BIVARIATE TRUNCATED MOMENT SEQUENCES WITH THE COLUMN RELATION $XY = X^m + q(X)$, WITH q OF DEGREE $m-1$

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Abstract. When the algebraic variety associated with a truncated moment sequence is finite, solving the moment problem follows a well-defined procedure. However, moment problems involving infinite algebraic varieties are more complex and less well-understood. Recent studies suggest that certain bivariate moment sequences can be transformed into equivalent univariate sequences, offering a valuable approach for solving these problems. In this paper, we focus on addressing the truncated moment problem (TMP) for specific rational plane curves. For a curve of general degree we derive an equivalent Hankel positive semidefinite completion problem. For cubic curves, we solve this problem explicitly, which resolves the TMP for one of the four types of cubic curves, up to affine linear equivalence. For the quartic case we simplify the completion problem to a feasibility question of a three-variable system of inequalities.

1. Introduction

Given a real 2-dimensional multisequence of degree m, $\beta \equiv \beta^{(m)} = \{\beta_{00}, \beta_{10}, \beta_{01}, \cdots, \beta_{m,0}, \beta_{m-1,1}, \cdots, \beta_{1,m-1}, \beta_{0,m}\}$ with $\beta_{00} > 0$, the **truncated moment problem** (TMP) entails finding necessary and sufficient conditions for the existence of a positive Borel measure μ such that supp $\mu \subseteq \mathbb{R}^2$ and

$$\beta_{i,j} \equiv \beta_{(i,j)} = \int x^i y^j \, d\mu \ \ (0 \le i+j \le 2n; \ i,j \in \mathbb{Z}_+).$$

In this context, we refer to μ a **representing measure** (rm) for β or the moment matrix $M(n)$ defined below. When the order of a moment sequence is even, such as $m = 2n$ for some $n \in \mathbb{N}$, it is possible to define the **moment matrix** $M(n) \equiv M(n)(\beta^{(2n)})$ of β as follows:

$$M(n) \equiv M(n)(\beta^{(2n)}) := (\beta_{\mathbf{i}+\mathbf{j}})_{\mathbf{i},\,\mathbf{j} \in \mathbb{Z}_+^2 : |\mathbf{i}|,\,|\mathbf{j}| \le n},$$

where $\mathbf{i}+\mathbf{j}$ stands for the coordinate-wise sum and $|\mathbf{i}|$ is the sum of coordinates of $\mathbf{i}$. To guarantee the existence of a representing measure for $\beta^{(2n)}$, it is essential that the matrix $M(n)$ is positive semidefinite. However, there are additional conditions that must be met. To examine these conditions, let $\mathbb{R}[x,y]_n$ denote the set of bivariate polynomials in $\mathbb{R}[x,y]$ with degree at most n. We arrange the columns of

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2020 *Mathematics Subject Classification.* Primary 47A57, 44A60; Secondary 15-04, 47A20, 32A60.

Key words and phrases. strong truncated moment problem, recursively generated, algebraic variety.

The second-named author was supported by the ARIS (Slovenian Research and Innovation Agency) research core funding No. P1-0228 and grant No. J1-50002.

$M(n)$ according to the monomials $\{\mathit{1}, X, Y, X^2, XY, Y^2, \ldots, X^n, \ldots, Y^n\}$ in degree-lexicographic order. For instance, a sextic moment matrix $M(3)$ would appear as follows:

$$
\begin{array}{c}
\mathit{1} \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \\ X^3 \\ X^2Y \\ XY^2 \\ Y^3
\end{array}
\!\!\!
\overset{\begin{array}{cccccccccc} \mathit{1} & X & Y & X^2 & XY & Y^2 & X^3 & X^2Y & XY^2 & Y^3 \end{array}}{
\left(\begin{array}{c|cc|ccc|cccc}
\beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{03} \\ \hline
\beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{40} & \beta_{31} & \beta_{22} & \beta_{13} \\
\beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} & \beta_{31} & \beta_{22} & \beta_{13} & \beta_{04} \\ \hline
\beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} & \beta_{50} & \beta_{41} & \beta_{32} & \beta_{23} \\
\beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} & \beta_{41} & \beta_{32} & \beta_{23} & \beta_{32} \\
\beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} & \beta_{32} & \beta_{23} & \beta_{32} & \beta_{23} \\ \hline
\beta_{30} & \beta_{40} & \beta_{31} & \beta_{50} & \beta_{41} & \beta_{32} & \beta_{60} & \beta_{51} & \beta_{42} & \beta_{33} \\
\beta_{21} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{23} & \beta_{51} & \beta_{42} & \beta_{33} & \beta_{24} \\
\beta_{12} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{13} & \beta_{42} & \beta_{33} & \beta_{24} & \beta_{15} \\
\beta_{03} & \beta_{13} & \beta_{04} & \beta_{23} & \beta_{13} & \beta_{05} & \beta_{33} & \beta_{24} & \beta_{15} & \beta_{06}
\end{array}\right)} .
$$

When the matrix $M(n)$ exhibits a column relation, it can be written as $p(X,Y) = \mathbf{0}$ for some polynomial $p(x,y) = \sum_{ij} a_{ij} x^i y^j \in \mathbb{R}[x,y]_n$, where each monomial is replaced by the column of $M(n)$ indexed with this monomial and $\mathbf{0}$ stands for a zero vector; this concept is known as functional calculus. For convenience, we sometimes refer to this column dependence relation as that of the moment sequence. Although column relations themselves are not polynomials, they can be interpreted as such and provide vital information about a representing measure for $M(n)$. Specifically, if β has a representing measure, the following condition must be satisfied [7]:

$$p(X,Y) = \mathbf{0} \implies (pq)(X,Y) = \mathbf{0} \text{ for every } q \in \mathbb{R}[x,y] \text{ with } \deg(pq) \le n.$$

In this case, $M(n)$ is said to be **recursively generated** (rg).

Let $\mathcal{Z}(p) := \{(x,y) \in \mathbb{R}^2 : p(x,y) = 0\}$ denote the zero set of the polynomial p. The following result helps determine the support of a representing measure for a truncated moment sequence:

Proposition 1.1. ([6, Proposition 3.1]) *Suppose μ is a representing measure for $\beta^{(2n)}$. For $p \in \mathbb{R}[x,y]_n$,*

$$\text{supp } \mu \subseteq \mathcal{Z}(p) \iff p(X,Y) = \mathbf{0}.$$

Now, we aim to define a set contained the support of representing measures; the **algebraic variety** of $\beta \equiv \beta^{(2n)}$ is given by

$$\mathcal{V}_\beta \equiv \mathcal{V}\left(M(n)\right) := \bigcap\nolimits_{p(x,y)=0,\, \deg\, p \le n} \mathcal{Z}(p). \tag{1}$$

Using Proposition 1.1 the following holds: if $M(n)$ admits a representing measure μ, then

$$\text{supp } \mu \subseteq \mathcal{V}_\beta$$

and

$$\text{rank } M(n) \le \text{card supp } \mu \le \text{card}\, \mathcal{V}_\beta.$$

The second inequality in the above is known as the **variety condition**.

Here, we introduce the **Riesz functional** defined on the space of all monomials. For a given moment sequence $\beta \equiv \beta^{(m)}$, the linear functional Λ_β is given by

$$\Lambda_\beta \left(\sum_{0 \le i+j \le m} a_{i,j} x^i y^j \right) = \sum_{0 \le i+j \le m} a_{i,j} \beta_{i,j},$$

where all $a_{i,j}$ are real numbers and $i, j \in \mathbb{Z}_+$. We say that Λ_β is **K-positive** for a closed set $K \in \mathbb{R}^2$ if

$$\Lambda_\beta(p) \geq 0 \text{ for all } p \in \mathbb{R}[x,y]_m \text{ such that } p|_K \geq 0.$$

If, in addition, the conditions $p|_K \geq 0$ and $p|_K \not\equiv 0$ imply $\Lambda_\beta(p) > 0$, then Λ_β is said to be **strictly K-positive**. When $K = \mathbb{R}^2$, we simply refer to Λ_β as **positive** rather than K-positive. The K-positivity of Λ_β is a necessary condition for β to have a **K-representing measure**, i.e., a rm supported on K. Conversely, M. Riesz's classical theorem shows that K-positivity is also sufficient to guarantee the existence of K-representing measures for infinite moment sequences. This result was later extended to $\mathbb{R}^n$ by E. K. Haviland. Similar results are available for the truncated moment problem, see the reference [9].

One of the most significant results in truncated moment theory is the Flat Extension Theorem. This theorem states that if $M(n)$ has a rank-preserving positive extension $M(n+1)$, then $\beta^{(2n)}$ possesses a rank $M(n)$-atomic representing measure [6]. The extension $M(n+1)$ is referred to as a **flat extension**. A notable special case arises when $\operatorname{rank} M(n) = \operatorname{rank} M(n-1)$. In this situation, $M(n)$ is referred to as **flat**, and $\beta^{(2n)}$ has a unique rank $M(n)$-atomic representing measure.

We now provide a brief overview of how to find a flat extension of $M(n)$. Notice that each rectangular block of $M(n)$ with the same degree moments forms a Hankel matrix. To construct an extension $M(n+1)$, consider the following form:

$$M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix},$$

where B and C are Hankel matrices with some new moments of degree $2n-1$ and $2n$, respectively. To ensure that a prospective moment matrix $M(n+1)$ is positive semidefinite, we need the following classical result:

Theorem 1.2. ([1, 21]) *Let A, B, and C be matrices of complex numbers, with A and C being square matrices. Then*

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \iff \begin{cases} A \geq 0, \\ B = AW \text{ for some } W \\ C \geq W^*AW. \end{cases} \iff \begin{cases} A \geq 0, \\ B = AW \text{ for some } W, \\ C \geq B^*A^\dagger B, \end{cases}$$

where $A^\dagger$ stands for the Moore-Penrose inverse of A. Moreover,

$$\operatorname{rank} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \operatorname{rank} A \iff C = B^*A^\dagger B.$$

Although finding the positive extension may appear straightforward, verifying that the C-block is indeed a Hankel matrix is not trivial.

Recently, many intriguing interactions between moment theory and algebraic geometry have been uncovered. Solving truncated moment problems can be interpreted as finding the roots of a system of multivariate polynomial equations. By Richter's result [17] (see also [20, Theorem 1.24]), up to recently more often credited to Bayer and Teichmann [2], if a moment sequence $\beta^{(2n)}$ has one or more representing measures, then at least one of these measures must be finitely atomic.

Consequently, if a real sequence $\beta^{(2n)}$ is associated with a finitely atomic representing measure μ, it can be expressed as

$$\mu = \sum_{\ell=1}^{r} \rho_\ell \delta_{(x_\ell, y_\ell)},$$

where $r \le \dim \mathbb{R}[x,y]_{2n}$. Our task is to determine positive numbers $\rho_1, \ldots, \rho_r$ (referred to as *densities*) and points $(x_1, y_1), \ldots, (x_r, y_r)$ (referred to as *atoms*) of the measure μ, such that for $i, j \in \mathbb{Z}_+$ and $0 \le i + j \le 2n$,

$$\beta_{i,j} = \rho_1 x_1^i y_1^j + \cdots + \rho_\ell x_\ell^i y_\ell^j.$$

Degree-One Transformation. The following discusses a method for simplifying the moment problem using invertible affine linear transformations (alt), specifically the invariance of moment problems under degree-one transformations. The complex version of this approach is detailed in [8], and we adopt the same notation to develop its real counterpart.

For $a, b, c, d, e, f \in \mathbb{R}$ with $bf \ne ce$, define

$$\Psi(x,y) \equiv (\Psi_1(x,y), \Psi_2(x,y)) := (a + bx + cy, d + ex + fy) \quad \text{for } x, y \in \mathbb{R}.$$

If Λ_β represents the Riesz functional associated with a given $\beta \equiv \beta^{(2n)}$, then we can construct a new equivalent moment sequence $\tilde{\beta}^{(2n)}$ with $\tilde{\beta}_{i,j} := \Lambda_\beta(\Psi_1^i \Psi_2^j)$ for $i, j \in \mathbb{Z}_+$ and $0 \le i + j \le 2n$. It follows that $\Lambda_{\tilde{\beta}}(p) = \Lambda_\beta(p \circ \Psi)$ for all $p \in \mathbb{R}[x,y]_n$. For more details, refer to [8].

Truncated Moment Sequences with an Infinite Algebraic Variety. The moment matrix $M(n)(\beta^{(2n)})$ (or the moment sequence) is said to be p**-pure** if its only column relations are those recursively derived from a polynomial $p \in \mathbb{R}[x,y]_n$. Thus, $\mathcal{V}_\beta$ is precisely $\mathcal{Z}(p)$; in other words, the algebraic variety of β is infinite. When the algebraic variety associated with a truncated moment sequence is finite, a clear procedure exists for solving the moment problem [13]. However, concrete solutions for $M(n)$ with $n \ge 3$ are scarce and challenging to study [14, 22, 23, 25, 26, 27, 28].

Main Results. In this paper we focus on the TMP on curves of the form

$$xy = q_m x^m + q(x) + \alpha y, \quad \text{where } q_m \in \mathbb{R} \setminus \{0\},\ q(x) \in \mathbb{R}[x]_{m-1},\ \alpha \in \mathbb{R}.$$

By applying the alt $(x,y) \mapsto (x + \alpha, q_m y)$, it suffices to solve the TMP on curves with the simpler form

$$xy = x^m + r(x), \quad \text{where } r(x) \in \mathbb{R}[x]_{m-1}. \tag{2}$$

These curves have a parametrization $(x(t), y(t)) = (t, t^{m-1} + \frac{r(t)}{t})$, $t \in \mathbb{R}$, $t \ne 0$. The TMP on any rational curve is equivalent to a univariate TMP, where some moments are missing and the measure must vanish in certain points. This observation simplifies solving the original TMP, since univariate TMPs are easier to tackle and related technique was already exploited in [23, 25, 26, 27, 28]. To adress the TMP on (2), the solution to the the strong Hamburger TMP [27] is required, i.e., the $\mathbb{R}$–rm of the univariate sequence must vanish in $\{0\}$. The original motivation for this paper was the cubic relation of type (2) with $m = 3$, since after applying an alt, every cubic relation has one of four canonical forms, with type (2) and $m = 3$ being one of them. The other three types are $y = q(x)$, $y^2 = q(x)$ and $xy^2 + ay = q(x)$ for some $q \in \mathbb{R}[x]_3$ and $a \in \mathbb{R}$. A concrete solution to $y = q(x)$ is known [14], while concrete solutions to the other two types are known for $y^2 = x^3$ [25] and $xy^2 = 1$ [27].

Our first main result (see Theorem 2.2) is the solution to the TMP on (2) in terms of the corresponding univariate TMP with gaps. The solution is a Hankel positive semidefinite (psd) completion problem, i.e., the question is when missing anti-diagonals of a partially defined Hankel matrix can be chosen so that the completion is psd and satisties two additional constraints, coming from the solution to the strong Hamburger TMP. Our second main result (see Theorems 3.1 and 3.2) solves this completion problem corresponding to the cubic case ($m = 3$ in (2)) in terms of concrete numerical conditions. We also bound the number of atoms in a representing measure with the lowest number of atoms and demonstrate the solution on a numerical example (see Example 3.4). Our third main result (see Theorems 4.1 and 4.2) solves the completion problem for the quartic case ($m = 4$ in (2)) in terms of feasibility of a three-variable system of inequalities.

2. TMP ON $xy = x^m + \sum_{s=0}^{m-1} q_s x^s$ WITH $q_0 \neq 0$

In this section we prove that the $\mathcal{Z}(p)$–TMP for $p(x, y) = xy - x^m - \sum_{s=0}^{m-1} q_s x^s$, where each $q_s \in \mathbb{R}$ and $q_0 \neq 0$, is equivalent to the Hankel positive semidefinite completion problem (see Theorem 2.2).

Let p be as in the first paragraph. By using an alt we may assume that $q_m = 1$. Let $\beta \equiv \left\{\beta^{(2n)}_{i,j}\right\}$ be a sequence with for $i, j \in \mathbb{Z}_+$, $i + j \leq 2n$. Let us reorder the indices

$$-2n, -2n + 1, \ldots, -1, 0, 1, \ldots, 2(m - 1)n - 1, 2(m - 1)n$$

in the following way:

$$\begin{aligned}
&\text{Row } 0: \quad 0, 1, \ldots, 2n,\\
&\text{Row } 1: \quad -1, 2n + 1, \ldots, (2n - 1) + m - 1,\\
&\text{Row } 2: \quad -2, (2n - 1) + m, \ldots, (2n - 2) + (m - 1)2,\\
&\quad\vdots\\
&\text{Row } k: \quad -k, (2n - k + 1) + (m - 1)(k - 1) + 1, \ldots, (2n - k) + (m - 1)k,\\
&\quad\vdots\\
&\text{Row } 2n: \quad -2n, (2n - 1) + (m - 1)(2n - 1) + 1, \ldots, (m - 1)2n.
\end{aligned} \tag{3}$$

Now we adapt Row 1 to Row $2n$, while rewriting Row 0, in the following way

$$\text{Row } k: \quad -k, h(k), h(k) + 1, \ldots, (2n - k) + (m - 1)k,$$

where

$$h(k) := \max\{2n - k + 1 + (m - 1)(k - 1), (m - 1)k\} + 1 \tag{4}$$

Remark 2.1. The reason for this adaptation is the fact that expressing y from the equality $p(x, y) = 0$ and then raising to the power of the index of the row the relation will be of the form

$$y^k = \Big(\sum_{s=0}^{m} q_s x^{s-1}\Big)^k = q_m^k x^{(m-1)k} + \sum_{i=-k}^{(m-1)k} r_i x^i$$

for some $r_i \in \mathbb{R}$. Multiplying this relation with $x, \dots, x^{2n-k}$, we can successively express $x^{(m-1)k+j}$, $j = 1, \dots, 2n-k$, out of the equations obtained. If $(m-1)k$ is larger than $(2n-k+1)+(m-1)(k-1)$, then some powers of x will be missing in this procedure. The missing powers will be precisely $(2n-k+1)+(m-1)(k-1)+1, \dots, h(k)$.

As explained in Remark 2.1, the adaptation may result in the loss of some indices in each row. Let $\mathcal{I}$ be the set of indices remaining in the sequence after this adaptation. We define a map f on $\mathcal{I}$ by the rule

$$\begin{aligned} f(s) &\equiv (f_1(s), f_2(s)) \\ &:= \left\{ \begin{array}{rl} \big(s - \#(s)(m-1), \#(s)\big), & \text{if } s \geq 0, \\ \big(0, \#(s)\big), & \text{if } s < 0, \end{array} \right. \end{aligned} \tag{5}$$

where the index s is contained in Row $\#(s)$.

Expressing y from the relation $p(x,y) = 0$, we see that for $i, j \in \mathbb{Z}_+$ we have

$$\begin{aligned} & x^i \left(\sum_{s=0}^{m} q_s x^{s-1} \right)^j \\ &= x^i \left(\sum_{\substack{k_0+\ldots+k_m=j, \\ k_0,\ldots,k_m \in \mathbb{Z}_+}} \frac{j!}{k_0! \cdots k_m!} q_m^{k_0} q_{m-1}^{k_1} \cdots q_0^{k_m} x^{\sum_{s=0}^{m}(m-1-s)k_s} \right) \\ &= \sum_{\substack{k_0+\ldots+k_m=j, \\ k_0,\ldots,k_m \in \mathbb{Z}_+}} \frac{j!}{k_0! \cdots k_m!} q_m^{k_0} q_{m-1}^{k_1} \cdots q_0^{k_m} x^{\sum_{s=0}^{m}(m-1-s)k_s+i} \\ &= \sum_{t=-j}^{(m-1)j} q_{j,t} x^{t+i}, \end{aligned} \tag{6}$$

where

$$q_{j,t} := \sum_{\substack{k_0+\ldots+k_m=j, \\ k_0,\ldots,k_m \in \mathbb{Z}_+, \\ \sum_{s=0}^{m}(m-1-s)k_s=t}} \frac{j!}{k_0! \cdots k_m!} q_m^{k_0} q_{m-1}^{k_1} \cdots q_0^{k_m}$$

Note that

$$q_{j,(m-1)j} = q_m^j = 1 \quad \text{and} \quad q_{j,-j} = q_0^j.$$

Hence,

$$x^{(m-1)j+i} = x^i \left(\sum_{s=0}^{m} q_s x^{s-1} \right)^j - \sum_{t=-j}^{(m-1)j-1} q_{j,t} x^{t+i}. \tag{7}$$

and

$$x^{-j} = \frac{1}{q_0^j} \left[x^i \left(\sum_{s=0}^{m} q_s x^{s-1} \right)^j - \sum_{t=-j+1}^{(m-1)j} q_{j,t} x^{t+i} \right]. \tag{8}$$

Using the relations (6)–(8) we define a number γ_s for every $s \in \mathcal{I}$ following the order (3) by

$$\gamma_s := \begin{cases} \beta_{f(s)} - \displaystyle\sum_{t=-f_2(s)}^{(m-1)f_2(s)-1} q_{f_2(s),t}\gamma_{t+f_1(s)}, & \text{if } s \geq 0, \\ \dfrac{1}{q_0^{f_2(s)}}\Big(\beta_{f(s)} - \displaystyle\sum_{t=-f_2(s)+1}^{(m-1)f_2(s)} q_{f_2(s),t}\gamma_t\Big), & \text{if } s < 0. \end{cases} \tag{9}$$

Namely, we define γ_s in the order

$$\underbrace{\gamma_0, \gamma_1, \ldots, \gamma_{2n}}_{j=0}, \underbrace{\gamma_{-1}, \gamma_{h(1)}, \gamma_{h(1)+1} \cdots, \gamma_{2n+m}}_{j=1}, \ldots, \underbrace{\gamma_{-2n+1}, \gamma_{h(2n-1)+1}}_{j=2n-1}, \underbrace{\gamma_{-2n}}_{j=2n}. \tag{10}$$

If s does not appear in $\mathcal{I}$, then we call γ_s a **free moment**. If $s \in \mathcal{I}$ and in the definition of γ_s there also exist γ_j which are free moments, then γ_s is not uniquely determined and we call it an **auxiliary moment**. If γ_s is not free or auxiliary, then it is called a **fully–determined moment**.

Let $k \in \mathbb{N}$. For $v = (v_0, \ldots, v_{2k}) \in \mathbb{R}^{2k+1}$ we define the corresponding Hankel matrix as

$$A_v := (v_{i+j})_{i,j=0}^k = \begin{pmatrix} v_0 & v_1 & v_2 & \cdots & v_k \\ v_1 & v_2 & \iddots & \iddots & v_{k+1} \\ v_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & v_{2k-1} \\ v_k & v_{k+1} & \cdots & v_{2k-1} & v_{2k} \end{pmatrix}.$$

The main result of this section is the following solution to the $\mathcal{Z}(p)$–TMP for β.

Theorem 2.2. *Let $p(x, y) = xy - \sum_{s=0}^m q_s x^s$ with $q_i \in \mathbb{R}$, $q_0 \neq 0$ $q_m = 1$. Given a sequence $\beta \equiv \left\{\beta_{i,j}^{(2n)}\right\}$ for $i, j \in \mathbb{Z}_+$, $i + j \leq 2n$, let*

$$\gamma \equiv \gamma^{(-2n,2(m-1)n)} = \big(\gamma_{-2n}, \gamma_{-2n+1}, \ldots, \gamma_{-1}, \gamma_0, \gamma_1, \ldots, \gamma_{2(m-1)n}\big)$$

be defined by the procedure above and suppose $\gamma_{\mathbf{j_1}}, \ldots, \gamma_{\mathbf{j_p}}$ are free moments. Then the following are equivalent:

(i) *β admits a $\mathcal{Z}(p)$–representing measure.*

(ii) *γ has a $\mathbb{R}$–representing measure μ for some choice of real values of free moments such that $\mu(\{0\}) = 0$.*

(iii) *There is a choice of real values of free moments such that A_γ is positive semidefinite and one of the following holds:*

  (a) *A_γ is positive definite.*

  (b) $\operatorname{rank} A_\gamma = \operatorname{rank} A_{\gamma^{(-2n,2(m-1)n-2)}} = \operatorname{rank} A_{\gamma^{(-2n+2,2(m-1)n)}}$.

Proof. The equivalence $(ii) \Leftrightarrow (iii)$ is [27, Theorem 3.1]. It remains to prove the equivalence $(i) \Leftrightarrow (ii)$. By [17] (see also [20, Theorem 1.24]), it suffices to prove $(i) \Leftrightarrow (ii)$ for finitely atomic measures, and hence it is enough to establish the following claim.

Claim. Let $r \in \mathbb{N}$. A sequence γ admits a r–atomic $\mathbb{R}$–rm vanishing in $\{0\}$ if and only if β admits a r–atomic $\mathcal{Z}(p)$–rm.

Proof of Claim. First we prove the forward implication. Let $\mu_x = \sum_{\ell=1}^{r} \rho_\ell \delta_{x_\ell}$ be a $\mathbb{R}$–rm for γ where $x_\ell \in \mathbb{R} \setminus \{0\}$ and $\rho_\ell > 0$ for each ℓ. We will prove that $\mu = \sum_{\ell=1}^{r} \rho_\ell \delta_{(x_\ell, y_\ell)}$, where $y_\ell = \sum_{s=0}^{m} q_s x^{s-1}$, is a $\mathcal{Z}(p)$–rm for β. We use induction on the index j in $\beta_{i,j}$, where $i + j \leq 2n$:

Base of induction: For $j = 0$, we see that

$$\beta_{i,0} = \gamma_i = \sum_{\ell=1}^{r} \rho_\ell x_\ell^i = \sum_{\ell=1}^{r} \rho_\ell x_\ell^i y_\ell^0,$$

where we used (9) in the first equality and $\gamma_i = \int x^i d\mu_x$ in the second.

Induction step: Assume that the Claim holds for every $j \leq j_0 - 1$ for some $1 \leq j_0 \leq 2n$. Let us prove its validity for j_0. We consider two cases separately.

Case 1: *(i, j_0) is in the image of f.*
Let $s = f^{-1}((i, j_0))$. Then we have

$$\begin{aligned}
\beta_{i,j_0} &= \sum_{t=-j}^{(m-1)j_0} q_{j_0,t}\gamma_{t+i} \\
&= \sum_{t=-j_0}^{(m-1)j} \Big(q_{j_0,t} \Big(\sum_{\ell=1}^{r} \rho_\ell x_\ell^{t+i} \Big) \Big) \\
&= \sum_{\ell=1}^{r} \Big(\rho_\ell \sum_{t=-j_0}^{(m-1)j_0} q_{j_0,t} x_\ell^{t+i} \Big) \\
&= \sum_{\ell=1}^{r} \Big(\rho_\ell x_\ell^i \sum_{t=-j_0}^{(m-1)j_0} q_{j_0,t} x_\ell^{t} \Big) \\
&= \sum_{\ell=1}^{r} \rho_\ell x_\ell^i y_\ell^{j_0},
\end{aligned}$$

where we used (9) in the first equality, $\gamma_{t+i} = \int x^{t+i} d\mu_x$ in the second, we interchanged the order of summation in the third, factored out x_ℓ^i from the inner sum in the fourth and used (6) for $i = 0$ in the fifth.

Case 2: *(i, j_0) is not in the image of f.*
Since (i, j_0) is not in the image of f, this means that

$$i \neq 0 \quad \text{and} \quad i + (m-1)j_0 \leq (2n - j_0 + 1) + (m-1)(j_0 - 1). \tag{11}$$

Indeed, the first condition in (11) is clear, since $f(-j) = (0, j)$ for every $0 \leq j \leq 2n$, while the second inequality implies that $f(i + (m-1)j_0) = (i + m - 1, j_0 - 1)$. If (i, j_0) was in the image of f, then $f^{-1}((i, j_0)) = i + (m-1)j_0$. The second inequality in (11) is equivalent to

$$i \leq -m + 2n - j_0 + 2. \tag{12}$$

Since the moment sequence must be rg, we must have

$$\beta_{i,j_0} = \sum_{s=0}^{m} q_s \beta_{i-1+s, j_0-1}. \tag{13}$$

Since $0 \le i-1+s$ for each s, $0 \le j_0 - 1$ and

$$i+s+j_0-2 \le i+m+j_0-2 \le -m+2n-j_0+2+m+j_0-2 = 2n,$$

where we used $s \le m$ in the first inequality and (12) in the second, it follows that each β_{i-1+s,j_0-1} in (13) is a part of the original sequence. We now see that

$$\begin{aligned}
\beta_{i,j_0} &= \sum_{s=0}^{m} q_s \beta_{i-1+s,j_0-1} \\
&= \sum_{s=0}^{m} q_s \Big(\sum_{\ell=1}^{r} \rho_\ell x_\ell^{i-1+s} y_\ell^{j_0-1} \Big) \\
&= \sum_{\ell=1}^{r} \rho_\ell x_\ell^{i} y_\ell^{j_0-1} \Big(\sum_{s=0}^{m} \rho_s x_\ell^{s-1} \Big) \\
&= \sum_{\ell=1}^{r} \rho_\ell x_\ell^{i} y_\ell^{j_0-1} y_\ell \\
&= \sum_{\ell=1}^{r} \rho_\ell x_\ell^{i} y_\ell^{j_0},
\end{aligned}$$

where we used (13) in the first equality, induction hypothesis in the second, rearranged the double sum in the third and used (6) for $i = 0$, $j = 1$ in the fourth equality. This concludes the induction step and proves the forward implication.

It remains to prove the backward implication of Claim. Let $\mu = \sum_{\ell=1}^{r} \rho_\ell \delta_{(x_\ell,y_\ell)}$ be a $\mathcal{Z}(p)$–rm for γ, where $(x_\ell, y_\ell) \in \mathcal{Z}(p)$ and $\rho_\ell > 0$ for each ℓ. We will prove that $\mu_x = \sum_{\ell=1}^{r} \rho_\ell \delta_{x_\ell}$ is a rm for β which by construction vanishes on $\{0\}$ (since each $x_\ell \ne 0$). We use induction on the index i in γ_i according to the ordering (3). For $i = 0$, we have $\gamma_0 = \beta_{0,0} = \sum_{\ell=0}^{r} \rho_\ell x_\ell^0$ and the statement holds. Assume now that the statement holds up to some index s_0 in (3) and prove it for s_1. We consider two cases separately.

Case 1: *γ_{s_1} is a free moment.*
In this case we are able to define $\gamma_{s_1} = \sum_{\ell=1}^{r} \rho_\ell x_\ell^{s_1}$.

Case 2: *γ_{s_1} is not a free moment.*

In this case, γ_{s_1} is fully-determined or auxiliary moment, but in both cases $s_1 \in \mathcal{I}$. Let us write $f(s_1) = (i_1, j_1)$. If $s_1 \geq 0$, then

$$\begin{aligned}
\gamma_{s_1} &= \beta_{i_1,j_1} - \sum_{t=-j_1}^{(m-1)j_1-1} q_{j_1,t}\gamma_{t+i_1} \\
&= \sum_{\ell=1}^{r} \rho_\ell x_\ell^{i_1} y_\ell^{j_1} - \sum_{t=-j_1}^{(m-1)j_1-1} q_{j_1,t}\Big(\sum_{\ell=1}^{r} \rho_\ell x_\ell^{t+i_1}\Big) \\
&= \sum_{\ell=1}^{r} \rho_\ell x_\ell^{i_1} \Big(y_\ell^{j_1} - \sum_{t=-j_1}^{(m-1)j_1-1} q_{j_1,t} x_\ell^{t}\Big) \\
&= \sum_{\ell=1}^{r} \rho_\ell x_\ell^{i_1} x_\ell^{(m-1)j_1} = \sum_{\ell=1}^{r} \rho_\ell x_\ell^{(m-1)j_1+i_1} = \sum_{\ell=1}^{r} \rho_\ell x_\ell^{s_1},
\end{aligned}$$

where we used (9) in the first equality, induction hypothesis and the definition of free moments in the second, rearranged the terms in the third, (6) in the fourth and definition of (i_1, j_1) in the last.

If $s_1 < 0$, then $i_1 = 0$ and

$$\begin{aligned}
\gamma_{s_1} &= \frac{1}{q_0^{j_1}}\Big(\beta_{0,j_1} - \sum_{t=-j_1+1}^{(m-1)j_1} q_{j_1,t}\gamma_t\Big) \\
&= \frac{1}{q_0^{j_1}}\Big[\sum_{\ell=1}^{r} \rho_\ell y_\ell^{j_1} - \sum_{t=-j_1+1}^{(m-1)j_1} q_{j_1,t}\Big(\sum_{\ell=1}^{r} \rho_\ell x_\ell^{t}\Big)\Big] \\
&= \sum_{\ell=1}^{r} \rho_\ell \frac{1}{q_0^{j_1}}\Big(y_\ell^{j_1} - \sum_{t=-j_1+1}^{(m-1)j_1} q_{j_1,t} x_\ell^{t}\Big) \\
&= \sum_{\ell=1}^{r} \rho_\ell x_\ell^{-j_1},
\end{aligned}$$

where we used (9) in the first equality, induction hypothesis and the definition of free moments in the second, rearranged the terms in the third and (6) in the last. This proves the backward implication of Claim and concludes the proof of the equivalence $(i) \Leftrightarrow (ii)$ of the theorem. □

3. Concrete solution to the TMP on $p(x, y) = xy - x^3 - \sum_{i=0}^{2} q_i x^i$, $q_0 \neq 0$

In this section, we derive explicit numerical conditions for the existence of free moments in Theorem 2.2 above for $m = 3$, solving the TMP concretely. Let us see why $q_0 \neq 0$ is given; if not, the TMP would involve a reducible column dependency, which could already be solved using known results. We also show that the Carathéodory number of the moment sequence of degree $2n$ is $3n$, which represents the minimum number of atoms needed to achieve a representing measure. Moreover, if a $\mathcal{Z}(p)$–representing measure exists, then it is (rank $M(n)$)–atomic. The main results are Theorem 3.1, which is the solution to p-pure cases, and Theorem 3.2, which solves singular cases.

Assume the notation from Section 2 and let $\gamma \equiv \gamma^{(-2n,4n)}$ be defined by (9) for $m = 3$. We have (see (4))

$$h(k) = \max\{2n+k-1, 2k\} + 1 = \begin{cases} 2n+k, & \text{if } k < 2n, \\ 4n+1, & \text{if } k = 2n. \end{cases}$$

Hence, $\mathcal{I} = \{-2n, -2n+1, \ldots, 4n-1\}$ and the only free moment is $\gamma_{\mathbf{4n}}$. We define γ_s in the order

$$\underbrace{\gamma_0, \gamma_1, \ldots, \gamma_{2n}}_{j=0}, \underbrace{\gamma_{-1}, \gamma_{2n+1}}_{j=1}, \ldots, \underbrace{\gamma_{-k}, \gamma_{2n+k}}_{j=k}, \ldots, \underbrace{\gamma_{-2n+1}, \gamma_{4n-1}}_{j=2n-1}, \underbrace{\gamma_{-2n}}_{j=2n} \tag{14}$$

by (9). The only auxiliary moment is γ_{-2n}. Except $\gamma_{\mathbf{4n}}$ and γ_{-2n} all the other moments are fully–determined. Namely, we may rewrite moments for $j = 0$,

$$\gamma_0 = \beta_{0,0}, \quad \gamma_1 = \beta_{1,0}, \quad \ldots, \quad \gamma_{2n} = \beta_{2n,0};$$

for $j = 1$,

$$\gamma_{-1} = \frac{1}{q_0}\Big(\beta_{0,1} - \sum_{t=0}^{2} q_{1,t}\gamma_t\Big),$$
$$\gamma_{2n+1} = \beta_{2n-1,1} - \sum_{t=-1}^{1} q_{1,t}\gamma_{t+2n-1};$$
$$\vdots$$

for $j = k$,

$$\gamma_{-k} = \frac{1}{q_0^k}\Big(\beta_{0,k} - \sum_{t=-k+1}^{2k} q_{k,t}\gamma_t\Big),$$
$$\gamma_{2n+k} = \beta_{2n-k,k} - \sum_{t=-k}^{2k-1} q_{k,t}\gamma_{t+2n-k};$$
$$\vdots$$

for $j = 2n-1$,

$$\gamma_{-2n+1} = \frac{1}{q_0^{2n-1}}\Big(\beta_{0,2n-1} - \sum_{t=-2n+2}^{4n-2} q_{2n-1,t}\gamma_t\Big),$$
$$\gamma_{4n-1} = \beta_{1,2n-1} - \sum_{t=-2n+1}^{4n-3} q_{2n-1,t}\gamma_t;$$

for $j = 2n$,

$$\gamma_{-2n}(\gamma_{\mathbf{4n}}) = \frac{1}{q_0^{2n}}\Big(\beta_{0,2n} - \sum_{t=-2n+1}^{4n-1} q_{2n,t}\gamma_t - \gamma_{\mathbf{4n}}\Big) =: D - q_0^{-2n}\gamma_{\mathbf{4n}}.$$

We introduce a new variable $\mathbf{t}$ for $\gamma_{\mathbf{4n}}$ and write

$$A_{\gamma(\mathbf{t})} = \begin{array}{c} T^{-n} \\ T^{-n+1} \\ \vdots \\ T^{-1} \\ \mathit{1} \\ T \\ \vdots \\ T^{2n} \end{array} \overset{\begin{array}{cccccccc} T^{-n} & T^{-n+1} & \cdots & T^{-1} & \mathit{1} & T & \cdots & T^{2n} \end{array}}{\begin{pmatrix} \gamma_{-2n}(\mathbf{t}) & \gamma_{-2n+1} & \cdots & \gamma_{-n+1} & \gamma_{-n} & \gamma_{-n+1} & \cdots & \gamma_{n} \\ \gamma_{-2n+1} & \gamma_{-2n+2} & \cdots & \gamma_{-n+2} & \gamma_{-n+1} & \gamma_{-n+2} & \cdots & \gamma_{n+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \gamma_{-n-1} & \gamma_{-n} & \cdots & \gamma_{-2} & \gamma_{-1} & \gamma_{0} & \cdots & \gamma_{2n-1} \\ \gamma_{-n} & \gamma_{-n+1} & \cdots & \gamma_{-1} & \gamma_{0} & \gamma_{1} & \cdots & \gamma_{2n} \\ \gamma_{-n+1} & \gamma_{-n+2} & \cdots & \gamma_{0} & \gamma_{1} & \gamma_{2} & \cdots & \gamma_{2n+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \gamma_{n} & \gamma_{n+1} & \cdots & \gamma_{2n-1} & \gamma_{2n} & \gamma_{2n+1} & \cdots & \mathbf{t} \end{pmatrix}}$$

for the corresponding Hankel matrix. For $i \leq j$ we write

$$\vec{T}^{(i,j)} := \begin{pmatrix} T^i & T^{i+1} & \cdots & T^j \end{pmatrix}.$$

Now the matrix $A_{\gamma(\mathbf{t})}$ has the form

$$A_{\gamma(\mathbf{t})} = \begin{array}{c} T^{-n} \\ (\vec{T}^{(-n+1,2n-1)})^T \\ T^{2n} \end{array} \overset{\begin{array}{ccc} T^{-n} & \vec{T}^{(-n+1,2n-1)} & T^{2n} \end{array}}{\begin{pmatrix} D - q_0^{-2n}\mathbf{t} & b^T & \gamma_n \\ b & A_{\widetilde{\gamma}} & c \\ \gamma_n & c^T & \mathbf{t} \end{pmatrix}},$$

where

$$\begin{aligned} b^T &= \begin{pmatrix} \gamma_{-2n+1} & \cdots & \gamma_{-n-1} & \gamma_{-n} & \gamma_{-n+1} & \cdots & \gamma_{2n-1} \end{pmatrix}, \\ c^T &= \begin{pmatrix} \gamma_{n+1} & \cdots & \gamma_{2n-1} & \gamma_{2n} & \gamma_{2n+1} & \cdots & \gamma_{4n-1} \end{pmatrix}, \\ \widetilde{\gamma} &= (\gamma_{-2n+2}, \gamma_{-2n+3}, \ldots, \gamma_{4n-2}). \end{aligned}$$

The following theorem is a solution to the p-pure TMP.

Theorem 3.1 (Pure case)**.** *Let $p(x,y) = xy - \sum_{s=0}^{3} q_s x^s$ with $q_i \in \mathbb{R}$, $q_0 \neq 0$ $q_3 = 1$. Given a p-pure sequence $\beta \equiv \left\{\beta_{i,j}^{(2n)}\right\}$ for $i, j \in \mathbb{Z}_+$, $i + j \leq 2n$, let*

$$\gamma(\mathbf{t}) = (\gamma_{-2n}(\mathbf{t}), \gamma_{-2n+1}, \ldots, \gamma_{-1}, \gamma_0, \gamma_1, \ldots, \gamma_{4n-1}, \mathbf{t})$$

be defined by the procedure above. Assume the notation above. Let us define

$$\begin{aligned} t_{\min} &:= c^T A_{\widetilde{\gamma}}^{-1} c, \\ t_{\max} &:= q_0^{2n}(D - b^T A_{\widetilde{\gamma}}^{-1} b), \\ E &:= b^T A_{\widetilde{\gamma}}^{-1} c c^T A_{\widetilde{\gamma}}^{-1} b - 2 b^T A_{\widetilde{\gamma}}^{-1} c \gamma_n + \gamma_n^2. \end{aligned} \tag{15}$$

Then the following are equivalent:

(i) β admits a representing measure.
(ii) β admits a $(3n)$–atomic representing measure.
(iii) $t_{\min} < t_{\max}$ and $(t_{\max} - t_{\min})^2 \geq 4 q_0^{2n} E$.

The following theorem is a solution to the singular $\mathcal{Z}(p)$–TMP with a finite algebraic variety.

Theorem 3.2 (Singular case)**.** *Let $p(x,y) = xy - \sum_{s=0}^{3} q_s x^s$ with $q_i \in \mathbb{R}$, $q_0 \neq 0$ $q_3 = 1$. Given a sequence $\beta \equiv \left\{\beta_{i,j}^{(2n)}\right\}$ for $i, j \in \mathbb{Z}_+$, $i + j \leq 2n$, with* rank $M(n) <$ *$3n$, let*

$$\gamma(\mathbf{t}) = (\gamma_{-2n}(\mathbf{t}), \gamma_{-2n+1}, \ldots, \gamma_{-1}, \gamma_0, \gamma_1, \ldots, \gamma_{4n-1}, \mathbf{t})$$

be defined by the procedure above. Assume the notation above. Then the following are equivalent:

(i) β *admits a representing measure.*
(ii) β *admits a* $(\operatorname{rank} M(n))$*–atomic representing measure.*
(iii) $c = A_{\widetilde{\gamma}} w$ *for some* $w \in \mathbb{R}^{3k-1}$ and $\operatorname{rank} A_{\gamma(t_0)} = \operatorname{rank} A_{\widetilde{\gamma}}$ *for* $t_0 := c^T A_{\widetilde{\gamma}}^{+} c$, *where* $A_{\widetilde{\gamma}}^{+}$ *stands for the Moore-Penrose inverse of* $A_{\widetilde{\gamma}}$.

Proof of Theorem 3.1. Before we prove the equivalences of the theorem, we derive a few claims. Let us denote by $(A_{\gamma(\mathbf{t})})|_{\vec{T}^{(i,j)}}$ the restriction of $A_{\gamma(\mathbf{t})}$ to a principal submatrix on rows and columns labelled by elements from $\vec{T}^{(i,j)}$.

Claim 1. $A_1 := (A_{\gamma(t)})|_{\vec{T}^{(-n,2n-1)}} \succeq 0 \iff t \leq t_{\max}$.

Proof of Claim 1. We see that

$$A_1 = \begin{array}{c} \\ T^{-n} \\ (\vec{T}^{(-n+1,2n-1)})^T \end{array} \begin{array}{c} \begin{array}{cc} T^{-n} & \vec{T}^{(-n+1,2n-1)} \end{array} \\ \begin{pmatrix} D - q_0^{-2n} t & b^T \\ b & A_{\widetilde{\gamma}} \end{pmatrix} \end{array}.$$

Since $A_{\widetilde{\gamma}}$ is positive definite, Theorem 1.2 implies the following:

$$A_1 \succeq 0 \iff D - q_0^{-2n} t \geq b^T A_{\widetilde{\gamma}}^{-1} b \iff t \leq t_{\max},$$

which proves Claim 1. □

Claim 2. $A_2 := (A_{\gamma(t)})|_{\vec{T}^{(-n+1,2n)}} \succeq 0 \iff t_{\min} \leq t$.

Proof of Claim 2. We see that

$$A_2 = \begin{array}{c} \\ (\vec{T}^{(-n+1,2n-1)})^T \\ T^{2n} \end{array} \begin{array}{c} \begin{array}{cc} \vec{T}^{(-n+1,2n-1)} & T^{2n} \end{array} \\ \begin{pmatrix} A_{\widetilde{\gamma}} & c \\ c^T & t \end{pmatrix} \end{array}.$$

Since $A_{\widetilde{\gamma}}$ is positive definite, Theorem 1.2 implies the following:

$$A_2 \succeq 0 \iff t \geq c^T A_{\widetilde{\gamma}}^{-1} c = t_{\min},$$

which proves Claim 2. □

Claim 3. Let $\mathbf{t} = t_{\min} + \mathbf{w}$ for $\mathbf{w} > 0$ and

$$Q(\mathbf{w}) := -\frac{\mathbf{w}^2}{q_0^{2n}} + \frac{t_{\max} - t_{\min}}{q_0^{2n}} \mathbf{w} - E \tag{16}$$

be a quadratic polynomial. Then

$$\begin{aligned} A_{\gamma(t)} \succeq 0 &\iff Q(w) \geq 0 \text{ and } w \leq t_{\max} - t_{\min} & (17)\\ &\iff (t_{\max} - t_{\min})^2 \geq 4 q_0^{2n} E. & (18) \end{aligned}$$

Proof of Claim 3. In particular, for $A_{\gamma(t)} \succeq 0$ we must have $A_1 \succeq 0$ and $A_2 \succeq 0$. By Claim 1, it follows that $t \leq t_{\max}$ which is equivalent to $w \leq t_{\max} - t_{\min}$. Since

$t > t_{\min}$, we know that $A_2 \succ 0$ and then by [30, Formula (0.7.2)] we have

$$A_2^{-1} = \begin{pmatrix} A_{\tilde{\gamma}}^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{w} \begin{pmatrix} A_{\tilde{\gamma}}^{-1} cc^t A_{\tilde{\gamma}}^{-1} & -A_{\tilde{\gamma}}^{-1} c \\ -c^T A_{\tilde{\gamma}}^{-1} & 1 \end{pmatrix}.$$

Using Theorem 1.2, we see that

$$\begin{aligned} & A_\gamma \succeq 0 \\ \iff & D - \frac{t_{\min} + w}{q_0^{2n}} \geq \begin{pmatrix} b^T & \gamma_n \end{pmatrix} A_2^{-1} \begin{pmatrix} b \\ \gamma_n \end{pmatrix} \\ \iff & -\frac{w}{q_0^{2n}} + \left(D - b^T A_{\tilde{\gamma}}^{-1} b - \frac{t_{\min}}{q_0^{2n}}\right) - \frac{1}{w}(b^T A_{\tilde{\gamma}}^{-1} cc^t A_{\tilde{\gamma}}^{-1} b - 2b^T A_{\tilde{\gamma}}^{-1} c\gamma_n + \gamma_n^2) \geq 0 \\ \iff & Q(w) \geq 0, \end{aligned}$$

where the last equivalence follows after multiplying by w (which is positive) and the definition of Q. This proves the equivalence in (17). Since $Q(w_0) = \frac{(t_{\max} - t_{\min})^2}{4q_0^{-2n}} - E$ is a maximum of Q attained in $w_0 = \frac{t_{\max} - t_{\min}}{2}$, this gives the equivalence (18). □

Let us now prove $(i) \Rightarrow (iii)$. Since β has a representing measure, then by Theorem 2.2 there exists γ_{4n} such that γ has a representing measure. In particular, this means there is $t \in \mathbb{R}$, such that $A_\gamma \succeq 0$. By Claims 1 and 2, this particularly implies that $t_{\min} \leq t_{\max}$ holds. Next let us show that the inequality is strict. Assume on the contrary that $t_{\min} = t_{\max}$. This means that γ_{4k} must be precisely $t_{\min} = t_{\max}$. Hence, the first and the last column of A_γ are in the span of the intermediate ones and rank $A_\gamma = 3n - 1$, whence γ has a $(3n-1)$–atomic representing measure by [27, Theorem 3.1]. But then β also admits a $(3n-1)$–atomic representing measure, which is a contradiction with rank $M(n) = 3n$, because β is p-pure. This proves that $t_{\min} < t_{\max}$ holds. By Claim 3, also the second inequality in (iii) holds.

Next we prove $(iii) \Rightarrow (ii)$. To prove that β admits a representing measure containing rank $M(n) = 3n$ atoms, we have to show by [27, Theorem 3.1] there exists a choice of t such that

$$A_{\gamma(t)} \succeq 0 \quad \text{and} \quad 3n = \text{rank } A_{\gamma(t)} = \text{rank } A_1 = \text{rank } A_2. \tag{19}$$

By Claims 1 and 2 above, $t \in [t_{\min}, t_{\max}]$. If t is equal to one of $t_{\min}$ or $t_{\max}$, then (19) cannot hold due to singularity of A_1 or A_2, and so $t \in (t_{\min}, t_{\max})$. By assumption in (iii) and the equivalences in Claim 3, there exists $w \in (0, t_{\max} - t_{\min})$ such that $Q(w) = 0$ with Q as in (16). For this w we see that $A_\gamma \succeq 0$ and rank $A_\gamma = 3n$. Since $t := t_{\min} + w \in (t_{\min}, t_{\max})$ also the other two rank conditions in (19) hold. This concludes the proof of $(iii) \Rightarrow (ii)$.

Finally, $(ii) \Rightarrow (i)$ is trivial. □

Proof of Theorem 3.2. Since rank $M(n) < 3n$, there must be another column relation not recursively generated by $XY = X^3 + q_2X^2 + q_1X + q_0$. Each additional relation is of the form

$$\sum_{\substack{i,j \in \mathbb{Z}_+, \\ i+j \leq n}} \alpha_{i,j} X^i Y^j = \mathbf{0}, \quad \alpha_{i,j} \in \mathbb{R}. \tag{20}$$

We distinguish between two cases based on additional relations.

Case 1: *There exists an additional relation* (20) *with* $\alpha_{0,n} = 0$.

The column X^iY^j of $M(n)$ corresponds to the linear combination $\sum_{t=-j}^{2j} q_{j,t}T^{t+i}$ of columns T^ℓ of $A_{\gamma(\mathbf{t})}$, where $q_{j,t}$ are as in (6) for $m = 3$. If $(i, j) \neq (0, n)$, then the exponent $t+i$ can run only from $-n+1$ to $2n-1$. So the relation (20) in Case 1 gives a relation between the columns of $A_{\tilde{\gamma}}$. But then [27, Theorem 3.1] implies that $\gamma(t)$ has a $\mathbb{R}$–rm vanishing in $\{0\}$ for some $t \in \mathbb{R}$ if and only if rank $A_{\tilde{\gamma}} =$ rank $A_{\gamma(t)}$. In particular, rank $(A_{\gamma(t)})|_{\vec{T}^{(-n+1,2n)}} =$ rank $A_{\tilde{\gamma}}$ and by an analogous proof as for Claim 2 in Thereom 3.1, t must be equal to $c^T A_{\tilde{\gamma}}^\dagger c$ and $c = A_{\tilde{\gamma}} w$ for some $w \in \mathbb{R}^{3n-1}$. By Theorem 2.2 and [27, Theorem 3.1], the equivalences of Theorem 3.2 in this case follow.

Case 2: *For every additional relation* (20)*, we have* $\alpha_{0,n} \neq 0$.

Using the relation coming form $Y^n = \left(\sum_{i=0}^3 X^3 + q_2X^2 + q_1X + q_0\right)^n$ and the additional relation (20) containing Y^n nontrivially, we get a nontrivial relation among columns T^ℓ, $\ell = -n, \ldots, 2n-1$ of $A_{\gamma(t)}$. But then by [27, Theorem 3.1] for the existence of a $\mathbb{R}$–rm vanishing in $\{0\}$ for $\gamma(t)$, $t \in \mathbb{R}$, there must be a nontrivial relation among columns T^ℓ, $\ell = -n+1, \ldots, 2n$, containing T^{2n} nontrivially (due to rg). This further implies rank $(A_{\gamma(t)})|_{\vec{T}^{(-n+1,2n)}} =$ rank $A_{\tilde{\gamma}}$ and by the same arguments as in Case 1, the equivalences of Theorem 3.2 in this case follow. □

Remark 3.3. Recently, the Carathéodory number of real plane cubics with smooth projectivization was studied in [3] using tools from algebraic geometry. The main results show (see [3, Section 6]), that the Carathéodory number is at most $3n+1$ for degree $2n$ p-pure sequences and characterize in terms of the number of connected components of $\mathcal{Z}(p)$, when it is $3n$. Note that the cubic curve studied in this section does not satisfy projective smoothness assumption and hence the result about the Carathéodory number from Theorem 3.1 does not follow from [3].

Asymptotic estimates for Carathéodory number on affine plane curves have been recently studied also in [11] and [18].

The following example demonstrates the solution to the $\mathcal{Z}(p)$–TMP for $m = 4$.

Example 3.4. Consider $\beta \equiv \beta^{(8)}$ with moments generated by the 14-atomic representing measure $\mu = \sum_{\ell=1}^{14} \rho_\ell \delta_{(x_\ell,y_\ell)}$, where $\rho_\ell = \frac{1}{14}$, $x_\ell = \ell$, and $y_\ell = \frac{(x_\ell+1)(x_\ell+2)(x_\ell+4)}{x_\ell}$ for $\ell = 1, \ldots, 14$. The moments are given by

$$\begin{aligned}
&\beta_{00} = 1,\ \beta_{10} = \frac{15}{2},\ \beta_{01} = \frac{88829303}{630630},\ \ldots,\\
&\beta_{80} = \frac{443370241}{2},\ \ldots,\\
&\beta_{08} = \frac{22487477336665209271315822126590856880864213410143767741 77}{23730165424120344378453143258050 5468750}.
\end{aligned}$$

Using *Mathematica*, we find the row-reduced form of the moment matrix $M(4)(\beta)$, which shows that it is both positive semidefinite and p-pure, where $p(x, y) = xy - x^3 - 7x^2 - 14x + 8$. The columns X^3, X^4, and X^3Y in $\mathcal{C}_{M(4)}$ are linearly dependent, and so rank $M(4) = 12$. Following the procedure form Section 2, we obtain the associated univariate strong moment sequence $\gamma \equiv \gamma^{(-8,16)}$ as follows:

$$
\begin{aligned}
\gamma_{-8} &= \tfrac{140083717080719587571499472609943956948732559159463163881 7-2373016542412034437845314325805054687 50\beta_{2,7}}{3981261110361986276316941303192577638400000000},\\
\gamma_{-7} &= \tfrac{7957323812886914290800315153314061845 09}{11048010629265141181920694037053440000000},\\
\gamma_{-6} &= \tfrac{4455708392992197620203912120814 93}{6131652030894184250150235340800000},\\
&\ \vdots\\
\gamma_{14} &= \tfrac{2405869901763265}{2},\\
\gamma_{15} &= \tfrac{32512083310326375}{2},\\
\gamma_{16} &= \tfrac{25963365785343570521437 50\beta_{2,7}-1475429646468458910782485055142974987757631 7}{25963365785343570521437 50},
\end{aligned}
$$

where $\beta_{2,7}$ is a parameter.

A calculation shows that no value of $\beta_{2,7}$ such that

$$
\text{rank } A_{(\gamma_{-8},\ldots,\gamma_{16})} = \text{rank } A_{(\gamma_{-8},\ldots,\gamma_{14})} = \text{rank } A_{(\gamma_{-6},\ldots,\gamma_{16})}.
$$

Another possibility for having a representing measure is $A_\gamma \succ 0$, which corresponds to the following approximation:

$$
5.9031917636064208814 \times 10^{18} < \beta_{2,7} < 5.9031917636066715225 \times 10^{18}, \qquad (21)
$$

In this case, β supports infinitely many 13-atomic representing measures. Alternatively, if $A_\gamma \succeq 0$, rank $A_\gamma = 12$ and A_γ is recursively generated in both directions, this occurs at the endpoints of the inequality in (21). In particular, if $\beta_{2,7} \approx 5.9031917636064208814 \times 10^{18}$, then the zeros of the generating function are given by

$$
\begin{aligned}
&t_1 \approx 9.35449 \times 10^{-11}, && t_2 \approx 1, && t_3 \approx 2.00001,\\
&t_4 \approx 3.00159, && t_5 \approx 4.03688, && t_6 \approx 5.23594,\\
&t_7 \approx 6.70231, && t_8 \approx 8.37317, && t_9 \approx 10.0875,\\
&t_{10} \approx 11.6566, && t_{11} \approx 12.9415, && t_{12} \approx 13.9981.
\end{aligned}
$$

Solving the Vandermonde equation in this case, the densities are

$$
\begin{aligned}
&\rho_1 \approx -0.000301331, && \rho_2 \approx 0.0700711, && \rho_3 \approx 0.0747554,\\
&\rho_4 \approx 0.0588108, && \rho_5 \approx 0.0738502, && \rho_6 \approx 0.0903296,\\
&\rho_7 \approx 0.111516, && \rho_8 \approx 0.122776, && \rho_9 \approx 0.119394,\\
&\rho_{10} \approx 0.102687, && \rho_{11} \approx 0.0814315, && \rho_{12} \approx 0.0720834.
\end{aligned}
$$

We have demonstrated that β admits a 12-atomic rm $\sum_{\ell=1}^{12} \rho_\ell \delta_{(t_\ell, s_\ell)}$, where

$$
s_\ell = \frac{(t_\ell + 1)(t_\ell + 2)(t_\ell + 4)}{t_\ell},
$$

which differs from the initial measure μ, used to generate β.

4. More concrete solutions to the TMP on $p(x,y) = xy - x^4 - q_3x^3 - q_2x^2 - q_1x - q_0,\ q_0 \neq 0$

In this section, we derive more concrete numerical conditions for the existence of free moments in Theorem 2.2 to solve the TMP for $m = 4$. The main results are Theorem 4.1, which characterizes the existence of a positive definite completion of the corresponding Hankel matrix from Section 2 in terms of a system of inequalities, while Theorem 4.2 solves the $\mathcal{Z}(p)$–TMP for the cases without positive definite completion.

Assume the notations introduced in Section 2 and let $\gamma \equiv \gamma^{(-2n,6n)}$ be defined by (9) for $m = 4$. We have (see (4))

$$h(k) = \max\{2n + 2(k-1), 3k\} + 1 = \begin{cases} 2n + 2k - 1, & \text{if } k \le 2n - 2, \\ 6n - 2, & \text{if } k = 2n - 1, \\ 6n + 1, & \text{if } k = 2n, \end{cases}$$

Hence, $\mathcal{I} = \{-2n, -2n+1, \dots, 6n-4, 6n-2\}$ and the free moments in γ are $\gamma_{\mathbf{6n-3}}$, $\gamma_{\mathbf{6n-1}}$, $\gamma_{\mathbf{6n}}$. We define γ_s in the order

$$\begin{aligned} &\underbrace{\gamma_0, \gamma_1, \dots, \gamma_{2n}}_{j=0}, \underbrace{\gamma_{-1}, \gamma_{2n+1}, \gamma_{2n+2}}_{j=1}, \dots, \underbrace{\gamma_{-k}, \gamma_{2n+2k-1}, \gamma_{2n+2k}}_{j=k}, \cdots \\ &\qquad\qquad \cdots, \underbrace{\gamma_{-2n+2}, \gamma_{6n-5}, \gamma_{6n-4}}_{j=2n}, \underbrace{\gamma_{-2n+1}, \gamma_{6n-2}}_{j=2n-1}, \underbrace{\gamma_{-2n}}_{j=2n} \end{aligned} \tag{22}$$

by (9). The auxiliary moments are $\gamma_{-2n+1}, \gamma_{6n-2}$, and γ_{-2n}. Namely, we may rewrite moments for $j = 0$,

$$\gamma_0 = \beta_{0,0}, \quad \gamma_1 = \beta_{1,0}, \quad \dots, \quad \gamma_{2n} = \beta_{2n,0};$$

for $j = 1$,

$$\begin{aligned} \gamma_{-1} &= \frac{1}{q_0}\Big(\beta_{0,1} - \sum_{t=0}^{3} q_{1,t}\gamma_t\Big), \\ \gamma_{2n+1} &= \beta_{2n-2,1} - \sum_{t=-1}^{2} q_{1,t}\gamma_{t+2n-2}, \\ \gamma_{2n+2} &= \beta_{2n-1,1} - \sum_{t=-1}^{2} q_{1,t}\gamma_{t+2n-1}; \\ &\vdots \end{aligned}$$

for $j = 2n$,

$$\begin{aligned} \gamma_{-k} &= \frac{1}{q_0^k}\Big(\beta_{0,k} - \sum_{t=-k+1}^{3k} q_{k,t}\gamma_t\Big), \\ \gamma_{2n+2k-1} &= \beta_{2n-k-1,k} - \sum_{t=-k}^{3k-1} q_{k,t}\gamma_{t+2n-k-1}, \\ \gamma_{2n+2k} &= \beta_{2n-k,1} - \sum_{t=-k}^{3k-1} q_{1,t}\gamma_{t+2n-k}; \end{aligned}$$

for $j=2n-1$,

$$\gamma_{-2n+1}=\frac{1}{q_0^{2n-1}}\Big(\beta_{0,2n-1}-\sum_{t=-2n+2}^{6n-4}q_{2n-1,t}\gamma_t-\gamma_{\mathbf{6n-3}}\Big),$$
$$\gamma_{6n-2}=\beta_{1,2n-1}-\sum_{t=-2n+1}^{6n-5}q_{2n-1,t}\gamma_{t+1}-q_{2n-1,6n-4}\gamma_{\mathbf{6n-3}};$$

for $j=2n$,

$$\gamma_{-2n}=\frac{1}{q_0^{2n}}\Big(\beta_{0,2n}-\sum_{t=-2n+1}^{6n-4}q_{2n,t}\gamma_t-q_{2n,6n-4}\gamma_{\mathbf{6n-3}}-q_{2n,6n-3}\gamma_{6n-2}-$$
$$q_{2n,6n-2}\gamma_{\mathbf{6n-1}}-\gamma_{\mathbf{6n}}\Big).$$

We need to introduce new variables $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$ for $\gamma_{\mathbf{6n-3}}$, $\gamma_{\mathbf{6n-1}}$, $\gamma_{\mathbf{6n}}$, respectively. Let $\underline{\mathbf{t}}:=(\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3)$. Then $A_{\gamma(\underline{\mathbf{t}})}$ is of the form

$$\begin{array}{c} \\ T^{-n} \\ T^{-n+1} \\ \vdots \\ \mathit{1} \\ \vdots \\ T^{3n-2} \\ T^{3n-1} \\ T^{3n} \end{array}\begin{array}{c} \begin{array}{ccccccc} T^{-n} & T^{-n+1} & \cdots & 1 & \cdots & T^{3n-2} & T^{3n-1} & T^{3n} \end{array} \\ \begin{pmatrix} \gamma_{-2n}(\underline{\mathbf{t}}) & \gamma_{-2n+1}(\mathbf{t}_1) & \cdots & \gamma_{-n} & \cdots & \gamma_{2n-2} & \gamma_{2n-1} & \gamma_{2n} \\ \gamma_{-2n+1}(\mathbf{t}_1) & \gamma_{-2n+2} & \cdots & \gamma_{-n+1} & \cdots & \gamma_{2n-1} & \gamma_{2n} & \gamma_{2n+1} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ \gamma_{-n} & \gamma_{-n+1} & \cdots & \gamma_0 & \cdots & \gamma_{3n-2} & \gamma_{3n-1} & \gamma_{3n} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ \gamma_{2n-2} & \gamma_{2n-1} & \cdots & \gamma_{3n-2} & \cdots & \gamma_{6n-4} & \mathbf{t}_1 & \gamma_{6n-2}(\mathbf{t}_1) \\ \gamma_{2n-1} & \gamma_{2n} & \cdots & \gamma_{3n-1} & \cdots & \mathbf{t}_1 & \gamma_{6n-2}(\mathbf{t}_1) & \mathbf{t}_2 \\ \gamma_{2n} & \gamma_{2n+1} & \cdots & \gamma_{3n} & \cdots & \gamma_{6n-2}(\mathbf{t}_1) & \mathbf{t}_2 & \mathbf{t}_3 \end{pmatrix} \end{array}$$

for the corresponding Hankel matrix, where

$$\gamma_{-2n}(\underline{\mathbf{t}}) =: C+D\mathbf{t}_1+E\mathbf{t}_2-q_0^{-2n}\mathbf{t}_3,$$
$$\gamma_{-2n+1}(\mathbf{t}_1) =: F-q_0^{-2n+1}\mathbf{t}_1,$$
$$\gamma_{6n-2}(\mathbf{t}_1) =: G-H\mathbf{t}_1.$$

For $i\le j$ we write

$$\vec{T}^{(i,j)}:=\begin{pmatrix}T^i & T^{i+1} & \cdots & T^j\end{pmatrix}.$$

4.1. **Existence of a positive definite completion** $A_{\gamma(\underline{t})}$**.** In this subsection, we will characterize the existence of t_1,t_2,t_3 such that $A_{\gamma(t_1,t_2,t_3)}$ is positive definite. The latter is a sufficient condition for the existence of a $\mathcal{Z}(p)$–rm for β by [5, Theorem 3.9] and Theorem 2.2 above.

Assume that

$$(A_{\gamma(\underline{\mathbf{t}})})|_{\vec{T}^{(-n+1,3n-2)}} \text{ is positive definite.} \tag{23}$$

We then focus on the submatrix

$$F_1(\mathbf{t}_1):=(A_{\gamma(\underline{\mathbf{t}})})|_{\vec{T}^{(-n+1,3n-1)}}. \tag{24}$$

Note that

$$p(\mathbf{t}_1):=\det\big(F_1(\mathbf{t}_1)\big)=c_2\mathbf{t}_1^2+c_1\mathbf{t}_1+c_0$$

with $c_0, c_1, c_2 \in \mathbb{R}$. Assuming that $(A_{\gamma(\underline{\mathbf{t}})})|_{\vec{T}^{(-n+1,3n-2)}} \succ 0$, it follows that $c_2 < 0$. For the existence of a positive definite completion $A_{\gamma(\underline{t})}$, the first necessary condition is the following:

$$p(\mathbf{t_1}) \text{ has a real zero.} \tag{25}$$

Assume (25) is satisfied. Let $(t_1)_-, (t_1)_+ \in \mathbb{R}$, with $(t_1)_- \leq (t_1)_+$, be real zeroes of $p(\mathbf{t}_1)$. Then $F_1(\mathbf{t}_1)$ is positive definite on the interval $((t_1)_-, (t_1)_+)$, and positive semidefinite but not definite in $((t_1)_-, (t_1)_+)$. The question is, whether there exists a choice of

$$t_1 \in ((t_1)_-, (t_1)_+), \tag{26}$$

such that there are $t_2, t_3 \in \mathbb{R}$ with $A_{\gamma(\underline{t})}$ being positive definite.

Second, assuming (26) holds, we observe the submatrix

$$F_2(\underline{\mathbf{t}}) := \big(A_{\gamma(\underline{\mathbf{t}})}\big)|_{\vec{T}^{(-n+1,3n)}}. \tag{27}$$

By Theorem 1.2, we see that

$$F_2(\underline{t}) \succeq 0 \quad \Longleftrightarrow \quad t_3 \geq \begin{pmatrix} z_1^T & \gamma_{6n-2}(t_1) & t_2 \end{pmatrix} \Big(F_1(t_1)\Big)^{-1} \begin{pmatrix} z_1 \\ \gamma_{6n-2}(t_1) \\ t_2 \end{pmatrix} \tag{28}$$

with $z_1 := \begin{pmatrix} \gamma_{2n+1} & \gamma_{2n+2} & \cdots & \gamma_{6n-4} & t_1 \end{pmatrix}^T$. Writing

$$\begin{aligned} \widetilde{\gamma} &:= (\gamma_{-2n+2}, \gamma_{-2n+1}, \ldots, \gamma_{6n-4}), \\ c_1 &:= \begin{pmatrix} \gamma_{2n} & \gamma_{2n+1} & \cdots & \gamma_{6n-4} & t_1 \end{pmatrix}^T, \\ w_1 &:= \gamma_{6n-2}(t_1) - c_1^T A_{\widetilde{\gamma}}^{-1} c_1, \end{aligned}$$

we have

$$\Big(F_1(t_1)\Big)^{-1} = \begin{pmatrix} A_{\widetilde{\gamma}}^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{w_1} \begin{pmatrix} A_{\widetilde{\gamma}}^{-1} c_1 c_1^t A_{\widetilde{\gamma}}^{-1} & -A_{\widetilde{\gamma}}^{-1} c_1 \\ -c_1^T A_{\widetilde{\gamma}}^{-1} & 1 \end{pmatrix}.$$

Using this in the inequality (28), we know that $F_2(\underline{t}) \succ 0$ is equivalent to

$$\begin{aligned} t_3 > \begin{pmatrix} z_1^T & \gamma_{6n-2}(t_1) \end{pmatrix} \left(A_{\widetilde{\gamma}}^{-1} + \frac{1}{w_1} A_{\widetilde{\gamma}}^{-1} c_1 c_1^T A_{\widetilde{\gamma}}^{-1} \right) \begin{pmatrix} z_1 \\ \gamma_{6n-2}(t_1) \end{pmatrix} \\ - \frac{2}{w_1} \begin{pmatrix} z_1^T & \gamma_{6n-2}(t_1) \end{pmatrix} A_{\widetilde{\gamma}}^{-1} c_1 + \frac{t_2^2}{w_1}. \end{aligned} \tag{29}$$

Finally, assuming that (26) and (29) hold, we now examine the entire matrix $A_{\gamma(\underline{\mathbf{t}})}$. By Thorem 1.2, we see that

$$A_{\gamma(\underline{t})} \succeq 0 \quad \Longleftrightarrow \quad \gamma_{-2n}(\underline{t}) \geq \begin{pmatrix} z_2^T & \gamma_{2n} \end{pmatrix} (F_2(\underline{t}))^{-1} \begin{pmatrix} z_2 \\ \gamma_{2n} \end{pmatrix}, \tag{30}$$

where $z_2 := \begin{pmatrix} \gamma_{-2n+1}(t_1) & \gamma_{-2n+2} & \cdots & \gamma_{2n-1} \end{pmatrix}^T$. Writing

$$\begin{aligned} B &:= F_1(t_1), \\ c_2 &:= \begin{pmatrix} \gamma_{2n+1} & \cdots & \gamma_{6n-2}(t_1) & t_2 \end{pmatrix}^T, \\ w_2 &:= t_3 - c_2^T (F_1(t_1))^{-1} c_2, \end{aligned}$$

we see that

$$(F_2(\underline{t}))^{-1} = \begin{pmatrix} B^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{w_2} \begin{pmatrix} B^{-1} c_2 c_2^t B^{-1} & -B^{-1} c_2 \\ -c_2^T B^{-1} & 1 \end{pmatrix}.$$

Using this in the inequality (30), it follows that $A_{\gamma(\underline{t})} \succ 0$ is equivalent to

$$\gamma_{-2n}(\underline{t}) > z_2^T \left(B^{-1} + \frac{1}{w_2} B^{-1} c_2 c_2^T B^{-1} \right) z_2 - \frac{2\gamma_{2n}}{w_2} z_2^T B^{-1} c_2 + \frac{\gamma_{2n}^2}{w_2}. \tag{31}$$

By Theorem 2.2, the arguments above give sufficient conditions to solve the p-pure TMP.

Theorem 4.1 (Purely pure case)**.** *Let $p(x,y) = xy - \sum_{s=0}^4 q_s x^s$ with $q_i \in \mathbb{R}$, $q_0 \neq 0$ $q_4 = 1$. Let $\beta \equiv \left\{\beta_{i,j}^{(2n)}\right\}$ for $i,j \in \mathbb{Z}_+$, $i+j \leq 2n$, be a p-pure sequence. Assume the notation above and* (23) *holds. If there exists a triple $(t_1,t_2,t_3) \in \mathbb{R}^3$ such that* (25)*,* (26)*,* (29)*,* (31) *hold, then β admits a $\mathcal{Z}(p)$–representing measure.*

4.2. **Existence of positive semidefinite completion $A_{\gamma(\underline{t})}$ with a $\mathbb{R}$–rm vanishing in $\{0\}$.** In this subsection, we study the existence of a $\mathbb{R}$–rm for $\gamma(\underline{t})$ vanishing in $\{0\}$ in case β is not p-pure or a triple $(t_1,t_2,t_3) \in \mathbb{R}^3$ satisfying the conditions in Theorem 4.1 does not exist. The main result is Theorem 4.2 below.

We say a column relation in $A_{\gamma(t)}$ of the form

$$\sum_{i=i_1}^{i_2} a_i T^i = \mathbf{0}, \tag{32}$$

where $a_i \in \mathbb{R}$, $-n \leq i_1 < i_2 \leq 3n$, $a_{i_1} \neq 0$, $a_{i_2} \neq 0$, **propagates through** $A_{\gamma(\underline{t})}$, if

$$\begin{aligned} \sum_{i=i_1}^{i_2} a_i T^{i-j} &= \mathbf{0} \quad \text{for } j = 1,\ldots,n-i_1, \\ \sum_{i=i_1}^{i_2} a_i T^{i+j} &= \mathbf{0} \quad \text{for } j = 1,\ldots,3n-i_2, \end{aligned} \tag{33}$$

are also relations of $A_{\gamma(\underline{t})}$.

Theorem 4.2 (Singular case)**.** *Let $p(x,y) = xy - \sum_{s=0}^4 q_s x^s$ with $q_i \in \mathbb{R}$, $q_0 \neq 0$ $q_4 = 1$. Let $\beta \equiv \left\{\beta_{i,j}^{(2n)}\right\}$ for $i,j \in \mathbb{Z}_+$, $i+j \leq 2n$, be a sequence such that there does not exist a triple $(t_1,t_2,t_3) \in \mathbb{R}^3$ satisfying the conditions in Theorem 4.1. Assume the notation of Section 4, and Subsection 4.1 above. We write $C := (A_{\gamma(\underline{\mathbf{t}})})|_{\vec{T}^{(-n+1,3n-2)}}$. Then β admits a $\mathcal{Z}(p)$–representing measure if and only if one of the following holds:*

- *(i) $C \succeq 0$, $C \not\succ 0$ and a relation* (32) *satisfied in C propagates through $A_{\gamma(\underline{t})}$ for $\underline{t} \in \mathbb{R}^3$, which is uniquely determined using* (33)*.*
- *(ii) $C \succ 0$,* (25) *holds, and the relation* (32) *satisfied in $F_1((t_1)_-)$, for F_1 defined by* (24)*, propagates through $A_{\gamma(\underline{t})}$ for $(t_2,t_3) \in \mathbb{R}^2$, uniquely determined using* (33)*.*
- *(iii) $C \succ 0$,* (25) *holds, and the relation* (32) *satisfied in $F_1((t_1)_+)$, for F_1 defined by* (24)*, propagates through $A_{\gamma(\underline{t})}$ for $(t_2,t_3) \in \mathbb{R}^2$, uniquely determined using* (33)*.*
- *(iv) $C \succ 0$,* (25) *holds, $t_1 \in ((t_1)_,(t_1)_+)$, t_3 is equal to the right hand side of* (29) *for some t_2, and the relation* (32) *satisfied in $F_2(\underline{t})$, for F_2 defined by* (27)*, propagates through $A_{\gamma(\underline{t})}$.*

(v) $C \succ 0$, *(25) holds,* $t_1 \in ((t_1)_{,}(t_1)_+)$, t_3 *satisfies* (29) *for some* t_2, $\gamma_{-2n}(\underline{t})$ *is equal to the right hand side of* (31)*, and in the relation* (32)*, satisfied in* $A_{\gamma(\underline{t})}$*, we have* $i_1 = -n$ *and* $i_2 = 3n$*.*

Proof. By assumption, there does not exist a triple $(t_1, t_2, t_3) \in \mathbb{R}^3$ such that $A_{\gamma(\underline{t})} \succ 0$. We distinguish a few cases based on the point where non-definiteness occurs.

Case 1: $C \succeq 0$ *and* $C \not\succ 0$.

In this case, there is a relation of the form (32), where $a_i \in \mathbb{R}$, $-n+1 \leq i_1 < i_2 \leq 3n-2$, $a_{i_1} \neq 0$, $a_{i_2} \neq 0$, among the columns of C. By the extension principle [12, Proposition 2.4], this relation must hold in any positive semidefinite completion $A_{\gamma(\underline{t})}$. By [27, Theorem 3.1], the existence of a $\mathbb{R}$–rm canishing in $\{0\}$ is equivalent to well–definedness of the completion $A_{\gamma(\underline{t})}$ determined by propagating the relation (32) in both directions by (33). This gives (i).

Case 2: $C \succ 0$*;* (25) *holds and* $(t_1)_-$ *is defined as in Subsection 4.1.*

In this case, there is a relation of the form (32) among columns of $F_1((t_1)_-)$, where $F_1(\mathbf{t}_1)$ is defined by (24), with $i_2 = 3n-1$. As in Case 1 above, well–definedness of the relations (33) characterizes the existence of a $(\mathbb{R} \setminus \{0\})$–rm for $\gamma(\underline{t})$. This shows (ii).

Case 3: $C \succ 0$*;* (25) *holds and* $t_1 = (t_1)_+$ *with* $(t_1)_+$ *defined as in Subsection 4.1.*

This case is analogous to Case 2 and completes (iii).

Case 4: $C \succ 0$*;* (25) *holds,* $t_1 \in ((t_1)_-, (t_1)_+)$ *and* t_3 *is equal to the right hand side of* (29) *for some* t_2*.*

In this case, there is a relation of the form (32) among columns of $F_2(\underline{t})$, where $F_2(\underline{\mathbf{t}})$ is defined by (27), with $i_2 = 3n$. As in Case 1 above, well–definedness of the relations (33) characterizes the existence of a $(\mathbb{R}\setminus\{0\})$–rm for $\gamma(\underline{t})$. This proves (iv).

Case 5: $C \succ 0$*;* (25) *holds,* $t_1 \in ((t_1)_-, (t_1)_+)$ *and* t_3 *satisfies* (29) *for some* t_2 *and* $\gamma_{-2n}(\underline{t})$ *is equal to the right hand side of* (31)*.*

In this case, there is a relation of the form (32) among columns of $A_{\gamma(\underline{t})}$ with $i_1 = -n$; a $\mathbb{R}$–rm vanishing in $\{0\}$ for $A_{\gamma(\underline{t})}$ exists only if $i_2 = 3n$. Otherwise, the second type relations from (33) would need to hold contradicting to positive definiteness of $F_2(\underline{t})$. This verifies (v). ☐

Acknowledgment. Example 3.4 was obtained using calculations with the software tool *Mathematica* [24].

References

[1] A. Albert, Conditions for positive and nonnegative definiteness in terms of pseudoinverses, *SIAM J. Appl. Math.* 17 (1969), 434–440.

[2] C. Bayer and J. Teichmann, The proof of Tchakaloff's Theorem, *Proc. Amer. Math. Soc.* 134 (2006), 3035–3040.

[3] L. Baldi, G. Blekherman and R. Sinn, Nonnegative polynomials and moment problems on algebraic curves, arXiv preprint `https://arxiv.org/pdf/2407.06017`.

[4] T.S. Chihara, *An introduction to orthogonal polynomials*, Gordon and Breach Science Publishers, New York-London-Paris, 1978.

[5] R. Curto and L. Fialkow, A. Recursiveness, positivity, and truncated moment problems, *Houston J. Math.* 17 (1991), no. 4, 603–635.
[6] R. Curto and L. Fialkow, Solution of the truncated complex moment problem for flat data, *Mem. Amer. Math. Soc.* 119 (1996), no. 568, x+52 pp.
[7] R. Curto and L. Fialkow, Flat extensions of positive moment matrices: recursively generated relations, *Mem. Amer. Math. Soc.* 136 (1998), no. 648, x+56 pp.
[8] R. Curto and L. Fialkow, Solution of the singular quartic moment problem, *J. Operator Theory* 48 (2002), 315–354.
[9] R. Curto and L. Fialkow, An analogue of the Riesz-Haviland theorem for the truncated moment problem, J. Funct. Anal. 255 (2008), no. 10, 2709–2731.
[10] R. Curto and S. Yoo, Concrete solution to the nonsingular quartic binary moment problem, *Proc. Amer. Math. Soc.* 144 (2016), no. 1, 249–258.
[11] P.J. di Dio and M. Kummer, The multidimensional truncated moment problem: Carathéodory numbers from Hilbert functions, *Math. Ann.* 380 (2021), no. 1, 267—291.
[12] L. Fialkow, Positivity, extensions and the truncated complex moment problem, *Contemporary Math.* 185 (1995), 133–150.
[13] L. Fialkow, Truncated multivariable moment problems with finite variety, *J. Operator Theory* 60 (2008), 343–377.
[14] L. Fialkow, Solution of the truncated moment problem with variety $y = x^3$, *Trans. Amer. Math. Soc.* 363 (2011), 3133–3165
[15] E.K. Haviland, On the momentum problem for distributions in more than one dimension, Amer. J. Math. 57 (1935), 562–568.
[16] E.K. Haviland, On the momentum problem for distributions in more than one dimension, Part II, Amer. J. Math. 58 (1936), 164–168.
[17] H. Richter, Parameterfreie Abschätzung und Realisierung von Erwartungswerten, *Bl. der Deutsch. Ges. Versicherungsmath.* 3 (1957), 147–161.
[18] C. Riener and M. Schweighofer, Optimization approaches to quadrature:a new characterization of Gaussian quadrature on the line and quadrature with few nodes on plane algebraic curves, on the plane and in higher dimensions, *J. Complex.* 45 (2018) 22–54.
[19] M. Riesz, Sur le problème des moments, Troisième Note, Arkiv för Matematik, Astronomi och Fysik, 17 (1923), no. 16, 1–52.
[20] K. Schmüdgen, *The Moment Problem*, Springer, 2017. 225–234.
[21] J.L. Smul'jan, An operator Hellinger integral (Russian), *Mat. Sb.* 91 (1959), 381–430.
[22] S. Yoo, Sextic moment problems with a reducible cubic column relation, *Integral Equations Operator Theory* 88 (2017), no. 1, 45–63.
[23] S. Yoo and A. Zalar, The Truncated Moment Problem on Reducible Cubic Curves I: Parabolic and Circular Type Relations, *Complex Anal. Oper. Theory* 18 (2024), no. 5, Paper No. 111.
[24] Wolfram Research, Inc., *Mathematica*, Version 12.3.1, Champaign, IL, 2021.
[25] A. Zalar, The truncated Hamburger moment problems with gaps in the index set, *Integral Equations Operator Theory* 93 (2021), no. 3, Paper No. 22, 36 pp.
[26] A. Zalar, The truncated moment problem on the union of parallel lines, *Linear Algebra Appl.* 649 (2022), 186–239.
[27] A. Zalar, The strong truncated Hamburger moment problem with and without gaps, *J. Math. Anal. Appl.* 516 (2022), no. 2, Paper No. 126563, 21 pp.
[28] A. Zalar, The truncated moment problem on curves $y = q(x)$ and $yx^\ell = 1$, *Linear Multilinear Algebra* 72 (2024), no. 12, 1922–1966.
[29] F. Zhang, *Matrix Theory: Basic Results and Techniques*, 2nd edition, Springer, New York, NY, 2011. 420 pp.
[30] F. Zhang, *The Schur Complement and Its Applications*, Springer-Verlag, New York, 2005.

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