CONSTRUCTIVE APPROACH TO THE TRUNCATED MOMENT PROBLEM ON REDUCIBLE CUBIC CURVES: HYPERBOLIC TYPE RELATIONS

SEONGUK YOO AND ALJAŽ ZALAR[2]

ABSTRACT. In this paper, we solve *constructively* the bivariate truncated moment problem (TMP) of even degree on reducible cubic curves, where the conic part is a hyperbola. According to the classification from our previous work [YZ24], these represent three out of nine possible canonical forms of reducible cubic curves after applying an affine linear transformation. The TMP on the union of three parallel lines, the circular and the parabolic type TMP were solved constructively in [Zal22a, YZ24], while in this paper we consider three cases of hyperbolic type, i.e., a type without real self-intersection points, a type with a simple real self-intersection point and a type with a double real self-intersection point. In all cases, we also establish bounds on the number of atoms in a minimal representing measure.

1. INTRODUCTION

Let $\mathbb{Z}_+$ stand for nonnegative integers. Given a real 2–dimensional sequence

$$\beta \equiv \beta^{(2k)} = \{\beta_{0,0}, \beta_{1,0}, \beta_{0,1}, \ldots, \beta_{2k,0}, \beta_{2k-1,1}, \ldots, \beta_{1,2k-1}, \beta_{0,2k}\}$$

of degree $2k$ and a closed subset K of $\mathbb{R}^2$, the **truncated moment problem (K–TMP)** supported on K for $\beta^{(2k)}$ asks to characterize the existence of a positive Borel measure μ on $\mathbb{R}^2$ with support in K, such that

$$\beta_{i,j} = \int_K x^i y^j d\mu \quad \text{for} \quad i, j \in \mathbb{Z}_+,\ i + j \leq 2k. \tag{1.1}$$

If such a measure exists, we say that $\beta^{(2k)}$ has a representing measure supported on K and μ is its K–**representing measure (K–rm).**

In the degree-lexicographic order $\mathit{1}, X, Y, X^2, XY, Y^2, \ldots, X^k, X^{k-1}Y, \ldots, Y^k$ of rows and columns, the corresponding moment matrix to β is equal to

$$\mathcal{M}(k) = \mathcal{M}(k;\beta) := \begin{pmatrix} \mathcal{M}[0,0](\beta) & \mathcal{M}[0,1](\beta) & \cdots & \mathcal{M}[0,k](\beta) \\ \mathcal{M}[1,0](\beta) & \mathcal{M}[1,1](\beta) & \cdots & \mathcal{M}[1,k](\beta) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M}[k,0](\beta) & \mathcal{M}[k,1](\beta) & \cdots & \mathcal{M}[k,k](\beta) \end{pmatrix}, \tag{1.2}$$

where

$$\mathcal{M}[i,j](\beta) := \begin{pmatrix} \beta_{i+j,0} & \beta_{i+j-1,1} & \beta_{i+j-2,2} & \cdots & \beta_{i,j} \\ \beta_{i+j-1,1} & \beta_{i+j-2,2} & \beta_{i+j-3,3} & \cdots & \beta_{i-1,j+1} \\ \beta_{i+j-2,2} & \beta_{i+j-3,3} & \beta_{i+j-4,4} & \cdots & \beta_{i-2,j+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{j,i} & \beta_{j-1,i+1} & \beta_{j-2,i+2} & \cdots & \beta_{0,i+j} \end{pmatrix}.$$

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Let $\mathbb{R}[x,y]_{\leq k} := \{p \in \mathbb{R}[x,y] \colon \deg p \leq k\}$ stand for the set of real polynomials in variables x, y of total degree at most k. For every $p(x,y) = \sum_{i,j} a_{ij} x^i y^j \in \mathbb{R}[x,y]_{\leq k}$ we define its **evaluation** $p(X,Y)$ on the columns of the matrix $\mathcal{M}(k)$ by replacing each capitalized monomial X^iY^j in $p(X,Y) = \sum_{i,j} a_{ij} X^i Y^j$ by the column of $\mathcal{M}(k)$, indexed by this monomial. Then $p(X,Y)$ is a vector from the linear span of the columns of $\mathcal{M}(k)$. If this vector is the zero one, i.e., all coordinates are equal to 0, then $p(X,Y)$ is a **column relation** of $\mathcal{M}(k)$. A column relation $p(X,Y)$ is **nontrivial**, if $p \not\equiv 0$. The matrix $\mathcal{M}(k)$ is **recursively generated (rg)** if for $p, q, pq \in \mathbb{R}[x,y]_{\leq k}$ such that $p(X,Y)$ is a column relation of $\mathcal{M}(k)$, it follows that $(pq)(X,Y)$ is also a column relation of $\mathcal{M}(k)$. The matrix $\mathcal{M}(k)$ is p**–pure** if the only column relation of $\mathcal{M}(k)$ are those determined recursively by p. In this case the TMP for β is called p**–pure**.

For $p \in \mathbb{R}[x,y]$ we denote by $\mathcal{Z}(p) := \{(x,y) \in \mathbb{R}^2 \colon p(x,y) = 0\}$ the zero set of p and by $\deg p$ its total degree.

A **concrete solution** to the TMP is a set of necessary and sufficient conditions for the existence of a K–representing measure μ, that can be tested in numerical examples. Among necessary conditions, $\mathcal{M}(k)$ must be positive semidefinite (psd) and rg [CF04, Fia95], and by [CF96] if the support $\operatorname{supp}(\mu)$ of μ is a subset of $\mathcal{Z}(p)$ for a polynomial $p \in \mathbb{R}[x,y]_{\leq k}$, then p is a column relation of $\mathcal{M}(k)$. The bivariate $\mathcal{Z}(p)$–TMP (*not necessarily p–pure*) is concretely solved in the following cases: (i) $\deg p = 1$ [CF08], (ii) $\deg p = 2$ [CF02, CF04, CF05, Fia15], (iii) p is irreducible with $\deg p = 3$ [KZ25+] and (iv) p is reducible, $\deg p = 3$ and p has a special form [Zal22a, YZ24]. The bivariate *p–pure* TMP is concretely solved also for: (v) $p(x,y) = xy + q(x) - x^4$ with $\deg q = 3$ [YZ24+], (vi) $p(x,y) = y - x^4$ [FZ25+] and (vii) p is reducible with $\deg p = 3$ and β is *purely pure* (i.e., the corresponding linear functional is strictly positive on nonzero polynomials, positive on $\mathcal{Z}(p)$) [KZ25+]. For a more detailed description and some other less concrete solutions to the TMP on plane algebraic curves we refer the reader to [YZ24, p. 3] or [KZ25+, p. 2–3], while for a recent development in the area of moment problems to a monograph [Sch17].

A **constructive solution** to the K–TMP is a solution, where not only the existence of a K–rm is characterized, but a concrete K–rm is explicitly constructed.

The motivation for this paper was to solve the TMP *constructively* on reducible cubic curves of hyperbolic type, according to the classification of [YZ24, Proposition 3.1]. By applying an affine linear transformation, each TMP on reducible cubic curve is equivalent to the TMP on one of nine canonical cases of the form $yc(x,y) = 0$, where $c \in \mathbb{R}[x,y]$, $\deg c = 2$. In [Zal22a], the case of three parallel lines is solved constructively, while in [YZ24], the solutions to the *circular type* (the curve is a line and a circle touching at a double real point) and the *parabolic type* relations (the curve is a line and a parabola that intersect tangentially at a real point) are presented. In this paper, we solve the TMP constructively for the cases $c(x,y) = 1 - xy$, $c(x,y) = x + y - xy$ and $c(x,y) = ay + x^2 - y^2$, $a \in \mathbb{R} \setminus \{0\}$, which are called in [YZ24] the *hyperbolic type* 1, 2 and 3 *relations*, respectively. We also characterize the number of atoms in a minimal representing measure, i.e., a measure with the minimal number of atoms in the support. The question of bounds on the cardinality of minimal representing measures in the TMP, supported on algebraic curves, which is always finite by [Ric57] (or [Sch17, Theorem 1.24]), has attracted a recent attention of several authors (see [RS18, dDS18, dDK21, Zal24, BBS24+, RTT25+]).

In terms of the self-intersection points of the cubic $yc(x,y) = 0$, we can classify the hyperbolic types from the previous paragraph into a type without real self-intersection points (type 1), a type with a single real self-intersection point (type 2) and a type with a double real self-intersection point (type 3). To prove our main results, we follow the idea presented in [Zal22a, YZ24], which characterizes the existence of a decomposition of β into the sum

$\beta^{(\ell)}+\beta^{(c)}$, where $\beta^{(\ell)}=\{\beta^{(\ell)}_{i,j}\}_{i,j\in\mathbb{Z}_+,\, i+j\leq 2k}$ and $\beta^{(c)}=\{\beta^{(c)}_{i,j}\}_{i,j\in\mathbb{Z}_+,\, i+j\leq 2k}$ admit a $\mathbb{R}$–rm and a $\mathcal{Z}(c)$–rm, respectively. The crucial property of the forms of the cubic, which makes this idea realizable, is that the line is equal to $y=0$. This ensures that all but two moments of $\beta^{(\ell)}$ and $\beta^{(c)}$ are not already determined by the original sequence, i.e., $\beta^{(\ell)}_{2k-1,0}$, $\beta^{(\ell)}_{2k,0}$, $\beta^{(c)}_{2k-1,0}$, $\beta^{(c)}_{2k,0}$ in the hyperbolic type 1 case (as in the case of three parallel lines [Zal22a]) , $\beta^{(\ell)}_{0,0}$, $\beta^{(\ell)}_{2k,0}$, $\beta^{(c)}_{0,0}$, $\beta^{(c)}_{2k,0}$ in the hyperbolic type 2 case (as in the parabolic type case [YZ24, Section 6]) and $\beta^{(\ell)}_{0,0}$, $\beta^{(\ell)}_{1,0}$, $\beta^{(c)}_{0,0}$, $\beta^{(c)}_{1,0}$ in the hyperbolic type 3 case (as in the circular type case [YZ24, Section 5]). Then, by an involved analysis, the characterization of the existence of a decomposition $\beta=\beta^{(\ell)}+\beta^{(c)}$ can be done in all three cases. We mention that the analysis in the hyperbolic type cases is more demanding than in the corresponding cases with the same positions of the free moments from [Zal22a, YZ24] stated in parentheses, since the solution to the TMP on a hyperbola (see Subsection 2.7) contains more linear algebraic requirements than in the case of other conics.

1.1. **Reader's Guide.** The paper is organized as follows. In Section 2 we fix notation and present some preliminary results needed to establish our main results. In Section 3 we recall the approach for solving the TMP constructively on reducible cubic curves in the canonical form $yc(x,y)=0$ developed in [YZ24, Section 4]. In Sections 4–6 we solve constructively the TMP for reducible cubic curves of hyperbolic types 1–3, respectively, and characterize the cardinality of minimal representing measures (see Theorems 4.1, 5.2, 5.6, 6.1 and 6.7). Numerical examples demonstrating the main results are also given (see Subsections 4.3, 5.3 and 6.3).

2. Preliminaries

We write $\mathbb{R}^{n\times m}$ for the set of $n\times m$ real matrices. For a matrix M we call the linear span of its columns a **column space** and denote it by $\mathcal{C}(M)$. The set of real symmetric matrices of size n will be denoted by S_n. For a matrix $A\in S_n$ the notation $A\succ 0$ (resp. $A\succeq 0$) means A is positive definite (pd) (resp. positive semidefinite (psd)). We write $\mathbf{0}_{t_1,t_2}$ for a $t_1\times t_2$ matrix with only zero entries and $\mathbf{0}_t=\mathbf{0}_{t,t}$ for short, where $t_1,t_2,t\in\mathbb{N}$. The notation $E^{(\ell)}_{i,j}$, $\ell\in\mathbb{N}$, stands for the usual $\ell\times\ell$ coordinate matrix with the only nonzero entry at position (i,j), equal to 1.

In the rest of this section let $k\in\mathbb{N}$ and $\beta=\beta^{(2k)}=\{\beta_{i,j}\}_{i,j\in\mathbb{Z}_+,\, i+j\leq 2k}$ be a bivariate sequence of degree $2k$.

2.1. **Moment matrix.** Let $\mathcal{M}(k)$ be the moment matrix of β (see (1.2)). Let Q_1,Q_2 be subsets of the set $\{X^iY^j\colon i,j\in\mathbb{Z}_+,\ i+j\leq k\}$. We denote by $\mathcal{M}(k)_{Q_1,Q_2}$ the submatrix of $\mathcal{M}(k)$ consisting of the rows indexed by the elements of Q_1 and the columns indexed by the elements of Q_2. In case $Q:=Q_1=Q_2$, we write $\mathcal{M}(k)_Q:=\mathcal{M}(k)_{Q,Q}$ for short.

2.2. **Affine linear transformations.** The existence of representing measures is invariant under invertible affine linear transformations of the form

$$\phi(x,y)=(\phi_1(x,y),\phi_2(x,y)):=(a+bx+cy,d+ex+fy),\ (x,y)\in\mathbb{R}^2, \tag{2.1}$$

$a,b,c,d,e,f\in\mathbb{R}$ with $bf-ce\neq 0$. Namely, let $L_\beta:\mathbb{R}[x,y]_{\leq 2k}\to\mathbb{R}$ be a **Riesz functional** of the sequence β defined by

$$L_\beta(p):=\sum_{\substack{i,j\in\mathbb{Z}_+,\\ i+j\leq 2k}} a_{i,j}\beta_{i,j},\qquad \text{where}\quad p=\sum_{\substack{i,j\in\mathbb{Z}_+,\\ i+j\leq 2k}} a_{i,j}x^iy^j.$$

We define $\widetilde{\beta}=\{\widetilde{\beta}_{i,j}\}_{i,j\in\mathbb{Z}_+,\, i+j\leq 2k}$ by

$$\widetilde{\beta}_{i,j}=L_\beta(\phi_1(x,y)^i\cdot\phi_2(x,y)^j).$$

By [CF04, Proposition 1.9], β admits a r–atomic rm supported on K if and only if $\widetilde{\beta}$ admits a r–atomic rm supported on $\phi(K)$. We write $\widetilde{\beta} = \phi(\beta)$ and $\mathcal{M}(k;\widetilde{\beta}) = \phi(\mathcal{M}(k;\beta))$.

2.3. **Generalized Schur complements.** Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}$$

be a real matrix where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$. The **generalized Schur complement** [Zha05] of A (resp. D) in M is defined by

$$M/A = D - CA^{\dagger}B \quad (\text{resp. } M/D = A - BD^{\dagger}C),$$

where $A^{\dagger}$ (resp. $D^{\dagger}$) stands for the Moore-Penrose inverse of A (resp. D).

The following lemma will be used in the proofs of our main results.

Lemma 2.1. *Let $n, m \in \mathbb{N}$ and*

$$M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \in S_{n+m},$$

where $A \in S_n$, $B \in \mathbb{R}^{n\times m}$ and $C \in S_m$. If $\operatorname{rank} M = \operatorname{rank} A$, then the matrix equation

$$\begin{pmatrix} A \\ B^T \end{pmatrix} W = \begin{pmatrix} B \\ C \end{pmatrix}, \tag{2.2}$$

where $W \in \mathbb{R}^{n\times m}$, is solvable and the solutions are precisely the solutions of the matrix equation $AW = B$. In particular, $W = A^{\dagger}B$ satisfies (2.2).

The following theorem is a characterization of psd 2×2 block matrices.

Theorem 2.2 ([Alb69]). *Let*

$$M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \in S_{n+m}$$

be a real symmetric matrix where $A \in S_n$, $B \in \mathbb{R}^{n\times m}$ and $C \in S_m$. Then:

(1) The following conditions are equivalent:
  (a) $M \succeq 0$.
  (b) $C \succeq 0$, $\mathcal{C}(B^T) \subseteq \mathcal{C}(C)$ and $M/C \succeq 0$.
  (c) $A \succeq 0$, $\mathcal{C}(B) \subseteq \mathcal{C}(A)$ and $M/A \succeq 0$.

(2) If $M \succeq 0$, then

$$\operatorname{rank} M = \operatorname{rank} A + \operatorname{rank} M/A = \operatorname{rank} C + \operatorname{rank} M/C.$$

2.4. **Partially positive semidefinite matrices and their completions.** A **partial matrix** $A = (a_{i,j})_{i,j=1}^n$ is a matrix of real numbers $a_{i,j} \in \mathbb{R}$, where some of the entries are not specified.

A partial symmetric matrix $A = (a_{i,j})_{i,j=1}^n$ is **partially positive semidefinite (ppsd)** (resp. **partially positive definite (ppd)**) if the following two conditions hold:

(1) $a_{i,j}$ is specified if and only if $a_{j,i}$ is specified and $a_{i,j} = a_{j,i}$.
(2) All fully specified principal minors of A are psd (resp. pd).

For $n \in \mathbb{N}$ write $[n] := \{1, 2, \ldots, n\}$. We denote by A_{Q_1,Q_2} the submatrix of $A \in \mathbb{R}^{n\times n}$ consisting of the rows indexed by the elements of $Q_1 \subseteq [n]$ and the columns indexed by the elements of $Q_2 \subseteq [n]$. In case $Q := Q_1 = Q_2$, we write $A_Q := A_{Q,Q}$ for short.

Lemma 2.3 ([YZ24, Lemma 2.4]). *Let $A(\mathbf{x})$ be a partially positive semidefinite symmetric matrix of size $n \times n$ with the missing entries in the positions (i,j) and (j,i), $1 \le i < j \le n$. Let*

$$A_1 = (A(\mathbf{x}))_{[n]\setminus\{i,j\}}, \ a = (A(\mathbf{x}))_{[n]\setminus\{i,j\},\{i\}}, \ b = (A(\mathbf{x}))_{[n]\setminus\{i,j\},\{j\}}, \ \alpha = (A(\mathbf{x}))_{i,i}, \ \gamma = (A(\mathbf{x}))_{j,j}.$$

Let

$$A_2 = (A(\mathbf{x}))_{[n]\setminus\{j\}} = \begin{pmatrix} A_1 & a \\ a^T & \alpha \end{pmatrix} \in S_{n-1}, \qquad A_3 = (A(\mathbf{x}))_{[n]\setminus\{i\}} = \begin{pmatrix} A_1 & b \\ b^T & \gamma \end{pmatrix} \in S_{n-1},$$

and

$$x_\pm := b^T A_1^\dagger a \pm \sqrt{(A_2/A_1)(A_3/A_1)} \in \mathbb{R}.$$

Then:

(1) $A(x_0)$ is positive semidefinite if and only if $x_0 \in [x_-, x_+]$.

(2)

$$\operatorname{rank} A(x_0) = \begin{cases} \max\left\{\operatorname{rank} A_2, \operatorname{rank} A_3\right\}, & \text{for } x_0 \in \{x_-, x_+\}, \\ \max\left\{\operatorname{rank} A_2, \operatorname{rank} A_3\right\} + 1, & \text{for } x_0 \in (x_-, x_+). \end{cases}$$

(3) The following statements are equivalent:

(a) $x_- = x_+$.

(b) $A_2/A_1 = 0$ or $A_3/A_1 = 0$.

(c) $\operatorname{rank} A_2 = \operatorname{rank} A_1$ or $\operatorname{rank} A_3 = \operatorname{rank} A_1$.

2.5. **Extension principle.**

Proposition 2.4 ([Fia95, Proposition 2.4] or [Zal22a, Lemma 2.4])**.** *Let $\mathcal{A} \in S_n$ be positive semidefinite, Q a subset of the set $\{1, \ldots, n\}$ and $\mathcal{A}_Q$ the restriction of $\mathcal{A}$ to the rows and columns from the set Q. If $\mathcal{A}_Q v = 0$ for a nonzero vector v, then $\mathcal{A}\widehat{v} = 0$, where $\widehat{v}$ is a vector with the only nonzero entries in the rows from Q and such that the restriction $\widehat{v}_Q$ to the rows from Q equals to v.*

2.6. **(Strong) Hamburger TMP.**

In this subsection we recall the solutions to the univariate TMP and its strong version, since it will be essentially used in the proofs of our main results.

Let $k \in \mathbb{N}$ and $\gamma := (\gamma_0, \ldots, \gamma_{2k}) \in \mathbb{R}^{2k+1}$. We say that γ is **$\mathbb{R}$–representable** if there is a positive Borel measure μ on $\mathbb{R}$ such that $\gamma_i = \int_{\mathbb{R}} x^i \, d\mu$ for $0 \le i \le 2k$. Characterizing the existence of the $\mathbb{R}$–rm for γ is called the **truncated Hamburger moment problem (THMP)** or also the **$\mathbb{R}$–TMP**.

We define the Hankel matrix corresponding to γ by

$$A_\gamma := (\gamma_{i+j})_{i,j=0}^k = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \iddots & \iddots & \gamma_{k+1} \\ \gamma_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \gamma_{2k-1} \\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k-1} & \gamma_{2k} \end{pmatrix} \in S_{k+1}. \tag{2.3}$$

For $m \le k$ we denote the upper left–hand corner $(\gamma_{i+j})_{i,j=0}^m \in S_{m+1}$ of A_γ of size $m+1$ by $A_\gamma(m)$, while the lower right–hand corner of A_γ of size $m+1$ by $A_\gamma[m]$.

The solution to the THMP is the following.

Theorem 2.5 ([CF91, Theorems 3.9–3.10])**.** *For $k \in \mathbb{N}$ and $\gamma = (\gamma_0, \ldots, \gamma_{2k}) \in \mathbb{R}^{2k+1}$ with $\gamma_0 > 0$, the following statements are equivalent:*

(1) There exists a $\mathbb{R}$–representing measure for γ.

(2) There exists a $(\operatorname{rank} A_\gamma)$–atomic $\mathbb{R}$–representing measure for γ.

(3) A_γ *is positive semidefinite and one of the following holds:*
   (a) $A_\gamma(k-1)$ *is positive definite.*
   (b) $\operatorname{rank} A_\gamma(k-1) = \operatorname{rank} A_\gamma$.

Let $k_1, k_2 \in \mathbb{N}$ and

$$\widetilde{\gamma} := (\widetilde{\gamma}_{-2k_1}, \widetilde{\gamma}_{-2k_1+1}, \widetilde{\gamma}_{-2k_2+2}, \ldots, \widetilde{\gamma}_{2k_2-1}, \widetilde{\gamma}_{2k_2}) \in \mathbb{R}^{2k_1+2k_2+1}. \tag{2.4}$$

We say that $\widetilde{\gamma}$ is **strongly $\mathbb{R}$–representable** if there is a positive Borel measure μ on $\mathbb{R} \setminus \{0\}$ such that $\widetilde{\gamma}_i = \int_{\mathbb{R}} x^i \, d\mu$ for $-2k_1 \le i \le 2k_2$. Characterizing the existence of the $(\mathbb{R} \setminus \{0\})$–rm for $\widetilde{\gamma}$ is called **the strong truncated Hamburger moment problem (STHMP)**.

The solution to the STHMP is the following.

Theorem 2.6. *Let $k_1, k_2 \in \mathbb{N}$ and $\widetilde{\gamma}$ as in (2.4) with $\gamma_{-2k_1} > 0$. Define $\gamma := (\gamma_0, \gamma_1, \ldots, \gamma_{2k_1+2k_2}) \in \mathbb{R}^{2k_1+2k_2+1}$ by $\gamma_i := \widetilde{\gamma}_{i-2k_1}$. The following statements are equivalent:*
(1) *There exists a $(\mathbb{R} \setminus \{0\})$–representing measure for $\widetilde{\gamma}$.*
(2) *There exists a $(\operatorname{rank} A_\gamma)$–atomic $(\mathbb{R} \setminus \{0\})$–representing measure for γ.*
(3) A_γ *is positive semidefinite and one of the following holds:*
   (a) A_γ *is positive definite.*
   (b) $\operatorname{rank} A_\gamma = \operatorname{rank} A_\gamma(k_1+k_2-1) = \operatorname{rank} A_\gamma[k_1+k_2-1]$.

Let $k \in \mathbb{N}$. We say a sequence $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{2k}) \in \mathbb{R}^{2k+1}$ is $(\mathbb{R} \setminus \{0\})$**–representable** if there is a positive Borel measure μ on $\mathbb{R} \setminus \{0\}$ such that $\gamma_i = \int_{\mathbb{R}\setminus\{0\}} x^i \, d\mu$ for $0 \le i \le 2k$.

Note that Theorem 2.6 above characterizes when a given sequence γ is $(\mathbb{R}\setminus\{0\})$–representable.

Remark 2.7. The matrix version of Theorem 2.6 appears in [Sim06] using involved operator theory as the main tool. A proof of the scalar version using linear algebra techniques is [Zal22b, Theorems 3.1].

2.7. **Hyperbolic TMP.** We will need the following solution to the hyperbolic TMP (see [Zal22b, Corollary 3.5] and Remark 2.9 below).

Theorem 2.8. *Let $p(x,y) = xy - 1$ nd $\beta := \beta^{(2k)} = \{\beta_{i,j}\}_{i,j \in \mathbb{Z}_+, i+j \le 2k}$, where $k \ge 2$. Let*

$$\mathcal{B} = \{Y^k, Y^{k-1}, \ldots, Y, \mathit{1}, X, X^2, \ldots, X^k\}. \tag{2.5}$$

Then the following statements are equivalent:
(1) β *has a $\mathcal{Z}(p)$–representing measure.*
(2) β *has a $(\operatorname{rank} \mathcal{M}(k))$–atomic $\mathcal{Z}(p)$–representing measure.*
(3) $\mathcal{M}(k)$ *is positive semidefinite, the relations $\beta_{i+1,j+1} = \beta_{i,j}$ hold for every $i, j \in \mathbb{Z}_+$ with $i+j \le 2k-2$ and one of the following statements holds:*
   (a) $\mathcal{M}(k)_{\mathcal{B}}$ *is positive definite.*
   (b) $\operatorname{rank} \mathcal{M}(k) = \operatorname{rank} \mathcal{M}(k)_{\mathcal{B}\setminus\{Y^k\}} = \operatorname{rank} \mathcal{M}(k)_{\mathcal{B}\setminus\{X^k\}}$.

Remark 2.9. The first solution to the hyperbolic TMP is [CF05, Theorem 1.5], which contains a condition called *variety condition*. To apply the solution to the hyperbolic TMP, when solving the TMP on a reducible cubic with an irreducible component equivalent to the hyperbola $xy = 1$ after applying an invertible affine linear transformation, it is not easy to check the variety condition symbolically. Theorem 2.8 does not contain the variety condition, but only linear algebraic conditions. Theorem 2.8 is a slight improvement of [Zal22b, Corollary 3.5]. Namely, instead of (3) the statement in [Zal22b, Corollary 3.5] reads:

(3′) $\mathcal{M}(k)$ is positive semidefinite, recursively generated and if $\operatorname{rank}\mathcal{M}(k)_{\mathcal{B}} = 2k$, then

$$\operatorname{rank}\mathcal{M}(k)_{\mathcal{B}\setminus\{X^k\}} = \operatorname{rank}\mathcal{M}(k)_{\mathcal{B}\setminus\{Y^k\}} = 2k.$$

Furthermore, in the proof of [Zal22b, Corollary 3.5] it is shown that (3′) is equivalent to a variant of (3) (labelled (A) in the proof), in which all rg relations are assumed, not only *pure ones*, i.e.,

$$\beta_{i+1,j+1} = \beta_{i,j} \text{ holds for every } i, j \in \mathbb{Z}_+ \text{ with } i + j \leq 2k - 2, \tag{2.6}$$

or equivalently

$$X^{i+1}Y^{j+1} = X^iY^j \text{ hold for all } i, j \in \mathbb{Z}_+ \text{ with } i + j \leq k - 2. \tag{2.7}$$

The improvement of Theorem 2.8 compared to [Zal22b, Corollary 3.5] lies in the replacement of the assumption $\mathcal{M}(k)$ *is rg* by the seemingly weaker assumption that $\mathcal{M}(k)$ *satisties all pure rg relations* (i.e., (2.6)). We now explain why this can be done. Due to the existence of the relations (2.7), it is sufficient to assume that other relations are among columns and rows, indexed by $\mathcal{B}$ (see (2.5)). Defining $v = (\beta_{0,2k}, \beta_{0,2k-1}, \beta_{0,2k-2}, \ldots, \beta_{0,0}, \beta_{1,0}, \beta_{2,0}, \ldots, \beta_{2k,0}) \in \mathbb{R}^{4k+1}$, note that $\mathcal{M}(k)_{\mathcal{B}} = A_v$ (see (2.3)) is a singular psd Hankel matrix, which is rg in the sense of a univariate sequence [Zal22b, Section 2] by the assumptions in (3b) and [Zal22b, Proposition 2.1.(4),(5)]. By [Zal22b, Theorem 3.1], v has a representing measure supported on $\mathbb{R}\setminus\{0\}$, where $\beta_{0,i}$ corresponds to the moment of x^{-i} and $\beta_{j,0}$ corresponds to the moment of x^j. By [Zal22b, Claim in the proof of Corollary 3.5], β has a $\mathcal{Z}(xy-1)$–rm.

Corollary 2.10. *Let* $p(x, y) = x + y - xy$ *and* $\beta := \beta^{(2k)} = \{\beta_{i,j}\}_{i,j\in\mathbb{Z}_+, i+j\leq 2k}$, *where* $k \geq 2$. *Let*

$$\mathcal{B} = \{Y^k, Y^{k-1}, \ldots, Y, \mathit{1}, X, X^2, \ldots, X^k\}. \tag{2.8}$$

Then the following statements are equivalent:

(1) β *has a* $\mathcal{Z}(p)$*–representing measure.*

(2) β *has a* $(\operatorname{rank}\mathcal{M}(k))$*–atomic* $\mathcal{Z}(p)$*–representing measure.*

(3) $\mathcal{M}(k)$ *is positive semidefinite, the relations* $\beta_{i+1,j+1} = \beta_{i+1,j} + \beta_{i,j+1} = 0$ *hold for every* $i, j \in \mathbb{Z}_+$ *with* $i + j \leq 2k - 2$ *and one of the following statements holds:*

  (a) $\mathcal{M}(k)_{\mathcal{B}}$ *is positive definite.*

  (b) $\operatorname{rank}\mathcal{M}(k) = \operatorname{rank}\mathcal{M}(k)_{\mathcal{B}\setminus\{Y^k\}} = \operatorname{rank}\mathcal{M}(k)_{\mathcal{B}\setminus\{X^k\}}$.

Proof. Note that applying an affine linear transformation $\phi(x, y) = (x + 1, y + 1)$ (see Section 2.2) to β we obtain a new sequence $\widetilde{\beta}^{(2k)} = \{\widetilde{\beta}_{i,j}\}_{i,j\in\mathbb{Z}_+, i+j\leq 2k}$ satisfying the relations $\widetilde{\beta}_{i+1,j+1} = \widetilde{\beta}_{i,j}$ for $i, j \in \mathbb{Z}_+$ with $i + j \leq 2k - 2$. Since the existence of a $\mathcal{Z}(p)$–rm for β is equivalent to the existence of a $\mathcal{Z}(xy-1)$–rm for $\widetilde{\beta}$, Corollary 2.10 follows by Theorem 2.8. □

Corollary 2.11. *Let* $p(x, y) = ay + x^2 - y^2$, $a \in \mathbb{R}\setminus\{0\}$, *and* $\beta := \beta^{(2k)} = \{\beta_{i,j}\}_{i,j\in\mathbb{Z}_+, i+j\leq 2k}$, *where* $k \geq 2$. *Let*

$$\mathcal{B}' = \{YX^{k-1}, YX^{k-2}, \ldots, Y, \mathit{1}, X, X^2, \ldots, X^k\}. \tag{2.9}$$

Then the following statements are equivalent:

(1) β *has a* $\mathcal{Z}(p)$*–representing measure.*

(2) β *has a* $(\operatorname{rank}\mathcal{M}(k))$*–atomic* $\mathcal{Z}(p)$*–representing measure.*

(3) $\mathcal{M}(k)$ *is positive semidefinite, the relations* $\beta_{i,j+2} = \beta_{i+2,j} + a\beta_{i,j+1}$ *hold for every* $i, j \in \mathbb{Z}_+$ *with* $i + j \leq 2k - 2$ *and one of the following statements holds:*

  (a) $\mathcal{M}(k)_{\mathcal{B}'}$ *is positive definite.*

  (b) $\operatorname{rank}\mathcal{M}(k) = \operatorname{rank}\mathcal{M}(k)_{\mathcal{B}'\setminus\{YX^{k-1}\}} = \operatorname{rank}\mathcal{M}(k)_{\mathcal{B}'\setminus\{X^k\}}$.

Proof. Note that applying an affine linear transformation $\phi(x,y) = \big(\frac{2}{a}(x+\frac{a}{2}-y), \frac{2}{a}(x+\frac{a}{2}+y)\big)$ (see Section 2.2) to β we obtain a new sequence $\widetilde{\beta}^{(2k)} = \{\widetilde{\beta}_{i,j}\}_{i,j\in\mathbb{Z}_+, i+j\leq 2k}$ satisfying the relations $\widetilde{\beta}_{i+1,j+1} = \widetilde{\beta}_{i,j}$ for $i,j \in \mathbb{Z}_+$ with $i+j \leq 2k-2$. Since the existence of a $\mathcal{Z}(p)$–rm for β is equivalent to the existence of a $\mathcal{Z}(xy-1)$–rm for $\widetilde{\beta}$, Corollary 2.11 follows by Theorem 2.8. □

3. Common approach to all cases

In this section we recall the constructive approach to solving the TMP on reducible cubic curves in the canonical form $yc(x,y) = 0$ developed in [YZ24, Section 4].

Let

$$\mathcal{C} = \{\mathit{1}, X, Y, X^2, XY, Y^2, \ldots, X^k, X^{k-1}Y, \ldots, Y^k\} \tag{3.1}$$

be the set of columns and rows of the moment matrix $\mathcal{M}(k)$ in the degree-lexicographic order. Let

$$p(x,y) = y \cdot c(x,y) \in \mathbb{R}[x,y]_{\leq 3} \tag{3.2}$$

be a polynomial of degree 3 in one of the canonical forms from [YZ24, Proposition 3.1], were $c(x,y)$ a polynomial of degree 2. A given 2–dimensional sequence $\beta = \{\beta_{i,j}\}_{i,j\in\mathbb{Z}_+, i+j\leq 2k}$ of degree $2k$, $k\in\mathbb{N}$, will have a $\mathcal{Z}(p)$–rm if and only if it can be decomposed as

$$\beta = \beta^{(\ell)} + \beta^{(c)}, \tag{3.3}$$

where

$$\begin{aligned} \beta^{(\ell)} &:= \{\beta^{(\ell)}_{i,j}\}_{i,j\in\mathbb{Z}_+, i+j\leq 2k} \quad \text{has a measure on } y=0,\\ \beta^{(c)} &:= \{\beta^{(c)}_{i,j}\}_{i,j\in\mathbb{Z}_+, i+j\leq 2k} \quad \text{has a measure on the conic } c(x,y)=0, \end{aligned}$$

and the sum in (3.3) is the component-wise sum. On the level of moment matrices, (3.3) is equivalent to

$$\mathcal{M}(k;\beta) = \mathcal{M}(k;\beta^{(\ell)}) + \mathcal{M}(k;\beta^{(c)}). \tag{3.4}$$

Note that if β has a $\mathcal{Z}(p)$–rm, then the matrix $\mathcal{M}(k;\beta)$ satisfies the relation $p(X,Y) = \mathbf{0}$ and by rg also

$$(x^i y^j p)(X,Y) = \mathbf{0} \quad \text{for } i,j = 0,\ldots,k-3 \text{ such that } i+j \leq k-3. \tag{3.5}$$

Let $\mathcal{T} \subseteq \mathcal{C}$ be a subset, such that $\{1, X, \ldots, X^k\} \subseteq \mathcal{T}$ and the columns from $\mathcal{T}$ span the column space $\mathcal{C}(\mathcal{M}(k;\beta))$. We write $\vec{X}^{(0,k)} := (1, X, \ldots, X^k)$, $\mathcal{T}_1 = \mathcal{T} \setminus \{1, X, \ldots, X^k\}$ and

$$\widetilde{\mathcal{M}}(k;\beta) := Q\mathcal{M}(k;\beta)Q^T = \begin{array}{c} \\ (\vec{X}^{(0,k)})^T \\ (\vec{\mathcal{T}}_1)^T \\ (\overrightarrow{\mathcal{C}\setminus\mathcal{T}})^T \end{array} \begin{array}{c} \begin{array}{ccc} \vec{X}^{(0,k)} & \vec{\mathcal{T}}_1 & \overrightarrow{\mathcal{C}\setminus\mathcal{T}} \end{array} \\ \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ (A_{12})^T & A_{22} & A_{23} \\ (A_{13})^T & (A_{23})^T & A_{33} \end{pmatrix} \end{array}, \tag{3.6}$$

where $\vec{\mathcal{T}}_1$ and $\overrightarrow{\mathcal{C}\setminus\mathcal{T}}$ are tuples of elements of $\mathcal{T}_1$ and $\mathcal{C}\setminus\mathcal{T}$, arranged in some order, and Q is the appropriate permutation matrix. In this new order of rows and columns, (3.4) becomes equivalent to

$$\widetilde{\mathcal{M}}(k;\beta) = \widetilde{\mathcal{M}}(k;\beta^{(\ell)}) + \widetilde{\mathcal{M}}(k;\beta^{(c)}), \tag{3.7}$$

where $\widetilde{\mathcal{M}}(k;\beta^{(\ell)}) := Q\mathcal{M}(k;\beta^{(\ell)})Q^T$ and $\widetilde{\mathcal{M}}(k;\beta^{(c)}) := Q\mathcal{M}(k;\beta^{(c)})Q^T$. By the form of the atoms we know that $\widetilde{\mathcal{M}}(k;\beta^{(c)})$ and $\widetilde{\mathcal{M}}(k;\beta^{(\ell)})$ will have forms

$$\begin{aligned}
\widetilde{\mathcal{M}}(k;\beta^{(c)}) &= \begin{array}{c} \\ (\vec{X}^{(0,k)})^T \\ (\vec{\mathcal{T}}_1)^T \\ (\overrightarrow{\mathcal{C}\setminus\mathcal{T}})^T \end{array} \begin{array}{c} \begin{array}{ccc} \vec{X}^{(0,k)} & \vec{\mathcal{T}}_1 & \overrightarrow{\mathcal{C}\setminus\mathcal{T}} \end{array} \\ \begin{pmatrix} A & A_{12} & A_{13} \\ (A_{12})^T & A_{22} & A_{23} \\ (A_{13})^T & (A_{23})^T & A_{33} \end{pmatrix} \end{array}, \\
\widetilde{\mathcal{M}}(k;\beta^{(\ell)}) &= \begin{array}{c} \\ (\vec{X}^{(0,k)})^T \\ (\vec{\mathcal{T}}_1)^T \\ (\overrightarrow{\mathcal{C}\setminus\mathcal{T}})^T \end{array} \begin{array}{c} \begin{array}{ccc} \vec{X}^{(0,k)} & \vec{\mathcal{T}}_1 & \overrightarrow{\mathcal{C}\setminus\mathcal{T}} \end{array} \\ \begin{pmatrix} A_{11}-A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}
\end{aligned} \tag{3.8}$$

for some Hankel matrix $A \in S_{k+1}$. Define matrix functions

$$\mathcal{F} : S_{k+1} \to S_{\frac{(k+1)(k+2)}{2}} \qquad \text{and} \qquad \mathcal{H} : S_{k+1} \to S_{k+1}$$

by

$$\begin{aligned}
\mathcal{F}(\mathbf{A}) &= \begin{array}{c} \\ (\vec{X}^{(0,k)})^T \\ (\vec{\mathcal{T}}_1)^T \\ (\overrightarrow{\mathcal{C}\setminus\mathcal{T}})^T \end{array} \begin{array}{c} \begin{array}{ccc} \vec{X}^{(0,k)} & \vec{\mathcal{T}}_1 & \overrightarrow{\mathcal{C}\setminus\mathcal{T}} \end{array} \\ \begin{pmatrix} \mathbf{A} & A_{12} & A_{13} \\ (A_{12})^T & A_{22} & A_{23} \\ (A_{13})^T & (A_{23})^T & A_{33} \end{pmatrix} \end{array}, \\
\mathcal{H}(\mathbf{A}) &= \begin{array}{c} \\ (\vec{X}^{(0,k)})^T \end{array} \begin{array}{c} \vec{X}^{(0,k)} \\ \begin{pmatrix} A_{11}-\mathbf{A} \end{pmatrix} \end{array}.
\end{aligned} \tag{3.9}$$

Using (3.8), (3.7) becomes equivalent to

$$\widetilde{\mathcal{M}}(k;\beta) = \mathcal{F}(A) + \mathcal{H}(A) \oplus \mathbf{0}_{\frac{k(k+1)}{2}} \tag{3.10}$$

for some Hankel matrix $A \in S_{k+1}$.

Lemma 3.1 ([YZ24, Lemma 4.1]). *Let $k \in \mathbb{N}, k \geq 3$. Assume the notation above. Then the sequence $\beta = \{\beta_{i,j}\}_{i,j\in\mathbb{Z}_+, i+j\leq 2k}$ has a $\mathcal{Z}(p)$–representing measure if and only if there exist a Hankel matrix $A \in S_{k+1}$, such that:*

(1) The sequence with the moment matrix $\mathcal{F}(A)$ has a $\mathcal{Z}(c)$–representing measure.
(2) The sequence with the moment matrix $\mathcal{H}(A)$ has a $\mathbb{R}$–representing measure.

Lemma 3.2 ([YZ24, Lemma 4.2]). *Let $k \in \mathbb{N}, k \geq 3$. Assume the notation above and the sequence $\beta = \{\beta_{i,j}\}_{i,j\in\mathbb{Z}_+, i+j\leq 2k}$ admits a $\mathcal{Z}(p)$–representing measure. Let*

$$A := A_{\left(\beta^{(c)}_{0,0},\beta^{(c)}_{1,0},\ldots,\beta^{(c)}_{2k,0}\right)} \in S_{k+1}$$

be a Hankel matrix (see (2.3)) such that $\mathcal{F}(A)$ admits a $\mathcal{Z}(c)$–representing measure and $\mathcal{H}(A)$ admits a $\mathbb{R}$–representing measure. Let $c(x,y)$ be of the form

$$\begin{aligned} &c(x,y) = a_{00} + a_{10}x + a_{20}x^2 + a_{01}y + a_{02}y^2 + a_{11}xy \quad \textit{with } a_{ij} \in \mathbb{R} \\ &\textit{and exactly one of the coefficients } a_{00}, a_{10}, a_{20} \textit{ is nonzero.} \end{aligned} \tag{3.11}$$

If:

(1) $a_{00} \neq 0$, *then*
$$\beta_{i,0}^{(c)} = -\frac{1}{a_{0,0}}(a_{01}\beta_{i,1} + a_{0,2}\beta_{i,2} + a_{11}\beta_{i+1,1}) \quad \textit{for } i = 0, \ldots, 2k-2.$$
(2) $a_{10} \neq 0$, *then*
$$\beta_{i,0}^{(c)} = -\frac{1}{a_{1,0}}(a_{01}\beta_{i,1} + a_{0,2}\beta_{i,2} + a_{11}\beta_{i+1,1}) \quad \textit{for } i = 1, \ldots, 2k-1.$$
(3) $a_{20} \neq 0$, *then*
$$\beta_{i,0}^{(c)} = -\frac{1}{a_{2,0}}(a_{01}\beta_{i,1} + a_{0,2}\beta_{i,2} + a_{11}\beta_{i+1,1}) \quad \textit{for } i = 2, \ldots, 2k.$$

Lemma 3.2 states that if c is as in (3.11), all but two entries of the Hankel matrix A from Lemma 3.1 are uniquely determined by β. The following lemma gives the smallest candidate for A in Lemma 3.1 with respect to the usual Loewner order of matrices.

Lemma 3.3 ([YZ24, Lemma 4.3])**.** *Assume the notation above and let $\beta = \{\beta_{i,j}\}_{i,j\in\mathbb{Z}_+, i+j\leq 2k}$, where $k \geq 3$, be a sequence of degree $2k$. Assume that $\widetilde{\mathcal{M}}(k;\beta)$ is positive semidefinite and satisfies the column relations* (3.5)*. Then:*

(1) $\mathcal{F}(A) \succeq 0$ *for some* $A \in S_{k+1}$ *if and only if* $A \succeq A_{12}(A_{22})^{\dagger}(A_{12})^T$.

(2) $\mathcal{F}\big(A_{12}(A_{22})^{\dagger}(A_{12})^T\big) \succeq 0$ *and* $\mathcal{H}\big(A_{12}(A_{22})^{\dagger}(A_{12})^T\big) \succeq 0$.

(3) $\mathcal{F}\big(A_{12}(A_{22})^{\dagger}(A_{12})^T\big)$ *satisfies the column relations* $(x^iy^jc)(X,Y) = 0$ *for* $i,j \in \mathbb{Z}_+$ *such that* $i+j \leq k-2$.

(4) We have that
$$\begin{aligned}\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta) &= \operatorname{rank} A_{22} + \operatorname{rank}\big(A_{11} - A_{12}(A_{22})^{\dagger}(A_{12})^T\big) \\ &= \operatorname{rank} \mathcal{F}\big(A_{12}(A_{22})^{\dagger}(A_{12})^T\big) + \operatorname{rank} \mathcal{H}\big(A_{12}(A_{22})^{\dagger}(A_{12})^T\big).\end{aligned}$$

Remark 3.4. By Lemmas 3.1–3.3, solving the $\mathcal{Z}(p)$–TMP for $\beta = \{\beta_{i,j}\}_{i,j\in\mathbb{Z}_+, i+j\leq 2k}$, where $k \geq 3$, with p being of the form $yc(x,y)$ and c as in (3.11), the natural procedure is the following:

(1) First compute $A_{\min} := A_{12}(A_{22})^{\dagger}(A_{12})^T$. By Lemma 3.3.(3), there is one entry of $A_{\min}$, which might need to be changed to obtain a Hankel structure. Namely, in the notation (3.11), if:
  (a) $a_{00} \neq 0$, then the value of $(A_{\min})_{k,k}$ must be changed to $(A_{\min})_{k-1,k+1}$.
  (b) $a_{10} \neq 0$, then the value of $(A_{\min})_{1,k+1}$ must be changed to $(A_{\min})_{2,k}$.
  (c) $a_{20} \neq 0$, then the value of $(A_{\min})_{2,2}$ must be changed to $(A_{\min})_{3,1}$.

Let $\widehat{A}_{\min}$ be the matrix obtained from $A_{\min}$ after performing the change described above.

(2) Study if $\mathcal{F}(\widehat{A}_{\min})$ and $\mathcal{H}(\widehat{A}_{\min})$ admit a $\mathcal{Z}(c)$–rm and a $\mathbb{R}$–rm, respectively. If the answer is yes, β admits a $\mathcal{Z}(p)$–rm. Otherwise by Lemma 3.2, there are two antidiagonals of the Hankel matrix $\widehat{A}_{\min}$, which can by varied so that the matrices $\mathcal{F}(\widehat{A}_{\min})$ and $\mathcal{H}(\widehat{A}_{\min})$ will admit the corresponding measures. Namely, in the notation (3.11), if:
  (a) $a_{00} \neq 0$, then the last two antidiagonals of $\widehat{A}_{\min}$ can be changed.
  (b) $a_{10} \neq 0$, then the left–upper and the right–lower corner of $\widehat{A}_{\min}$ can be changed.
  (c) $a_{20} \neq 0$, then the first two antidiagonals of $\widehat{A}_{\min}$ can be changed.

To solve the $\mathcal{Z}(p)$–TMP for β one needs to characterize, when it is possible to change these antidiagonals in such a way to obtain a matrix $\breve{A}_{\min}$, such that $\mathcal{F}(\breve{A}_{\min})$ and $\mathcal{H}(\breve{A}_{\min})$ admit a $\mathcal{Z}(c)$–rm and a $\mathbb{R}$–rm, respectively.

4. Hyperbolic type 1 relation: $p(x,y)=y(1-xy)$.

In this section we solve constructively the $\mathcal{Z}(p)$–TMP for the sequence $\beta=\{\beta_{i,j}\}_{i,j\in\mathbb{Z}_+,i+j\leq 2k}$ of degree $2k$, $k\geq 3$, where $p(x,y)$ is as in the title of the section. The main result is Theorem 4.1, which characterizes concrete numerical conditions for the existence of a $\mathcal{Z}(p)$–rm for β and also the number of atoms needed in a minimal $\mathcal{Z}(p)$–rm. A numerical example demonstrating the main result is presented in Subsection 4.3.

Assume the notation from Section 3. If β admits a $\mathcal{Z}(p)$–TMP, then $\mathcal{M}(k;\beta)$ must satisfy the relations

$$Y^{2+j}X^{i+1}=Y^{1+j}X^{i}\quad \text{for } i,j\in\mathbb{Z}_+ \text{ such that } i+j\leq k-3. \tag{4.1}$$

On the level of moments the relations (4.1) mean that

$$\beta_{i+1,j+2}=\beta_{i,j+1}\quad \text{for } i,j\in\mathbb{Z}_+ \text{ such that } i+j\leq 2k-3. \tag{4.2}$$

In the presence of all column relations (4.1), the column space $\mathcal{C}(\mathcal{M}(k;\beta))$ is spanned by the columns in the tuple

$$\vec{\mathcal{T}}:=(\underbrace{Y^k,Y^{k-1},\ldots,Y}_{\vec{Y}^{(k,1)}},\underbrace{YX,YX^2,\ldots,YX^{k-1}}_{Y\vec{X}^{(1,k-1)}},\underbrace{\mathit{1},X,\ldots,X^k}_{\vec{X}^{(0,k)}}). \tag{4.3}$$

Let

(4.4) P be a permutation matrix such that moment matrix $\widehat{\mathcal{M}}(k):=P\mathcal{M}(k;\beta)P^T$ has rows and columns indexed in the order $\vec{Y}^{(k,1)},Y\vec{X}^{(1,k-1)},\vec{X}^{(0,k)},Y^2\vec{X}^{(1,k-2)},\ldots,Y^{k-1}X$,

where $Y^j\vec{X}^{(1,k-j)}:=(Y^jX,Y^jX^2,\ldots,Y^jX^{k-j})$ for $1\leq j\leq k-1$. Let $\widehat{\mathcal{M}}(k)_{\vec{\mathcal{T}}}$ be the restriction of the moment matrix $\widehat{\mathcal{M}}(k)$ to the rows and columns in the tuple $\vec{\mathcal{T}}$ and write

$$\widehat{\mathcal{M}}(k)_{\vec{\mathcal{T}}}=\begin{array}{c} \\ (\vec{Y}^{(k,1)})^T\\ (Y\vec{X}^{(1,k-1)})^T\\ (\vec{X}^{(0,k)})^T\end{array}\begin{array}{c}\begin{array}{ccc}\vec{Y}^{(k,1)} & Y\vec{X}^{(1,k-1)} & \vec{X}^{(0,k)}\end{array}\\ \begin{pmatrix} B_{11} & B_{12} & B_{13}\\ (B_{12})^T & B_{22} & B_{23}\\ (B_{13})^T & (B_{23})^T & B_{33}\end{pmatrix}\end{array}$$

$$=\begin{array}{c} \\ (\vec{Y}^{(k,1)})^T\\ (Y\vec{X}^{(1,k-1)})^T\\ (\vec{X}^{(0,k-1)})^T\\ X^k\end{array}\begin{array}{c}\begin{array}{cccc}\vec{Y}^{(k,1)} & Y\vec{X}^{(1,k-1)} & \vec{X}^{(0,k-1)} & X^k\end{array}\\ \begin{pmatrix} B_{11} & B_{12} & B_{13}^{(0,k-1)} & b_{13}^{(k)}\\ (B_{12})^T & B_{22} & B_{23}^{(0,k-1)} & b_{23}^{(k)}\\ (B_{13}^{(0,k-1)})^T & (B_{23}^{(0,k-1)})^T & B_{33}^{(0,k-1)} & b_{33}^{(k)}\\ (b_{13}^{(k)})^T & (b_{23}^{(k)})^T & (b_{33}^{(k)})^T & \beta_{2k,0}\end{pmatrix}\end{array}\in S_{3k}.$$

We also write

$$\widehat{\mathcal{M}}(k)_{\vec{\mathcal{T}}}=:\begin{pmatrix} R & m_{12}\\ (m_{12})^T & \beta_{2k,0}\end{pmatrix}. \tag{4.5}$$

Next we define the matrix $\mathcal{N}(k)$, which extends the restriction of $\widehat{\mathcal{M}}(k)$ to rows and columns in $(\vec{Y}^{(k,1)},Y\vec{X}^{(1,k-1)},\vec{X}^{(0,k-1)})$, with a row and a column YX^k. Namely, it contains the only candidates for the corresponding moments, which are generated by the atoms in any $\mathcal{Z}(p)$–rm. The reason for introducing precisely $\mathcal{N}(k)$ is the fact, that together with $\mathcal{M}(k)$, they contain crucial information needed to characterize the existence of the solution to the $\mathcal{Z}(p)$–TMP (see

Theorem 4.1 below). So,

$$
\text{(4.6)}\qquad \mathcal{N}(k)=\begin{array}{c}\\ (\vec{Y}^{(k,1)})^T\\ (Y\vec{X}^{(1,k-1)})^T\\ (\vec{X}^{(0,k-1)})^T\\ YX^k\end{array}\begin{array}{c}\begin{array}{cccc}\vec{Y}^{(k,1)} & Y\vec{X}^{(1,k-1)} & \vec{X}^{(0,k-1)} & YX^k\end{array}\\ \begin{pmatrix} B_{11} & B_{12} & B_{13}^{(0,k-1)} & a\\ (B_{12})^T & B_{22} & B_{23}^{(0,k-1)} & b\\ (B_{13}^{(0,k-1)})^T & (B_{23}^{(0,k-1)})^T & B_{33}^{(0,k-1)} & c\\ a^T & b^T & c^T & \beta_{2k-1,1}\end{pmatrix}\end{array}\in S_{3k},
$$

where

$$
a=\begin{pmatrix}\beta_{0,1} & \beta_{1,1} & \cdots & \beta_{k-1,1}\end{pmatrix}^T,\qquad b=\begin{pmatrix}\beta_{k,1} & \beta_{k+1,1} & \cdots & \beta_{2k-2,1}\end{pmatrix}^T,
$$
$$
c=\begin{pmatrix}\beta_{k,1} & \beta_{k+1,1} & \cdots & \beta_{2k-1,1}\end{pmatrix}^T.
$$

In the definition of a, b, c we used the fact that if the $\mathcal{Z}(p)$–rm for β exists, then the relations (4.2) hold also for $i + j \le 2k - 1$. In particular, we used the relations $\beta_{2k-1,2} = \beta_{2k-2,1}$, $\beta_{k,k+1} = \beta_{k-1,k}$ and $\beta_{2k,2} = \beta_{2k-1,1}$. These relations also imply that

$$
\text{(4.7)}\qquad \begin{pmatrix}B_{12} & a\end{pmatrix}=B_{13}^{(0,k-1)},\qquad \begin{pmatrix}B_{22} & b\end{pmatrix}=B_{23}^{(0,k-1)}\qquad\text{and}\qquad \begin{pmatrix}b^T & \beta_{2k-1,1}\end{pmatrix}=c^T.
$$

We also write

$$
\text{(4.8)}\qquad \mathcal{N}(k)=:\begin{pmatrix}R & n_{12}\\ (n_{12})^T & \beta_{2k-1,1}\end{pmatrix},
$$

where note that R is the same as in (4.5) above.

Next we define two additional matrices F_1 and F_2, needed in the statement of the solution to the $\mathcal{Z}(p)$–TMP:

- F_1 the restriction of $\widetilde{\mathcal{M}}(k;\beta^{(\ell)})$ (see (3.8)) to rows and columns in $\vec{X}^{(0,k-1)}$. Namely, it contains the only candidates for the corresponding moments, which are generated by the atoms in any $\mathcal{Z}(p)$–rm, that are supported on the line $y = 0$.
- F_2 is the restriction of $\mathcal{N}(k)$ to the rows and columns in $(\vec{Y}^{(k,1)}, Y\vec{X}^{(1,k)})$.

So

$$
\text{(4.9)}\quad F_1:=B_{33}^{(0,k-1)}-\begin{pmatrix}(B_{23}^{(0,k-1)})^T & c\end{pmatrix},\qquad F_2:=\begin{pmatrix}B_{11} & B_{13}^{(0,k-1)}\\ (B_{13}^{(0,k-1)})^T & \begin{pmatrix}(B_{23}^{(0,k-1)})^T & c\end{pmatrix}\end{pmatrix},
$$

where we used (4.7) in the equalities. Define real numbers

$$
\text{(4.10)}\qquad
\begin{aligned}
t' &= (n_{12})^T R^\dagger\, m_{12},\\
u' &= \beta_{2k,0}-(w_1)^T(F_1)^\dagger\, w_1,\\
u'' &= (w_2)^T(F_2)^\dagger\, w_2,
\end{aligned}
$$

where

$$
w_1=\begin{pmatrix}\beta_{k,0}-\beta_{k+1,1} & \beta_{k+1,0}-\beta_{k+2,1} & \cdots & \beta_{2k-2,0}-\beta_{2k-1,1} & \beta_{2k-1,0}-t'\end{pmatrix}^T,
$$
$$
w_2=\begin{pmatrix}\beta_{1,1} & \beta_{2,1} & \cdots & \beta_{2k-1,1} & t'\end{pmatrix}^T.
$$

Note that:

- w_1 is the difference of the restriction of the column X^k of $\widehat{\mathcal{M}}(k)$ to the rows $\vec{X}^{(0,k-1)}$ and a vector that is the restriction of the only (up to the choice of t') potential column YX^{k+1} of the extension of $\widehat{\mathcal{M}}(k)$ to the rows $X^{(0,\vec{k}-1)}$, if the $\mathcal{Z}(p)$–rm for β exists.

- w_2 is the restriction to the rows $(\vec{Y}^{(k,1)}, \vec{X}^{(0,k-1)})$ of the only (up to the choice of t') potential column X^k of the matrix $\widetilde{\mathcal{M}}(k;\beta^{(c)})$ (see (3.8)), which is generated by the atoms lying on $xy=1$ if a $\mathcal{Z}(p)$–rm for β exists.
- t' is the only candidate for the moment of x^{2k-1} coming from the atoms in any $\mathcal{Z}(p)$–rm, that are supported on the conic part $\mathcal{Z}(xy-1)$ of $\mathcal{Z}(p)$, if one of $\widehat{\mathcal{M}}(k)_{\vec{\mathcal{T}}}$ or $\mathcal{N}(k)$ is not positive definite. Furthermore, u' and u'' are the only two candidates for the moment of x^{2k} from the conic part that need to be checked when deciding on the existence of a $\mathcal{Z}(p)$–rm for β.

Define the sequences

$$\begin{aligned}\gamma_1(\mathbf{t},\mathbf{u}) &:= (\beta_{0,0}-\beta_{1,1}, \beta_{1,0}-\beta_{2,1},\ldots,\beta_{2k-2,0}-\beta_{2k-1,1},\beta_{2k-1,0}-\mathbf{t},\beta_{2k,0}-\mathbf{u}),\\ \gamma_2(\mathbf{t},\mathbf{u}) &:= (\beta_{0,2k},\beta_{0,2k-1},\ldots,\beta_{0,1},\beta_{0,0},\beta_{1,0},\ldots,\beta_{2k-2,0},\mathbf{t},\mathbf{u}).\end{aligned} \tag{4.11}$$

Let $\mathcal{F}(\mathbf{A})$ and $\mathcal{H}(\mathbf{A})$ be as in (3.9) with $\vec{\mathcal{T}}_1 := (\vec{Y}^{(k,1)}, Y\vec{X}^{(1,k)})$. Define the matrix function

$$\mathcal{G}:\mathbb{R}^2\to S_{k+1},\qquad \mathcal{G}(\mathbf{t},\mathbf{u}) = \widehat{A}_{\min} + \mathbf{t}\big(E^{(k+1)}_{k,k+1}+E^{(k+1)}_{k+1,k}\big) + \mathbf{u}E^{(k+1)}_{k+1,k+1}, \tag{4.12}$$

where $\widehat{A}_{\min}$ is as in Remark 3.4.(1). We will prove that $A_{\gamma_1(t,u)}$, $A_{\gamma_2(t,u)}$ (see (2.3)) are equal to $\mathcal{H}(\mathcal{G}(t,u))$, $\mathcal{F}(\mathcal{G}(t,u))P^T)_{\vec{Y}^{(k,1)}\cup\vec{X}^{(0,k)}}$, which represent possible restrictions of $\widetilde{\mathcal{M}}(k;\beta^{(\ell)})$ and $\widetilde{\mathcal{M}}(k;\beta^{(c)})$ (see (3.8)) to the rows and columns in $\vec{X}^{(0,k)}$ and $(\vec{Y}^{(k,1)},\vec{X}^{(0,k)})$, respectively.

The solution to the $\mathcal{Z}(p)$–TMP is the following.

Theorem 4.1. *Let $p(x,y) = y(xy-1)$ and $\beta := \beta^{(2k)} = (\beta_{i,j})_{i,j\in\mathbb{Z}_+, i+j\le 2k}$, where $k\ge 3$. Assume the notation above.*

Then the following statements are equivalent:

1. *β has a $\mathcal{Z}(p)$–representing measure.*
2. *$\mathcal{M}(k;\beta)$ and $\mathcal{N}(k)$ are positive semidefinite, the relations (4.2) hold and one of the following statements holds:*
   (a) *$\widehat{\mathcal{M}}(k)_{\vec{\mathcal{T}}}$ and $\mathcal{N}(k)$ are positive definite.*
   (b) *$\gamma_1(t',u)$ and $\gamma_2(t',u)$ are $\mathbb{R}$–representable and $(\mathbb{R}\setminus\{0\})$–representable, respectively, for some $u\in\{u',u''\}$.*

Moreover, assume a $\mathcal{Z}(p)$–representing measure for β exists. If the rank inequality

$$\operatorname{rank}\mathcal{N}(k)\le\operatorname{rank}\mathcal{M}(k;\beta)$$

holds, then there is a $(\operatorname{rank}\mathcal{M}(k;\beta))$–atomic $\mathcal{Z}(p)$–representing measure; otherwise there is a $(\operatorname{rank}\mathcal{M}(k;\beta)+1)$–atomic one.

Remark 4.2. The implication (2a) $\Rightarrow$ (1) of Theorem 4.1 already follows from [KZ25+, Theorem 8.9], which characterizes all positive polynomials on $\mathcal{Z}(p)$, together with [dDS18, Proposition 2 and Corollary 6], which states that strictly positive Riesz functional implies the existence of a $\mathcal{Z}(p)$–rm. However, here we present a constructive proof for this implication, which also shows that a minimal measure is $3k$–atomic.

As explained in Remark 3.4, the existence of a $\mathcal{Z}(p)$–rm for β is equivalent to the existence of a pair $(t_0,u_0)\in\mathbb{R}^2$, such that $\mathcal{H}(\mathcal{G}(t_0,u_0))$ and $\mathcal{F}(\mathcal{G}(t_0,u_0))$ admit a $\mathcal{Z}(xy-1)$–rm and $\mathbb{R}$–rm, respectively. Note that the solutions to the $\mathcal{Z}(xy-1)$–TMP and $\mathbb{R}$–TMP are Theorems 2.5 and 2.8, respectively. For the forms of $\mathcal{H}(\mathcal{G}(t_0,u_0))$, $\mathcal{F}(\mathcal{G}(t_0,u_0))$ and also in other parts of the proof of Theorem 4.1, we need the following lemma.

Lemma 4.3. *Assume the notation above. Let* $\vec{\mathcal{T}}_1 := (\vec{Y}^{(k,1)}, Y\vec{X}^{(1,k)})$ *and* $\mathcal{M}(\mathbf{Z},\mathbf{t})$ *be a function on* $S_{k+1}\times\mathbb{R}$ *defined by*

(4.13)

$$\mathcal{M}(\mathbf{Z},\mathbf{t}) = \begin{array}{cc} & \begin{array}{cc} \vec{\mathcal{T}}_1 & \qquad\qquad \vec{X}^{(0,k)} \end{array} \\ \begin{array}{c} (\vec{\mathcal{T}}_1)^T \\[2em] (\vec{X}^{(0,k)})^T \end{array} & \begin{pmatrix} F_2 & \begin{pmatrix} B_{13}^{(0,k-1)} & b_{13}^{(k)} \\ B_{23}^{(0,k-1)} & b_{23}^{(k)} \\ c^T & \mathbf{t} \end{pmatrix} \\ \begin{pmatrix} (B_{13}^{(0,k-1)})^T & (B_{23}^{(0,k-1)})^T & c \\ (b_{13}^{(k)})^T & (b_{23}^{(k)})^T & \mathbf{t} \end{pmatrix} & \mathbf{Z} \end{pmatrix} \end{array}.$$

Assume that there exists $t_0\in\mathbb{R}$ *such that* $\mathcal{M}(B_{33},t_0)\succeq 0$. *Let*

$$Z_0 := \begin{pmatrix} (B_{13}^{(0,k-1)})^T & (B_{23}^{(0,k-1)})^T & c \\ (b_{13}^{(k)})^T & (b_{23}^{(k)})^T & t_0 \end{pmatrix} (F_2)^{\dagger} \begin{pmatrix} B_{13}^{(0,k-1)} & b_{13}^{(k)} \\ B_{23}^{(0,k-1)} & b_{23}^{(k)} \\ c^T & t_0 \end{pmatrix}. \tag{4.14}$$

Then the following statements hold:

1. *(1)* $\mathcal{M}\big(Z_0,t_0\big)\succeq 0$ *and* $B_{33}-Z_0\succeq 0$.
2. *(2)* $\mathcal{M}(Z_0,t_0)$ *satisfies the column relations* $YX^{i+1}=X^i$ *for* $i=0,\ldots,k-1$. *Hence,*
$$Z_0=\mathcal{G}(t_0,u_0) \qquad \textit{for some } u_0\in\mathbb{R}.$$
3. *(3)* $\operatorname{rank}\mathcal{N}(k)=\operatorname{rank}F_1+\operatorname{rank}F_2$.

Proof. By the equivalence between (1a) and (1c) of Theorem 2.2 used for the pair $(M,A)=(\mathcal{M}(\mathbf{Z},t_0),F_2)$, Lemma 4.3.(1) follows.

Relations (4.1) and definitions of $\beta_{2k-1,2}$, $\beta_{k,k+1}$ and $\beta_{2k,2}$ imply that the restriction

$$\big(\mathcal{M}(Z_0,t_0)\big)_{\vec{\mathcal{T}}_1,(\vec{\mathcal{T}}_1,\vec{X}^{(0,k)})} = \begin{array}{cc} & \begin{array}{cc} \vec{\mathcal{T}}_1 & \quad \vec{X}^{(0,k)} \end{array} \\ (\vec{\mathcal{T}}_1)^T & \begin{pmatrix} F_2 & \begin{pmatrix} B_{13}^{(0,k-1)} & b_{13}^{(k)} \\ B_{23}^{(0,k-1)} & b_{23}^{(k)} \\ c^T & t_0 \end{pmatrix} \end{pmatrix} \end{array}$$

satisfies the relations $YX^{i+1}=X^i$ for $i=0,\ldots,k-1$. By Lemma 2.1, the restriction

$$\big(\mathcal{M}(Z_0,t_0)\big)_{\vec{X}^{(0,k)},(\vec{\mathcal{T}}_1,\vec{X}^{(0,k)})} = \begin{array}{cc} & \begin{array}{cc} \vec{\mathcal{T}}_1 & \qquad\qquad \vec{X}^{(0,k)} \end{array} \\ (\vec{X}^{(0,k)})^T & \begin{pmatrix} \begin{pmatrix} (B_{13}^{(0,k-1)})^T & (B_{23}^{(0,k-1)})^T & c \\ (b_{13}^{(k)})^T & (b_{23}^{(k)})^T & t_0 \end{pmatrix} & Z_0 \end{pmatrix} \end{array}.$$

also satisfies the relations $YX^{i+1}=X^i$ for $i=0,\ldots,k-1$, which proves Lemma 4.3.(2).

Permuting the rows and columns of $\mathcal{N}(k)$ to the order $(\vec{\mathcal{T}}_1,\vec{X}^{(0,k-1)})$, with a permutation matrix P_1, we get

$$P_1\mathcal{N}(k)(P_1)^T = \begin{pmatrix} F_2 & \begin{pmatrix} B_{13}^{(0,k-1)} \\ B_{23}^{(0,k-1)} \\ c^T \end{pmatrix} \\ \begin{pmatrix} (B_{13}^{(0,k-1)})^T & (B_{23}^{(0,k-1)})^T & c \end{pmatrix} & B_{33}^{(0,k-1)} \end{pmatrix}. \tag{4.15}$$

By Theorem 2.2.(2), used for the pair $(M,A)=(P_1\mathcal{N}(k)(P_1)^T,F_2)$, noticing that

$$\big(P_1\mathcal{N}(k)(P_1)^T\big)/F_2 = B_{33}^{(0,k-1)} - \Big((B_{23}^{(0,k-1)})^T \quad c\Big) = F_1, \tag{4.16}$$

Lemma 4.3.(3) follows. $\square$

Remark 4.4. Note that the restriction of $\mathcal{M}(B_{33},\mathbf{t})$ to $(\vec{Y}^{(k,1)}, Y\vec{X}^{(1,k-1)}, \vec{X}^{(0,k)})$ is $\widehat{\mathcal{M}}(k)_{\vec{\mathcal{T}}}$, while to $(\vec{Y}^{(k,1)}, Y\vec{X}^{(1,k)}, \vec{X}^{(0,k-1)})$ it is $P_1\mathcal{N}(k)P_1^T$ with P_1 as in (4.15).

Using Lemma 4.3, the existence of $t_0 \in \mathbb{R}$ such that $\mathcal{M}(B_{33}, t_0) \succeq 0$, implies that

$$
\mathcal{H}(\mathcal{G}(t,u)) = B_{33} - \begin{pmatrix} \begin{pmatrix} (B_{23}^{(0,k-1)})^T & c \end{pmatrix} & \begin{pmatrix} b' \\ t \end{pmatrix} \\ \begin{pmatrix} (b')^T & t \end{pmatrix} & u \end{pmatrix}
$$

$$
\begin{aligned}
&= \begin{array}{cc} & \begin{array}{cc} \vec{X}^{(0,k-1)} & X^k \end{array} \\ \begin{array}{c} (\vec{X}^{(0,k-1)})^T \\ X^k \end{array} & \begin{pmatrix} B_{33}^{(0,k-1)} - \begin{pmatrix} (B_{23}^{(0,k-1)})^T & c \end{pmatrix} & b_{33}^{(k)} - \begin{pmatrix} b' \\ t \end{pmatrix} \\ b_{33}^{(k)} - \begin{pmatrix} (b')^T & t \end{pmatrix} & \beta_{2k,0} - u \end{pmatrix} \end{array} \\
&= \begin{array}{cc} & \begin{array}{cc} \vec{X}^{(0,k-1)} & X^k \end{array} \\ \begin{array}{c} (\vec{X}^{(0,k-1)})^T \\ X^k \end{array} & \begin{pmatrix} F_1 & b_{33}^{(k)} - \begin{pmatrix} b' \\ t \end{pmatrix} \\ b_{33}^{(k)} - \begin{pmatrix} (b')^T & t \end{pmatrix} & \beta_{2k,0} - u \end{pmatrix} \end{array},
\end{aligned} \tag{4.17}
$$

where

$$
b' = \begin{pmatrix} \beta_{k+1,1} & \beta_{k+2,1} & \cdots & \beta_{2k-1,1} \end{pmatrix}^T, \tag{4.18}
$$

and

$$
\begin{aligned}
\mathcal{F}(\mathcal{G}(t,u))_{(\vec{Y}^{(k,1)},\vec{X}^{(0,k)})} &= \begin{array}{cc} & \begin{array}{ccc} \vec{Y}^{(k,1)} & \vec{X}^{(0,k-1)} & X^k \end{array} \\ \begin{array}{c} (\vec{Y}^{(k,1)})^T \\ (\vec{X}^{(0,k-1)})^T \\ X^k \end{array} & \begin{pmatrix} B_{11} & B_{13}^{(0,k-1)} & b_{13}^{(k)} \\ (B_{13}^{(0,k-1)})^T & \begin{pmatrix} (B_{23}^{(0,k-1)})^T & c \end{pmatrix} & \begin{pmatrix} b' \\ t \end{pmatrix} \\ (b_{13}^{(k)})^T & \begin{pmatrix} (b')^T & t \end{pmatrix} & u \end{pmatrix} \end{array} \\
&= \begin{array}{cc} & \begin{array}{cc} (\vec{Y}^{(k,1)}, \vec{X}^{(0,k-1)}) & X^k \end{array} \\ \begin{array}{c} (\vec{Y}^{(k,1)}, \vec{X}^{(0,k-1)})^T \\ X^k \end{array} & \begin{pmatrix} F_2 & \begin{pmatrix} b_{13}^{(k)} \\ b' \\ t \end{pmatrix} \\ \begin{pmatrix} (b_{13}^{(k)})^T & (b')^T & t \end{pmatrix} & u \end{pmatrix} \end{array}.
\end{aligned}
$$

4.1. **Proof of the implication (1) $\Rightarrow$ (2) of Theorem 4.1.** We denote by $\mathcal{M}^{(\mu)}(k+1)$ the moment matrix associated to the sequence generated by some finitely atomic $\mathcal{Z}(p)$–rm μ for β, which exists by [Ric57]. The following statements hold:

- The moment matrix $\mathcal{M}^{(\mu)}(k+1)$ is psd.
- The extension of $\widehat{\mathcal{M}}(k)_{\vec{\mathcal{T}}\setminus\{X^k\}}$ with a row and column YX^k is equal to the matrix $\mathcal{N}(k)$ due to the relation $Y^2X^k = YX^{k-1}$, which is satisfied by the moment matrix $\mathcal{M}^{(\mu)}(k+1)$.
- The matrix $\mathcal{N}(k)$ is psd as the restriction of $\mathcal{M}^{(\mu)}(k+1)$.

We separate two subcases.

Case 1: $\widehat{\mathcal{M}}(k)_{\vec{\mathcal{T}}}$ and $\mathcal{N}(k)$ are positive definite. This is Theorem 4.1.(2a).

Case 2: At least one of $\widehat{\mathcal{M}}(k)_{\vec{\mathcal{T}}}$ and $\mathcal{N}(k)$ is not positive definite. The restriction $\mathcal{M}^{(\mu)}(k+1)_{(\vec{\mathcal{T}},YX^k)}$ is of the form (see (4.5), (4.8))

$$\mathcal{M}^{(\mu)}(k+1)_{(\vec{\mathcal{T}},YX^k)} = \begin{pmatrix} R & m_{12} & n_{12} \\ (m_{12})^T & \beta_{2k,0} & \beta_{2k,1}(\mu) \\ (n_{12})^T & \beta_{2k,1}(\mu) & \beta_{2k-1,1} \end{pmatrix}.$$

Claim 1. $\beta_{2k,1}(\mu) = t'$, where t' is as in (4.10).

Proof of Claim 1. By definition of t' and by Lemma 2.3, used for

$$A(\mathbf{x}) := \begin{pmatrix} R & m_{12} & n_{12} \\ (m_{12})^T & \beta_{2k,0} & \mathbf{x} \\ (n_{12})^T & \mathbf{x} & \beta_{2k-1,1} \end{pmatrix},$$

we have $A(t') \succeq 0$. We separate two cases according to invertibility of $\mathcal{N}(k)$.

Case (i): $\mathcal{N}(k)$ is invertible. It follows that $\operatorname{rank} \widehat{\mathcal{M}}(k)_{\vec{\mathcal{T}}} < \operatorname{rank} \mathcal{N}(k)$ and hence by Lemma 2.3, there is no other $t \in \mathbb{R}$ except t' such that $A(t) \succeq 0$. Hence, $\beta_{2k,1}(\mu)$ must be equal to t'.

Case (ii): $\mathcal{N}(k)$ is singular. The singularity of $\mathcal{N}(k)$ implies, by Lemma 4.3.(3), that

(4.19) at least one of the matrices F_1 and F_2 is singular.

Let $\mathcal{M}(\mathbf{Z}, \mathbf{t})$ be as in (4.13). Note that $\mathcal{M}(B_{33}, \mathbf{t})$ is obtained by permuting rows and columns of $A(\mathbf{t})$. In particular, $\mathcal{M}(B_{33}, t') \succeq 0$. Now let t_0 be any real number such that $\mathcal{M}(B_{33}, t_0) \succeq 0$. By Lemma 4.3, it follows that

$$\mathcal{M}(Z_0, t_0) \succeq 0 \quad \text{and} \quad B_{33} - Z_0 \succeq 0 \tag{4.20}$$

for Z_0 as in (4.14), and

$$B_{33} - Z_0 = \mathcal{H}(\mathcal{G}(t_0, u_0)) = \begin{pmatrix} F_1 & \begin{pmatrix} * \\ t_0 \\ * \end{pmatrix} \\ \begin{pmatrix} * & t_0 \end{pmatrix} & * \end{pmatrix}. \tag{4.21}$$

Now we separate possible cases in (4.19).

Case (ii).(I): F_1 is singular. Since $B_{33} - Z_0$ is psd by (4.20) and has the form (4.21), it follows, by [CF91, Theorem 2.4(ii)], that t_0 is uniquely determined by F_1. Hence, Claim 1 holds in this case.

Case (ii).(II): F_2 is singular. The restriction of $\mathcal{M}(Z_0, t_0)$ to the rows and columns in $(\vec{\mathcal{T}}_1, X^k)$ is a psd Hankel matrix of the form

$$\mathcal{M}(Z_0, t_0)_{(\vec{\mathcal{T}}_1, X^k)} = \begin{pmatrix} F_2 & \begin{pmatrix} * \\ t_0 \\ * \end{pmatrix} \\ \begin{pmatrix} * & t_0 \end{pmatrix} & * \end{pmatrix}. \tag{4.22}$$

As in the Case (ii).(I), [CF91, Theorem 2.4(ii)] implies that t_0 is uniquely determined by F_2. Hence, Claim 1 holds also in this case. ∎

As explained in the paragraph following the statement of Theorem 4.1, there exist $t_0, u_0 \in \mathbb{R}$ such that $\mathcal{F}(\mathcal{G}(t_0, u_0))$ and $\mathcal{H}(\mathcal{G}(t_0, u_0))$ admit a $\mathcal{Z}(xy-1)$–rm and a $\mathbb{R}$–rm, respectively. By Claim 1, we have that $t_0 = t'$. Note that the right-lower corner of Z_0 is precisely u'' (see (4.10)).

By definitions (4.10) of u' and u'', u' is the largest number such that $\mathcal{H}(\mathcal{G}(t',u')) \succeq 0$ and u'' is the smallest number such that $\mathcal{F}(\mathcal{G}(t',u'')) \succeq 0$. In particular, $u'' \leq u'$ and $u_0 \in [u'',u']$. Note that $\mathcal{H}(\mathcal{G}(t',u')) = A_{\gamma_1(t',u')}$ admits a $\mathbb{R}$–rm by Theorem 2.5, since the last column is in the span of the previous ones. We have

$$\begin{aligned}\mathcal{F}(\mathcal{G}(t',u'))_{\vec{\mathcal{T}}} &= \mathcal{F}\big(\mathcal{G}(t',u_0)+(u'-u_0)E^{(k+1)}_{1,1}\big)_{\vec{\mathcal{T}}}\\ &= \mathcal{F}(\mathcal{G}(t',u_0))_{\vec{\mathcal{T}}}+(u'-u_0)E^{(3k)}_{2k+1,2k+1}\\ &\succeq \mathcal{F}(\mathcal{G}(t',u_0))_{\vec{\mathcal{T}}}.\end{aligned} \tag{4.23}$$

Since $\mathcal{F}(\mathcal{G}(t',u_0))$ admits a $\mathcal{Z}(xy-1)$–rm and $\mathcal{F}(\mathcal{G}(t',u_0))_{(\vec{Y}^{(k,1)},\vec{X}^{(0,k)})} = A_{\gamma_2(t',u_0)}$, it follows that $\gamma_2(t',u_0)$ is $(\mathbb{R}\setminus\{0\})$–representable. From now on we separate two cases according to the invertibility of F_2.

Case 2.1: F_2 is invertible. We separate two cases according to the invertibility of $A_{\gamma_2(t',u_0)}$.

Case 2.1.1: $A_{\gamma_2(t',u_0)} \succ 0$. It follows that $\operatorname{rank}\mathcal{F}(\mathcal{G}(t',u_0))_{(\vec{Y}^{(k,1)},\vec{X}^{(0,k)})} = 2k+1$ and hence $\operatorname{rank}\mathcal{F}(\mathcal{G}(t',u'))_{(\vec{Y}^{(k,1)},\vec{X}^{(0,k)})} = 2k+1$ by (4.23). By Theorem 2.6, $\gamma_2(t',u')$ is $(\mathbb{R}\setminus\{0\})$–representable.

Case 2.1.2: $A_{\gamma_2(t',u_0)} \succeq 0$ and $A_{\gamma_2(t',u_0)} \not\succ 0$. It follows that $A_{\gamma_2(t',u_0)}$ satisfies Theorem 2.6.(3b) and hence

$$2k = \operatorname{rank}\mathcal{F}(\mathcal{G}(t',u_0))_{(\vec{Y}^{(k,1)},\vec{X}^{(0,k-1)})} = \operatorname{rank}\mathcal{F}(\mathcal{G}(t',u_0))_{(\vec{Y}^{(k-1,1)},\vec{X}^{(0,k)})}, \tag{4.24}$$

where we also used invertibility of F_2 in the first equality. If $u_0 = u'$, (4.24) implies that $\gamma_2(t',u')$ is $(\mathbb{R}\setminus\{0\})$–representable. Otherwise $u' > u_0$ and (4.23), (4.24) imply that

$$\operatorname{rank}\mathcal{F}(\mathcal{G}(t',u'))_{(\vec{Y}^{(k,1)},\vec{X}^{(0,k)})} = 2k+1,$$

which again implies that $\gamma_2(t',u')$ is $(\mathbb{R}\setminus\{0\})$–representable by Theorem 2.6.

Case 2.2: F_2 is singular. If $\gamma_2(t',u_0)$ is $(\mathbb{R}\setminus\{0\})$–representable, then in particular it is $\mathbb{R}$–representable. But due to singularity of F_2, u'' is the only candidate for u_0 by Theorem 2.5. Hence, $\gamma_1(t',u'')$ is also $\mathbb{R}$–representable.

This concludes the proof of the implication (1) $\Rightarrow$ (2) of Theorem 4.1.

4.2. **Proof of the implication (2) $\Rightarrow$ (1) of Theorem 4.1.** We separate two cases according to the assumptions in (2).

Case 1: (2a) of Theorem 4.1 holds. By Lemma 2.3, used for $A(\mathbf{x}) = \mathcal{M}(B_{33},\mathbf{x})$ (as in (4.13)), there exist $t_\ell \in \mathbb{R}$, $\ell = 1,2$, such that (see also Remark 4.4):

$$\begin{aligned}&\mathcal{M}(B_{33},t) \succeq 0 \quad \text{for every } t\in[t_1,t_2],\\ &\operatorname{rank}\mathcal{M}(B_{33},t_\ell) = \operatorname{rank}\mathcal{M}(k) = \operatorname{rank}\mathcal{N}(k) \quad \text{for } \ell=1,2,\\ &\operatorname{rank}\mathcal{M}(B_{33},t) = \operatorname{rank}\mathcal{M}(k)+1 = \operatorname{rank}\mathcal{N}(k)+1 \quad \text{for } t\in(t_1,t_2).\end{aligned} \tag{4.25}$$

Let $t_0 \in [t_1,t_2]$. By Lemma 4.3.(1), we have $\mathcal{M}(Z_0,t_0) \succeq 0$, where Z_0 is as in (4.14). By Theorem 2.2.(2), used for the pair $(M,A) = \big(\mathcal{M}(Z_0,t_0),F_2\big)$, we have

$$\operatorname{rank}\mathcal{M}(Z_0,t_0) = \operatorname{rank}F_2. \tag{4.26}$$

By Lemma 4.3.(2),

$$\mathcal{C}\big(\mathcal{M}(Z_0,t_0)\big) = \mathcal{C}\big(\mathcal{M}(Z_0,t_0)_{(\vec{Y}^{(k,1)},Y\vec{X}^{(1,k)})}\big) = \mathcal{C}\big(\mathcal{M}(Z_0,t_0)_{(\vec{Y}^{(k,1)},\vec{X}^{(0,k-1)})}\big),$$

which implies that

$$\operatorname{rank}\left(\mathcal{M}(Z_0,t_0)_{(\vec{Y}^{(k,1)},\vec{X}^{(0,k-1)})}\right)=\operatorname{rank}\mathcal{M}(Z_0,t_0). \tag{4.27}$$

By Lemma 4.3.(2), it follows that $Z_0=\mathcal{G}(t_0,u(t_0))$ for some $u(t_0)\in\mathbb{R}$ and by Lemma 4.3.(1), $B_{33}\succeq Z_0$. Hence, $\mathcal{H}(Z_0)\succeq 0$. By (4.16),

$$F_1=B_{33}^{(0,k-1)}-\begin{pmatrix}(B_{23}^{(0,k-1)})^T & c\end{pmatrix}\succ 0.$$

By the equivalence between (1a) and (1c) of Theorem 2.2, used for the pair $\big(\mathcal{H}(\mathcal{G}(t_0,u(t_0))),F_1\big)$ (see (4.17) for the 2×2 block decomposition of $\mathcal{H}(\mathcal{G}(t_0,u(t_0)))$, it follows that

$$\delta_0:=(\beta_{2k,0}-u(t_0))-\left(b_{33}^{(k)}-\begin{pmatrix}b'\\ t_0\end{pmatrix}\right)^T(F_1)^{-1}\left(b_{33}^{(k)}-\begin{pmatrix}b'\\ t_0\end{pmatrix}\right)\geq 0 \tag{4.28}$$

and

$$\operatorname{rank}\mathcal{H}\big(\mathcal{G}(t_0,u(t_0))\big)=\begin{cases}\operatorname{rank}F_1, & \text{if } \delta_0=0,\\ \operatorname{rank}F_1+1, & \text{if } \delta_0>0.\end{cases}=\begin{cases}k, & \text{if } \delta_0=0,\\ k+1, & \text{if } \delta_0>0.\end{cases} \tag{4.29}$$

By Theorem 2.2.(2), used for the pair $(M,A)=\big(\mathcal{M}(B_{33},t_0),F_2\big)$, we have

$$\operatorname{rank}\mathcal{M}(B_{33},t_0)=\operatorname{rank}F_2+\operatorname{rank}\mathcal{H}\big(\mathcal{G}((t_0,u(t_0)))\big). \tag{4.30}$$

Since $\operatorname{rank}F_2=2k$ by the invertibility of $\mathcal{N}(k)$ and $\operatorname{rank}\mathcal{M}(B_{33},t_\ell)=3k$, $\ell=1,2$, by (4.25), it follows that

$$\operatorname{rank}\mathcal{H}\big(\mathcal{G}(t_\ell,u(t_\ell))\big)=k,\quad \ell=1,2. \tag{4.31}$$

Note that $A_{\gamma_1(t_\ell,u(t_\ell))}=\mathcal{H}\big(\mathcal{G}(t_\ell,u(t_\ell))\big)$, $\ell=1,2$, where $\gamma_1(\mathbf{t},\mathbf{u})$ is as in (4.11). By (4.29) and (4.31), $\operatorname{rank}A_{\gamma_1(t_\ell,u(t_\ell))}(k-1)=\operatorname{rank}A_{\gamma_1(t_\ell,u(t_\ell))}$, $\ell=1,2$. By Theorem 2.5, $\gamma_1(t_\ell,u(t_\ell))$, $\ell=1,2$, has a k–atomic $\mathbb{R}$–rm. Note that in $\mathcal{F}\big(\mathcal{G}(t_\ell,u(t_\ell))\big)$, $\ell=1,2$,

$$X^k\in\operatorname{span}\{Y^k,Y^{k-1},\ldots,Y,\mathit{1},X,X^2,\ldots,X^{k-1}\}. \tag{4.32}$$

Further,

$$\mathcal{F}(\mathcal{G}(t_\ell,u(t_\ell)))_{(\vec{Y}^{(k,1)},\vec{X}^{(0,k)})}=A_{\gamma_2(t_\ell,u(t_\ell))},$$

where $\gamma_2(\mathbf{t},\mathbf{u})$ is as in (4.11). Since (4.32) holds, the sequences $\gamma_2(t_\ell,u(t_\ell))$, $\ell=1,2$, are $\mathbb{R}$–representable. Since

$$F_2=A_{\gamma_2(t_1,u(t_1))}(2k-1)=A_{\gamma_2(t_2,u(t_2))}(2k-1)$$

is invertible, by [Zal22b, Proposition 2.5], at least one of $\gamma_2(t_1,u(t_1))$ or $\gamma_2(t_2,u(t_2))$ is $(\mathbb{R}\setminus\{0\})$–representable. By Lemma 3.1, β has a $(3k)$–atomic $\mathcal{Z}(p)$–rm, which concludes the proof of the implication (2) $\Rightarrow$ (1) in this case.

Case 2: (2b) of Theorem 4.1 holds. By Lemma 2.3, used for $A(\mathbf{x})=\mathcal{M}(B_{33},\mathbf{x})$, for t' as in (4.10), we have $\mathcal{M}(B_{33},t')\succeq 0$ and

$$\operatorname{rank}\mathcal{M}(B_{33},t')=\max\big(\operatorname{rank}\widehat{\mathcal{M}}(k),\operatorname{rank}\mathcal{N}(k)\big).$$

By Lemma 4.3, it follows that $A_{\gamma_2(t',u'')}=\mathcal{F}(\mathcal{G}(t',u''))_{(\vec{Y}^{(k,1)},\vec{X}^{(0,k)})}$ and $A_{\gamma_1(t',u'')}=\mathcal{H}\big(\mathcal{G}(t',u'')\big)$ are psd, and

$$\operatorname{rank}\mathcal{M}(B_{33},t')=\operatorname{rank}A_{\gamma_2(t',u'')}+\operatorname{rank}A_{\gamma_1(t',u'')}. \tag{4.33}$$

We separate two subcases according to the invertibility of F_2.

Case 2.1: F_2 is invertible. Note that $\mathcal{G}(t',u'')$ is equal to Z_0 from (4.14) with $t_0=t'$. By definition (4.10), u' is the largest such that $\mathcal{H}(\mathcal{G}(t',u'))\succeq 0$. Thus, $u'\geq u''$. We have

$\mathcal{F}(\mathcal{G}(t',u')) \succeq \mathcal{F}(\mathcal{G}(t',u''))$ (see the inequality (4.23) above) and

$$\begin{aligned}\operatorname{rank}\mathcal{F}(\mathcal{G}(t',u')) &= \begin{cases} \operatorname{rank}\mathcal{F}(\mathcal{G}(t',u'')), & \text{if } u'=u'',\\ \operatorname{rank}\mathcal{F}(\mathcal{G}(t',u''))+1, & \text{if } u'>u'',\end{cases}\\ &= \begin{cases} 2k, & \text{if } u'=u'',\\ 2k+1, & \text{if } u'>u'',\end{cases}\end{aligned} \tag{4.34}$$

where we used the fact that F_2 is invertbile in the second equality. Note that

$$A_{\gamma_2(t',u)} = \mathcal{F}(\mathcal{G}(t',u))_{(\vec{Y}^{(k,1)},\vec{X}^{(0,k)})}.$$

If $u' = u''$, then by definition of u'', we also have $\operatorname{rank} A_{\gamma_2(t',u')}(2k-1) = \operatorname{rank} A_{\gamma_2(t',u')}$. Otherwise $u' > u''$ and $\operatorname{rank} A_{\gamma_2(t',u')} = 2k+1$. Since $A_{\gamma_1(t',u')} = \mathcal{H}(\mathcal{G}(t',u'))$ satisfies the equality $\operatorname{rank} A_{\gamma_1(t',u')} = \operatorname{rank} A_{\gamma_1(t',u')}(k-1)$ by definition of u', it admits a $(\operatorname{rank} A_{\gamma_1(t',u')})$–atomic $\mathbb{R}$–rm by Theorem 2.5. Using (4.33) and in the case $u' > u''$ also rank equalities $\operatorname{rank} A_{\gamma_2(t',u'')} = \operatorname{rank} A_{\gamma_2(t',u')} - 1$ (by (4.34)) and $\operatorname{rank} A_{\gamma_1(t',u'')} = \operatorname{rank} A_{\gamma_1(t',u')} + 1$ (by definition of u'), it follows that β admits a $(\operatorname{rank}\mathcal{M}(B_{33},t'))$–atomic $\mathcal{Z}(p)$–rm. This proves the implication (2) $\Rightarrow$ (1) in this case.

Case 2.2: F_2 is singular. Note that $A_{\gamma_2(t',u'')} = \mathcal{F}(\mathcal{G}(t',u''))_{(\vec{Y}^{(k,1)},\vec{X}^{(0,k)})}$ satisfies the equality $\operatorname{rank} A_{\gamma_2(t',u'')}(2k-1) = \operatorname{rank} A_{\gamma_2(t',u'')}$ by definition of u''. Moreover, $\gamma_1(t',u'')$ admits a $(\operatorname{rank} A_{\gamma_1(t',u'')})$–atomic $\mathbb{R}$–rm. Since also $\operatorname{rank}\mathcal{M}(B_{33},t') = \operatorname{rank}\mathcal{F}(\mathcal{G}(t',u'')) + \operatorname{rank} A_{\gamma_1(t',u'')}$, it follows that β admits a $(\operatorname{rank}\mathcal{M}(B_{33},t'))$–atomic $\mathcal{Z}(p)$–rm.

This concludes the proof of the implication (2) $\Rightarrow$ (1) of Theorem 4.1. Note also that the moreover part of the theorem follows from the proof of this implication.

4.3. **Example.** [1] The sequence β is said to be p**–purely pure**, if it is p–pure and also the matrix $\mathcal{N}(k)$ is invertible. By Theorem 4.1.(2a), a p–purely pure sequence β, such that $\mathcal{M}(k;\beta)$ and $\mathcal{N}(k)$ are psd, admits a $\mathcal{Z}(p)$–rm. The following example shows that, in contrary to the TMP on the union of three parallel lines [Zal22a], in this hyperbolic type case, a p–pure sequence β, such that $\mathcal{M}(k;\beta)$ and $\mathcal{N}(k)$ are psd, does not necessarily admit a $\mathcal{Z}(p)$–rm.

Let β be a bivariate degree 6 sequence given by

$\beta_{00} = 1,$
$\beta_{10} = \frac{3}{4}, \beta_{01} = 0$
$\beta_{20} = 3, \beta_{11} = \frac{1}{2}, \beta_{02} = \frac{5}{16},$
$\beta_{30} = \frac{9}{2}, \beta_{21} = 0, \beta_{12} = 0, \beta_{03} = 0,$
$\beta_{40} = \frac{17}{64}, \beta_{31} = \frac{5}{4}, \beta_{22} = \frac{1}{2}, \beta_{13} = \frac{5}{16}, \beta_{04} = \frac{17}{64},$
$\beta_{50} = \frac{69}{2}, \beta_{41} = 0, \beta_{32} = 0, \beta_{23} = 0, \beta_{14} = 0, \beta_{05} = 0,$
$\beta_{60} = \frac{231}{2}, \beta_{51} = \frac{17}{4}, \beta_{42} = \frac{5}{4}, \beta_{33} = \frac{1}{2}, \beta_{24} = \frac{5}{16}, \beta_{15} = \frac{17}{64}, \beta_{06} = \frac{81}{256}.$

We will prove below that β does not have a $\mathbb{R}^2$–rm. It is easy to check that $\widehat{\mathcal{M}}(3)$ is psd and satisfies only one column relation $Y^2X = Y$, while the matrix $\mathcal{N}(3)$ is psd and has only one column relation $YX^3 = 5YX - 4Y^2$. The sequences $\gamma_1(\mathbf{t},\mathbf{u})$ and $\gamma_2(\mathbf{t},\mathbf{u})$ (see (4.11)) are

[1] The *Mathematica* file with numerical computations can be found on the link https://github.com/ZalarA/TMP_cubic_reducible.

equal to

$$\gamma_1(\mathbf{t},\mathbf{u}) = \Big(\frac{1}{2},\frac{3}{4},\frac{7}{4},\frac{9}{2},\frac{49}{2},\frac{69}{2}-\mathbf{t},\frac{231}{2},-\mathbf{u}\Big),$$
$$\gamma_2(\mathbf{t},\mathbf{u}) = \Big(\frac{81}{256},0,\frac{17}{64},0,\frac{5}{16},0,\frac{1}{2},0,\frac{5}{4},0,\frac{17}{4},\mathbf{t},\mathbf{u}\Big).$$

Computing t', u', u'' (see (4.10)) we get $t'=0$, $u'=\frac{659}{40}$ and $u''=\frac{65}{4}$. Since $A_{\gamma_2(t',u')}$ satisfies $\operatorname{rank} A_{\gamma_2(t',u')} = 6$ and $\operatorname{rank} A_{\gamma_2(t',u')}(6) = \operatorname{rank} A_{\gamma_2(t',u')}[6] = 5$, Theorem 2.6 implies that $\gamma_2(t',u')$ is not $(\mathbb{R}\setminus\{0\})$–representable. Since $A_{\gamma_2(t',u'')}$ satisfies $\operatorname{rank} A_{\gamma_2(t',u'')} = 5$ and $\operatorname{rank} A_{\gamma_2(t',u'')}[6] = 4$, Theorem 2.6 implies that $\gamma_2(t',u'')$ is not $(\mathbb{R}\setminus\{0\})$–representable. So neither of $\gamma_2(t',u')$ or $\gamma_2(t',u'')$ is $(\mathbb{R}\setminus\{0\})$–representable, which implies, by Theorem 4.1, that β does not admit a $\mathcal{Z}(p)$–rm.

5. Hyperbolic type 2 relation: $p(x,y) = y(x+y-xy)$

In this section we solve constructively the $\mathcal{Z}(p)$–TMP for the sequence $\beta = \{\beta_{i,j}\}_{i,j\in\mathbb{Z}_+,i+j\leq 2k}$ of degree $2k$, $k\geq 3$, where $p(x,y)$ is as in the title of the section. The main results are Theorem 5.2, which characterizes concrete numerical conditions for the existence of a $\mathcal{Z}(p)$–rm for β and Theorem 5.6, which characterizes the number of atoms needed in a minimal $\mathcal{Z}(p)$–rm. A numerical example demonstrating the main results is presented in Subsection 5.3.

Remark 5.1. In the classification from [YZ24, Proposition 3.1], in the hyperbolic type 2 relation, $c(x,y)$ is equal to $x+y+axy$, $a\in\mathbb{R}\setminus\{0\}$. However, after applying an affine linear transformation (see Subsection 2.2) $\phi(x,y) = (-ax,-ay)$ we can assume that $a=-1$.

5.1. **Existence of a representing measure.** Assume the notation from Section 3. If β admits a $\mathcal{Z}(p)$–TMP, then $\mathcal{M}(k;\beta)$ must satisfy the relations

$$Y^{2+j}X^{1+i} = Y^{1+j}X^{1+i} + Y^{2+j} \quad \text{for } i,j\in\mathbb{Z}_+ \text{ such that } i+j\leq k-3. \tag{5.1}$$

On the level of moments the relations (5.2) mean that

$$\beta_{i+1,j+2} = \beta_{i+1,j+1} + \beta_{i,2+j} \quad \text{for } i,j\in\mathbb{Z}_+ \text{ such that } i+j\leq 2k-3. \tag{5.2}$$

In the presence of all column relations (5.2), the column space $\mathcal{C}(\mathcal{M}(k;\beta))$ is spanned by the columns in the tuple

$$\vec{\mathcal{T}} := (\underbrace{Y^k, Y^{k-1},\ldots,Y}_{\vec{Y}^{(k,1)}}, \underbrace{YX-Y, YX^2-YX,\ldots,YX^{k-1}-YX^{k-2}}_{\vec{\mathcal{T}}_2}, \vec{X}^{(0,k)}). \tag{5.3}$$

where $\vec{X}^{(i,j)} := (X^i, X^{i+1},\ldots,X^j)$, $0\leq i\leq j\leq k$ and $X^0 := \mathit{1}$. Let

> P be a permutation matrix such that moment matrix $\widehat{\mathcal{M}}(k) := P\mathcal{M}(k;\beta)P^T$ has rows and columns indexed in the order $\vec{\mathcal{T}}, \vec{\mathcal{C}}\setminus\vec{\mathcal{T}}$.

Let $\widehat{\mathcal{M}}(k)_{\vec{\mathcal{T}}}$ be the restriction of the moment matrix $\widehat{\mathcal{M}}(k)$ to the rows and columns in the tuple $\vec{\mathcal{T}}$:

$$\widehat{\mathcal{M}}(k)_{\vec{\mathcal{T}}} := \begin{array}{c} \\ (\vec{Y}^{(k,1)})^T \\ (\vec{\mathcal{T}}_2)^T \\ (\vec{X}^{(0,k)})^T \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \vec{Y}^{(k,1)} & \vec{\mathcal{T}}_2 & \vec{X}^{(0,k)} \end{array} \\ \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ (B_{12})^T & B_{22} & B_{23} \\ (B_{13})^T & (B_{23})^T & B_{33} \end{pmatrix} \end{array}$$

$$
= \begin{array}{cc}
 & \begin{array}{cccc} \vec{Y}^{(k,1)} & \vec{\mathcal{T}}_2 & \vec{X}^{(0,k-1)} & X^k \end{array} \\
\begin{array}{c} (\vec{Y}^{(k,1)})^T \\ (\vec{\mathcal{T}}_2)^T \\ (\vec{X}^{(0,k-1)})^T \\ X^k \end{array} &
\begin{pmatrix}
B_{11} & B_{12} & B_{13}^{(0,k-1)} & b_{13}^{(k)} \\
(B_{12})^T & B_{22} & B_{23}^{(0,k-1)} & b_{23}^{(k)} \\
(B_{13}^{(0,k-1)})^T & (B_{23}^{(0,k-1)})^T & (B_{33}^{(0,k-1)}) & b_{33}^{(k)} \\
(b_{13}^{(k)})^T & (b_{23}^{(k)})^T & (b_{33}^{(k)})^T & \beta_{2k,0}
\end{pmatrix}
\end{array}
$$

$$
= \begin{array}{cc}
 & \begin{array}{ccccc} \vec{Y}^{(k,1)} & \vec{\mathcal{T}}_2 & \mathit{1} & \vec{X}^{(1,k-1)} & X^k \end{array} \\
\begin{array}{c} (\vec{Y}^{(k,1)})^T \\ (\vec{\mathcal{T}}_2)^T \\ \mathit{1} \\ (\vec{X}^{(1,k-1)})^T \\ X^k \end{array} &
\begin{pmatrix}
B_{11} & B_{12} & b_{13}^{(0)} & B_{13}^{(1,k-1)} & b_{13}^{(k)} \\
(B_{12})^T & B_{22} & b_{23}^{(0)} & B_{23}^{(1,k-1)} & b_{23}^{(k)} \\
(b_{13}^{(0)})^T & (b_{23}^{(0)})^T & \beta_{0,0} & (b_{33;0}^{(1,k-1)})^T & \beta_{k,0} \\
(B_{13}^{(1,k-1)})^T & (B_{23}^{(1,k-1)})^T & b_{33;0}^{(1,k-1)} & B_{33}^{(1,k-1)} & b_{33;k}^{(1,k-1)} \\
(b_{13}^{(k)})^T & (b_{23}^{(k)})^T & \beta_{k,0} & (b_{33;k}^{(1,k-1)})^T & \beta_{2k,0}
\end{pmatrix}
\end{array}.
$$

Let $\widetilde{\mathcal{M}}(k;\beta)$ be as in (3.6) with $\vec{\mathcal{T}}_1 := (\vec{Y}^{(k,1)}, \vec{\mathcal{T}}_2)$ and define

$$
A_{\min} := A_{12}(A_{22})^{\dagger}(A_{12})^T \quad \text{and} \quad \widehat{A}_{\min} := A_{\min} + \eta\big(E_{1,k+1}^{(k+1)} + E_{k+1,1}^{(k+1)}\big), \tag{5.4}
$$

where $\eta := (A_{\min})_{2,k} - (A_{\min})_{1,k+1}$. See Remark 3.4 for the explanation of these definitions. Let $\mathcal{F}(\mathbf{A})$ and $\mathcal{H}(\mathbf{A})$ be as in (3.9). Define the matrix function

$$
\mathcal{G} : [0,\infty)^2 \to S_{k+1}, \qquad \mathcal{G}(\mathbf{t},\mathbf{u}) = \widehat{A}_{\min} + \mathbf{t}E_{1,1}^{(k+1)} + \mathbf{u}E_{k+1,k+1}^{(k+1)}. \tag{5.5}
$$

Next we define the sequences $\gamma_1(\mathbf{t},\mathbf{u})$, $\gamma_2(\mathbf{t},\mathbf{u})$:

$$
\begin{aligned}
\gamma_1(\mathbf{t},\mathbf{u}) := \big(&\beta_{0,0} - (A_{\min})_{1,1} - \mathbf{t}, \beta_{1,0} - \beta_{1,1} + \beta_{0,1}, \beta_{2,0} - \beta_{2,1} + \beta_{1,1}, \ldots, \\
&\beta_{2k-1,0} - \beta_{2k-1,1} + \beta_{2k-2,1}, \beta_{2k,0} - (A_{\min})_{k+1,k+1} - \mathbf{u}\big), \\
\gamma_2(\mathbf{t},\mathbf{u}) := \big(&(A_{\min})_{1,1} + \mathbf{t}, \beta_{1,1} - \beta_{0,1}, \beta_{2,1} - \beta_{1,1}, \ldots, \beta_{2k-1,1} - \beta_{2k-2,1}, \\
&(A_{\min})_{k+1,k+1} + \mathbf{u}\big).
\end{aligned} \tag{5.6}
$$

Observe that

(5.7)

$$
\mathcal{H}(\mathcal{G}(\mathbf{t},\mathbf{u})) =
$$

$$
= \begin{array}{cc}
 & \begin{array}{ccc} \mathit{1} & \vec{X}^{(1,k-1)} & X^k \end{array} \\
\begin{array}{c} \mathit{1} \\ (\vec{X}^{(1,k-1)})^T \\ X^k \end{array} &
\begin{pmatrix}
\beta_{0,0} - (A_{\min})_{1,1} - \mathbf{t} & (b_{33;0}^{(1,k-1)})^T - (b_{23}^{(0)})^T & \beta_{k,0} - \beta_{k,1} + \beta_{k-1,1} \\
b_{33;0}^{(1,k-1)} - b_{23}^{(0)} & B_{33}^{(1,k-1)} - B_{23}^{(1,k-1)} & b_{33;k}^{(1,k-1)} - b_{23}^{(k)} \\
\beta_{k,0} - \beta_{k,1} + \beta_{k-1,1} & (b_{33;k}^{(1,k-1)})^T - (b_{23}^{(k)})^T & \beta_{2k,0} - (A_{\min})_{k+1,k+1} - \mathbf{u}
\end{pmatrix}
\end{array}
$$

$$
= A_{\gamma_1(\mathbf{t},\mathbf{u})}
$$

and

$$\mathcal{F}(\mathcal{G}(\mathbf{t},\mathbf{u}))_{(\vec{Y}^{(k,1)},\vec{X}^{(0,k)})}$$

$$(5.8)\qquad = \begin{array}{c} \\ (\vec{Y}^{(k,1)})^T \\ \mathit{1} \\ (\vec{X}^{(1,k-1)})^T \\ X^k \end{array} \begin{array}{c} \begin{array}{cccc} \vec{Y}^{(k,1)} & \mathit{1} & \vec{X}^{(1,k-1)} & X^k \end{array} \\ \begin{pmatrix} B_{11} & b_{13}^{(0)} & B_{13}^{(1,k-1)} & b_{13}^{(k)} \\ (b_{13}^{(0)})^T & (A_{\min})_{1,1}+\mathbf{t} & (b_{23}^{(0)})^T & \beta_{k,1}-\beta_{k-1,1} \\ (B_{13}^{(1,k-1)})^T & b_{23}^{(0)} & B_{23}^{(1,k-1)} & b_{23}^{(k)} \\ (b_{13}^{(k)})^T & \beta_{k,1}-\beta_{k-1,1} & (b_{23}^{(k)})^T & (A_{\min})_{k+1,k+1}+\mathbf{u} \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \\ (\vec{Y}^{(k,1)})^T \\ (\vec{X}^{(0,k)})^T \end{array} \begin{array}{c} \begin{array}{cc} \vec{Y}^{(k,1)} & \vec{X}^{(0,k)} \end{array} \\ \begin{pmatrix} B_{11} & B_{13} \\ (B_{13})^T & A_{\gamma_2(\mathbf{t},\mathbf{u})} \end{pmatrix} \end{array}.$$

By Lemmas 3.1–3.3 and Remark 3.4, the existence of a $\mathcal{Z}(p)$–rm for β is equivalent to:

$$(5.9)\qquad \begin{aligned} &\widetilde{\mathcal{M}}(k;\beta)\succeq 0, \text{ the relations (5.2) hold and} \\ &\text{there exists } (\tilde{t}_0,\tilde{u}_0)\in\mathbb{R}^2 \text{ such that } \mathcal{F}(\mathcal{G}(\tilde{t}_0,\tilde{u}_0)) \text{ and } \mathcal{H}(\mathcal{G}(\tilde{t}_0,\tilde{u}_0)) \\ &\text{admit a } \mathcal{Z}(x+y-xy)\text{–rm and a } \mathbb{R}\text{–rm, respectively.} \end{aligned}$$

We also write

$$\mathcal{H}(\widehat{A}_{\min}) =$$

$$(5.10)\qquad = \begin{array}{c} \\ \mathit{1} \\ \vec{X}^{(1,k-1)} \\ X^k \end{array} \begin{array}{c} \begin{array}{ccc} \mathit{1} & \vec{X}^{(1,k-1)} & X^k \end{array} \\ \begin{pmatrix} \beta_{0,0}-(A_{\min})_{1,1} & (b_{33;0}^{(1,k-1)})^T-(b_{23}^{(0)})^T & \beta_{k,0}-(A_{\min})_{k+1,1} \\ b_{33;0}^{(1,k-1)}-b_{23}^{(0)} & B_{33}^{(1,k-1)}-B_{23}^{(1,k-1)} & b_{33;k}^{(1,k-1)}-b_{23}^{(k)} \\ \beta_{k,0}-(A_{\min})_{k+1,1} & (b_{33;k}^{(1,k-1)})^T-(b_{23}^{(k)})^T & \beta_{2k,0}-(A_{\min})_{k+1,k+1} \end{pmatrix} \end{array}$$

$$=: \begin{array}{c} \\ \mathit{1} \\ (\vec{X}^{(1,k-1)})^T \\ X^k \end{array} \begin{array}{c} \begin{array}{ccc} \mathit{1} & \vec{X}^{(1,k-1)} & X^k \end{array} \\ \begin{pmatrix} \beta_{0,0}-(A_{\min})_{1,1} & (h_{12})^T & \beta_{k,0}-(A_{\min})_{k+1,1} \\ h_{12} & H_{22} & h_{23} \\ \beta_{k,0}-(A_{\min})_{k+1,1} & (h_{23})^T & \beta_{2k,0}-(A_{\min})_{k+1,k+1} \end{pmatrix} \end{array},$$

and

(5.11)

$$\begin{aligned} K &:= \mathcal{H}(\widehat{A}_{\min})/H_{22} \\ &= \begin{pmatrix} \beta_{0,0}-(A_{\min})_{1,1} & \beta_{k,0}-(A_{\min})_{2,k} \\ \beta_{k,0}-(A_{\min})_{2,k} & \beta_{2k,0}-(A_{\min})_{k+1,k+1}. \end{pmatrix} - \begin{pmatrix} (h_{12})^T \\ (h_{23})^T \end{pmatrix} (H_{22})^\dagger \begin{pmatrix} h_{12} & h_{23} \end{pmatrix} \\ &:= \begin{pmatrix} \beta_{0,0}-(A_{\min})_{1,1}-(h_{12})^T(H_{22})^\dagger h_{12} & \beta_{k,0}-(A_{\min})_{2,k}-(h_{12})^T(H_{22})^\dagger h_{23} \\ \beta_{k,0}-(A_{\min})_{2,k}-(h_{23})^T(H_{22})^\dagger h_{12} & \beta_{2k,0}-(A_{\min})_{k+1,k+1}-(h_{12})^T(H_{22})^\dagger h_{12} \end{pmatrix}. \end{aligned}$$

Let

$$(5.12)\qquad \begin{aligned} t_{\max} &:= \beta_{0,0}-(A_{\min})_{1,1}-(h_{12})^T(H_{22})^\dagger h_{12}, \\ u_{\max} &:= \beta_{2k,0}-(A_{\min})_{k+1,k+1}-(h_{12})^T(H_{22})^\dagger h_{12}, \\ k_{12} &:= \beta_{k,0}-(A_{\min})_{2,k}-(h_{23})^T(H_{22})^\dagger h_{12}. \end{aligned}$$

Note that

$$K = \begin{pmatrix} t_{\max} & k_{12} \\ k_{12} & u_{\max} \end{pmatrix} = \mathcal{H}(A_{\min})/H_{22} + \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix}. \tag{5.13}$$

Write

$$\mathcal{B} := \{Y^k, Y^{k+1}, \ldots, Y, \mathit{1}, X, X^2, \ldots, X^k\}.$$

We say the matrix $A \in S_{k+1}$ satisfies **the property (Hyp)** if $\mathcal{F}(A)$ is positive semidefinite and one of the following holds:

$$\underbrace{\operatorname{rank}\mathcal{F}(A) = \operatorname{rank}\mathcal{F}(A)_{\mathcal{B}\setminus\{X^k\}} = \operatorname{rank}\mathcal{F}(A)_{\mathcal{B}\setminus\{Y^k\}}}_{(\mathrm{Hyp})_1} \quad \text{or} \quad \underbrace{\operatorname{rank}\mathcal{F}(A) = 2k+1}_{(\mathrm{Hyp})_2}. \tag{5.14}$$

The solution to the $\mathcal{Z}(p)$–TMP is the following.

Theorem 5.2. *Let $p(x,y) = y(x+y-xy)$ and $\beta := \beta^{(2k)} = \{\beta_{i,j}\}_{i,j\in\mathbb{Z}_+, i+j\leq 2k}$, where $k \geq 3$. Assume the notation above. Then the following statements are equivalent:*

(1) β has a $\mathcal{Z}(p)$–representing measure.

(2) $\mathcal{M}(k;\beta)$ is positive semidefinite, the relations (5.2) hold and there exists a pair $(\tilde{t},\tilde{u})$ such that $\gamma_1(\tilde{t},\tilde{u})$ is $\mathbb{R}$–representable and $A_{\gamma_2(\tilde{t},\tilde{u})}$ satisfies (Hyp), where:

(a) If $u_{\max} = 0$,

$$(\tilde{t},\tilde{u}) \in \{(0,0), (t_{\max}, 0)\}.$$

(b) If $u_{\max} > 0$ and $k_{12} = 0$,

$$(\tilde{t},\tilde{u}) \in \left\{(0,0), \left(\frac{\eta^2}{u_{\max}}, u_{\max}\right), (t_{\max}, u_{\max})\right\}.$$

(c) If $u_{\max} > 0$ and $k_{12} \neq 0$,

$$(\tilde{t},\tilde{u}) \in \left\{(t_{-,\eta^2}, u_{-,\eta^2}), (t_{+,\eta^2}, u_{+,\eta^2}), \left(t_{\max} - \frac{|k_{12}|\sqrt{t_{\max}}}{\sqrt{u_{\max}}}, u_{\max} - \frac{|k_{12}|\sqrt{u_{\max}}}{\sqrt{t_{\max}}}\right)\right\},$$

where writing $B := k_{12}^2 - t_{\max}u_{\max} - \eta^2$ we have

$$u_{\pm,\eta^2} = \frac{-B \pm \sqrt{B^2 - 4t_{\max}u_{\max}\eta^2}}{2t_{\max}} \quad \text{and} \quad t_{\pm,\eta^2} = \frac{\eta^2}{u_{\pm,\eta^2}}.$$

Before we prove Theorem 5.2 we need few lemmas. Their statements and the proofs coincide verbatim with [YZ24, Theorem 6.1, Claims 1–3], but we state them for easier readability.

Let

$$\mathcal{R}_1 = \{(t,u) \in \mathbb{R}^2 : \mathcal{F}(\mathcal{G}(t,u)) \succeq 0\} \quad \text{and} \quad \mathcal{R}_2 = \{(t,u) \in \mathbb{R}^2 : \mathcal{H}(\mathcal{G}(t,u)) \succeq 0\}.$$

Claims 1 and 2 below describe ranks of $\mathcal{F}(\mathcal{G}(t,u))$ and $\mathcal{H}(\mathcal{G}(t,u))$ for various choices of (t,u) in $\mathcal{R}_1$ and $\mathcal{R}_2$.

Lemma 5.3 ([YZ24, Theorem 6.1, Claim 1])**.** *Assume that $\widetilde{\mathcal{M}}(k;\beta) \succeq 0$. Then*

$$\mathcal{R}_1 = \{(t,u) \in \mathbb{R}^2 : t \geq 0, u \geq 0, tu \geq \eta^2\}. \tag{5.15}$$

If $(t,u) \in \mathcal{R}_1$, we have

$$\operatorname{rank}\mathcal{F}(\mathcal{G}(t,u)) = \begin{cases} \operatorname{rank}\mathcal{F}(A_{\min}), & \text{if } \eta = t = u = 0, \\ \operatorname{rank}\mathcal{F}(A_{\min}) + 1, & \text{if } (\eta = t = 0, u > 0) \text{ or} \\ & (\eta = u = 0, t > 0) \text{ or } (\eta \neq 0, tu = \eta^2), \\ \operatorname{rank}\mathcal{F}(A_{\min}) + 2, & \text{if } tu > \eta^2. \end{cases} \tag{5.16}$$

Define the matrix function

$$\begin{aligned}\mathcal{K}(\mathbf{t},\mathbf{u}) &:= \mathcal{H}(\mathcal{G}(\mathbf{t},\mathbf{u}))/H_{22} = \mathcal{H}(\widehat{A}_{\min})/H_{22} - \begin{pmatrix}\mathbf{t} & 0\\ 0 & \mathbf{u}\end{pmatrix}\\ &= K - \begin{pmatrix}\mathbf{t} & 0\\ 0 & \mathbf{u}\end{pmatrix} = \begin{pmatrix}t_{\max}-\mathbf{t} & k_{12}\\ k_{12} & u_{\max}-\mathbf{u}\end{pmatrix}.\end{aligned} \tag{5.17}$$

Lemma 5.4 ([YZ24, Theorem 6.1, Claim 2]). *Assume that $\widetilde{\mathcal{M}}(k;\beta)\succeq 0$. Then*

$$\begin{aligned}\mathcal{R}_2 &= \big\{(t,u)\in\mathbb{R}^2\colon \mathcal{K}(t,u)\succeq 0\big\}\\ &= \big\{(t,u)\in\mathbb{R}^2\colon t\le t_{\max}, u\le u_{\max}, (t_{\max}-t)(u_{\max}-u)\ge k_{12}^2\big\}.\end{aligned} \tag{5.18}$$

If $(t,u)\in\mathcal{R}_2$, we have

(5.19)

$$\operatorname{rank}\mathcal{H}(\mathcal{G}(t,u)) = \begin{cases}\operatorname{rank} H_{22}, & \text{if } k_{12}=0, t=t_{\max}, u=u_{\max},\\ \operatorname{rank} H_{22}+1, & \text{if } (t_{\max}-t)(u_{\max}-u)=k_{12}^2, (t\neq t_{\max} \text{ or } u\neq u_{\max}),\\ \operatorname{rank} H_{22}+2, & \text{if } (t_{\max}-t)(u_{\max}-u)>k_{12}^2.\end{cases}$$

Lemma 5.5 ([YZ24, Theorem 6.1, Claim 3]). *If $(t,u)\in\mathcal{R}_2\cap(\mathbb{R}_+)^2$, then*

$$tu\le(\sqrt{t_{\max}u_{\max}}-\operatorname{sign}(k_{12})k_{12})^2 =: p_{\max}.$$

The equality is achieved if:

- $k_{12}=0$, *only in the point* $(t,u)=(t_{\max},u_{\max})$.
- $k_{12}\neq 0$, *only in point* $(t_{p_{\max}},u_{p_{\max}})=(t_{\max}-\frac{|k_{12}|\sqrt{t_{\max}}}{\sqrt{u_{\max}}}, u_{\max}-\frac{|k_{12}|\sqrt{u_{\max}}}{\sqrt{t_{\max}}})$.

Moreover, if $k_{12}\neq 0$, then for every $p\in[0,p_{\max}]$ there exists a point $(\tilde{t},\tilde{u})\in\mathcal{R}_2\cap(\mathbb{R}_+)^2$ such that $\tilde{t}\tilde{u}=p$ and $(t_{\max}-\tilde{t})(u_{\max}-\tilde{u})=k_{12}^2$.

Proof of Theorem 5.2. The implication (2) $\Rightarrow$ (1) is trivial, since (2) immediately implies (5.9). It remains to prove the implication (1) $\Rightarrow$ (2). By (5.9), there exists $(\tilde{t}_0,\tilde{u}_0)$, such that $\gamma_1(\tilde{t}_0,\tilde{u}_0)$ is $\mathbb{R}$–representable and $A_{\gamma_2(\tilde{t}_0,\tilde{u}_0)}$ satisfies (Hyp). We separate two cases according to H_{22} (see (5.10)) being positive definite or not.

Case 1: H_{22} is not positive definite. By Theorem 2.5, it follows that the only option for $\tilde{u}_0$ is $u_{\max}$. By Lemma 5.4, we have $k_{12}=0$. Applying Theorem 2.5 again, for any $t\in[0,t_{\max}]$, the sequence $\gamma_1(t,u_{\max})$ is $\mathbb{R}$–representable. We separate two cases according to the value of $u_{\max}$.

Case 1.1: $u_{\max}=0$. By Lemma 5.3, $\eta=0$. This and the definition of $A_{\min}$ implies that for any $t\in[0,t_{\max}]$, $\operatorname{rank}\mathcal{F}(\mathcal{G}(t,0))_{\mathcal{B}\setminus\{X^k\}}=\operatorname{rank}\mathcal{F}(\mathcal{G}(t,0))$. If $\tilde{t}_0>0$, then

$$\begin{aligned}\operatorname{rank}\mathcal{F}(A_{\min})+1 \underbrace{=}_{(5.16)} \operatorname{rank}\mathcal{F}(\mathcal{G}(\tilde{t}_0,0)) &= \operatorname{rank}\mathcal{F}(\mathcal{G}(\tilde{t}_0,0))_{\mathcal{B}\setminus\{Y^k\}}\\ &\le \operatorname{rank}\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{Y^k\}}+1\le\operatorname{rank}\mathcal{F}(A_{\min})+1,\end{aligned} \tag{5.20}$$

where in the second equality the assumption that $A_{\gamma_2(\tilde{t}_0,0)}$ satisfies (Hyp) was used. It follows that all inequalities in (5.20) must be equalities. In particular, we have $\operatorname{rank}\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{Y^k\}}=\operatorname{rank}\mathcal{F}(A_{\min})$. It follows that $A_{\min}=A_{\gamma_2(0,0)}$ satisfies (Hyp) and $(0,0)$ is a good choice for $(\tilde{t},\tilde{u})$ in Theorem 5.2.(2).

Case 1.2: $u_{\max} > 0$**.** By Lemma 5.3, $\tilde{t}_0\tilde{u}_0 \geq \eta^2$. If $\eta \neq 0$, then $\tilde{t}_0 > 0$. If $\eta = 0$, then

$$\begin{aligned}2k+1 > \operatorname{rank}\mathcal{F}(A_{\min})+1 = \operatorname{rank}\mathcal{F}(\mathcal{G}(0,0))+1 \underbrace{=}_{(5.16)} \operatorname{rank}\mathcal{F}(\mathcal{G}(0,u_{\max}))\\ \underbrace{>}_{u_{\max}>0} \operatorname{rank}\mathcal{F}(\mathcal{G}(0,u_{\max}))_{\mathcal{B}\setminus\{X^k\}}.\end{aligned}$$

Hence, $(0, u_{\max})$ cannot satisfy (Hyp) and thus $\tilde{t}_0 > 0$. We separate two cases according to the product $\tilde{t}_0 u_{\max}$, which must be at least η^2, by Lemma 5.3.

Case 1.2.1: $\tilde{t}_0 u_{\max} = \eta^2$**.** In this case $(\frac{\eta^2}{u_{\max}}, u_{\max})$ is a good choice for $(\tilde{t}, \tilde{u})$ in Theorem 5.2.(2).

Case 1.2.2: $\tilde{t}_0 u_{\max} > \eta^2$**.** We separate two cases according to the rank of $\mathcal{F}(\mathcal{G}(\tilde{t}_0, u_{\max}))$.

Case 1.2.2.1: $\operatorname{rank}\mathcal{F}(\mathcal{G}(\tilde{t}_0, u_{\max})) = 2k+1$**.** The inequality $\mathcal{F}(\mathcal{G}(\tilde{t}_0, u_{\max})) \preceq \mathcal{F}(\mathcal{G}(t_{\max}, u_{\max}))$ implies that $\operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\max}, u_{\max})) = 2k+1$ and thus $(t_{\max}, u_{\max})$ satisfies (Hyp). Therefore $(t_{\max}, u_{\max})$ is a good choice for $(\tilde{t}, \tilde{u})$ in Theorem 5.2.(2).

Case 1.2.2.2: $\operatorname{rank}\mathcal{F}(\mathcal{G}(\tilde{t}_0, u_{\max})) < 2k+1$**.** Then

$$\begin{aligned}\operatorname{rank}\mathcal{F}(A_{\min})+2 \underbrace{=}_{(5.16)} \operatorname{rank}\mathcal{F}(\mathcal{G}(\tilde{t}_0, u_{\max})) = \operatorname{rank}\mathcal{F}(\mathcal{G}(\tilde{t}_0, u_{\max}))_{\mathcal{B}\setminus\{Y^k\}}\\ \leq \operatorname{rank}\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{Y^k\}}+2 \leq \operatorname{rank}\mathcal{F}(A_{\min})+2,\end{aligned} \tag{5.21}$$

where in the second equality we used the assumption that $A_{\gamma_2(\tilde{t}_0, u_{\max})}$ satisfies (Hyp). It follows that all inequalities in (5.21) must be equalities. Since

$$\mathcal{F}(\mathcal{G}(\tilde{t}_0, u_{\max}))_{\mathcal{B}\setminus\{Y^k\}} \preceq \mathcal{F}(\mathcal{G}(t_{\max}, u_{\max}))_{\mathcal{B}\setminus\{Y^k\}},$$

it follows that

$$\operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\max}, u_{\max})) = \operatorname{rank}\mathcal{F}(A_{\min})+2 = \operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\max}, u_{\max}))_{\mathcal{B}\setminus\{Y^k\}}.$$

Similarly, $\operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\max}, u_{\max})) = \operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\max}, u_{\max}))_{\mathcal{B}\setminus\{X^k\}}$. Therefore $(t_{\max}, u_{\max})$ satisfies (Hyp) and is a good choice for $(\tilde{t}, \tilde{u})$ in Theorem 5.2.(2).

Case 2: H_{22} **is positive definite.** We separate three cases according to the value of the pair (k_{12}, η).

Case 2.1: $k_{12} = \eta = 0$**.** By definition of $t_{\max}, u_{\max}$ and Theorem 2.5, $\gamma_1(t,u)$ is $\mathbb{R}$–representable for every

$$(t,u) \in [0, t_{\max}) \times [0, u_{\max}] \quad \text{and} \quad (t,u) = (t_{\max}, u_{\max}). \tag{5.22}$$

We separate two cases acccording to the rank of $\mathcal{F}(A_{\min})$.

Case 2.1.1: $\operatorname{rank}\mathcal{F}(A_{\min}) < 2k-1$**.** If $t_{\max} = 0$, by Lemma 5.3, $\operatorname{rank}\mathcal{F}(\mathcal{G}(\tilde{t}_0, \tilde{u}_0)) \leq \operatorname{rank}\mathcal{F}(A_{\min})+2$. Since $(\tilde{t}_0, \tilde{u}_0)$ satisfies (Hyp) and $\operatorname{rank}\mathcal{F}(A_{\min}) < 2k-1$, it follows that $(\tilde{t}_0, \tilde{u}_0)$ satisfies $(\text{Hyp})_1$. We separate four options depending on the sign of $\tilde{t}_0$ and $\tilde{u}_0$.

Case 2.1.1.1: $\tilde{t}_0 = \tilde{u}_0 = 0$**.** This means $(0,0)$ is a good choice for $(\tilde{t}, \tilde{u})$ in Theorem 5.2.(2).

Case 2.1.1.2: $\tilde{t}_0 > 0$ **and** $\tilde{u}_0 > 0$**.** Then

$$\begin{aligned}\operatorname{rank}\mathcal{F}(A_{\min})+2 \underbrace{=}_{(5.16)} \operatorname{rank}\mathcal{F}(\mathcal{G}(\tilde{t}_0,\tilde{u}_0)) &= \operatorname{rank}\mathcal{F}(\mathcal{G}(\tilde{t}_0,\tilde{u}_0))_{\mathcal{B}\setminus\{Y^k\}} \\ &\leq \operatorname{rank}\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{Y^k\}}+2 \leq \operatorname{rank}\mathcal{F}(A_{\min})+2,\end{aligned} \tag{5.23}$$

where in the second equality we used the assumption that $A_{\gamma_2(\tilde{t}_0,\tilde{u}_0)}$ satisfies (Hyp). It follows that all inequalities in (5.23) must be equalities. Since

$$\mathcal{F}(\mathcal{G}(\tilde{t}_0,\tilde{u}_0))_{\mathcal{B}\setminus\{Y^k\}} \preceq \mathcal{F}(\mathcal{G}(t_{\max},u_{\max}))_{\mathcal{B}\setminus\{Y^k\}},$$

it follows that

$$\operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\max},u_{\max})) = \operatorname{rank}\mathcal{F}(A_{\min})+2 = \operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\max},u_{\max}))_{\mathcal{B}\setminus\{Y^k\}}.$$

Similarly,

$$\operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\max},u_{\max})) = \operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\max},u_{\max}))_{\mathcal{B}\setminus\{X^k\}}.$$

Therefore $(t_{\max},u_{\max})$ satisfies (Hyp)$_1$ and is a good choice for $(\tilde{t},\tilde{u})$ in Theorem 5.2.(2).

Case 2.1.1.3: $\tilde{t}_0 = 0$ **and** $\tilde{u}_0 > 0$**.** Then

$$\operatorname{rank}\mathcal{F}(\mathcal{G}(0,0))_{\mathcal{B}\setminus\{X^k\}} = \operatorname{rank}\mathcal{F}(\mathcal{G}(0,\tilde{u}_0))_{\mathcal{B}\setminus\{X^k\}} \underbrace{<}_{\tilde{u}_0>0} \operatorname{rank}\mathcal{F}(\mathcal{G}(0,\tilde{u}_0)),$$

where in the equality we used the observation $\tilde{u}_0$ occurs only in the column X^k. Hence, $\gamma_2(0,\tilde{u}_0)$ cannot satisfy (Hyp). So this case does not occur.

Case 2.1.1.4: $\tilde{t}_0 > 0$ **and** $\tilde{u}_0 = 0$**.** Then

$$\begin{aligned}\operatorname{rank}\mathcal{F}(A_{\min})+1 \underbrace{=}_{(5.16)} \operatorname{rank}\mathcal{F}(\mathcal{G}(\tilde{t}_0,0)) &= \operatorname{rank}\mathcal{F}(\mathcal{G}(\tilde{t}_0,0))_{\mathcal{B}\setminus\{Y^k\}} \\ &\leq \operatorname{rank}\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{Y^k\}}+1 \leq \operatorname{rank}\mathcal{F}(A_{\min})+1,\end{aligned} \tag{5.24}$$

where in the second equality we used the assumption that $A_{\gamma_2(\tilde{t}_0,0)}$ satisfies (Hyp). It follows that all inequalities in (5.24) must be equalities. In particular, $\operatorname{rank}\mathcal{F}(A_{\min}) = \operatorname{rank}\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{Y^k\}}$. Similarly, $\operatorname{rank}\mathcal{F}(A_{\min}) = \operatorname{rank}\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{X^k\}}$. Therefore $(0,0)$ satisfies (Hyp)$_1$ and is a good choice for $(\tilde{t},\tilde{u})$ in Theorem 5.2.(2).

Case 2.1.2: $\operatorname{rank}\mathcal{F}(A_{\min}) = 2k-1$**.** The reasoning in the case $\tilde{t}_0 = \tilde{u}_0 = 0$ is the same as in Case 2.1.1.1, in the case $\tilde{t}_0 = 0$ and $\tilde{u}_0 > 0$ the same as in Case 2.1.1.3, and in the case $\tilde{t}_0 > 0$ and $\tilde{u}_0 = 0$ the same as in Case 2.1.1.4 above. Assume that $\tilde{t}_0 > 0$ and $\tilde{u}_0 > 0$. By Lemma 5.3, $\operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\max},u_{\max})) = 2k+1$ and $A_{\gamma_2}(t_{\max},u_{\max})$ satisfies (Hyp). Together with (5.22), it follows that $(t_{\max},u_{\max})$ is a good choice for $(\tilde{t},\tilde{u})$ in Theorem 5.2.(2).

Case 2.2: $k_{12} = 0$ **and** $\eta \neq 0$**.** Since $\eta \neq 0$, it follows that $u_{\max} > 0$ (using Lemma 5.3). But then as in Case 1.2 above, one of $(\frac{\eta^2}{u_{\max}},u_{\max})$, $(t_{\max},u_{\max})$ is a good choice for $(\tilde{t},\tilde{u})$ in Theorem 5.2.(2).

Case 2.3: $k_{12} \neq 0$ **and** $\eta \neq 0$**.** Let $p_{\max}$ be as in Lemma 5.5. We separete two cases according to the value of $p_{\max}$.

Case 2.3.1: $p_{\max} = \eta^2$**.** In this case $\mathcal{R}_1 \cap \mathcal{R}_2 = \{(t_{p_{\max}},u_{p_{\max}})\}$, where $(t_{p_{\max}},u_{p_{\max}})$ is as in Lemma 5.5. and thus $(t_{p_{\max}},u_{p_{\max}})$ is the only candidate for $(\tilde{t}_0,\tilde{u}_0)$ in (5.9).

Case 2.3.2: $p_{\max} > \eta^2$. We separate two cases depending on $\operatorname{rank}\mathcal{F}(A_{\min})$.

Case 2.3.2.1: $\operatorname{rank}\mathcal{F}(A_{\min}) = 2k-1$**.** Then by Lemma 5.3, $\operatorname{rank}\mathcal{F}(\mathcal{G}(t',u')) = 2k+1$ for every (t',u') such that $t'>0, u'>0$, $t'u'>\eta^2$ and hence $A_{\gamma_2}(t',u')$ satisfies $(\mathrm{Hyp})_2$. By Lemma 5.5, (t',u') equal to $(t_{p_{\max}}, u_{p_{\max}})$ satisfies $(\mathrm{Hyp})_2$, and $\gamma_1(t',u')$ is $\mathbb{R}$–representable, since $A_{\gamma_1(t',u'}(k) \succ 0$ (by $t'<t_{\max}$). Hence, this (t',u') is a good choice for $(\tilde{t},\tilde{u})$ in Theorem 5.2.(2).

Case 2.3.2.2: $\operatorname{rank}\mathcal{F}(A_{\min}) < 2k-1$**.** By Lemma 5.3, $\operatorname{rank}\mathcal{F}(\mathcal{G}(t',u')) < 2k+1$ for every $(t',u')\in\mathcal{R}_1$ and hence $A_{\gamma_2}(t',u')$ cannot satisfy $(\mathrm{Hyp})_2$ for any $(t',u')\in\mathcal{R}_1$. Thus $(\tilde{t}_0,\tilde{u}_0)$ satisfies $(\mathrm{Hyp})_1$. We have

$$
\begin{aligned}
\operatorname{rank}\mathcal{F}(A_{\min}) + \operatorname{rank}\begin{pmatrix}\tilde{t}_0 & \eta\\ \eta & \tilde{u}_0\end{pmatrix} \underbrace{=}_{(5.16)} & \operatorname{rank}\mathcal{F}(\mathcal{G}(\tilde{t}_0,\tilde{u}_0))\\
&= \operatorname{rank}\mathcal{F}(\mathcal{G}(\tilde{t}_0,\tilde{u}_0))_{\mathcal{B}\setminus\{Y^k\}}\\
&\le \operatorname{rank}\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{Y^k\}} + \operatorname{rank}\begin{pmatrix}\tilde{t}_0 & \eta\\ \eta & \tilde{u}_0\end{pmatrix}\\
&\le \operatorname{rank}\mathcal{F}(A_{\min}) + \operatorname{rank}\begin{pmatrix}\tilde{t}_0 & \eta\\ \eta & \tilde{u}_0\end{pmatrix},
\end{aligned}
\tag{5.25}
$$

where in the second equality we used the assumption that $A_{\gamma_2(\tilde{t}_0,\tilde{u}_0)}$ satisfies (Hyp). It follows that equalities hold for all inequalities in (5.25). In particular,

$$\operatorname{rank}\mathcal{F}(A_{\min}) = \operatorname{rank}\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{Y^k\}}. \tag{5.26}$$

Similarly,

$$\operatorname{rank}\mathcal{F}(A_{\min}) = \operatorname{rank}\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{X^k\}}. \tag{5.27}$$

By Lemma 5.5, there is a point $(\tilde{t},\tilde{u})\in\mathcal{R}_2\cap(\mathbb{R}_+)^2$, such that

$$\tilde{t}\tilde{u} = \eta^2 \quad\text{and}\quad (t_{\max}-\tilde{t})(u_{\max}-\tilde{u}) = k_{12}^2. \tag{5.28}$$

Moreover, there are exactly two such points $(\tilde{t},\tilde{u})$ satisfying (5.28):

$$
\begin{aligned}
\Big(t_{\max}-\frac{\eta^2}{u}\Big)\Big(u_{\max}-u\Big) = k_{12}^2 \quad &\Leftrightarrow \quad (t_{\max}u-\eta^2)(u_{\max}-u) = k_{12}^2 u\\
&\Leftrightarrow \quad t_{\max}u^2 + (k_{12}^2 - t_{\max}u_{\max} - \eta^2)u + \eta^2 u_{\max} = 0\\
&\Leftrightarrow \quad u_{\pm,\eta^2} = \frac{-B\pm\sqrt{B^2-4t_{\max}u_{\max}\eta^2}}{2t_{\max}},
\end{aligned}
$$

where $B = k_{12}^2 - t_{\max}u_{\max} - \eta^2$. Clearly, $t_{\max}u_{\max} \ge k_{12}^2$, and since $\eta^2>0$, it follows that $B<0$. A short computation shows that

$$0 = B^2 - 4t_{\max}u_{\max}\eta^2 \quad\Leftrightarrow\quad \eta^2 \in \{(\sqrt{t_{\max}}\sqrt{u_{\max}}+k_{12})^2, (\sqrt{t_{\max}}\sqrt{u_{\max}}-k_{12})^2\}. \tag{5.29}$$

We have

$$p_{\max} = (\sqrt{t_{\max}}\sqrt{u_{\max}} - |k_{12}|)^2 < (\sqrt{t_{\max}}\sqrt{u_{\max}} + |k_{12}|)^2.$$

Since $\eta^2 < p_{\max}$, (5.29) implies that $B^2 - 4t_{\max}u_{\max}\eta^2 \neq 0$. Therefore

$$0 < u_{-,\eta^2} < u_{+,\eta^2}.$$

Let $t_{\pm,\eta^2} := \frac{\eta^2}{u_{\pm,\eta^2}}$. Note that

$$\begin{aligned}\operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\pm,\eta^2},u_{\pm,\eta^2}))_{\mathcal{B}\setminus\{X^k\}} \underbrace{=}_{t_{\pm,\eta^2}>0} \operatorname{rank}\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{X^k\}}+1 &\underbrace{=}_{(5.27)} \operatorname{rank}\mathcal{F}(\mathcal{G}(A_{\min}))+1\\ &\underbrace{=}_{(5.16)} \operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\pm,\eta^2},u_{\pm,\eta^2})).\end{aligned}$$

So $A_{\gamma_2(t_{\pm,\eta^2},u_{\pm,\eta^2})}$ satisfies $(\text{Hyp})_1$ if and only if

$$\operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\pm,\eta^2},u_{\pm,\eta^2}))_{\mathcal{B}\setminus\{Y^k\}} = \operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\pm,\eta^2},u_{\pm,\eta^2})). \tag{5.30}$$

Since $t_{\pm,\eta^2}>0$ and $u_{\pm,\eta^2}>0$, it follows that

$$\begin{aligned}\operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\pm,\eta^2},u_{\pm,\eta^2}))_{\mathcal{B}\setminus\{X^k,Y^k\}} &> \operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\pm,\eta^2},u_{\pm,\eta^2}))_{\mathcal{B}\setminus\{\mathit{1},X^k,Y^k\}},\\ \operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\pm,\eta^2},u_{\pm,\eta^2}))_{\mathcal{B}\setminus\{\mathit{1},Y^k\}} &> \operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\pm,\eta^2},u_{\pm,\eta^2}))_{\mathcal{B}\setminus\{\mathit{1},X^k,Y^k\}}.\end{aligned}$$

If

$$\operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\pm,\eta^2},u_{\pm,\eta^2}))_{\mathcal{B}\setminus\{Y^k\}} = \operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\pm,\eta^2},u_{\pm,\eta^2}))_{\mathcal{B}\setminus\{\mathit{1},X^k,Y^k\}}+2, \tag{5.31}$$

then (5.30) holds. Indeed, in this case

$$(A_{\min})_{\{1,X^k\}} = \left(\mathcal{F}(A_{\min})_{\{\mathit{1},X^k\},\mathcal{B}\setminus\{\mathit{1},X^k,Y^k\}}\right)\left(\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{\mathit{1},X^k,Y^k\}}\right)^{\dagger}\left(\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{\mathit{1},X^k,Y^k\},\{\mathit{1},X^k\}}\right),$$

whence

$$\begin{aligned}\operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\pm,\eta^2},u_{\pm,\eta^2}))_{\mathcal{B}\setminus\{Y^k\}} &= \operatorname{rank}\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{Y^k\}}+1\\ &\underbrace{=}_{(5.26)} \operatorname{rank}\mathcal{F}(A_{\min})+1 \underbrace{=}_{(5.16)} \operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\pm,\eta^2},u_{\pm,\eta^2})).\end{aligned}$$

Write

$$\mathcal{T}_3 := \{Y^{k-1},Y^{k-2},\ldots,Y,X,X^2,\ldots,X^{k-1}\}. \tag{5.32}$$

Assume now that (5.31) does not hold for both $(t_{-,\eta^2},u_{-,\eta^2})$ and $(t_{+,\eta^2},u_{+,\eta^2})$. Therefore there are relations

(5.33)
$$\begin{aligned}\mathcal{F}(\mathcal{G}(t_{+,\eta^2},u_{+,\eta^2}))_{\mathcal{B},\{\mathit{1}\}}+\alpha_+\mathcal{F}(\mathcal{G}(t_{+,\eta^2},u_{+,\eta^2}))_{\mathcal{B},\{X^k\}}+\mathcal{F}(\mathcal{G}(t_{+,\eta^2},u_{+,\eta^2}))_{\mathcal{B},\mathcal{T}_3}v_+ &= 0,\\ \mathcal{F}(\mathcal{G}(t_{-,\eta^2},u_{-,\eta^2}))_{\mathcal{B},\{\mathit{1}\}}+\alpha_-\mathcal{F}(\mathcal{G}(t_{-,\eta^2},u_{-,\eta^2}))_{\mathcal{B},\{X^k\}}+\mathcal{F}(\mathcal{G}(t_{-,\eta^2},u_{-,\eta^2}))_{\mathcal{B},\mathcal{T}_3}v_- &= 0,\end{aligned}$$

for some $\alpha_+,\alpha_-\in\mathbb{R}$, $v_+,v_-\in\mathbb{R}^{2k-1}$ in $\mathcal{F}(\mathcal{G}(t_{+,\eta^2},u_{+,\eta^2}))$ and $\mathcal{F}(\mathcal{G}(t_{-,\eta^2},u_{-,\eta^2}))$, respectively. Since $\mathcal{F}(\mathcal{G}(t_{\pm,\eta^2},u_{\pm,\eta^2}))\succeq\mathcal{F}(A_{\min})\succeq 0$, the relations (5.33) must hold also in $\mathcal{F}(A_{\min})$. Subtracting these relations we get

$$(\alpha_+-\alpha_-)\mathcal{F}(A_{\min})_{\mathcal{B},\{X^k\}}+\mathcal{F}(A_{\min})_{\mathcal{B},\mathcal{T}_3}(v_+-v_-)=0. \tag{5.34}$$

If $\alpha_+=\alpha_-$, then $v_+-v_-\in\ker\mathcal{F}(A_{\min})_{\mathcal{B},\mathcal{T}_3}$ and hence $\mathcal{F}(A_{\min})_{\mathcal{B},\mathcal{T}_3}v_+=\mathcal{F}(A_{\min})_{\mathcal{B},\mathcal{T}_3}v_-$. But observing the first entries of the left hand side vectors in (5.33), this cannot hold since

$$\mathcal{F}(\mathcal{G}(t_{+,\eta^2},u_{+,\eta^2}))_{\{\mathit{1}\}} = t_{+,\eta^2} \neq t_{-,\eta^2} = \mathcal{F}(\mathcal{G}(t_{-,\eta^2},u_{-,\eta^2}))_{\{\mathit{1}\}}.$$

So $\alpha_+\neq\alpha_-$ and from (5.34) it follows that in $\mathcal{F}(A_{\min})$, the column X^k is linearly dependent from the columns in $\mathcal{T}_3$. Using one of the relations (5.33) for $\mathcal{F}(A_{\min})$, the same holds for the column $\mathit{1}$. But then

$$\operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\pm,\eta^2},u_{\pm,\eta^2}))_{\mathcal{B}\setminus\{Y^k\}} = \operatorname{rank}\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{Y^k\}}+1 = \operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\pm,\eta^2},u_{\pm,\eta^2})), \tag{5.35}$$

and (5.30) holds for both points $(t_{\pm,\eta^2},u_{\pm,\eta^2})$. Note that $\gamma_1(t_{\pm,\eta^2},u_{\pm,\eta^2})$ is $\mathbb{R}$–representable, since $A_{\gamma_1(t_{\pm,\eta^2},u_{\pm,\eta^2})}(k)\succ 0$ (by $t_{\pm,\eta^2}<t_{\max}$). So at least one of $(t_{\pm,\eta^2},u_{\pm,\eta^2})$ is a good choice

for $(\tilde{t}_0, \tilde{u}_0)$ in (5.9).

This concludes the proof of the implication (1) $\Rightarrow$ (2). $\square$

5.2. **Cardinality of a minimal representing measure.** It remains to characterize the cardinality of a minimal $\mathcal{Z}(p)$–rm for β in Theorem 5.2.

Let $\mathcal{T}_4 := \{Y^k, \ldots, Y, X, \ldots, X^{k-1}\}$, $\vec{\mathcal{T}_4} := (\vec{Y}^{(k,1)}, \vec{X}^{(1,k-1)})$ and $\mathcal{T}_5 = \{1, X^k\} \cup \mathcal{T}_4$. Write

$$\mathcal{F}(\mathcal{G}(\mathbf{t},\mathbf{u}))_{(1,\vec{\mathcal{T}_4},X^k,\overrightarrow{\mathcal{C}\setminus\mathcal{T}_5})} \tag{5.36}$$
$$= \begin{array}{c} \\ \mathit{1} \\ (\vec{\mathcal{T}_4})^T \\ X^k \\ \overrightarrow{\mathcal{C}\setminus\mathcal{T}_5} \end{array} \begin{array}{c} \begin{array}{cccc} \mathit{1} & \vec{\mathcal{T}_4} & X^k & \overrightarrow{\mathcal{C}\setminus\mathcal{T}_5} \end{array} \\ \begin{pmatrix} (A_{\min})_{1,1}+\mathbf{t} & (f_{12})^T & (A_{\min})_{2,k} & (f_{14})^T \\ f_{12} & F_{22} & f_{23} & F_{24} \\ (A_{\min})_{2,k} & (f_{23})^T & (A_{\min})_{k+1,k+1}+\mathbf{u} & (f_{34})^T \\ f_{14} & (F_{24})^T & f_{34} & F_{44} \end{pmatrix} \end{array}.$$

Note that

$$f_{12}^T = \begin{pmatrix} (b_{13}^{(0)})^T & (b_{23}^{(0)})^T \end{pmatrix}, \quad (A_{\min})_{2,k} = \beta_{k,1} - \beta_{k-1,1}, \quad F_{22} = \begin{pmatrix} B_{11} & B_{13}^{(1,k-1)} \\ (B_{13}^{(1,k-1)})^T & B_{23}^{(1,k-1)} \end{pmatrix}.$$

The following theorem characterizes the cardinality of a minimal measure in case β admits a $\mathcal{Z}(p)$–rm.

Theorem 5.6. *Let $p(x,y) = y(x+y-xy)$ and $\beta = (\beta_{i,j})_{i,j\in\mathbb{Z}_+, i+j\le 2k}$, where $k \ge 3$, admits a $\mathcal{Z}(p)$–representing measure. Assume the notation above. The following statements hold:*

1. *There exists at most $(\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta)+2)$–atomic $\mathcal{Z}(p)$–representing measure.*
2. *There is no $\mathcal{Z}(p)$–representing measure with less than $\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta)+2$ atoms if and only if $\eta \neq 0$, $k_{12} \neq 0$, $\operatorname{rank} \mathcal{H}(A_{\min}) = k$ and $A_{\min}$ does not satisfy (Hyp).*
3. *There does not exist a $(\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta))$–atomic $\mathcal{Z}(p)$–representing measure if and only if any of the following holds:*
   - *(a) $F_{22} \succ 0$, $H_{22} \not\succ 0$, $\operatorname{rank} \mathcal{H}(A_{\min}) = \operatorname{rank} H_{22} + 1$ and $t_{\max}u_{\max} > \eta^2 > 0$.*
   - *(b) $F_{22} \succ 0$, $H_{22} \succ 0$ and one of the following holds:*
     - *(i) $\eta = 0$, $k_{12} \neq 0$, $\operatorname{rank} \mathcal{H}(A_{\min}) = k+1$ and $A_{\min}$ does not satisfy (Hyp)$_1$.*
     - *(ii) $\eta \neq 0$, $k_{12} \neq 0$, $A_{\min}$ satisfies (Hyp)$_1$ and $\operatorname{rank} \mathcal{H}(A_{\min}) = k$.*
     - *(iii) $\eta \neq 0$, $k_{12} \neq 0$ and $A_{\min}$ does not satisfy (Hyp)$_1$.*

In particular, a p–pure sequence β with a measure admits at most $(3k+1)$–atomic $\mathcal{Z}(p)$–representing measure.

Proof. By Lemma 3.3.(4),

$$\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta) = \operatorname{rank} \mathcal{F}(A_{\min}) + \operatorname{rank} \mathcal{H}(A_{\min}). \tag{5.37}$$

By (5.9), there exists a pair $(\tilde{t}_0, \tilde{u}_0) \in \mathbb{R}^2$ such that $\mathcal{F}(\mathcal{G}(\tilde{t}_0, \tilde{u}_0))$ and $\mathcal{H}(\mathcal{G}(\tilde{t}_0, \tilde{u}_0))$ admit a $\mathcal{Z}(x+y-xy)$–rm and a $\mathbb{R}$–rm, respectively. In the proof we will separate the following cases:

- **Case 1:** F_{22} **is not pd.**
- **Case 2:** F_{22} **is pd,** H_{22} **is not pd and** $u_{\max} = 0$**.**
- **Case 3:** F_{22} **is pd,** H_{22} **is not pd and** $u_{\max} > 0$**.**
- **Case 4:** F_{22} **and** H_{22} **are pd,** $\eta = 0$**.**

- **Case 5:** F_{22} **and** H_{22} **are pd,** $\eta \neq 0$**.**

Case 1: F_{22} **is not pd.** Note that the matrix $A_{\gamma_2(t,u)}$ can satisfy (Hyp) only if it satisfies (Hyp)$_1$:

$$\operatorname{rank} \mathcal{F}(A_{\gamma_2(t,u)}) = \operatorname{rank} \mathcal{F}(A_{\gamma_2(t,u)})_{\mathcal{B}\setminus\{X^k\}} = \operatorname{rank} \mathcal{F}(A_{\gamma_2(t,u)})_{\mathcal{B}\setminus\{Y^k\}}. \tag{5.38}$$

Since F_{22} is not pd, $\mathcal{F}(\mathcal{G}(\tilde{t}_0, \tilde{u}_0))$ satisfies a nontrivial column relation of the form

$$\sum_{i=1}^{k} \delta_i Y^i + \sum_{j=1}^{k-1} \xi_j X^j = 0 \qquad \text{for some } \delta_i, \xi_j \in \mathbb{R}, \text{ not all zero.} \tag{5.39}$$

Since $\mathcal{F}(\mathcal{G}(\tilde{t}_0, \tilde{u}_0))$ satisfies the column relation $XY = X + Y$, it follows by recursive generation, that its extensions, generated by a representing measure, must satisfy column relations

$$\begin{aligned} Y^2X &= Y(YX) = Y(X+Y) = Y^2 + YX = Y^2 + Y + X, \\ Y^3X &= Y(Y^2X) = Y(Y^2+Y+X) = Y^3 + Y^2 + YX = Y^3 + Y^2 + Y + X, \\ &\vdots \\ Y^iX &= Y^i + Y^{i-1} + \ldots + Y + X \quad \text{for } i \geq 1. \end{aligned} \tag{5.40}$$

Multiplying (5.39) with X and using (5.40), we get a column relation

$$\sum_{i=1}^{k} \Big(\sum_{j=i}^{k} \delta_j\Big) Y^i + \Big(\sum_{j=1}^{k} \delta_j\Big) X + \sum_{j=2}^{k} \xi_{j-1} X^j = 0, \tag{5.41}$$

We separate two subcases according to the values of ξ_j and $\sum_{j=1}^k \delta_j$.

Case 1.1: $\xi_j \neq 0$ **for some** j **or** $\sum_{j=1}^k \delta_j \neq 0$**.** Multiplying (5.39) with X^ℓ for ℓ large enough, we will eventually get a column relation (5.41) with a nonzero coefficient at X^k. But this means X^k must be in the span of the columns $Y^k, \ldots, Y, X, \ldots, X^{k-1}$. In particular, $\tilde{u}_0 = 0$. By (5.15), this implies that $\eta = 0$ and thus $\widehat{A}_{\min} = A_{\min}$. Moreover, since $\gamma_2(\tilde{t}_0, 0)$ satisfies (Hyp), it follows that $\gamma_2(0,0)$ also satisfies (Hyp). The $\mathbb{R}$–representability of $\gamma_1(\tilde{t}_0, 0)$ implies that $\gamma_1(0,0)$ is also $\mathbb{R}$–representable. So $\mathcal{F}(A_{\min})$ and $\mathcal{H}(A_{\min})$ admit a $\mathcal{Z}(x+y-xy)$–rm and a $\mathbb{R}$–rm, respectively. By Theorem 2.5 and Corollary 2.10, there also exist a $(\operatorname{rank} \mathcal{F}(A_{\min}))$–atomic and a $(\operatorname{rank} \mathcal{H}(A_{\min}))$–atomic rms. By (5.37), β has a $(\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta))$–atomic $\mathcal{Z}(p)$–rm.

Case 1.2: $\xi_j = 0$ **for all** j **and** $\sum_{j=1}^k \delta_j = 0$**.** (5.41) implies that there is a column relation of the form $\sum_{i=2}^k \delta_i^{(2)} Y^i = 0$ in $\mathcal{F}(\mathcal{G}(\tilde{t}_0, \tilde{u}_0))$ for some $\delta_i^{(2)} \in \mathbb{R}$, not all zero (observe that the coefficients at Y and X in (5.41) are both $\sum_{j=1}^k \delta_j$). Mutliplying $\sum_{i=2}^k \delta_i^{(2)} Y^i = 0$ with X we get a relation of the form (5.41) with $\xi_j = 0$ for all j and $\delta_j^{(2)}$ instead of δ_j. If $\sum_{j=1}^k \delta_j^{(2)} = \sum_{j=2}^k \delta_j^{(2)} \neq 0$, then the coefficient at X is nonzero and we can proceed as in Case 1.1 above. Otherwise the coefficients at X, Y and Y^2 are all zero. Hence, we got a new column relation of the form $\sum_{i=3}^k \delta_i^{(3)} Y^i = 0$ in $\mathcal{F}(\mathcal{G}(\tilde{t}_0, \tilde{u}_0))$ for some $\delta_i^{(3)} \in \mathbb{R}$, not all zero. Proceeding with this procedure inductively we either eventually come into Case 1.1 or end with a relation of the form $\alpha Y^i = 0$, $\alpha \neq 0$, $i > 0$, which holds in $\mathcal{F}(\mathcal{G}(\tilde{t}_0, \tilde{u}_0))$. But this means all atoms in the conic part of $\mathcal{Z}(p)$ also lie on the line $y = 0$. So a $\mathcal{Z}(p)$-rm for β is a $\mathcal{Z}(y)$–rm for β and, by Theorem 2.5, β has a $(\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta))$–atomic $\mathcal{Z}(p)$–rm.

Case 2: F_{22} is pd, H_{22} is not pd and $u_{\max} = 0$. Since $u_{\max} = 0$, it follows that $\tilde{u}_0 = 0$. By (5.15), this implies that $\eta = 0$. Analogously as in Case 1.1 above we conclude that $\gamma_2(0,0)$ satisfies (Hyp) and $\gamma_1(0,0)$ is $\mathbb{R}$–representable, which implies the existence of a $(\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta))$–atomic $\mathcal{Z}(p)$–rm for β.

Case 3: F_{22} is pd, H_{22} is not pd, $u_{\max} > 0$. Since H_{22} is not pd, Theorem 2.5 implies that $\tilde{u}_0 = u_{\max}$. But then (5.18) implies that $k_{12} = 0$. We separate two subcases according to the value of η.

Case 3.1: $\eta = 0$. Since $\gamma_2(\tilde{t}_0, u_{\max})$ satisfies (Hyp) and $u_{\max} > 0$, it follows that $\tilde{t}_0 > 0$. But then also $\gamma_2(t_{\max}, u_{\max})$ satisfies (Hyp). Since $0 < \tilde{t}_0 \le t_{\max}$, (5.16) implies that

$$\operatorname{rank} \mathcal{F}(\mathcal{G}(t_{\max}, u_{\max})) = \operatorname{rank} \mathcal{F}(A_{\min}) + 2. \tag{5.42}$$

Moreover, $\gamma_1(t_{\max}, u_{\max})$ is also $\mathbb{R}$–representable and (5.19) implies that

$$\operatorname{rank} \mathcal{H}(\mathcal{G}(t_{\max}, u_{\max})) = \operatorname{rank} H_{22} = \operatorname{rank} \mathcal{H}(\mathcal{G}(0,0)) - 2 = \operatorname{rank} \mathcal{H}(A_{\min}) - 2. \tag{5.43}$$

By (5.37), (5.42) and (5.43), there exists a $(\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta))$–atomic $\mathcal{Z}(p)$–rm for β.

Case 3.2: $\eta \neq 0$. Since $\eta \neq 0$, it follows, by (5.15), that $\tilde{t}_0 > 0$ and hence $t_{\max} > 0$. Since $k_{12} = 0$, (5.18) implies that $\mathcal{H}(\mathcal{G}(t_{\max}, u_{\max}))$ is psd and $\gamma_1(t_{\max}, u_{\max})$ is $\mathbb{R}$–representable. Moreover, since $A_{\gamma_2(\tilde{t}_0, u_{\max})}$ satisfies (Hyp), also $A_{\gamma_2(t_{\max}, u_{\max})}$ satisfies (Hyp). Indeed, either $\tilde{t}_0 = t_{\max}$ or $\tilde{t}_0 < t_{\max}$. In the latter case, $t_{\max}u_{\max} > \tilde{t}_0 u_{\max} \geq \eta^2$ and $(t_{\max}, u_{\max})$ satisfies $(\text{Hyp})_2$. Hence, $(t_{\max}, u_{\max})$ is a good choice for $(\tilde{t}_0, \tilde{u}_0)$ in (5.9). By (5.19), $t_{\max} > 0$, $u_{\max} > 0$ and $k_{12} = 0$, imply that

$$\operatorname{rank} \mathcal{H}(\mathcal{G}(0,0)) = \operatorname{rank} \mathcal{H}(\widehat{A}_{\min}) = \operatorname{rank} H_{22} + 2.$$

We have

$$\begin{aligned}
&\operatorname{rank} \mathcal{H}(\mathcal{G}(t_{\max}, u_{\max})) + \operatorname{rank} \mathcal{F}(\mathcal{G}(t_{\max}, u_{\max})) = \\
&= \begin{cases} \operatorname{rank} H_{22} + \operatorname{rank} \mathcal{F}(A_{\min}) + 1, & \text{if } t_{\max}u_{\max} = \eta^2, \\ \operatorname{rank} H_{22} + \operatorname{rank} \mathcal{F}(A_{\min}) + 2, & \text{if } t_{\max}u_{\max} > \eta^2, \end{cases}
\end{aligned} \tag{5.44}$$

where we used (5.16) and (5.19) in the equality. By (5.13),

$$\mathcal{H}(A_{\min})/H_{22} = \begin{pmatrix} t_{\max} & -\eta \\ -\eta & u_{\max} \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $\operatorname{rank} \mathcal{H}(A_{\min}) = \operatorname{rank} H_{22} + i$ for some $i \in \{1,2\}$. We separate these two cases.

Case 3.2.1: $\operatorname{rank} \mathcal{H}(A_{\min}) = \operatorname{rank} H_{22} + 2$**.** We have

$$\begin{aligned}
&\operatorname{rank} \mathcal{H}(\mathcal{G}(t_{\max}, u_{\max})) + \operatorname{rank} \mathcal{F}(\mathcal{G}(t_{\max}, u_{\max})) \\
&= \begin{cases} \operatorname{rank} \widetilde{\mathcal{M}}(k;\beta) - 1, & \text{if } t_{\max}u_{\max} = \eta^2, \\ \operatorname{rank} \widetilde{\mathcal{M}}(k;\beta), & \text{if } t_{\max}u_{\max} > \eta^2, \end{cases}
\end{aligned}$$

where we used (5.44) and (5.37) in the equality. The case $t_{\max}u_{\max} = \eta^2$ cannot happen, since this would imply β has a $(\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta) - 1)$–atomic $\mathcal{Z}(p)$–rm, which is not possible. Hence, there is a $(\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta))$–atomic $\mathcal{Z}(p)$–rm for β.

Case 3.2.2: $\operatorname{rank} \mathcal{H}(A_{\min}) = \operatorname{rank} H_{22} + 1$**.** In this case we have

$$\begin{aligned}
&\operatorname{rank} \mathcal{H}(\mathcal{G}(t_{\max}, u_{\max})) + \operatorname{rank} \mathcal{F}(\mathcal{G}(t_{\max}, u_{\max})) \\
&\underbrace{=}_{(5.37)} \begin{cases} \operatorname{rank} \widetilde{\mathcal{M}}(k;\beta), & \text{if } t_{\max}u_{\max} = \eta^2, \\ \operatorname{rank} \widetilde{\mathcal{M}}(k;\beta) + 1, & \text{if } t_{\max}u_{\max} > \eta^2, \end{cases}.
\end{aligned} \tag{5.45}$$

where we used (5.16) and (5.19) in the equality. Hence, β has a $(\operatorname{rank}\widetilde{\mathcal{M}}(k;\beta))$–atomic rm if $t_{\max}u_{\max}=\eta^2$ and $(\operatorname{rank}\widetilde{\mathcal{M}}(k;\beta)+1)$–atomic rm if $t_{\max}u_{\max}>\eta^2$. It remains to show that in the case $t_{\max}u_{\max}>\eta^2$, there does not exist a $(\operatorname{rank}\widetilde{\mathcal{M}}(k;\beta))$–atomic rm. Since H_{22} is not pd and $u_{\max}>0$, if $\mathcal{H}(\mathcal{G}(t',u'))$ has a $\mathbb{R}$–rm, then $u'=u_{\max}$. Since $\eta\neq 0$, we know $\mathcal{F}(\mathcal{G}(t',u_{\max}))$ with a $\mathcal{Z}(x+y-xy)$–rm is at least $(\operatorname{rank}\mathcal{F}(A_{\min})+1)$–atomic (see (5.16)). If $t'\neq t_{\max}$, then, by (5.19), $\operatorname{rank}\mathcal{H}(\mathcal{G}(t',u_{\max}))=\operatorname{rank}H_{22}+1$. Hence,

$$\begin{aligned}\operatorname{rank}\mathcal{H}(\mathcal{G}(t',u_{\max}))+\operatorname{rank}\mathcal{F}(\mathcal{G}(t',u_{\max}))&\geq(\operatorname{rank}H_{22}+1)+(\operatorname{rank}\mathcal{F}(A_{\min})+1)\\&\underbrace{=}_{(5.37)}\operatorname{rank}\widetilde{\mathcal{M}}(k;\beta)+1.\end{aligned}$$

Case 4: F_{22} and H_{22} are pd, $\eta=0$. We separate two cases according to the value of $u_{\max}$.

Case 4.1: $u_{\max}=0$. Since $u_{\max}=0$, it follows that $\tilde{u}_0=0$ and by (5.18), $k_{12}=0$. But then $\gamma_1(0,0)$ is $\mathbb{R}$–representable. Similarly, since $\gamma_2(\tilde{t}_0,0)$ satisfies (Hyp), it follows that $\gamma_2(0,0)$ also satisfies (Hyp). So $\mathcal{F}(A_{\min})$ and $\mathcal{H}(A_{\min})$ admit a $\mathcal{Z}(x+y-xy)$–rm and a $\mathbb{R}$–rm, respectively. Then (5.37) implies the existence of a $(\operatorname{rank}\widetilde{\mathcal{M}}(k;\beta))$–atomic $\mathcal{Z}(p)$–rm for β.

Case 4.2: $u_{\max}>0$. If $t_{\max}=0$, then $\tilde{t}_0=0$. Since $\eta=0$, $\gamma_2(0,\tilde{u}_0)$ can satisfy (Hyp) only if $\tilde{u}_0=0$ (see (5.16)). But $\gamma_1(0,0)$ cannot be $\mathbb{R}$–representable if $t_{\max}=0$, since then also $u_{\max}$ should be 0 (see Theorem 2.5). It follows that $t_{\max}>0$. We separate two subcases according to the value of k_{12}.

Case 4.2.1: $k_{12}=0$. In this case $\gamma_1(t_{\max},u_{\max})$ is $\mathbb{R}$–representable by Theorem 2.5 and $\gamma_2(t_{\max},u_{\max})$ satisfies (Hyp). We have

$$\begin{aligned}\operatorname{rank}\mathcal{H}(\mathcal{G}(t_{\max},u_{\max}))+\operatorname{rank}\mathcal{F}(\mathcal{G}(t_{\max},u_{\max}))&=\operatorname{rank}H_{22}+(\operatorname{rank}\mathcal{F}(A_{\min})+2)\\&=\operatorname{rank}\mathcal{H}(A_{\min})+\operatorname{rank}\mathcal{F}(A_{\min})\\&=\operatorname{rank}\widetilde{\mathcal{M}}(k;\beta),\end{aligned}\tag{5.46}$$

where we used (5.16), (5.19) in the first, (5.16) in the second and (5.37) in the third equality. So there exists a $(\operatorname{rank}\widetilde{\mathcal{M}}(k;\beta))$–atomic $\mathcal{Z}(p)$–rm for β.

Case 4.2.2: $k_{12}\neq 0$. We separate two cases according to $\operatorname{rank}\mathcal{H}(A_{\min})$, which can be either k or $k+1$ (using (5.19) and H_{22} is pd, $k_{12}\neq 0$).

Case 4.2.2.1: $\operatorname{rank}\mathcal{H}(A_{\min})=k$. By (5.19), it follows that $t_{\max}u_{\max}=k_{12}^2$ (plug $t=0$ and $u=0$ in (5.19)) and hence $\mathcal{R}_2=\{(0,0)\}$. This implies that under the assumptions of this case $(0,0)$ is the only candidate for $(\tilde{t}_0,\tilde{u}_0)$, which means that there exists a $(\operatorname{rank}\widetilde{\mathcal{M}}(k;\beta))$–atomic $\mathcal{Z}(p)$–rm for β.

Case 4.2.2.2: $\operatorname{rank}\mathcal{H}(A_{\min})=k+1$. By (5.19), it follows that $t_{\max}u_{\max}>k_{12}^2$ (plug $t=0$ and $u=0$ in (5.19)). We separate two cases according to whether $(0,0)$ is a good choice for $(\tilde{t}_0,\tilde{u}_0)$ in (5.9) or not.

Case 4.2.2.2.1: $(0,0)$ is a good choice for $(\tilde{t}_0,\tilde{u}_0)$ in (5.9). In this case there exists a $(\operatorname{rank}\widetilde{\mathcal{M}}(k;\beta))$–atomic $\mathcal{Z}(p)$–rm for β.

Case 4.2.2.2.2: $(0,0)$ is not a good choice for $(\tilde{t}_0,\tilde{u}_0)$ in (5.9). Note that $A_{\gamma_2(t,u)}$ satisfies (Hyp) precisely for $t>0$ and $u>0$. By Lemma 5.5, there is a point $(\tilde{t},\tilde{u})\in\mathcal{R}_2\cap(\mathbb{R}_+)^2$, such that $0<\tilde{t}\tilde{u}$ (since $p_{\max}>0$) and $(t_{\max}-\tilde{t})(u_{\max}-\tilde{u})=k_{12}^2$. By Theorem 2.5, $\gamma_1(\tilde{t},\tilde{u})$ is $\mathbb{R}$–representable, since $A_{\gamma_1(\tilde{t},\tilde{u})}$ is psd and $A_{\gamma_1(\tilde{t},\tilde{u})}(k)$ is pd. Note also that $\gamma_2(\tilde{t},\tilde{u})$ satisfies

(Hyp). Hence, $(\tilde{t}, \tilde{u})$ is a good choice for $(\tilde{t}_0, \tilde{u}_0)$ in (5.9). We have

$$\begin{aligned} \operatorname{rank} \mathcal{H}(\mathcal{G}(\tilde{t}, \tilde{u})) + \operatorname{rank} \mathcal{F}(\mathcal{G}(\tilde{t}, \tilde{u})) &\underbrace{=}_{\substack{(5.16),\\(5.19)}} (\operatorname{rank} H_{22} + 1) + (\operatorname{rank} \mathcal{F}(A_{\min}) + 2) \\ &= \operatorname{rank} \mathcal{H}(A_{\min}) + \operatorname{rank} \mathcal{F}(A_{\min}) + 1 \\ &\underbrace{=}_{(5.37)} \operatorname{rank} \widetilde{\mathcal{M}}(k; \beta) + 1, \end{aligned}$$

where in the second equality we used the assumption $\operatorname{rank} \mathcal{H}(A_{\min}) = k + 1$. So there exists a $(\operatorname{rank} \widetilde{\mathcal{M}}(k; \beta) + 1)$–atomic $\mathcal{Z}(p)$–rm for β. It remains to show that there does not exist a $(\operatorname{rank} \widetilde{\mathcal{M}}(k; \beta))$–atomic rm. By the first sentence of this case above, note that if (t', u') is a good choice for $(\tilde{t}_0, \tilde{u}_0)$ in (5.9), then $\operatorname{rank} \mathcal{F}(\mathcal{G}(t', u')) = \operatorname{rank} \mathcal{F}(A_{\min}) + 2$. Since $k_{12} \neq 0$, it follows by (5.19) that $\operatorname{rank} \mathcal{H}(\mathcal{G}(t', u')) \geq \operatorname{rank} H_{22} + 1 = \operatorname{rank} \mathcal{H}(A_{\min}) - 1$. Hence,

$$\begin{aligned} \operatorname{rank} \mathcal{H}(\mathcal{G}(t', u')) + \operatorname{rank} \mathcal{F}(\mathcal{G}(t', u')) &\geq (\operatorname{rank} \mathcal{H}(A_{\min}) - 1) + (\operatorname{rank} \mathcal{F}(A_{\min}) + 2) \\ &\underbrace{=}_{(5.37)} \operatorname{rank} \widetilde{\mathcal{M}}(k; \beta) + 1, \end{aligned}$$

Case 5: F_{22} and H_{22} are pd, $\eta \neq 0$. We separate two cases according to the value of k_{12}.

Case 5.1: $k_{12} = 0$. As in Case 4.2.1 above, $(t_{\max}, u_{\max})$ is a good choice for $(\tilde{t}_0, \tilde{u}_0)$ in (5.9). Since $\eta \neq 0$, (5.15) implies that $t_{\max} > 0$ and $u_{\max} > 0$. Then (5.19) implies that $\operatorname{rank} \mathcal{H}(\widehat{A}_{\min}) = k+1$. By (5.13), $\mathcal{H}(A_{\min})/H_{22} = \begin{pmatrix} t_{\max} & -\eta \\ -\eta & u_{\max} \end{pmatrix}$ and hence $\operatorname{rank} \mathcal{H}(A_{\min}) = \{k, k+1\}$. We separate two cases according to $\operatorname{rank} \mathcal{H}(A_{\min})$.

Case 5.1.1: $\operatorname{rank} \mathcal{H}(A_{\min}) = k+1$**.** By the same computation as in (5.46), there is a $(\operatorname{rank} \widetilde{\mathcal{M}}(k; \beta))$–atomic $\mathcal{Z}(p)$–rm for β in this case.

Case 5.1.2: $\operatorname{rank} \mathcal{H}(A_{\min}) = k$**.** Since H_{22} is pd and $\operatorname{rank} \mathcal{H}(A_{\min}) = k$, it follows that $t_{\max} u_{\max} = \eta^2$. Hence, $(t_{\max}, u_{\max})$ is the only candidate for $(\tilde{t}_0, \tilde{u}_0)$ in (5.9). By the same computation as in (5.45), β has a $(\operatorname{rank} \widetilde{\mathcal{M}}(k; \beta))$–atomic $\mathcal{Z}(p)$–rm.

Case 5.2: $k_{12} \neq 0$. We separate two cases according to whether $A_{\min}$ satisfies $(\mathrm{Hyp})_1$ or not.

Case 5.2.1: $A_{\min}$ satisfies $(\mathrm{Hyp})_1$. By analogous reasoning as in Case 2.3.2.2 of the proof of Theorem 5.2, one of $(t_{\pm,\eta^2}, u_{\pm,\eta^2})$ is a good choice for $(\tilde{t}_0, \tilde{u}_0)$ in (5.9). By (5.18), $t_{\max} > 0$ and $u_{\max} > 0$ (since $k_{12} \neq 0$). By (5.13), $\mathcal{H}(A_{\min})/H_{22} = \begin{pmatrix} t_{\max} & k_{12} - \eta \\ k_{12} - \eta & u_{\max} \end{pmatrix}$. Since $\mathcal{H}(A_{\min})/H_{22}$ is not a zero matrix, we have $\operatorname{rank} \mathcal{H}(A_{\min}) \in \{k, k+1\}$. Let us define the number

$$R := \begin{cases} \operatorname{rank} \widetilde{\mathcal{M}}(k; \beta), & \text{if } \operatorname{rank} \mathcal{H}(A_{\min}) = k + 1, \\ \operatorname{rank} \widetilde{\mathcal{M}}(k; \beta) + 1, & \text{if } \operatorname{rank} \mathcal{H}(A_{\min}) = k. \end{cases} \tag{5.47}$$

We have

$$\begin{aligned} &\operatorname{rank} \mathcal{H}(\mathcal{G}(t_{\pm,\eta^2}, u_{\pm,\eta^2})) + \operatorname{rank} \mathcal{F}(\mathcal{G}(t_{\pm,\eta^2}, u_{\pm,\eta^2})) \\ \underbrace{=}_{\substack{(5.16),\\(5.19)}} &(\operatorname{rank} H_{22} + 1) + (\operatorname{rank} \mathcal{F}(A_{\min}) + 1) \underbrace{=}_{(5.37)} R. \end{aligned} \tag{5.48}$$

So there is an R–atomic $\mathcal{Z}(p)$–rm for β.

Case 5.2.2: $A_{\min}$ does not satisfy $(\mathrm{Hyp})_1$. Let R be as in (5.47). We will show that in this case there is a $(R+1)$–atomic $\mathcal{Z}(p)$–rm and there does not exist an R–atomic $\mathcal{Z}(p)$–rm. Since $\eta \neq 0$, if $\mathcal{F}(\mathcal{G}(t', u'))$ is psd, it follows that $t'u' \geq \eta^2$ by (5.15). By the same argument as in the first paragraph of Case 2.3.2.2 of the proof of Theorem 5.2, if one of $(t_{\pm,\eta^2}, u_{\pm,\eta^2})$ is a good choice for $(\tilde{t}_0, \tilde{u}_0)$ in (5.9), then $A_{\min}$ satisfies $(\mathrm{Hyp})_1$, which is a contradiction. If $\eta^2 = p_{\max}$ from Lemma 5.5, then $(t_{\pm,\eta^2}, u_{\pm,\eta^2})$ are the only candidates for a good choice for $(\tilde{t}_0, \tilde{u}_0)$ in (5.9) and hence β does not have a $\mathcal{Z}(p)$–rm. It follows that $\eta^2 < p_{\max}$. By Lemma 5.5, there exists $(\check{t}, \check{u}) \in \mathcal{R}_2 \cap (\mathbb{R}_+)^2$, such that $\check{t}\check{u} = \frac{\eta^2 + p_{\max}}{2}$ and $(t_{\max} - \check{t})(u_{\max} - \check{u}) = k_{12}^2$. But then $\operatorname{rank} \mathcal{F}(\mathcal{G}(\check{t}, \check{u})) = 2k+1$, and $\gamma_2(\check{t}, \check{u})$ satisfies (Hyp). Also, $\gamma_1(\check{t}, \check{u})$ is $\mathbb{R}$–representable, since $A_{\gamma_1(\check{t},\check{u})} \succeq 0$ and $A_{\gamma_1(\check{t},\check{u})}(k)$ is pd (since $\check{t} < t_{\max}$). Repeating the calculation (5.48) and using that $\operatorname{rank} \mathcal{F}(\mathcal{G}(\check{t}, \check{u})) = \operatorname{rank} \mathcal{F}(A_{\min}) + 2$ instead of $\operatorname{rank} \mathcal{F}(\mathcal{G}(t_{\pm,\eta^2}, u_{\pm,\eta^2})) = \operatorname{rank} \mathcal{F}(A_{\min}) + 1$, we get $\operatorname{rank} \mathcal{H}(\mathcal{G}(\check{t}, \check{u})) + \operatorname{rank} \mathcal{F}(\mathcal{G}(\check{t}, \check{u})) = R+1$ and β admits an $(R+1)$–atomic $\mathcal{Z}(p)$–rm. It remains to show that there does not exist an R–atomic rm. As above, if $\mathcal{F}(\mathcal{G}(t', u'))$ is psd and has a $\mathcal{Z}(x+y-xy)$–rm, it follows that $t'u' > \eta^2$, which means that $\operatorname{rank} \mathcal{F}(\mathcal{G}(t', u')) = \operatorname{rank} \mathcal{F}(A_{\min}) + 2$. Since $k_{12} \neq 0$, $\operatorname{rank} \mathcal{H}(\mathcal{G}(t', u')) \geq \operatorname{rank} H_{22} + 1$ by (5.19). Hence,

$$\operatorname{rank} \mathcal{H}(\mathcal{G}(t', u')) + \operatorname{rank} \mathcal{F}(\mathcal{G}(t', u')) \geq (\operatorname{rank} H_{22} + 1) + (\operatorname{rank} \mathcal{F}(A_{\min}) + 2) \underbrace{=}_{(5.37)} R + 1.$$

It remains to establish the moreover part. Note that in the case where $\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta) + 2$ atoms might be needed, $\mathcal{H}(A_{\min})$ is not pd. Since for a p–pure sequence β with $\widetilde{\mathcal{M}}(k;\beta) \succeq 0$, (5.37) implies that F_{22} and $\mathcal{H}(A_{\min})$ are pd, the existence of a $\mathcal{Z}(p)$–rm implies the existence of at most $(\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta) + 1)$–atomic $\mathcal{Z}(p)$–rm. $\square$

5.3. **Example.** [2] In this subsection we demonstrate the use of Theorems 5.2 and 5.6 on a numerical example.

Let β be a bivariate degree 6 sequence given by

$$\begin{aligned}
&\beta_{00} = 1,\\
&\beta_{10} = \tfrac{11}{50}, \beta_{01} = -\tfrac{13}{100}\\
&\beta_{20} = \tfrac{12397}{18000}, \beta_{11} = -\tfrac{11}{100}, \beta_{02} = \tfrac{2947}{18000},\\
&\beta_{30} = \tfrac{1001}{1250}, \beta_{21} = -\tfrac{383}{18000}\ \beta_{12} = \tfrac{967}{18000}, \beta_{03} = -\tfrac{1117}{10000},\\
&\beta_{40} = \tfrac{117670993}{64800000}, \beta_{31} = -\tfrac{1843}{90000}, \beta_{22} = \tfrac{73}{2250}, \beta_{13} = -\tfrac{2609}{45000}, \beta_{04} = \tfrac{7105993}{64800000},\\
&\beta_{50} = \tfrac{100001}{31250}, \beta_{41} = -\tfrac{295967}{64800000}, \beta_{32} = \tfrac{359}{30000}, \beta_{23} = -\tfrac{383}{15000}, \beta_{14} = \tfrac{3349033}{64800000}, \beta_{05} = -\tfrac{103093}{1000000},\\
&\beta_{60} = \tfrac{1540453883617}{233280000000}, \beta_{51} = -\tfrac{1469467}{324000000}, \beta_{42} = \tfrac{479473}{64800000}, \beta_{33} = -\tfrac{407}{30000}, \beta_{24} = \tfrac{1694473}{64800000},\\
&\qquad\qquad \beta_{15} = -\tfrac{16656967}{324000000}, \beta_{06} = \tfrac{23769383617}{233280000000}.
\end{aligned}$$

We will prove below that β admits a 9–atomic $\mathcal{Z}(p)$–rm by applying Theorems 5.2 and 5.6. It is easy to check that $\widehat{\mathcal{M}}(3)$ is psd and satisfies only one column relation $YX + Y^2 - XY^2 = \mathbf{0}$. It turns out that $\eta = -\frac{51255911}{6577059124404}$, $t_{\max} = \frac{1827880655851}{20096569546790}$, $u_{\max} = \frac{272763812083768883}{833444932244474880}$ and $k_{12} = -\frac{9}{55}$. Computing $t_{-,\eta^2}, u_{-,\eta^2}$ we get

$$u_{-,\eta^2} = -\frac{49(-1858396786907068917274071 1 + 1644264781101\sqrt{12774179995369398596952890 5})}{553977407042444727681998008 32},$$

$$t_{-,\eta^2} = \frac{49(18583967869070689172740711 + 1644264781101\sqrt{127741799953693985969528905})}{19933152434141890714714234 6748}.$$

[2]The *Mathematica* file with numerical computations can be found on the link https://github.com/ZalarA/TMP_cubic_reducible.

It is easy to check that $\mathcal{H}(\mathcal{G}(t_{-,\eta^2}, u_{-,\eta^2}))$ is psd of rank 3 and $\mathcal{H}(\mathcal{G}(t_{-,\eta^2}, u_{-,\eta^2}))(2) \succ 0$. Hence, $\mathcal{H}(\mathcal{G}(t_{-,\eta^2}, u_{-,\eta^2}))$ admits a 3–atomic $\mathbb{R}$–rm. Moreover, $\mathcal{F}(\mathcal{G}(t_{-,\eta^2}, u_{-,\eta^2}))$ satisfies $(\text{Hyp})_1$ and has rank 6. So it admits a 6–atomic $\mathcal{Z}(x+y-xy)$–rm, whence β has a 9–atomic $\mathcal{Z}(p)$–rm. This also follows from Theorem 5.6, since $F_{22} \succ 0$, $H_{22} \succ 0$, $A_{\min}$ satisfies $(\text{Hyp})_1$ with $\operatorname{rank} \mathcal{F}(A_{\min}) = \operatorname{rank} \mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{X^3\}} = \operatorname{rank} \mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{Y^3\}} = 5$ and $\operatorname{rank} \mathcal{H}(A_{\min}) = 4$.

6. Hyperbolic type 3 relation: $p(x,y) = y(ay + x^2 - y^2)$, $a \notin \mathbb{R} \setminus \{0\}$.

In this section we solve constructively the $\mathcal{Z}(p)$–TMP for the sequence $\beta = \{\beta_{i,j}\}_{i,j\in\mathbb{Z}_+, i+j\leq 2k}$ of degree $2k$, $k \geq 3$, where $p(x,y)$ is as in the title of the section. The main results are Theorem 6.1, which characterizes concrete numerical conditions for the existence of a $\mathcal{Z}(p)$–rm for β, and Theorem 6.7, which characterizes the number of atoms needed in a minimal $\mathcal{Z}(p)$–rm. A numerical example demonstrating the main results is presented in Subsection 6.3.

6.1. **Existence of a representing measure.** Assume the notation from Section 3. If β admits a $\mathcal{Z}(p)$–TMP, then $\mathcal{M}(k;\beta)$ must satisfy the relations

$$aY^{2+j}X^i + Y^{1+j}X^{2+i} = Y^{3+j}X^i \quad \text{for } i,j \in \mathbb{Z}_+ \text{ such that } i+j \leq k-3. \tag{6.1}$$

In the presence of all column relations (6.1), the column space $\mathcal{C}(\mathcal{M}(k;\beta))$ is spanned by the columns in the tuple

$$\vec{\mathcal{T}} := (\vec{X}^{(0,k)}, Y\vec{X}^{(0,k-1)}, Y^2\vec{X}^{(0,k-2)}), \tag{6.2}$$

where

$$Y^i\vec{X}^{(j,\ell)} := (Y^iX^j, Y^iX^{j+1}, \ldots, Y^iX^\ell) \quad \text{with } i,j,\ell \in \mathbb{Z}_+,\ j \leq \ell,\ i+\ell \leq k.$$

Let $\widetilde{\mathcal{M}}(k;\beta)$ be as in (3.6) and define

$$A_{\min} := A_{12}(A_{22})^{\dagger}(A_{12})^T \quad \text{and} \quad \widehat{A}_{\min} := A_{\min} + \eta E_{2,2}^{(k+1)}, \tag{6.3}$$

where

$$\eta := (A_{\min})_{1,3} - (A_{\min})_{2,2}.$$

See Remark 3.4 for the explanation of these definitions. Let $\mathcal{F}(\mathbf{A})$ and $\mathcal{H}(\mathbf{A})$ be as in (3.9). Write

$$\mathcal{H}(\widehat{A}_{\min}) := \begin{array}{c} \\ \mathit{1} \\ X \\ (\vec{X}^{(2,k)})^T \end{array} \begin{array}{c} \begin{array}{ccc} \mathit{1} & X & \vec{X}^{(2,k)} \end{array} \\ \begin{pmatrix} \beta_{0,0} - (A_{\min})_{1,1} & \beta_{1,0} - (A_{\min})_{1,2} & (h_{12}^{(1)})^T \\ \beta_{1,0} - (A_{\min})_{1,2} & \beta_{2,0} - (A_{\min})_{1,3} & (h_{12}^{(2)})^T \\ h_{12}^{(1)} & h_{12}^{(2)} & H_{22} \end{pmatrix} \end{array},$$

$$H_1 := \mathcal{H}(\widehat{A}_{\min})_{\{\mathit{1}\}\cup\vec{X}^{(2,k)}} = \begin{array}{c} \\ \mathit{1} \\ (\vec{X}^{(2,k)})^T \end{array} \begin{array}{c} \begin{array}{cc} \mathit{1} & \vec{X}^{(2,k)} \end{array} \\ \begin{pmatrix} \beta_{0,0} - (A_{\min})_{1,1} & (h_{12}^{(1)})^T \\ h_{12}^{(1)} & H_{22} \end{pmatrix} \end{array}, \tag{6.4}$$

$$H_2 := \mathcal{H}(\widehat{A}_{\min})_{\vec{X}^{(1,k)}} = \begin{array}{c} \\ X \\ (\vec{X}^{(2,k)})^T \end{array} \begin{array}{c} \begin{array}{cc} X & \vec{X}^{(2,k)} \end{array} \\ \begin{pmatrix} \beta_{2,0} - (A_{\min})_{1,3} & (h_{12}^{(2)})^T \\ h_{12}^{(2)} & H_{22} \end{pmatrix} \end{array}.$$

We define also the matrix function

$$\mathcal{G}: \mathbb{R}^2 \to S_{k+1}, \qquad \mathcal{G}(\mathbf{t},\mathbf{u}) = \widehat{A}_{\min} + \mathbf{t}E_{1,1}^{(k+1)} + \mathbf{u}\big(E_{1,2}^{(k+1)} + E_{2,1}^{(k+1)}\big). \tag{6.5}$$

Let

$$\mathcal{R}_1=\big\{(t,u)\in\mathbb{R}^2\colon \mathcal{F}(\mathcal{G}(t,u))\succeq 0\big\} \quad \text{and} \quad \mathcal{R}_2=\big\{(t,u)\in\mathbb{R}^2\colon \mathcal{H}(\mathcal{G}(t,u))\succeq 0\big\}. \tag{6.6}$$

Further, we introduce real numbers

$$\begin{aligned} t_0 &:= \beta_{0,0}-(A_{\min})_{1,1}-(h_{12}^{(1)})^T(H_{22})^{\dagger}h_{12}^{(1)},\\ u_0 &:= \beta_{1,0}-(A_{\min})_{1,2}-(h_{12}^{(1)})^T(H_{22})^{\dagger}h_{12}^{(2)}, \end{aligned} \tag{6.7}$$

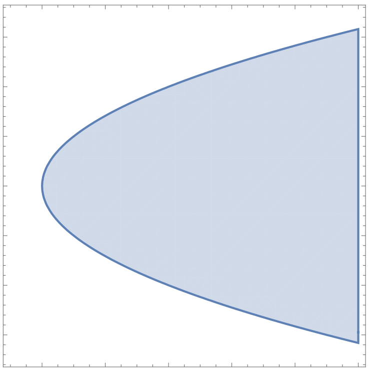
and a function

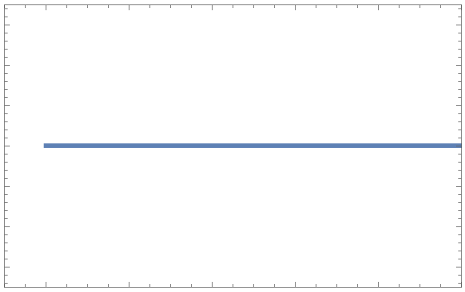
$$h(\mathbf{t})=\sqrt{(H_1/H_{22}-\mathbf{t})(H_2/H_{22})}. \tag{6.8}$$

It turns out [YZ24, Theorem 5.1, Claims 1–2] that

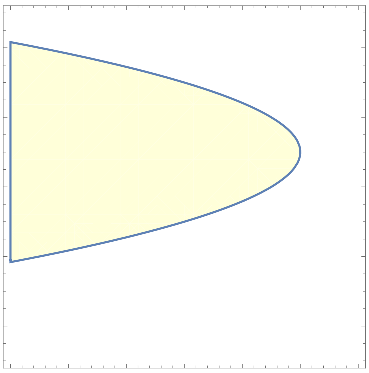
$$\begin{aligned} \mathcal{R}_1 &= \begin{cases} \big\{(t,u)\in\mathbb{R}^2\colon t\geq 0, u\in[-\sqrt{\eta t},\sqrt{\eta t}]\,\big\}, & \text{if } \eta\geq 0,\\ \varnothing, & \text{if } \eta<0,\end{cases}\\ \mathcal{R}_2 &= \begin{cases} \big\{(t,u)\in\mathbb{R}^2\colon t\leq t_0, u\in[u_0-h(t),u_0+h(t)]\big\}, & \text{if } H_2\succeq 0,\\ \varnothing, & \text{if } H_2\not\succeq 0.\end{cases} \end{aligned} \tag{6.9}$$

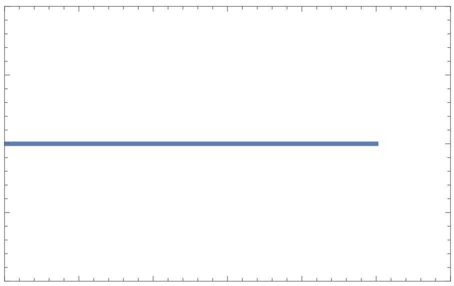
Therefore $\mathcal{R}_1$ has one of the following forms:





where the left case occurs if $\eta>0$, the right if $\eta=0$, while the case $\eta<0$ gives an empty set; and $\mathcal{R}_2$ can be one of the following:





where the left case occurs if $H_2/H_{22}>0$, the right if $H_2/H_{22}=0$, while the case $H_2/H_{22}<0$ gives an empty set.

By Lemmas 3.1–3.3 and Remark 3.4, the existence of a $\mathcal{Z}(p)$–rm for β is equivalent to:

$$\begin{aligned} &\widetilde{\mathcal{M}}(k;\beta)\succeq 0,\ \text{the relations (6.1) hold and}\\ &\text{there exists } (\tilde{t}_0,\tilde{u}_0)\in\mathcal{R}_1\cap\mathcal{R}_2 \text{ such that } \mathcal{F}(\mathcal{G}(\tilde{t}_0,\tilde{u}_0)) \text{ and } \mathcal{H}(\mathcal{G}(\tilde{t}_0,\tilde{u}_0))\\ &\text{admit a } \mathcal{Z}(ay+x^2-y^2)\text{–rm and a } \mathbb{R}\text{–rm, respectively.} \end{aligned} \tag{6.10}$$

Define the sequence

$$\begin{aligned}\gamma(\mathbf{t},\mathbf{u}) := \big(&\beta_{0,0}-(A_{\min})_{1,1}-\mathbf{t}, \beta_{1,0}-(A_{\min})_{1,2}-\mathbf{u}, \beta_{2,0}-\beta_{0,2}+a\beta_{0,1},\\ &\beta_{3,0}-\beta_{1,2}+a\beta_{1,1},\ldots,\beta_{2k,0}-\beta_{2k-2,2}+a\beta_{2k-2,1}\big).\end{aligned} \tag{6.11}$$

Note that $\mathcal{H}(\mathcal{G}(\mathbf{t},\mathbf{u})) = A_{\gamma(t,u)}$ (see (2.3) and Remark 3.4.(1)).

Write

$$\mathcal{B} := \{\mathit{1}, X, X^2, \ldots, X^k, Y, YX, \ldots, YX^{k-1}\}.$$

We say the matrix $A \in S_{k+1}$ satisfies **the property** $(\widetilde{\mathbf{Hyp}})$ if

$$\underbrace{\operatorname{rank}\mathcal{F}(A) = \operatorname{rank}\mathcal{F}(A)_{\mathcal{B}\setminus\{X^k\}} = \operatorname{rank}\mathcal{F}(A)_{\mathcal{B}\setminus\{YX^{k-1}\}}}_{(\widetilde{\text{Hyp}})_1} \quad\text{or}\quad \underbrace{\operatorname{rank}\mathcal{F}(A) = 2k+1}_{(\widetilde{\text{Hyp}})_2}. \tag{6.12}$$

We denote by $\partial\mathcal{R}_i$ and $\mathring{\mathcal{R}}_i$ the topological boundary and the interior of the set $\mathcal{R}_i$, respectively.

The solution to the $\mathcal{Z}(p)$–TMP is the following.

Theorem 6.1. *Let $p(x,y) = y(ay+x^2-y^2)$, $a \in \mathbb{R}\setminus\{0\}$, and $\beta = (\beta_{i,j})_{i,j\in\mathbb{Z}_+, i+j\le 2k}$, where $k \geq 3$. Assume also the notation above. Then the following statements are equivalent:*

1. *β has a $\mathcal{Z}(p)$–representing measure.*
2. *$\widetilde{\mathcal{M}}(k;\beta)$ is positive semidefinite, the relations (6.1) hold, $A_{\min}$ either satisfies $(\widetilde{Hyp})_1$ or the rank equality $\operatorname{rank}\mathcal{F}(A_{\min}) = 2k-1$, and one of the following statements is true:*
   - (a) *$\eta = 0$, $A_{\min}$ satisfies $(\widetilde{Hyp})_1$ and $\gamma(0,0)$ is $\mathbb{R}$–representable.*
   - (b) *$\eta > 0$ and one of the following holds:*
     - (i) *The set $\partial\mathcal{R}_1 \cap \partial\mathcal{R}_2$ has two elements and H_2 is positive definite.*
     - (ii) *$\partial\mathcal{R}_1\cap\partial\mathcal{R}_2 = \{(\tilde{t},\tilde{u})\}$ and there exist $(\hat{t},\hat{u}) \in \{((\tilde{t},\tilde{u}),(t_0,\tilde{u}))\}$ such that $\gamma(\hat{t},\hat{u})$ is $\mathbb{R}$–representable and $\mathcal{F}(\mathcal{G}(\hat{t},\hat{u}))$ satisfies $(\widetilde{Hyp})$.*

Before we prove Theorem 6.1 we need few lemmas. Their statements and the proofs coincide verbatim with [YZ24, Theorem 5.1, Claims 1–5], but we state them for easier readability.

Let $\mathcal{R}_1,\mathcal{R}_2$ be as in (6.6). Claims 1 and 2 below describe ranks of $\mathcal{F}(\mathcal{G}(t,u))$ and $\mathcal{H}(\mathcal{G}(t,u))$ for various choices of (t,u) in $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively.

Lemma 6.2 ([YZ24, Theorem 5.1, Claim 1])**.** *Assume that $\widetilde{\mathcal{M}}(k;\beta) \succeq 0$. Then $\mathcal{R}_1$ is as in (6.9) above. If $\eta \geq 0$, then we have*

$$\operatorname{rank}\mathcal{F}(\mathcal{G}(t,u)) = \begin{cases} \operatorname{rank}\mathcal{F}(A_{\min}), & \text{if } t=0, \eta=0,\\ \operatorname{rank}\mathcal{F}(A_{\min})+1, & \text{if } (t>0 \text{ or } \eta>0) \text{ and } u\in\{-\sqrt{\eta t},\sqrt{\eta t}\},\\ \operatorname{rank}\mathcal{F}(A_{\min})+2, & \text{if } t>0, \eta>0, u\in(-\sqrt{\eta t},\sqrt{\eta t}), \end{cases} \tag{6.13}$$

where $A_{\min}$ is as in (6.3).

Lemma 6.3 ([YZ24, Theorem 5.1, Claim 2])**.** *Assume that $\widetilde{\mathcal{M}}(k;\beta) \succeq 0$. Let $t_0, u_0, h(\mathbf{t})$ be as in (6.7), (6.8). If $H_2 \succeq 0$, then we have*

$$\operatorname{rank}\mathcal{H}(\mathcal{G}(t,u)) = \begin{cases} \operatorname{rank} H_2, & \text{for } t=t_0, u=u_0,\\ \operatorname{rank} H_{22}+1, & \text{for } t<t_0, u\in\{u_0-h(t),u_0+h(t)\},\\ \operatorname{rank} H_{22}+2, & \text{for } t<t_0, u\in(u_0-h(t),u_0+h(t)). \end{cases} \tag{6.14}$$

Lemma 6.4 ([YZ24, Theorem 5.1, Claim 3]). *Assume that* $\widetilde{\mathcal{M}}(k;\beta) \succeq 0$ *and* $\eta = 0$. *Then* $(0,0) \in \partial\mathcal{R}_1 \cap \mathcal{R}_2$.

Lemma 6.5 ([YZ24, Theorem 5.1, Claim 4]). *Assume that* $\widetilde{\mathcal{M}}(k;\beta) \succeq 0$ *and* $\eta > 0$. *Then:*

- *The set* $\partial\mathcal{R}_1 \cap \partial\mathcal{R}_2$ *has at most 2 elements.*
- $\mathcal{R}_1 \cap \mathcal{R}_2 \neq \varnothing$ *if and only if* $\partial\mathcal{R}_1 \cap \partial\mathcal{R}_2 \neq \varnothing$.
- *If* $\partial\mathcal{R}_1 \cap \partial\mathcal{R}_2$ *has two elements, then* $H_2/H_{22} > 0$.
- *If* $\partial\mathcal{R}_1 \cap \partial\mathcal{R}_2$ *has one element, which we denote by* $(\tilde{t},\tilde{u})$, *then one of the following holds:*
  - $\mathcal{R}_1 \cap \mathcal{R}_2 = \partial\mathcal{R}_1 \cap \partial\mathcal{R}_2$.
  - $\partial\mathcal{R}_2 = \mathcal{R}_2 = \{(t,u_0)\colon t \leq t_0\}$ *and* $\partial\mathcal{R}_1 \cap \partial\mathcal{R}_2 \subsetneq \mathcal{R}_1 \cap \mathcal{R}_2 = \{(t,u_0)\colon \tilde{t} \leq t \leq t_0\}$.

Lemma 6.6 ([YZ24, Theorem 5.1, Claim 5]). *Assume that* $\widetilde{\mathcal{M}}(k;\beta) \succeq 0$. *Let* H_2 *(see (6.4)) be positive definite,* $(t_1,u_1) \in \partial\mathcal{R}_2$, $(t_2,u_2) \in \partial\mathcal{R}_2$ *and* $u_1 \neq u_2$. *Then at least one of* $\mathcal{H}(\mathcal{G}(t_1,u_1))$ *or* $\mathcal{H}(\mathcal{G}(t_2,u_2))$ *admits a* $\mathbb{R}$*–representing measure.*

Proof of Theorem 6.1. First we prove the implication (1) $\Rightarrow$ (2). There exists $(\tilde{t}_0,\tilde{u}_0) \in \mathcal{R}_1 \cap \mathcal{R}_2$ satisfying (6.10). In particular, $\mathcal{R}_1 \neq \varnothing$ and by (6.9), $\eta \geq 0$. Since $\mathcal{F}(\mathcal{G}(\tilde{t}_0,\tilde{u}_0))$ has a $\mathcal{Z}(ay+x^2-y^2)$–rm, it follows, by Theorem 2.8, that $\mathcal{G}(\tilde{t}_0,\tilde{u}_0)$ satisfies $(\widetilde{\text{Hyp}})$. Note that

$$\mathcal{F}(\mathcal{G}(t,u)) = \mathcal{F}(A_{\min}) + \begin{pmatrix} t & u \\ u & \eta \end{pmatrix} \oplus \mathbf{0},$$

where $\mathbf{0}$ is a zero matrix of the appropriate size. Moreover, by definition of $A_{\min}$, we have

$$\begin{aligned}
\operatorname{rank}\mathcal{F}(\mathcal{G}(t,u)) &= \operatorname{rank}\mathcal{F}(A_{\min}) + \operatorname{rank}\begin{pmatrix} t & u \\ u & \eta \end{pmatrix},\\
\operatorname{rank}\mathcal{F}(\mathcal{G}(t,u))_{\mathcal{B}\setminus\{X^k\}} &\leq \operatorname{rank}\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{X^k\}} + \operatorname{rank}\begin{pmatrix} t & u \\ u & \eta \end{pmatrix}\\
&\leq \operatorname{rank}\mathcal{F}(A_{\min}) + \operatorname{rank}\begin{pmatrix} t & u \\ u & \eta \end{pmatrix},\\
\operatorname{rank}\mathcal{F}(\mathcal{G}(t,u))_{\mathcal{B}\setminus\{YX^{k-1}\}} &\leq \operatorname{rank}\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{YX^{k-1}\}} + \operatorname{rank}\begin{pmatrix} t & u \\ u & \eta \end{pmatrix}\\
&\leq \operatorname{rank}\mathcal{F}(A_{\min}) + \operatorname{rank}\begin{pmatrix} t & u \\ u & \eta \end{pmatrix}.
\end{aligned} \tag{6.15}$$

If $\mathcal{G}(\tilde{t}_0,\tilde{u}_0)$ satisfies $(\widetilde{\text{Hyp}})_1$, then all inequalities in (6.15) must be equalities and in particular, $A_{\min}$ satisfies $(\widetilde{\text{Hyp}})_1$. If $\mathcal{G}(\tilde{t}_0,\tilde{u}_0)$ satisfies $(\widetilde{\text{Hyp}})_2$, then clearly $\operatorname{rank}\mathcal{F}(A_{\min}) = 2k-1$. From now on we separate two cases according to the value of η.

Case 1: $\eta = 0$**.** For $\eta = 0$ we have $\widehat{A}_{\min} = A_{\min}$. By (6.9), we have $\tilde{u}_0 = 0$ and $\tilde{t}_0 \geq 0$. By Lemma 6.2, $\mathcal{G}(\tilde{t}_0,0)$ cannot satisfy $(\widetilde{\text{Hyp}})_2$ and hence it satisfies $(\widetilde{\text{Hyp}})_1$. But then by the explanation above, $A_{\min}$ satisfies $(\widetilde{\text{Hyp}})_1$ and by Corollary 2.11, it has a $\mathcal{Z}(ay+x^2-y^2)$–rm. By Theorem 2.5, $\mathbb{R}$–representability of $\gamma(\tilde{t}_0,0)$ implies $\mathbb{R}$–representability of $\gamma(0,0)$. This is Theorem 6.1.(2a).

Case 2: $\eta > 0$**.** By Lemma 6.5, $\partial\mathcal{R}_1 \cap \partial\mathcal{R}_2$ has one or two elements. We separate two cases according to the number of elements in $\partial\mathcal{R}_1 \cap \partial\mathcal{R}_2$.

Case 2.1: $\partial\mathcal{R}_1\cap\partial\mathcal{R}_2$ has two elements. By Lemma 6.5, $H_2/H_{22}>0$. Assume that $H_2\not\succ 0$. Then there is a nontrivial column relation among columns $X^2,\ldots,X^k$ in H_2. By Proposition 2.4, the same holds for $\mathcal{H}(\mathcal{G}(\tilde{t}_0,\tilde{u}_0))$. Let $\sum_{i=0}^{k-2}c_iX^{i+2}=\mathbf{0}$ be the nontrivial column relation in $\mathcal{H}(\mathcal{G}(\tilde{t}_0,\tilde{u}_0))$. But then $\mathcal{Z}(x^2\sum_{i=0}^{k-2}c_ix^i)=\mathcal{Z}(x\sum_{i=0}^{k-2}c_ix^i)$ and it follows by [CF96] that $\sum_{i=0}^{k-2}c_iX^{i+1}=\mathbf{0}$ is also a nontrivial column relation in $\mathcal{H}(\mathcal{G}(\tilde{t}_0,\tilde{u}_0))$. Inductively, this implies $H_2/H_{22}=0$, which is a contradiction. Hence, $H_2\succ 0$. This is the case of Theorem 6.1.(2(b)i).

Case 2.2: $\partial\mathcal{R}_1\cap\partial\mathcal{R}_2$ has one element. Let us denote this element by $(\tilde{t},\tilde{u})$. By Lemma 6.5, $\partial\mathcal{R}_1\cap\partial\mathcal{R}_2=\mathcal{R}_1\cap\mathcal{R}_2$ or $\partial\mathcal{R}_2=\mathcal{R}_2=\{(t,u_0)\colon t\leq t_0\}$ and $\partial\mathcal{R}_1\cap\partial\mathcal{R}_2\subsetneq\mathcal{R}_1\cap\mathcal{R}_2=\{(t,u_0)\colon \tilde{t}\leq t\leq t_0\}$. We separate two cases according to these two possibilities.

Case 2.2.1: $\partial\mathcal{R}_1\cap\partial\mathcal{R}_2=\mathcal{R}_1\cap\mathcal{R}_2$. In this case $(\tilde{t}_0,\tilde{u}_0)=(\tilde{t},\tilde{u})$ and hence $\gamma(\tilde{t},\tilde{u})$ is $\mathbb{R}$–representable, while by Corollary 2.11, $\mathcal{F}(\mathcal{G}(\tilde{t},\tilde{u}))$ satisfies $(\widetilde{\text{Hyp}})$. This is the case of Theorem 6.1.(2(b)ii).

Case 2.2.2: $\partial\mathcal{R}_2=\mathcal{R}_2=\{(t,u_0)\colon t\leq t_0\}$ and $\partial\mathcal{R}_1\cap\partial\mathcal{R}_2\subsetneq\mathcal{R}_1\cap\mathcal{R}_2=\{(t,u_0)\colon \tilde{t}\leq t\leq t_0\}$. By (6.9), it follows that $H_2/H_{22}=0$ (see definition (6.8) of $h(\mathbf{t})$). Since H_2 is not pd, Theorem 2.5 used for $\mathcal{H}(\mathcal{G}(\tilde{t}_0,\tilde{u}_0))$, implies that the last column of H_2 is in the span of the others. Hence, by Proposition 2.4, the same holds for $\mathcal{H}(\mathcal{G}(\tilde{t},\tilde{u}))$ and $\mathcal{H}(\mathcal{G}(t_0,\tilde{u}))$, whence $\gamma(\tilde{t},\tilde{u})$ and $\gamma(t_0,\tilde{u})$ are both $\mathbb{R}$–representable. By Lemma 6.2, $\operatorname{rank}\mathcal{F}(\mathcal{G}(\tilde{t},\tilde{u}))=\operatorname{rank}\mathcal{F}(A_{\min})+1$ and $\operatorname{rank}\mathcal{F}(\mathcal{G}(t,\tilde{u}))=\operatorname{rank}\mathcal{F}(A_{\min})+2$ for $t\in(\tilde{t},t_0]$. If $\mathcal{G}(t,\tilde{u})$ satisfies $(\widetilde{\text{Hyp}})_2$ for some $t\in(\tilde{t},t_0]$, then it satisfies $\mathcal{G}(t,\tilde{u})$ satisfies $(\widetilde{\text{Hyp}})_2$ for every $t\in(\tilde{t},t_0]$. Similarly, by (6.15), if $\mathcal{G}(t,\tilde{u})$ satisfies $(\widetilde{\text{Hyp}})_1$ for some $t\in(\tilde{t},t_0]$, then it satisfies $\mathcal{G}(t,\tilde{u})$ satisfies $(\widetilde{\text{Hyp}})_1$ for every $t\in(\tilde{t},t_0]$. This holds because validity of $(\widetilde{\text{Hyp}})_1$ for one $t\in(\tilde{t},t_0]$, implies that all inequalities in (6.15) must be equalities, which is true only if

(6.16)

$$
\begin{aligned}
&(A_{\min})_{\{1,X\}}=\\
=&\mathcal{F}(\mathcal{G}(t,\tilde{u}))_{\{1,X\},\mathcal{B}\backslash\{1,X,X^k\}}\big(\mathcal{F}(\mathcal{G}(t,\tilde{u}))_{\mathcal{B}\backslash\{1,X,X^k\}}\big)^{\dagger}\mathcal{F}(\mathcal{G}(t,\tilde{u}))_{\mathcal{B}\backslash\{1,X,X^k\},\{1,X\}}\\
=&\mathcal{F}(\mathcal{G}(t,\tilde{u}))_{\{1,X\},\mathcal{B}\backslash\{1,X,YX^{k-1}\}}\big(\mathcal{F}(\mathcal{G}(t,\tilde{u}))_{\mathcal{B}\backslash\{1,X,YX^{k-1}\}}\big)^{\dagger}\mathcal{F}(\mathcal{G}(t,\tilde{u}))_{\mathcal{B}\backslash\{1,X,YX^{k-1}\},\{1,X\}}.
\end{aligned}
$$

But then (6.16) holds for every $t\in(\tilde{t},t_0]$ and consequently $(\widetilde{\text{Hyp}})_1$ holds for $\mathcal{G}(t,\tilde{u})$ for every $t\in(\tilde{t},t_0]$. If $\mathcal{G}(t_0,\tilde{u})$ does not satisfy $(\widetilde{\text{Hyp}})$, it does not admit a $\mathcal{Z}(ay+x^2-y^2)$–rm by Corollary 2.11, which further implies that $\mathcal{G}(\tilde{t},\tilde{u})$ satisfies $(\widetilde{\text{Hyp}})$. This is the case of Theorem 6.1.(2(b)ii).

This concludes the proof of the implication (1) $\Rightarrow$ (2) of Theorem 6.1.

It remains to prove the implication (2) $\Rightarrow$ (1) of Theorem 6.1. We separate four cases according to the assumptions in Theorem 6.1.(2).

Case 1: Theorem 6.1.(2a) holds. By Lemma 6.4, $(0,0)\in\mathcal{R}_1\cap\mathcal{R}_2$. Further, $\eta=0$ implies that $\widehat{A}_{\min}=A_{\min}=\mathcal{G}(0,0)$ and hence $\mathcal{G}(0,0)$ satisfies $(\widetilde{\text{Hyp}})_1$ by assumption. By Corollary 2.11, $\mathcal{F}(\mathcal{G}(0,0))$ admits a $\operatorname{rank}(\mathcal{F}(A_{\min}))$–atomic $\mathcal{Z}(ay+x^2-y^2)$–rm. By assumption, $\gamma(0,0)$ is $\mathbb{R}$–representable. Hence, $(0,0)$ is a good choice for $(\tilde{t}_0,\tilde{u}_0)$ in (6.10). This proves (1) in this case.

Case 2: Theorem 6.1.(2(b)i) holds. By assumption, $\partial\mathcal{R}_1 \cap \partial\mathcal{R}_2 = \{(t_1, u_1), (t_2, u_2)\}$ has two distinct elements. Hence, $\partial\mathcal{R}_2$ is not a half-line and $\mathcal{R}_1 \cap \mathcal{R}_2$ has a non–empty interior, which is equal to $\mathring{\mathcal{R}}_1 \cap \mathring{\mathcal{R}}_2$. Since $H_2 \succ 0$, it follows, by Lemma 6.3, that $\mathcal{H}(\mathcal{G}(t,u)) \succ 0$ for every $(t,u) \in \mathring{\mathcal{R}}_2$. Hence, $\gamma(t,u)$ is $\mathbb{R}$–representable for every $(t,u) \in \mathring{\mathcal{R}}_2$. We separate two cases according to which of the assumptions:

- $A_{\min}$ satisfies $(\widetilde{\text{Hyp}})_1$.
- $\operatorname{rank} \mathcal{F}(A_{\min}) = 2k-1$.

holds.

Case 2.1: $A_{\min}$ satisfies $(\widetilde{\text{Hyp}})_1$. By Lemma 6.2, we have

$$\operatorname{rank} \mathcal{F}(\mathcal{G}(t_1,u_1)) = \operatorname{rank} \mathcal{F}(\mathcal{G}(t_2,u_2)) = \operatorname{rank} \mathcal{F}(A_{\min}) + 1.$$

We will prove that

$$\operatorname{rank} \mathcal{F}(\mathcal{G}(t_i,u_i))_{\mathcal{B}\setminus\{X^k\}} = \operatorname{rank} \mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{X^k\}} + 1 \quad \text{for } i = 1,2. \tag{6.17}$$

If (6.17) is not true, then

$$(A_{\min})_{\{1,X\}} \neq \underbrace{\mathcal{F}(\mathcal{G}(t_i,u_i)_{\{1,X\},\mathcal{B}\setminus\{1,X,X^k\}}\big(\mathcal{F}(\mathcal{G}(t_i,u_i))_{\mathcal{B}\setminus\{1,X,X^k\}}\big)^{\dagger}\mathcal{F}(\mathcal{G}(t_i,u_i))_{\mathcal{B}\setminus\{1,X,X^k\},\{1,X\}}}_{\breve{A}_{\min}}.$$

Note that the definition of $\breve{A}_{\min}$ does not depend on i, because t_i and u_i do not appear in the corresponding restrictions of $\mathcal{F}(\mathcal{G}(t_i,u_i))$. Clearly,

$$\begin{pmatrix} t_i & u_i \\ u_i & \eta \end{pmatrix} \succeq (A_{\min})_{\{1,X\}} - \breve{A}_{\min} \succeq 0 \quad \text{for } i = 1,2,$$

whence

$$\ker\big((A_{\min})_{\{1,X\}} - \breve{A}_{\min}\big) \subseteq \ker \begin{pmatrix} t_i & u_i \\ u_i & \eta \end{pmatrix} \quad \text{for } i = 1,2. \tag{6.18}$$

Since (6.18) holds for $i = 1,2$, it follows that $\ker\big((A_{\min})_{\{1,X\}} - \breve{A}_{\min}\big) = \mathbb{R}^2$, which contradicts to $(A_{\min})_{\{1,X\}} \neq \breve{A}_{\min}$. Hence, (6.17) is true. Similarly,

$$\operatorname{rank} \mathcal{F}(\mathcal{G}(t_i,u_i))_{\mathcal{B}\setminus\{YX^{k-1}\}} = \operatorname{rank} \mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{YX^{k-1}\}} + 1 \quad \text{for } i = 1,2.$$

So $\mathcal{G}(t_i,u_i)$ satisfies $(\widetilde{\text{Hyp}})_1$ for $i = 1,2$. By Corollary 2.11, $\mathcal{F}(\mathcal{G}(t_i,u_i))$ admits a $\mathcal{Z}(ay+x^2-y^2)$–rm for $i = 1,2$. By Lemma 6.6, there is $j \in \{1,2\}$ such that $\mathcal{H}(\mathcal{G}(t_j,u_j))$ admits a $\mathbb{R}$–rm, whence (t_j,u_j) is a good choice for $(\tilde{t}_0,\tilde{u}_0)$ in (6.10). This proves (1) in this case.

Case 2.2: $\operatorname{rank} \mathcal{F}(A_{\min}) = 2k-1$**.** By Lemma 6.2, $\operatorname{rank} \mathcal{F}(\mathcal{G}(t,u)) = 2k+1$ for every $(t,u) \in \mathring{\mathcal{R}}_1$. By Corollary 2.11, $\mathcal{F}(\mathcal{G}(t,u))$ admits a $\mathcal{Z}(ay+x^2-y^2)$–rm for every $(t,u) \in \mathring{\mathcal{R}}_1$. Hence, $(t,u) \in \mathring{\mathcal{R}}_1 \cap \mathring{\mathcal{R}}_2$ is a good choice for $(\tilde{t}_0,\tilde{u}_0)$ in (6.10). This proves (1) in this case.

Case 3: Theorem 6.1.(2(b)ii) holds. Clearly, one of the points $(\tilde{t},\tilde{u})$ or $(t_0,\tilde{u})$ is a good choice for $(\tilde{t}_0,\tilde{u}_0)$ in (6.10). This proves (1) in this case.

This concludes the proof of the implication (2) $\Rightarrow$ (1) of Theorem 6.1. $\square$

6.2. **Cardinality of a minimal representing measure.** The following theorem characterizes the cardinality of a minimal measure in case β admits a $\mathcal{Z}(p)$–rm.

Theorem 6.7. *Let $p(x,y) = y(ay + x^2 - y^2)$, $a \in \mathbb{R} \setminus \{0\}$, and $\beta = (\beta_{i,j})_{i,j\in\mathbb{Z}_+, i+j\leq 2k}$, where $k \geq 3$, admits a $\mathcal{Z}(p)$–representing measure. Assume the notation above. The following statements hold:*

(1) *There exists at most $(\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta) + 2)$–atomic $\mathcal{Z}(p)$–representing measure for β.*

(2) *There is no $\mathcal{Z}(p)$–representing measure with less than $\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta) + 2$ atoms if and only if $A_{\min}$ does not satisfy $(\widetilde{\text{Hyp}})_1$, $\operatorname{rank} \mathcal{F}(A_{\min}) = 2k - 1$, $\eta > 0$, $\partial\mathcal{R}_1 \cap \partial\mathcal{R}_2$ has two elements, H_2 is positive definite and $\operatorname{rank} \mathcal{H}(A_{\min}) = k$.*

(3) *There exists a $\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta)$–atomic $\mathcal{Z}(p)$–representing measure for β if and only if any of the following holds:*

   (a) *$\eta = 0$.*

   (b) *$\eta > 0$, $\operatorname{card}(\partial\mathcal{R}_1 \cap \partial\mathcal{R}_2) = 2$, $\widehat{A}_{\min}$ satisfies $(\widetilde{\text{Hyp}})_1$, $\mathcal{H}(A_{\min})$ is positive definite.*

   (c) *$\eta > 0$, $\operatorname{card}(\partial\mathcal{R}_1 \cap \partial\mathcal{R}_2) = \operatorname{card}(\mathcal{R}_1 \cap \mathcal{R}_2) = 1$ and the equality $\operatorname{rank} \mathcal{H}(A_{\min}) = \operatorname{rank} H_{22} + 2$ holds.*

   (d) *$\eta > 0$, $\operatorname{card}(\partial\mathcal{R}_1 \cap \partial\mathcal{R}_2) = 1$, $\{(\tilde{t},\tilde{u})\} = \partial\mathcal{R}_1 \cap \partial\mathcal{R}_2 \subsetneq \mathcal{R}_1 \cap \mathcal{R}_2$, $\mathcal{F}(\mathcal{G}(\tilde{t},\tilde{u}))$ admits a $\mathcal{Z}(ay + x^2 - y^2)$–representing measure and $\mathcal{H}(A_{\min}) = \operatorname{rank} H_{22} + 2$.*

In particular, a p–pure sequence β with a measure admits at most $(3k+1)$–atomic $\mathcal{Z}(p)$–representing measure.

Proof of Theorem 6.7. By Lemma 3.3.(4),

$$\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta) = \operatorname{rank} \mathcal{F}(A_{\min}) + \operatorname{rank} \mathcal{H}(A_{\min}). \tag{6.19}$$

We observe again the proof of the implication (2) $\Rightarrow$ (1) of Theorem 6.1.

In the proof of the implication Theorem 6.1.(2a) $\Rightarrow$ (6.10) we established that $\mathcal{F}(A_{\min})$ and $\mathcal{H}(A_{\min})$ admit a $\operatorname{rank} \mathcal{F}(A_{\min})$–atomic and a $\operatorname{rank} \mathcal{H}(A_{\min})$–atomic rm*s*. Using (6.19) it follows that β has a $\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta)$–atomic $\mathcal{Z}(p)$–rm.

In the proof of the implication Theorem 6.1.(2(b)i) $\Rightarrow$ (6.10) we separated two cases:

Case 1: $A_{\min}$ satisfies $(\widetilde{\textbf{Hyp}})_1$. In this case we established that $\gamma(t',u')$ is $\mathbb{R}$–representable for some $(t',u') \in \partial\mathcal{R}_1 \cap \partial\mathcal{R}_2$, where $\operatorname{rank} \mathcal{F}(\mathcal{G}(t',u')) = \operatorname{rank} \mathcal{F}(A_{\min}) + 1$ and $\operatorname{rank} \mathcal{H}(\mathcal{G}(t',u')) = k$. Since $\mathcal{H}(A_{\min}) \succeq \operatorname{rank} \mathcal{H}(\mathcal{G}(t',u'))$, it follows that

$$\operatorname{rank} \mathcal{F}(\mathcal{G}(t',u')) + \operatorname{rank} \mathcal{H}(\mathcal{G}(t',u')) =$$
$$= \begin{cases} \operatorname{rank} \widetilde{\mathcal{M}}(k;\beta), & \text{if } \operatorname{rank} \mathcal{H}(\mathcal{G}(t',u')) = \operatorname{rank} \mathcal{H}(A_{\min}) - 1, \\ \operatorname{rank} \widetilde{\mathcal{M}}(k;\beta) + 1, & \text{if } \operatorname{rank} \mathcal{H}(\mathcal{G}(t',u')) = \operatorname{rank} \mathcal{H}(A_{\min}). \end{cases}$$

It remains to show that if there does not exist (t,u), which is a good choice for $(\tilde{t}_0,\tilde{u}_0)$ in (6.10), such that $\operatorname{rank} \mathcal{H}(\mathcal{G}(t,u)) = \operatorname{rank} \mathcal{H}(A_{\min}) - 1$, then there is no $\big(\operatorname{rank} \widetilde{\mathcal{M}}(k;\beta)\big)$–atomic $\mathcal{Z}(p)$–rm. Since $\eta > 0$, it follows, by Lemma 6.2, that $\operatorname{rank} \mathcal{F}(\mathcal{G}(t,u)) \geq \operatorname{rank} \mathcal{F}(A_{\min} + 1$ for any good choice (t,u). Since also $\operatorname{rank} \mathcal{H}(\mathcal{G}(t,u)) \geq \operatorname{rank} \mathcal{H}(A_{\min})$, it follows that $\operatorname{rank} \mathcal{F}(\mathcal{G}(t,u)) + \operatorname{rank} \mathcal{H}(\mathcal{G}(t,u)) \geq \operatorname{rank} \widetilde{\mathcal{M}}(k;\beta) + 1$.

Case 2: $\operatorname{rank} \mathcal{F}(A_{\min}) = 2k - 1$**.** If $A_{\min}$ does not satisfy $(\widetilde{\text{Hyp}})_1$, $\mathcal{G}(t,u)$ does not satisfy $(\widetilde{\text{Hyp}})_1$ for any $(t,u) \in \mathbb{R}^2$. So every (t,u) which is a good choice for $(\tilde{t}_0,\tilde{u}_0)$, must satisfy $\operatorname{rank} \mathcal{F}(\mathcal{G}(t,u)) = 2k+1 = \operatorname{rank} \mathcal{F}(A_{\min}) + 2$. By Lemma 6.6, there exists $(t',u') \in \mathring{\mathcal{R}}_1 \cap \partial\mathcal{R}_2$, such that $\mathcal{H}(\mathcal{G}(t',u'))$ admits a $\mathbb{R}$–rm and satisfies

$$\operatorname{rank}\mathcal{H}(\mathcal{G}(t',u')) = \operatorname{rank} H_2 = \begin{cases} \operatorname{rank}\mathcal{H}(A_{\min}), & \text{if } \operatorname{rank}\mathcal{H}(A_{\min}) = k, \\ \operatorname{rank}\mathcal{H}(A_{\min}) - 1, & \text{if } \operatorname{rank}\mathcal{H}(A_{\min}) = k+1. \end{cases}$$

Hence,

$$\operatorname{rank}\mathcal{F}(\mathcal{G}(t',u')) + \operatorname{rank}\mathcal{H}(\mathcal{G}(t',u')) =$$
$$= \begin{cases} \operatorname{rank}\widetilde{\mathcal{M}}(k;\beta) + 2, & \text{if } \operatorname{rank}\mathcal{H}(A_{\min}) = k, \\ \operatorname{rank}\widetilde{\mathcal{M}}(k;\beta) + 1, & \text{if } \operatorname{rank}\mathcal{H}(A_{\min}) = k+1. \end{cases}$$

If (t,u) is a good choice for $(\tilde{t}_0,\tilde{u}_0)$, then $\operatorname{rank}\mathcal{H}(\mathcal{G}(t,u)) \geq k$ (since $H_2 \succ 0$) and also $\operatorname{rank}\mathcal{F}(\mathcal{G}(t,u)) = \operatorname{rank}\mathcal{F}(A_{\min}) + 2$. So

$$\operatorname{rank}\mathcal{F}(\mathcal{G}(t,u)) + \operatorname{rank}\mathcal{H}(\mathcal{G}(t,u)) \geq$$
$$\geq \operatorname{rank}\mathcal{F}(A_{\min}) + 2 + \begin{cases} \operatorname{rank}\mathcal{H}(A_{\min}), & \text{if } \operatorname{rank}\mathcal{H}(A_{\min}) = k, \\ \operatorname{rank}\mathcal{H}(A_{\min}) - 1, & \text{if } \operatorname{rank}\mathcal{H}(A_{\min} = k+1. \end{cases}$$

So the measure cannot contain less atoms than the one in (t',u') above.

Under the assumption Theorem 6.1.(2(b)ii) we separate two cases:

Case 1: $\partial\mathcal{R}_1 \cap \partial\mathcal{R}_2 = \mathcal{R}_1 \cap \mathcal{R}_2 = \{(\tilde{t},\tilde{u})\}$**.** Under the assumptions of this case, $\gamma(\tilde{t},\tilde{u})$ is $\mathbb{R}$–representable. Hence, $(\tilde{t},\tilde{u})$ is a good choice for $(\tilde{t}_0,\tilde{u}_0)$ in (6.10). If $\operatorname{rank}\mathcal{H}(A_{\min}) = \operatorname{rank} H_{22} + 2$, then a $\big(\operatorname{rank}\widetilde{\mathcal{M}}(k;\beta)\big)$–atomic $\mathcal{Z}(p)$–rm exists. This is due to the rank equality

$$r := \operatorname{rank}\mathcal{F}(\mathcal{G}(\tilde{t},\tilde{u})) + \operatorname{rank}\mathcal{H}(\mathcal{G}(\tilde{t},\tilde{u})) \underbrace{=}_{\substack{(5.16)\\(5.19)}} \operatorname{rank}\mathcal{F}(A_{\min}) + 1 + \operatorname{rank} H_{22} + 1.$$

Hence, $r = \operatorname{rank}\widetilde{\mathcal{M}}(k;\beta)$ if and only if $\operatorname{rank}\mathcal{H}(A_{\min}) = \operatorname{rank} H_{22} + 2$. Otherwise we have $\operatorname{rank}\mathcal{H}(A_{\min}) = \operatorname{rank} H_2 = \operatorname{rank} H_{22} + 1$ (since $\eta > 0$ and $H_2/H_{22} = 0$) and $r = \operatorname{rank}\widetilde{\mathcal{M}}(k;\beta) + 1$.

Case 2: $(\tilde{t},\tilde{u}) =: \partial\mathcal{R}_1 \cap \partial\mathcal{R}_2 \subsetneq \mathcal{R}_1 \cap \mathcal{R}_2$**.** In this case it follows by Theorem 6.1 that one of the points $(\tilde{t},\tilde{u})$ or $(t_0,\tilde{u})$ is a good choice for $(\tilde{t}_0,\tilde{u}_0)$ in (6.10).

Assume that $(t_0,\tilde{u})$ is a good choice. Since $\operatorname{rank}\mathcal{H}(\mathcal{G}(t_0,\tilde{u})) = \operatorname{rank}\mathcal{H}(A_{\min}) - 1$, which is due to $\eta > 0$ and $H_2/H_{22} = 0$ (if $H_2/H_{22} > 0$, then (6.9) would imply that $\operatorname{card}\partial\mathcal{R}_1 \cap \partial\mathcal{R}_2 > 1$), and $\operatorname{rank}\mathcal{F}(\mathcal{G}(t_0,\tilde{u})) = \operatorname{rank}\mathcal{F}(A_{\min}) + 2$ (by Lemma 6.2), it follows that a $\big(\operatorname{rank}\widetilde{\mathcal{M}}(k;\beta) + 1\big)$–atomic $\mathcal{Z}(p)$–rm exists.

As in the proof of Case 1 above, if $(\tilde{t},\tilde{u})$ is a good choice for $(\tilde{t}_0,\tilde{u}_0)$ in (6.10), then a $\big(\operatorname{rank}\widetilde{\mathcal{M}}(k;\beta)\big)$–atomic $\mathcal{Z}(p)$–rm exists if and only if $\operatorname{rank}\mathcal{H}(A_{\min}) = \operatorname{rank} H_{22} + 2$. Otherwise the measure is $\big(\operatorname{rank}\widetilde{\mathcal{M}}(k;\beta) + 1\big)$–atomic.

It remains to show that if $(\tilde{t},\tilde{u})$ is not a good choice for $(\tilde{t}_0,\tilde{u}_0)$ in (6.10), there does not exist a $\big(\operatorname{rank}\widetilde{\mathcal{M}}(k;\beta)\big)$–atomic $\mathcal{Z}(p)$–rm. By Lemma 6.5, the candidates for a good choice are points (t,u_0) for $t \in (\tilde{t},t_0]$. But as in the second paragraph above, we have $\operatorname{rank}\mathcal{H}(\mathcal{G}(t,\tilde{u})) \geq \operatorname{rank}\mathcal{H}(A_{\min}) - 1$ and $\operatorname{rank}\mathcal{F}(\mathcal{G}(t,\tilde{u})) = \operatorname{rank}\mathcal{F}(A_{\min}) + 2$ for every such t. So

$$\operatorname{rank}\mathcal{H}(\mathcal{G}(t,\tilde{u})) + \operatorname{rank}\mathcal{F}(\mathcal{G}(t,\tilde{u})) \geq \operatorname{rank}\widetilde{\mathcal{M}}(k;\beta) + 1.$$

It remains to establish the moreover part. Note that in the case where $\operatorname{rank}\widetilde{\mathcal{M}}(k;\beta) + 2$ atoms might be needed, $\mathcal{H}(A_{\min})$ is not pd. Since for a p–pure sequence β with $\widetilde{\mathcal{M}}(k;\beta) \succeq 0$, (6.19) implies that $\mathcal{H}(A_{\min})$ is pd, the existence of a $\mathcal{Z}(p)$–rm implies the existence of at most $(\operatorname{rank}\widetilde{\mathcal{M}}(k;\beta)+1)$–atomic $\mathcal{Z}(p)$–rm. This concludes the proof of Theorem 6.7. □

6.3. **Example.** [3] In this subsection we demonstrate the use of Theorems 6.1 and 6.7 on a numerical example.

Let β be a bivariate degree 6 sequence given by

$$\begin{aligned}
&\beta_{00}=1,\\
&\beta_{10}=\tfrac{37}{54},\ \beta_{01}=\tfrac{2}{3}\\
&\beta_{20}=\tfrac{769}{648},\ \beta_{11}=\tfrac{25}{54},\ \beta_{02}=\tfrac{1201}{648},\\
&\beta_{30}=\tfrac{11737}{7776},\ \beta_{21}=\tfrac{337}{648}\ \beta_{12}=\tfrac{12025}{7776},\ \beta_{03}=\tfrac{913}{216},\\
&\beta_{40}=\tfrac{258721}{93312},\ \beta_{31}=\tfrac{4825}{7776},\ \beta_{22}=\tfrac{169153}{93312},\ \beta_{13}=\tfrac{9625}{2592},\ \beta_{04}=\tfrac{957985}{93312},\\
&\beta_{50}=\tfrac{5088937}{1119744},\ \beta_{41}=\tfrac{72097}{93312},\ \beta_{32}=\tfrac{2497225}{1119744},\ \beta_{23}=\tfrac{136801}{31104},\ \beta_{14}=\tfrac{10813225}{1119744},\ \beta_{05}=\tfrac{2326373}{93312},\\
&\beta_{60}=\tfrac{115846129}{13436928},\ \beta_{51}=\tfrac{1107625}{1119744},\ \beta_{42}=\tfrac{38072593}{13436928},\ \beta_{33}=\tfrac{2034025}{373248},\ \beta_{24}=\tfrac{156268657}{13436928},\ \beta_{15}=\tfrac{27728525}{1119744},\\
&\hfill\beta_{06}=\tfrac{826264081}{13436928}.
\end{aligned}$$

We will prove below that β admits a 9–atomic $\mathcal{Z}(p)$–rm by applying Theorems 6.1 and 6.7. It is easy to check that $\widetilde{\mathcal{M}}(3)$ is psd and satisfies only one column relation $2Y^2+X^2Y-Y^3=\mathbf{0}$. It turns out that $\eta=0$, $\operatorname{rank}\mathcal{F}(A_{\min})=\operatorname{rank}\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{X^k\}}=\operatorname{rank}\mathcal{F}(A_{\min})_{\mathcal{B}\setminus\{YX^{k-1}\}}=5$, whence $A_{\min}$ satisfies $(\widetilde{\text{Hyp}})_1$. By Theorem 6.1, β has a $\mathcal{Z}(p)$–rm. By Theorem 6.7, there is a $\operatorname{rank}\widetilde{M}(3)$–atomic $\mathcal{Z}(p)$–rm (i.e., 9–atomic).

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DEPARTMENT OF MATHEMATICS EDUCATION AND RINS, GYEONGSANG NATIONAL UNIVERSITY, JINJU, 52828, KOREA.

Email address: seyoo@gnu.ac.kr

ALJAŽ ZALAR, FACULTY OF COMPUTER AND INFORMATION SCIENCE, UNIVERSITY OF LJUBLJANA & FACULTY OF MATHEMATICS AND PHYSICS, UNIVERSITY OF LJUBLJANA & INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS, LJUBLJANA, SLOVENIA.

Email address: aljaz.zalar@fri.uni-lj.si