary or orthogonal equivalence proves the result. $\square$

Remark 3.3. The additional assumption in the real case is necessary. Over $\mathbb{C}$, every masa in $M_m(\mathbb{C})$ is unitarily equivalent to the diagonal algebra. Over $\mathbb{R}$, however, a maximal abelian transpose-closed subalgebra need not be orthogonally diagonalizable. For example, the algebra

$$\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\} \subseteq M_2(\mathbb{R})$$

is a maximal abelian real algebra and is closed under transpose, but it is not orthogonally equivalent to the real diagonal algebra. Indeed, it contains the rotation matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which has no real eigenvectors. Thus, in the real statement above, we restrict to the real diagonal masa, or to algebras orthogonally equivalent to it.

4. A direct two-sided construction for the diagonal algebra

Although the result for the diagonal algebra follows from Theorem 3.1, it is useful to give a direct and elementary two-sided construction. Unlike the orthogonal complex construction in [13, Theorem 2.7], the construction below applies without change over both $\mathbb{C}$ and $\mathbb{R}$ and produces matrices V and W satisfying simultaneously $V^*V = I_n$, $WV = I_n$, $W\mathcal{D}V = M_n(\mathbb{F})$.

Theorem 4.1. *Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$, and let $\mathcal{D}$ be the algebra of all diagonal matrices in $M_{n^2}(\mathbb{F})$. Then there exist matrices $V \in M_{n^2,n}(\mathbb{F})$, $W \in M_{n,n^2}(\mathbb{F})$, such that $V^*V = I_n$, $WV = I_n$, and $W\mathcal{D}V = M_n(\mathbb{F})$.*

Proof. Let $e_1, \dots, e_n$ be the standard column basis of $\mathbb{F}^n$. Order the pairs (i, j), $1 \leq i, j \leq n$, as

$$(1,1), (2,1), \dots, (n,1), (1,2), \dots, (n,2), \dots, (1,n), \dots, (n,n),$$

so that the pair (i, j) corresponds to the index $i + (j-1)n$.

Define $V \in M_{n^2,n}(\mathbb{F})$ by

$$\operatorname{row}_{i+(j-1)n}(V) := \frac{1}{\sqrt{n}} e_j^T.$$

Thus, for each fixed j, the row $\frac{1}{\sqrt{n}}e_j^T$ occurs n times. Since the entries of V are real, the computation is the same over $\mathbb{C}$ and over $\mathbb{R}$:

$$V^*V = \sum_{j=1}^{n}\sum_{i=1}^{n}\frac{1}{n}e_je_j^T = I_n.$$

Next define $W \in M_{n,n^2}(\mathbb{F})$ by

$$\operatorname{col}_{i+(j-1)n}(W) \equiv w_{ij} := \begin{cases} e_i, & i \neq j, \\ \sqrt{n}\, e_j - \sum_{k\neq j} e_k, & i = j. \end{cases}$$

For each fixed j, we have

$$\sum_{i=1}^{n} w_{ij} = \sqrt{n}\, e_j. \tag{4.1}$$

Therefore

$$WV = \sum_{j=1}^{n}\sum_{i=1}^{n} w_{ij}\frac{1}{\sqrt{n}}e_j^T = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} w_{ij}\right)\frac{1}{\sqrt{n}}e_j^T \overset{(4.1)}{=} \sum_{j=1}^{n} e_je_j^T = I_n.$$

It remains to prove that $W\mathcal{D}V = M_n(\mathbb{F})$. Let

$$D = \operatorname{diag}(d_1, \dots, d_{n^2}) \in \mathcal{D}.$$

Using the above ordering, write $d_{ij} = d_{i+(j-1)n}$. Then

$$WDV = \sum_{j=1}^{n}\sum_{i=1}^{n} d_{ij}w_{ij}\frac{1}{\sqrt{n}}e_j^T = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\sum_{i=1}^{n} d_{ij}w_{ij}e_j^T\right).$$

Fix j. Note that the vectors $w_{1j}, w_{2j}, \dots, w_{nj}$ form a basis of $\mathbb{F}^n$. Hence the matrices $w_{ij}e_j^T$, $i = 1, \dots, n$, span all matrices supported in the j-th column. Since this holds for every $j = 1, \dots, n$, the matrices $w_{ij}e_j^T$, $1 \le i, j \le n$, span all of $M_n(\mathbb{F})$. By varying the entries d_{ij}, we obtain $W\mathcal{D}V = M_n(\mathbb{F})$. □

5. A direct orthogonal construction in the complex case

The preceding section gave a direct two-sided construction for the diagonal algebra. Over $\mathbb{C}$, this yields an orthogonal compression by Theorem 2.2; over $\mathbb{R}$, the corresponding geometric conclusion is given by Theorem 2.3. We now give an alternative explicit proof that works directly in the complex setting. The main point is to construct explicitly an isometry from $\mathbb{C}^n$ into $\mathbb{C}^{n^2}$ whose compression of the diagonal algebra is all of $M_n(\mathbb{C})$. Extending this isometry to a unitary matrix then gives the desired unitary-coordinate formulation. Although this construction is more elaborate than the preceding two-sided one, it provides an explicit orthogonal compression and reveals additional structure in the compressed rank-one diagonal matrix units.

Theorem 5.1. *Let $\mathcal{A}$ be the algebra of all diagonal complex matrices in $M_{n^2}(\mathbf{C})$. Then there is a unitary matrix of size n^2 such that $U^*\mathcal{A}U$ has an $n\times n$ compression equal to the whole algebra $M_n(\mathbb{C})$.*

Proof. **Step 1: Construction and normalization of the isometry.** Let $C \in M_n(\mathbb{C})$ be the cycle matrix of order n, that is, the permutation matrix satisfying

$$Ce_j = e_{j-1} \quad (2 \le j \le n), \qquad Ce_1 = e_n.$$

Let $e \in \mathbb{C}^n$ be the vector whose entries are all equal to one, so that $E = \frac{1}{n}ee^*$ is a projection of rank one and $E^\perp = I - E$ is its orthogonal complement. (Here and in the rest of the proof I stands for the identity matrix of $M_n(\mathbb{C})$ as an abbreviation of I_n.) Observe that $E^\perp + rE$ is a diagonalizable matrix, different from I as soon as (real) $r \neq 1$ and invertible as soon as $r \neq 0$. Moreover, let $\omega = \frac{1+i\sqrt{3}}{2}$. Define an $n^2 \times n$ complex matrix in terms of $n \times n$ blocks

$$\widetilde{U} = \begin{pmatrix} E^\perp + \sqrt{2}E \\ I - \frac{\omega}{n}C \\ \vdots \\ I - \frac{\omega}{n}C^{n-1} \end{pmatrix}.$$

Denote by $\widetilde{U}_k \in M_n(\mathbb{C})$ the k-th block of $\widetilde{U}$. Clearly,

$$\widetilde{U}_1^*\widetilde{U}_1 = E^\perp + 2E = I + E, \tag{5.1}$$

and for $k = 2, \ldots, n$,

$$\widetilde{U}_k^*\widetilde{U}_k = I - \frac{\omega}{n}C^{k-1} - \frac{\bar{\omega}}{n}C^{-(k-1)} + \frac{1}{n^2}I.$$

Note that $\omega + \bar{\omega} = 1$ and $I + \sum_{k=1}^{n-1} C^k = nE$. After a straightforward computation we get the sum of the above products equal to

$$\sum_{k=1}^{n} \widetilde{U}_k^*\widetilde{U}_k = \alpha_n I, \qquad \text{where} \quad \alpha_n = n + \frac{1}{n} + \frac{n-1}{n^2}.$$

Thus the columns of $\widetilde{U}$ are orthogonal and have common squared norm α_n. Replacing $\widetilde{U}$ by $U = \alpha_n^{-1/2}\widetilde{U}$, we obtain $U^*U = I_n$. Hence U is an isometry. It remains to prove that $U^*\mathcal{A}U = M_n(\mathbb{C})$. Once this is shown, extending the columns of U to an orthonormal basis of $\mathbb{C}^{n^2}$ gives the required unitary matrix.

Let
$$\psi : \mathcal{A} \longrightarrow M_n(\mathbb{C}), \qquad \psi(A) = U^*AU,$$
and denote its range by
$$\mathcal{R} = \psi(\mathcal{A}) = \{U^*AU : A \in \mathcal{A}\}.$$
We shall prove that $\mathcal{R} = M_n(\mathbb{C})$.

Step 2: The range contains I and E. Let E_1, E_2, up to E_n, be the standard basis of diagonal matrices in $M_n(\mathbb{C})$. Then the standard basis of $\mathcal{A}$ is obtained by inserting these matrices into consecutive diagonal $n \times n$ blocks for $k = 1, \dots, n$. First observe that $\psi(I_{n^2}) = I$. Now let $L \in \mathcal{A}$ be the diagonal block matrix whose first $n \times n$ diagonal block is I_n and whose remaining diagonal blocks are zero. Then
$$\psi(L) = U^*LU = \alpha_n^{-1}\widetilde{U}_1^*\widetilde{U}_1 \overset{(5.1)}{=} \alpha_n^{-1}(I + E).$$
Since both $\psi(L)$ and I_n belong to $\mathcal{R}$, and $\mathcal{R}$ is a linear space, it follows that $E \in \mathcal{R}$.

Step 3: Isolating one off-diagonal entry up to diagonal terms. Choose E_j, $j = 1, \dots, n$, and insert it into the k-th diagonal $n \times n$ block, where $k = 2, \dots, n$. Since $U = \alpha_n^{-1/2}\widetilde{U}$, the image under ψ is α_n^{-1} times the following expression:
$$\begin{aligned}
\widetilde{U}_k^*E_j\widetilde{U}_k &= \left(I - \frac{\bar{\omega}}{n}C^{-(k-1)}\right)E_j\left(I - \frac{\omega}{n}C^{k-1}\right)\\
&= E_j - \frac{\omega}{n}E_jC^{k-1} - \frac{\bar{\omega}}{n}C^{-(k-1)}E_j + \frac{|\omega|^2}{n^2}C^{-(k-1)}E_jC^{k-1}\\
&\overset{|\omega|=1}{=} E_j - \frac{\omega}{n}E_jC^{k-1} - \frac{\bar{\omega}}{n}C^{-(k-1)}E_j + \frac{1}{n^2}C^{-(k-1)}E_jC^{k-1}.
\end{aligned}$$
Assume that $j + k - 1 \leq n$. Then:
$$E_jC^{k-1} = E_{j,j+k-1}, \quad C^{-(k-1)}E_j = E_{j+k-1,j}, \quad C^{-(k-1)}E_jC^{k-1} = E_{j+k-1}.$$
Thus, if $j + k - 1 \leq n$, then
$$\widetilde{U}_k^*E_j\widetilde{U}_k = E_j - \frac{\omega}{n}E_{j,j+k-1} - \frac{\bar{\omega}}{n}E_{j+k-1,j} + \frac{1}{n^2}E_{j+k-1}.$$
Otherwise, if $j + k - 1 > n$, the same computation holds with indices taken modulo n and hence
$$\widetilde{U}_k^*E_j\widetilde{U}_k = E_j - \frac{\omega}{n}E_{j,j+k-1-n} - \frac{\bar{\omega}}{n}E_{j+k-1-n,j} + \frac{1}{n^2}E_{j+k-1-n}.$$
Fix $1 \leq l < m \leq n$. We first take $j = l$ and $k = m - l + 1$. Then $2 \leq k \leq n$ and $j + k - 1 = m \leq n$. Hence the preceding calculation shows that $\mathcal{R}$ contains
$$A_{lm} = E_l - \frac{\omega}{n}E_{l,m} - \frac{\bar{\omega}}{n}E_{m,l} + \frac{1}{n^2}E_m.$$

Next, take $j = m$ and $k = n - m + l + 1$. Again $2 \le k \le n$, and now

$$j + k - 1 = m + (n - m + l + 1) - 1 = n + l > n.$$

Reducing the indices modulo n, the preceding calculation shows that $\mathcal{R}$ contains

$$B_{lm} = \frac{1}{n^2} E_l - \frac{\bar{\omega}}{n} E_{l,m} - \frac{\omega}{n} E_{m,l} + E_m.$$

Thus, for every $1 \le l < m \le n$, the range $\mathcal{R}$ contains two matrices whose only off-diagonal entries occur in the (l, m) and (m, l) positions. Taking suitable linear combinations of A_{lm} and B_{lm}, we may eliminate either the (m, l)-entry or the (l, m)-entry. Hence $\mathcal{R}$ contains matrices of the form

$$\overline{E}_{l,m} = E_{l,m} + \omega_n E_l + \bar{\omega}_n E_m, \qquad \underline{E}_{m,l} = E_{m,l} + \bar{\omega}_n E_l + \omega_n E_m, \tag{5.2}$$

where

$$\omega_n = \frac{-n\omega + \frac{1}{n}\bar{\omega}}{\omega^2 - \bar{\omega}^2} = -\frac{1}{2}\left(n + \frac{1}{n}\right) + \frac{i}{2\sqrt{3}}\left(n - \frac{1}{n}\right).$$

Indeed, to obtain $\overline{E}_{l,m}$, we take a linear combination $\alpha A_{lm} + \beta B_{lm}$ whose $E_{l,m}$-coefficient is 1 and whose $E_{m,l}$-coefficient is 0. Since the coefficients of $E_{l,m}$ and $E_{m,l}$ in $\alpha A_{lm} + \beta B_{lm}$ are respectively $-\frac{1}{n}(\alpha\omega + \beta\bar{\omega})$ and $-\frac{1}{n}(\alpha\bar{\omega} + \beta\omega)$, we need $\alpha\omega + \beta\bar{\omega} = -n$, $\alpha\bar{\omega} + \beta\omega = 0$. Solving this system gives the coefficient of E_l in this linear combination as ω_n as above. The construction of $\underline{E}_{m,l}$ is obtained in the same way, or by taking the conjugate-transpose form of the above expression.

Thus $\mathcal{R}$ contains, for every $1 \le l < m \le n$, an upper triangular matrix whose only off-diagonal entry is in the (l, m) position, and a lower triangular matrix whose only off-diagonal entry is in the (m, l) position, up to diagonal terms.

Step 4: Recovering the diagonal matrix units. Next, insert E_j into the first diagonal $n \times n$ block of $\mathcal{A}$. Its image under ψ is

$$R_j = \alpha_n^{-1}\left(E_j + (\sqrt{2} - 1)EE_j + (\sqrt{2} - 1)E_jE + (\sqrt{2} - 1)^2 EE_jE\right).$$

Since $EE_jE = \frac{1}{n}E$, this becomes

$$R_j = \alpha_n^{-1}\left(E_j + (\sqrt{2} - 1)EE_j + (\sqrt{2} - 1)E_jE + \frac{(\sqrt{2} - 1)^2}{n} E\right) \in \mathcal{R}. \tag{5.3}$$

We already know that I and E belong to $\mathcal{R}$. We now express EE_j and E_jE in terms of the matrices in (5.2).

Since $E = \frac{1}{n}ee^*$, we get

$$nEE_j = E_j + \sum_{i \ne j} E_{i,j} = E_j + \sum_{i<j} E_{i,j} + \sum_{i>j} E_{i,j}.$$

Using (5.2), this becomes

$$\begin{aligned} nEE_j &= E_j + \sum_{i<j} \left(\overline{E}_{i,j} - \omega_n E_i - \bar{\omega}_n E_j\right) + \sum_{i>j} \left(\underline{E}_{i,j} - \omega_n E_i - \bar{\omega}_n E_j\right) \\ &= (1+\omega_n)E_j + \sum_{i<j} \overline{E}_{i,j} + \sum_{i>j} \underline{E}_{i,j} - \omega_n I - (n-1)\bar{\omega}_n E_j. \end{aligned}$$

Similarly,

$$\begin{aligned} nE_jE &= E_j + \sum_{i\neq j} E_{j,i} \\ &= (1+\bar{\omega}_n)E_j + \sum_{i<j} \underline{E}_{j,i} + \sum_{i>j} \overline{E}_{j,i} - \bar{\omega}_n I - (n-1)\omega_n E_j. \end{aligned}$$

Substituting these two expressions into (5.3), we see that R_j is equal to a scalar multiple of E_j, plus a linear combination of matrices already known to belong to $\mathcal{R}$, namely I, E, $\overline{E}_{i,j}$, $\underline{E}_{i,j}$. The coefficient of E_j is $1+\frac{\sqrt{2}-1}{n}\left(2+(n-2)\left(n+\frac{1}{n}\right)\right)$, which is nonzero. Therefore $E_j \in \mathcal{R}$ for every $j = 1, \ldots, n$. Hence all diagonal matrix units belong to $\mathcal{R}$.

Step 5: Recovering all matrix units. Finally, once the diagonal matrix units $E_1, \ldots, E_n$ belong to $\mathcal{R}$, equation (5.2) implies that the off-diagonal matrix units also belong to $\mathcal{R}$. This proves Theorem 5.1. □

6. Dimensions of other blocks

Fix an orthogonal decomposition

$$\mathbb{F}^{n^2} = \operatorname{Im} P \oplus \operatorname{Im} Q, \qquad Q = I - P,$$

where P has rank n. Relative to this decomposition, we refer to the four block spaces $P\mathcal{A}P$, $P\mathcal{A}Q$, $Q\mathcal{A}P$, $Q\mathcal{A}Q$ as the northwest, northeast, southwest, and southeast block spaces, respectively. We show that if the northwest block space has the largest possible dimension n^2, then the dimensions of all the remaining block spaces are determined.

The conclusion is stronger than a mere dimension count. Since $\dim \mathcal{A} = n^2$, maximality of the northwest block implies that the compression map $A \longmapsto PAP$ is injective and identifies $\mathcal{A}$ linearly with the full matrix algebra on $\operatorname{Im} P$. We show that this forces the complementary diagonal compression $A \mapsto QAQ$ to remain injective, while each off-diagonal block map has precisely the scalar matrices as its kernel. The proof first treats the complementary diagonal block after simultaneously diagonalizing the algebra and then determines the off-diagonal dimensions using commutativity, normality, and self-adjointness.

Theorem 6.1. *Let $n \geq 2$, $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$, and let $\mathcal{A} \subseteq M_{n^2}(\mathbb{F})$ be a commutative self-adjoint algebra of dimension n^2. Let P be an orthogonal projection of rank n, and put $Q = I - P$. Suppose that*

$$\dim(P\mathcal{A}P) = n^2,$$

where $P\mathcal{A}P$ is regarded as a space of operators on $\operatorname{Im} P$. *Then*

$$\dim(Q\mathcal{A}Q) = n^2$$

and

$$\dim(P\mathcal{A}Q) = \dim(Q\mathcal{A}P) = n^2 - 1.$$

In the proof of Theorem 6.1 we will need the following lemma.

Lemma 6.2. *Let $n \geq 2$, let $\mathcal{D}$ be the diagonal algebra in $M_{n^2}(\mathbb{C})$, and let P be an orthogonal projection of rank n. Put $Q = I - P$. If the map*

$$\mathcal{D} \longrightarrow PM_{n^2}(\mathbb{C})P, \qquad D \longmapsto PDP,$$

is injective, then the map

$$\mathcal{D} \longrightarrow QM_{n^2}(\mathbb{C})Q, \qquad D \longmapsto QDQ,$$

is also injective.

Proof. Put $N = n^2$, and let $e_1, \ldots, e_N$ be the standard basis of $\mathbb{C}^N$. Set

$$u_j = Pe_j, \qquad v_j = Qe_j, \qquad j = 1, \ldots, N.$$

If

$$D = \operatorname{diag}(d_1, \ldots, d_N) \in \mathcal{D},$$

then

$$PDP = \sum_{j=1}^{N} d_j u_j u_j^* \qquad \text{and} \qquad QDQ = \sum_{j=1}^{N} d_j v_j v_j^*.$$

Consequently, the injectivity of $D \mapsto PDP$ is equivalent to the linear independence of the $N = n^2$ rank-one operators

$$u_1 u_1^*, \ldots, u_N u_N^*.$$

Suppose, towards a contradiction, that the map $D \mapsto QDQ$ is not injective. Then there is a nonzero diagonal matrix D such that

$$QDQ = 0.$$

Taking adjoints gives $QD^*Q = 0$. Consequently, $Q\left(\frac{D+D^*}{2}\right)Q = 0$ and $Q\left(\frac{D-D^*}{2i}\right)Q = 0$. Both matrices in parentheses are self-adjoint and diagonal, and at least one of them is nonzero. Replacing D by that matrix, we may assume that D is nonzero and self-adjoint.

Let (p, q, z) be the inertia of D, where p, q, and z are, respectively, the numbers of positive, negative, and zero diagonal entries of D. Thus

$$p + q + z = N = n^2. \tag{6.1}$$

Replacing D by $-D$ if necessary, we may assume that

$$p \geq q. \tag{6.2}$$

Let $\mathcal{K} = \ker D$. Since D is diagonal, $\mathcal{K}$ is the coordinate subspace spanned by the e_j corresponding to the zero diagonal entries of D, and $\dim \mathcal{K} = z$.

Recall that a subspace $\mathcal{L} \subseteq \mathbb{C}^N$ is called *totally isotropic* for a Hermitian form $[\cdot, \cdot]$ if $[x, y] = 0$ for all $x, y \in \mathcal{L}$. The equality $QDQ = 0$ therefore means that $\operatorname{Im} Q$ is totally isotropic for the Hermitian form

$$[x, y]_D = \langle Dx, y \rangle.$$

Indeed, if $x, y \in \operatorname{Im} Q$, then $Qx = x$ and $Qy = y$, and, since $Q = Q^*$,

$$[x, y]_D = \langle Dx, y \rangle = \langle DQx, Qy \rangle = \langle QDQx, y \rangle = 0.$$

After passing to the quotient by $\ker D$, this form is nondegenerate and has signature (p, q). A totally isotropic subspace for a nondegenerate Hermitian form of signature (p, q) has dimension at most $\min\{p, q\} = q$. Indeed, after identifying the underlying space with $\mathbb{C}^p \oplus \mathbb{C}^q$, the form is given by

$$[(x_+, x_-), (y_+, y_-)] = \langle x_+, y_+ \rangle - \langle x_-, y_- \rangle.$$

If $\mathcal{L}$ is totally isotropic, then the coordinate projection $\mathcal{L} \to \mathbb{C}^q$ is injective: if $(x_+, 0) \in \mathcal{L}$, then $0 = [(x_+, 0), (x_+, 0)] = \|x_+\|^2$, so $x_+ = 0$. Thus $\dim \mathcal{L} \leq q$. By symmetry, $\dim \mathcal{L} \leq p$, and therefore $\dim \mathcal{L} \leq \min\{p, q\}$. Therefore

$$\dim \operatorname{Im} Q - \dim(\operatorname{Im} Q \cap \mathcal{K}) \leq q. \tag{6.3}$$

Using $\dim \operatorname{Im} Q = N - n$ in (6.3), we obtain

$$\dim(\operatorname{Im} Q \cap \mathcal{K}) \geq N - n - q. \tag{6.4}$$

Now restrict P to $\mathcal{K}$. Its kernel is

$$\ker(P|_{\mathcal{K}}) = \mathcal{K} \cap \ker P = \mathcal{K} \cap \operatorname{Im} Q.$$

Hence

$$\begin{aligned} \dim P\mathcal{K} &= \dim \mathcal{K} - \dim(\mathcal{K} \cap \operatorname{Im} Q) \\ &\overset{(6.4)}{\leq} z - (N - n - q) \\ &\overset{(6.1)}{=} (N - p - q) - (N - n - q) \\ &= n - p. \end{aligned} \tag{6.5}$$

For each index j for which $d_j = 0$, we have $e_j \in \mathcal{K}$, and therefore $u_j = Pe_j \in P\mathcal{K}$. There are z such indices. For each of them, the range of $u_j u_j^*$ is contained in $P\mathcal{K}$, and $u_j u_j^*$ vanishes on $(P\mathcal{K})^\perp$. Hence these rank-one operators belong to the space of operators on $\operatorname{Im} P$ supported on $P\mathcal{K}$. If $r = \dim P\mathcal{K}$, this operator space is isomorphic to $M_r(\mathbb{C})$ and therefore has dimension r^2. On the other hand, the injectivity of $D \mapsto PDP$ implies that

$$u_1 u_1^*, \dots, u_N u_N^*$$

are linearly independent. Hence the z operators corresponding to the zero diagonal entries of D are also linearly independent. Therefore

$$z \le r^2 = \big(\dim P\mathcal{K}\big)^2 \le (n-p)^2. \tag{6.6}$$

On the other hand,

$$\begin{aligned} z-(n-p)^2 &\overset{(6.1)}{=} n^2-p-q-(n-p)^2 = p(2n-p-1)-q \\ &\overset{(6.2)}{\ge} p(2n-p-2). \end{aligned} \tag{6.7}$$

If $n \ge 3$, then $1 \le p \overset{(6.5)}{\le} n$ implies $2n-p-2 \ge n-2 > 0$. Using this in (6.7), gives $z > (n-p)^2$, contradicting (6.6).

It remains to consider $n = 2$. We have

$$1 \le p \le 2, \qquad 0 \le q \le p, \qquad z = 4-p-q.$$

The inequality $z \le (2-p)^2$ rules out every possibility except $p = q = 2$, $z = 0$. In this exceptional case, D is invertible and has signature $(2,2)$, while $\operatorname{Im} Q$ is a two-dimensional totally isotropic subspace. Since

$$\langle Dx, y\rangle = 0 \qquad (x, y \in \operatorname{Im} Q),$$

we have $D(\operatorname{Im} Q) \subseteq (\operatorname{Im} Q)^\perp = \operatorname{Im} P$. Both spaces have dimension 2, and D is invertible, so $D(\operatorname{Im} Q) = \operatorname{Im} P$. Consequently, $D^{-1}(\operatorname{Im} P) = \operatorname{Im} Q$. Therefore

$$\langle D^{-1}x, y\rangle = 0 \qquad (x, y \in \operatorname{Im} P),$$

because $D^{-1}x \in \operatorname{Im} Q$ and $\operatorname{Im} P \perp \operatorname{Im} Q$. Hence $PD^{-1}P = 0$. But D^{-1} is a nonzero diagonal matrix, contradicting the injectivity of the map $D \mapsto PDP$.

Thus no nonzero diagonal matrix can satisfy $QDQ = 0$. Therefore the map $D \longmapsto QDQ$ is injective. $\square$

With the preceding lemma in hand, we are now ready to prove the main result of this section.

Proof of Theorem 6.1. Let

$$\Phi_{11} : \mathcal{A} \longrightarrow PM_{n^2}(\mathbb{F})P, \qquad \Phi_{11}(A) = PAP,$$

and define the other three block maps by

$$\Phi_{12}(A) = PAQ, \qquad \Phi_{21}(A) = QAP, \qquad \Phi_{22}(A) = QAQ.$$

We divide the proof into three steps.

Step 1: The algebra is unital, and both diagonal block maps are injective.

We first observe that $\mathcal{A}$ is automatically unital.

Suppose first that $\mathbb{F} = \mathbb{C}$. Since $\mathcal{A}$ is commutative and self-adjoint, every element of $\mathcal{A}$ is normal. Indeed, if $A \in \mathcal{A}$, then $A^* \in \mathcal{A}$, and commutativity gives $AA^* = A^*A$. Since the elements of $\mathcal{A}$ are commuting normal matrices, they are simultaneously unitarily diagonalizable. Hence there exists a unitary matrix $U \in M_{n^2}(\mathbb{C})$ such that $U^*\mathcal{A}U \subseteq \mathcal{D}$, where $\mathcal{D}$ denotes the full diagonal algebra in $M_{n^2}(\mathbb{C})$. Since

$$\dim_{\mathbb{C}}(U^*\mathcal{A}U) = \dim_{\mathbb{C}} \mathcal{A} = n^2 = \dim_{\mathbb{C}} \mathcal{D},$$

we have $U^*\mathcal{A}U = \mathcal{D}$. In particular, $I \in U^*\mathcal{A}U$, and hence $I \in \mathcal{A}$.

Now suppose that $\mathbb{F} = \mathbb{R}$. Note that the complexification $\mathcal{A}_{\mathbb{C}} = \mathcal{A} + i\mathcal{A} \subseteq M_{n^2}(\mathbb{C})$ is a commutative self-adjoint complex algebra. Moreover,

$$\dim_{\mathbb{C}} \mathcal{A}_{\mathbb{C}} = \dim_{\mathbb{R}} \mathcal{A} = n^2.$$

By the complex argument, $I \in \mathcal{A}_{\mathbb{C}}$. Consequently, $I \in \mathcal{A}$.

We now consider the diagonal block maps. By hypothesis,

$$\dim(P\mathcal{A}P) = n^2 = \dim \mathcal{A}.$$

Therefore Φ_{11} is injective. Since

$$\dim PM_{n^2}(\mathbb{F})P = (\operatorname{rank} P)^2 = n^2,$$

the map Φ_{11} is in fact an isomorphism onto $PM_{n^2}(\mathbb{F})P$. Thus

$$P\mathcal{A}P = PM_{n^2}(\mathbb{F})P,$$

which is naturally identified with $M_n(\mathbb{F})$.

We next prove that Φ_{22} is also injective.

Suppose first that $\mathbb{F} = \mathbb{C}$. As shown above, there exists a unitary matrix U such that $U^*\mathcal{A}U = \mathcal{D}$. Put

$$P_0 = U^*PU, \qquad Q_0 = U^*QU = I - P_0.$$

Then P_0 is an orthogonal projection of rank n. The injectivity of Φ_{11} is equivalent to the injectivity of the map

$$\mathcal{D} \longrightarrow P_0M_{n^2}(\mathbb{C})P_0, \qquad D \longmapsto P_0DP_0.$$

By Lemma 6.2, the map

$$\mathcal{D} \longrightarrow Q_0M_{n^2}(\mathbb{C})Q_0, \qquad D \longmapsto Q_0DQ_0,$$

is also injective. Conjugating back by U, we conclude that Φ_{22} is injective. Therefore

$$\dim(Q\mathcal{A}Q) = \dim \mathcal{A} = n^2.$$

Now suppose that $\mathbb{F} = \mathbb{R}$. The complexification of Φ_{11} is the map

$$\Phi_{11,\mathbb{C}} : \mathcal{A}_{\mathbb{C}} \longrightarrow PM_{n^2}(\mathbb{C})P, \qquad Z \longmapsto PZP.$$

This map is injective. Indeed, if $Z = A + iB$, with $A, B \in \mathcal{A}$, and $PZP = 0$, then $PAP = 0$, $PBP = 0$. The injectivity of Φ_{11} gives $A = B = 0$. Thus $Z = 0$.

We may now apply the complex case to $\mathcal{A}_{\mathbb{C}}$. It follows that $Z \longmapsto QZQ$ is injective on $\mathcal{A}_{\mathbb{C}}$. Its restriction to the real subalgebra $\mathcal{A}$ is therefore also injective. Consequently, $\dim_{\mathbb{R}}(Q\mathcal{A}Q) = \dim_{\mathbb{R}} \mathcal{A} = n^2$.

Step 2: The kernel of the northeast block map consists precisely of the scalar matrices.

We claim that

$$\ker \Phi_{12} = \mathbb{F}I. \tag{6.8}$$

Since $I \in \mathcal{A}$ and $PIQ = PQ = 0$, we have $\mathbb{F}I \subseteq \ker \Phi_{12}$.

Conversely, let $A \in \mathcal{A}$ satisfy $PAQ = 0$. Thus $\operatorname{Im} Q$ is invariant under A. Since A is normal and $\operatorname{Im} Q$ is invariant under A, it is reducing for A. For completeness, write A relative to $\operatorname{Im} Q \oplus \operatorname{Im} P$ as $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$. Normality implies that $A_{12} = 0$. Hence the orthogonal complement $\operatorname{Im} P$ is also invariant under A, and therefore $QAP = 0$. It follows that A is block diagonal with respect to the orthogonal decomposition $\mathbb{F}^{n^2} = \operatorname{Im} P \oplus \operatorname{Im} Q$. Equivalently, $AP = PA$.

Let $B \in \mathcal{A}$. Since A and B commute and A commutes with P, we have

$$\begin{aligned}(PAP)(PBP) &= PAPBP \\ &= PABP \\ &= PBAP \\ &= PBPAP \\ &= (PBP)(PAP).\end{aligned}$$

Thus PAP commutes with every element of $P\mathcal{A}P$.

By Step 1,

$$P\mathcal{A}P = PM_{n^2}(\mathbb{F})P \cong M_n(\mathbb{F}).$$

The commutant of $M_n(\mathbb{F})$ consists only of the scalar matrices. Hence there exists $\lambda \in \mathbb{F}$ such that $PAP = \lambda P$. Therefore $P(A - \lambda I)P = 0$. Since $I \in \mathcal{A}$, we have $A - \lambda I \in \mathcal{A}$. The injectivity of Φ_{11} now gives $A - \lambda I = 0$. Thus $A = \lambda I$. Consequently, (6.8) holds.

By the rank-nullity theorem,

$$\dim(P\mathcal{A}Q) = \dim \mathcal{A} - \dim \ker \Phi_{12} = n^2 - 1.$$

Step 3: The southwest block has the same dimension.

Since $\mathcal{A}$ is self-adjoint, $(PAQ)^* = QA^*P$. As A ranges over $\mathcal{A}$, so does A^*. Therefore taking adjoints defines a conjugate-linear bijection in the complex case, and a linear bijection in the real case, between $P\mathcal{A}Q$ and $Q\mathcal{A}P$. Hence

$$\dim(Q\mathcal{A}P) = \dim(P\mathcal{A}Q) = n^2 - 1.$$

Combining the three steps proves the theorem. □

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