THE STRONG TRUNCATED HAMBURGER MOMENT PROBLEM WITH AND WITHOUT GAPS

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ABSTRACT. The strong truncated Hamburger moment problem (STHMP) of degree $(-2k_1, 2k_2)$ asks to find necessary and sufficient conditions for the existence of a positive Borel measure, supported on $\mathbb{R}$, such that $\beta_i = \int x^i d\mu$ $(-2k_1 \le i \le 2k_2)$. The first solution of the STHMP, covering also its matrix generalization, was established by Simonov [60], who used the operator approach and described all solutions in terms of self-adjoint extensions of a certain symmetric operator. Using the solution of the truncated Hamburger moment problem and the properties of Hankel matrices we give an alternative solution of the STHMP and describe concretely all minimal solutions, i.e., solutions having the smallest support. Then, using the equivalence with the STHMP of degree $(-2k, 2k)$, we obtain the solution of the 2–dimensional truncated moment problem (TMP) of degree $2k$ with variety $xy = 1$, first solved by Curto and Fialkow [22]. Our addition to their result is the fact previously known only for $k = 2$, that the existence of a measure is equivalent to the existence of a flat extension of the moment matrix. Further on, we solve the STHMP of degree $(-2k_1, 2k_2)$ with one missing moment in the sequence, i.e., β_{-2k_1+1} or β_{2k_2-1}, which also gives the solution of the TMP with variety $x^2y = 1$ as a special case, first studied by Fialkow in [33].

1. INTRODUCTION

Given a real sequence $\beta^{(-2k_1,2k_2)} = (\beta_{-2k_1}, \beta_{-2k_1+1}, \ldots, \beta_{2k_2-1}, \beta_{2k_2})$ of degree $(-2k_1, 2k_2)$, $k_1, k_2 \in \mathbb{Z}_+$, the **strong truncated Hamburger moment problem (STHMP)** for $\beta^{(-2k_1,2k_2)}$ asks to characterize the existence of a positive Borel measure μ on $\mathbb{R}$, such that

$$\beta_i = \int_{\mathbb{R}} x^i d\mu \quad (i \in \mathbb{Z},\ -2k_1 \le i \le 2k_2). \tag{1.1}$$

The STHMP of degree $(0, 2k)$ is the usual **truncated Hamburger moment problem (THMP)** of degree $2k$.

We denote by $M(n_1, n_2) = M(n_1, n_2)(\beta^{(-2k_1,2k_2)}) = (\beta_{i+j})_{i,j=n_1}^{n_2}$, $-k_1 \le n_1 \le n_2 \le k_2$ the moment matrix associated with $\beta^{(-2k_1,2k_2)}$, where the rows and columns are indexed by monomials X^i in the degree increasing order

$$X^{n_1}, X^{n_1+1}, \ldots, X^{-1}, 1, X, \ldots, X^{n_2-1}, X^{n_2}.$$

Let $\mathbb{R}[x^{-1}, x]_{r_1,r_2} = \left\{\sum_{i=-r_1}^{r_2} a_i x^i \colon a_i \in \mathbb{R},\ r_1, r_2 \in \mathbb{Z}_+\right\}$ stand for the set of Laurent polynomials in variables x^{-1}, x of degree at most r_1 in x^{-1} and at most r_2 in x. For every Laurent polynomial

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$p(x^{-1},x)=\sum_{i=-k_1}^{k_2} a_i x^i \in \mathbb{R}[x^{-1},x]_{k_1,k_2}$, we denote by

$$p(X^{-1},X)=\sum_{i=-k_1}^{-1} a_i (X^{-1})^i + a_0 1 + \sum_{j=1}^{k_2} a_j X^j \in \mathcal{C}_{M(-k_1,k_2)}$$

the vector from the column space $\mathcal{C}_{M(-k_1,k_2)}$ of the moment matrix $M(-k_1,k_2)$. Let $\mathbf{0}$ stand for the zero vector. We say that the matrix $M(-k_1,k_2)$ is **recursively generated (rg)** if for $p,q,pq \in \mathbb{R}[x^{-1},x]_{k_1,k_2}$ such that $p(X^{-1},X)=\mathbf{0}$, it follows that $(pq)(X,X^{-1})=\mathbf{0}$.

Given a real 2–dimensional sequence

$$\beta^{(2k)}=\{\beta_{0,0},\beta_{1,0},\beta_{0,1},\ldots,\beta_{2k,0},\beta_{2k-1,1},\ldots,\beta_{1,2k-1},\beta_{0,2k}\}$$

of degree $2k$ and a closed subset K of $\mathbb{R}^2$, the **truncated moment problem (TMP)** supported on K for $\beta^{(2k)}$ asks to characterize the existence of a positive Borel measure μ on $\mathbb{R}^2$ with support in K, such that

$$\beta_{i,j}=\int_K x^i y^j d\mu \quad (i,j\in\mathbb{Z}_+,\ 0\le i+j\le 2k). \tag{1.2}$$

If such a measure exists, we say that $\beta^{(2k)}$ has a representing measure supported on K and μ is its K–**representing measure**.

We denote by $M(k)=M(k)(\beta^{(2k)})=(\beta_{i,j})_{i,j=0}^{2k}$ the moment matrix associated with $\beta^{(2k)}$, where the rows and columns are indexed in the degree lexicographic order

$$1,X,Y,\ldots,X^{2k},X^{2k-1}Y,\ldots,XY^{2k-1},Y^{2k}.$$

Let $\mathbb{R}[x,y]_k$ stand for the set of polynomials in variables x,y of degree at most k. For every $p(x,y)=\sum_{i,j}a_{ij}x^iy^j\in\mathbb{R}[x,y]_k$, we denote by $p(X,Y)=\sum_{i,j}a_{ij}X^iY^j$ the vector from the column space $\mathcal{C}_{M(k)}$ of the matrix $M(k)$. Recall from [18], that β has a representing measure μ with the support $\operatorname{supp}\mu$ being a subset of $\mathcal{Z}_p:=\{(x,y)\in\mathbb{R}^2\colon p(x,y)=0\}$ if and only if $p(X,Y)=\mathbf{0}$ where $\mathbf{0}$ stands for the zero vector. We say that the matrix $M(k)$ is **recursively generated (rg)** if for $p,q,pq\in\mathbb{R}[x,y]_k$ such that $p(X,Y)=\mathbf{0}$, it follows that $(pq)(X,Y)=\mathbf{0}$. The **variety** of $\beta^{(2k)}$ is defined by

$$\mathcal{V}(\beta^{(2k)}):=\bigcap_{\substack{g\in\mathbb{R}[X,Y]_{\le k},\\ g(X,Y)=\mathbf{0}}}\mathcal{Z}_g,$$

where $\mathcal{Z}_g:=\{(x,y)\in\mathbb{R}^2\colon g(x,y)=0\}$.

A **concrete solution** to the TMP is a set of necessary and sufficient conditions for the existence of a K–representing measure, that can be tested in numerical examples. Among necessary conditions, $M(k)$ must be positive semidefinite (psd), rg and satisfies the **variety condition** [18, Proposition 3.1 and Corollary 3.7], which states that the inequality $\operatorname{rank} M(k)\le\operatorname{card}\mathcal{V}(\beta^{(2k)})$ holds. The celebrated **flat extension theorem** of Curto and Fialkow [18, Theorem 7.10], [23, Theorem 2.19] states that $\beta^{(2k)}$ admits a $\operatorname{rank} M(k)$–atomic representing measure if and only if $M(k)$ is psd and admits a rank-preserving extension to a moment matrix $M(k+1)$. Using the flat extension theorem as the main tool the 2–dimensional TMP has been concretely solved in the following cases: K is the variety defined by a polynomial $p(x,y)=0$ with $\deg p\le 2$ [19, 20, 21, 22, 34], $K=\mathbb{R}^2$, $k=2$ and $M(2)$ is invertible [28, 32], K is the variety $y=x^3$ [33], $M(k)$ has a special feature called *recursive determinateness* [24] and in the *extremal case* with the equality in the variety condition [25]. Some other special cases have been solved in [10, 11, 27, 35, 41]. In [33], Fialkow studied also the TMP for the curves of the form $y=g(x)$ and $yg(x)=1$, where $g\in\mathbb{R}[x]$ is a polynomial, and obtained the bound on the degree m for which the existence of a positive extension $M(m)$ of $M(k)$ is equivalent to the existence of a measure. In our previous work we derived some of the above results and solved new cases of the 2–dimensional TMP using the solution of the THMP or the THMP with some missing moments: K with variety $xy = 0$ can be solved with the use of the THMP twice [9, Section 6], K with variety $y = x^3$ or $y^2 = x^3$ are equivalent to the THMP of degree $6k$ with a missing moment β_{6k-1} or β_1 [63, Subsections 3.1, 4.1], while special cases of K with variety $y = x^4$ or $y^3 = x^4$ to the THMP of degree $8k$ without β_{8k-2} and β_{8k-1} or β_1 and β_2 [63, Subsections 3.2, 4.2].

By [61] the TMP is more general than the classical full moment problem (MP). For nice expositions on the full MP and the TMP see [2, 46, 59]. Haviland's solution [37] of the MP established the duality of the MP with positive polynomials and led to further investigations of the MP from the perspective of real algebraic geometry (see [43, 47, 48, 49, 50, 51, 53, 54, 55, 56, 58]). Further on, various generalizations of the TMP and MP have been introduced, e.g., matrix and operator MPs [4, 6, 16, 38, 39, 44, 45, 52, 62], tracial MPs [8, 9, 12, 13, 14], MP supported on $\mathbb{N}_0$ [40], MPs in infinitely many variables and on more general commutative algebras [3, 26, 29, 36], TMP with a signed representing measure [42].

In this article we first give an alternative solution to the STHMP of degree $(2k_1, 2k_2)$, which was first solved in the more general matrix case in [60] using the operator approach, describing all solutions in terms of self-adjoint extensions of a certain symmetric operator. Our approach uses the solution of the THMP and the properties of Hankel matrices, giving also a concrete description of all minimal solutions, i.e., solutions having the smallest support. As a corollary we obtain a new proof of the TMP of degree $2k$ with variety $xy = 1$, solved in [22]. In addition, it follows that the existence of a flat extension of the moment matrix is equivalent to the existence of a measure; for $k = 2$ this was first proved in [20, Proposition 5.3]. Then we solve the STHMP of degree $(-2k_1, 2k_2)$ with the missing moment β_{-2k_1+1} or β_{2k_2-1} by using the solutions of the THMP of degree $2k$ with the missing moment β_1 or β_{2k-1} from [63]. Finally, as a corollary to this we obtain the solution of the TMP with variety $x^2y = 1$.

1.1. **Reader's Guide.** The paper is organized as follows. In Section 2 we present some properties of psd Hankel matrices and the solution of the THMP. In Section 3 we first state the solution of the STHMP (see Theorem 3.1), give a proof based on the solution of the THMP in Subsection 3.1, explain the connection with Simonov's approach [60] in Subsection 3.2, and finally as a corollary obtain the solution of the nondegenerate hyperbolic TMP (see Corollary 3.5). Finally, in Section 4 we present the solutions of the STHMP of degree $(-2k_1, 2k_2)$ with the missing moment β_{-2k_1+1} (see Theorem 4.1) or β_{-2k_2-1} (see Corollary 4.2) and as a consequence solve the TMP for the variety $x^2y = 1$ (see Corollary 4.3).

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2. PRELIMINARIES

We write $M_{n,m}$ (resp. M_n) for the set of $n \times m$ (resp. $n \times n$) real matrices. For a matrix M we denote by $\mathcal{C}_M$ its column space. The set of real symmetric matrices of size n will be denoted by S_n. For a matrix $A \in S_n$ the notation $A \succ 0$ (resp. $A \succeq 0$) means A is positive definite (pd) (resp. positive semidefinite (psd)).

Let $k \in \mathbb{N}$. For

$$v = (v_0, \ldots, v_{2k}) \in \mathbb{R}^{2k+1},$$

we denote by

$$A_v := (v_{i+j})_{i,j=0}^k = \begin{pmatrix} v_0 & v_1 & v_2 & \cdots & v_k \\ v_1 & v_2 & \iddots & \iddots & v_{k+1} \\ v_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & v_{2k-1} \\ v_k & v_{k+1} & \cdots & v_{2k-1} & v_{2k} \end{pmatrix} \in S_{k+1}$$

the corresponding Hankel matrix. We denote by $\mathbf{v_j} := (v_{j+\ell})_{\ell=0}^k$ the $(j+1)$–th column of A_v, $0 \le j \le k$, i.e.,

$$A_v = \begin{pmatrix} \mathbf{v_0} & \cdots & \mathbf{v_k} \end{pmatrix}.$$

As in [17], the **rank** of v, denoted by $\operatorname{rank} v$, is defined by

$$\operatorname{rank} v = \begin{cases} k+1, & \text{if } A_v \text{ is nonsingular}, \\ \min\{i \colon \mathbf{v_i} \in \operatorname{span}\{\mathbf{v_0}, \ldots, \mathbf{v_{i-1}}\}\}, & \text{if } A_v \text{ is singular}. \end{cases}$$

If $\operatorname{rank} v < k+1$ we say that v is **singular**. Else v is **nonsingular**.

We denote

- the upper left-hand corner $(v_{i+j})_{i,j=0}^m \in S_{m+1}$ of A_v of size $m+1$ by $A_v(m)$.
- the lower right-hand corner $(v_{i+j})_{i,j=k-m}^k \in S_{m+1}$ of A_v of size $m+1$ by $A_v[m]$.

For a sequence $v = (v_0, \ldots, v_{2k})$ we denote by $v^{(\mathrm{rev})} := (v_{2k}, v_{2k-1}, \ldots, v_0)$ the **reversed sequence**. A sequence v is called

- **positively recursively generated (prg)** if for $r = \operatorname{rank} v$ the following two conditions hold:
  - $A_v(r-1) \succ 0$.
  - If $r < k+1$, denoting

$$(\varphi_0, \ldots, \varphi_{r-1}) := A_v(r-1)^{-1}(v_r, \ldots, v_{2r-1})^T, \tag{2.1}$$

    the equality

$$v_j = \varphi_0 v_{j-r} + \cdots + \varphi_{r-1} v_{j-1} \tag{2.2}$$

    holds for $j = r, \ldots, 2k$.
- **negatively recursively generated (nrg)** if for $r = \operatorname{rank} v^{(\mathrm{rev})}$ the following two conditions hold:
  - $A_v[r-1] \succ 0$.
  - If $r < k+1$, denoting

$$(\psi_0, \ldots, \psi_{r-1}) := A_v[r-1]^{-1}(v_{2k-2r+1}, \ldots, v_{2k-r})^T,$$

    the equality

$$v_{2k-r-j} = \psi_0 v_{2k-r+1-j} + \cdots + \psi_{r-1} v_{2k-j}, \tag{2.3}$$

    holds for $j = 0, \ldots, 2k-r$.
- **recursively generated (rg)** if it is prg and nrg,

Proposition 2.1. *Let $v = (v_0, \ldots, v_{2k}) \in \mathbb{R}^{2k+1}$, $v_0 > 0$, be a singular sequence of rank $r \le k$ such that $A_v \succeq 0$. Let φ_i be defined by (2.1). Then the following statements are true:*

(1) (2.2) *holds for $j = r, \ldots, 2k-1$.*
(2) (2.3) *holds for $j = 0, \ldots, 2k-r-1$.*

(3) *The polynomial* $p(x) := x^r - \sum_{i=0}^{r-1} \varphi_i x^i$ *has* r *distinct real zeroes.*
(4) *The following statements are equivalent:*
   (a) v *is prg.*
   (b) (2.2) *holds for* $j = 2k$.
   (c) $\operatorname{rank} A_v(k-1) = \operatorname{rank} A_v$.
   (d) *There exist real numbers* v_{2k+1} *and* v_{2k+2} *such that* $A_{\tilde{v}} \succeq 0$*, where* $\tilde{v} := (v, v_{2k+1}, v_{2k+2})$.
   (e) v_{2k+1} *and* v_{2k+2} *defined by* (2.2) *for* $j = 2k+1, 2k+2$ *are the unique real numbers such that* $A_{\tilde{v}} \succeq 0$ *and* $\operatorname{rank} A_v = \operatorname{rank} A_{\tilde{v}}$*, where* $\tilde{v} := (v, v_{2k+1}, v_{2k+2})$.
(5) *The following statements are equivalent:*
   (a) v *is nrg.*
   (b) (2.3) *holds for* $j = 2k - r + 1$.
   (c) $\operatorname{rank} A_v[k-1] = \operatorname{rank} A_v$.
   (d) *There exist real numbers* v_{-1} *and* v_{-2} *such that* $A_{\tilde{v}} \succeq 0$*, where* $\tilde{v} := (v_{-2}, v_{-1}, v)$.
   (e) v_{-2} *and* v_{-1} *defined by* (2.3) *for* $j = 2k-r+1, 2k-r+2$ *are the unique real numbers such that* $A_{\tilde{v}} \succeq 0$ *and* $\operatorname{rank} A_v = \operatorname{rank} A_{\tilde{v}}$*, where* $\tilde{v} := (v_{-2}, v_{-1}, v)$.

Proof. (1) is [17, Theorem 2.4(ii)]. (3) follows from [17, Remark 3.5]. (4) follows from [17, Theorem 2.6 and Remark 2.7]. Using (1) (resp. (4)) for $v^{(\text{rev})}$ we obtain (2) (resp. (5)). □

Remark 2.2. (1) Proposition 2.1.(1) implies that for a singular sequence v, the numbers φ_i could also be defined as the unique coefficients such that $\mathbf{v_r} = \varphi_0 \mathbf{v_0} + \cdots + \varphi_{r-1}\mathbf{v_{r-1}}$. Moreover,

$$\mathbf{v_j} = \varphi_0 \mathbf{v_{j-r}} + \cdots + \varphi_{r-1} \mathbf{v_{j-1}} \tag{2.4}$$

holds for $j = r+1, \ldots, k-1$.
(2) Proposition 2.1.(4) implies that v is prg if and only if (2.4) holds also for $j = k$.
(3) Proposition 2.1.(2) implies that for a singular sequence v, the numbers ψ_i could also be defined as the unique coefficients such that $\mathbf{v_{k-r}} = \psi_0 \mathbf{v_{k-r+1}} + \cdots + \psi_{r-1}\mathbf{v_k}$. Moreover,

$$\mathbf{v_{k-r-j}} = \psi_0 \mathbf{v_{k-r+1-j}} + \cdots + \psi_{r-1} \mathbf{v_{k-j}} \tag{2.5}$$

holds for $j = 1, \ldots, k-r-1$.
(4) Proposition 2.1.(5) implies that v is nrg if and only if (2.5) holds also for $j = k - r$.

Let $v = (v_0, \ldots, v_{2k}) \in \mathbb{R}^{2k+1}$ be a sequence with the Hankel matrix $A_v = \begin{pmatrix} \mathbf{v_0} & \cdots & \mathbf{v_k} \end{pmatrix}$. For a polynomial $g(x) = \sum_{i=0}^k \gamma_i x^i$, $\gamma_i \in \mathbb{R}$, we define the **evaluation** $g(v)$ by the rule $g(v) = \sum_{i=0}^k \gamma_i \mathbf{v}_i$. For a singular sequence v we call the polynomial p from Proposition 2.1.(3) the **generating polynomial** of v. We write $\mathbf{0} \in \mathbb{R}^{k+1}$ for the zero vector.

Proposition 2.3. *For a singular sequence* $v = (v_0, \ldots, v_{2k}) \in \mathbb{R}^{2k+1}$ *the following statements are equivalent:*

(1) v *is prg and* $\varphi_0 \neq 0$.
(2) v *is nrg and* $\psi_{r-1} \neq 0$.
(3) v *is rg.*
(4) v *is rg,* $\operatorname{rank} v = \operatorname{rank} v^{(\text{rev})}$*,* $\varphi_0 \neq 0$ *and*

$$(\psi_0, \psi_1, \ldots, \psi_{r-2}, \psi_{r-1}) = \left(-\frac{\varphi_1}{\varphi_0}, -\frac{\varphi_2}{\varphi_0}, \ldots, -\frac{\varphi_{r-1}}{\varphi_0}, \frac{1}{\varphi_0} \right). \tag{2.6}$$

Proof. First we prove the implication (1) $\Rightarrow$ (2). By definition of $\operatorname{rank} v = r$, the set $\{\mathbf{v}_0, \ldots, \mathbf{v}_{r-1}\}$ is linearly independent. Since v is prg, Proposition (2.1).(4) implies that (2.4) holds for $j = r, \ldots, k$. Since $\varphi_0 \neq 0$, it follows that for $i = 0, \ldots, k-r$

$$\mathbf{v}_i = -\sum_{j=1}^{r-1} \frac{\varphi_j}{\varphi_0} \mathbf{v}_{i+j} + \frac{1}{\varphi_0} \mathbf{v}_{i+r},$$

and inductively every set $\{\mathbf{v}_{i+1}, \ldots, \mathbf{v}_{i+r}\}$, $i = 0, \ldots, k-r$, is linearly independent. Hence, (2.5) is true with $\psi_j = -\frac{\varphi_{j+1}}{\varphi_0}$, $j = 0, \ldots, r-2$, and $\psi_{r-1} = \frac{1}{\varphi_0}$.

The proof of the implication (1) $\Leftarrow$ (2) is analoguous to the proof of (1) $\Rightarrow$ (2) only that the induction step is in the backward direction.

Since (1) and (2) are equivalent, the implication (1) $\Rightarrow$ (3) follows. The nontrivial part of the implication (1) $\Leftarrow$ (3) is $\varphi_0 \neq 0$. If $\varphi_0 = 0$, then since (2.4) holds for $j = r, \ldots, k$ and $\mathbf{v}_0, \ldots, \mathbf{v}_{r-1}$ are linearly independent, it follows that $\mathbf{v}_0 \notin \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Since v is singular, A_v is also singular, and consequently $A_{v^{(\mathrm{rev})}}$ and $v^{(\mathrm{rev})}$ are both singular. Since v is nrg, it follows from (2.5), used for $j = k - r$, that $\mathbf{v}_0 \in \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, which is a contradiction. Hence, $\varphi_0 \neq 0$.

The nontrivial implication of the equivalence (1) $\Leftrightarrow$ (4) is (1) $\Rightarrow$ (4). Note that the equalities $\operatorname{rank} v = \operatorname{rank} v^{(\mathrm{rev})}$ and (2.6) follow from the proof of the implication (1) $\Rightarrow$ (2). $\square$

For $x \in \mathbb{R}^m$ we use δ_x to denote the probability measure on $\mathbb{R}^m$ such that $\delta_x(\{x\}) = 1$. By a **finitely atomic positive measure** on $\mathbb{R}^m$ we mean a measure of the form $\mu = \sum_{j=0}^{\ell} \rho_j \delta_{x_j}$, where $\ell \in \mathbb{N}$, each $\rho_j > 0$ and each $x_j \in \mathbb{R}^m$. The points x_j are called **atoms** of the measure μ and the constants ρ_j the corresponding **densities**.

For $v := (v_1, \ldots, v_m) \in \mathbb{R}^m$ we denote by $V_v \in \mathbb{R}^{m \times m}$ the Vandermondo matrix

$$V_v := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_m \\ \vdots & \vdots & & \vdots \\ v_1^{m-1} & v_2^{m-1} & \cdots & v_m^{m-1} \end{pmatrix}.$$

The solution of the THMP of degree $2k$ is the following.

Theorem 2.4 ([17, Theorems 3.9 and 3.10]). *For $k \in \mathbb{N}$ and $\beta = (\beta_0, \ldots, \beta_{2k}) \in \mathbb{R}^{2k+1}$ with $\beta_0 > 0$, the following statements are equivalent:*

1. *There exists a $\mathbb{R}$–representing measure for β, i.e., supported on $\mathbb{R}$.*
2. *There exists a $(\operatorname{rank} \beta)$–atomic representing measure for β.*
3. *β is positively recursively generated.*
4. *$M(0,k) \succeq 0$ and $\operatorname{rank} M(0,k) = \operatorname{rank} \beta$.*
5. *One of the following statements holds:*
   - (a) *$M(0,k) \succ 0$.*
   - (b) *$M(0,k) \succeq 0$ and $\operatorname{rank} M(0,k) = \operatorname{rank} M(0,k-1)$.*

(i) *$r \leq k$, then the $\mathbb{R}$–representing measure μ is unique and of the form $\mu = \sum_{i=1}^{r} \rho_i \delta_{x_i}$, where $x_1, \ldots, x_r$ are the roots of the generating polynomial of β,*

$$\begin{pmatrix} \rho_1 & \cdots & \rho_r \end{pmatrix}^T := V_x^{-1} u,$$

$x = (x_1, \ldots, x_r)$ and $u = \begin{pmatrix} \beta_0 & \cdots & \beta_{r-1} \end{pmatrix}^T$.

(ii) $r = k+1$, *then there are infinitely many* $\mathbb{R}$*–representing measures for* β. *All* $(k+1)$*–atomic ones are obtained by choosing* $\beta_{2k+1} \in \mathbb{R}$ *arbitrarily, defining* $\beta_{2k+2} := u^T (M(0,k))^{-1} u$, *where* $u = \begin{pmatrix} \beta_{k+1} & \cdots & \beta_{2k+1} \end{pmatrix}^T$, *and use (i) for* $\widetilde{\beta} := (\beta_0, \ldots, \beta_{2k+1}, \beta_{2k+2}) \in \mathbb{R}^{2k+3}$.

For a vector $v \in \mathbb{R}^m$ we denote by $v(0:i) \in \mathbb{R}^{i+1}$ the projection on the first $i+1$ coordinates and by $v(i) := v(i:i) \in \mathbb{R}$ the $(i+1)$–th coordinate of v.

We will need the following proposition in the solution of the STHMP.

Proposition 2.5. *Let* $k \in \mathbb{N}$ *and* $\beta = (\beta_0, \ldots, \beta_{2k}) \in \mathbb{R}^{2k+1}$ *with* $\beta_0 > 0$ *be a real sequence such that* $M(0,k) \succ 0$. *Then:*

(1) All but at most one $(k+1)$*–atomic representing measures for* β *described in Theorem 2.4.(ii) are supported on* $\mathbb{R} \setminus \{0\}$ *and the corresponding sequences* $\widetilde{\beta}$ *are singular and recursively generated.*

(2) Denoting $M(0,k) = \begin{pmatrix} \mathbf{v_0} & \mathbf{v_1} & \cdots & \mathbf{v_k} \end{pmatrix}$ *the* $(k+1)$*–atomic representing measure for* β *with a nonzero density in* 0 *exists if and only if*

$$C := \begin{pmatrix} \mathbf{v_1}(0:k-1) & \cdots & \mathbf{v_k}(0:k-1) \end{pmatrix}$$

is invertible. In this case $\beta_{2k+1} = w^T C^{-1} w$, *where* $w = \begin{pmatrix} \beta_{k+1} & \cdots & \beta_{2k} \end{pmatrix}^T$ *and* β_{2k+2} *is as in Theorem 2.4.(ii).*

Proof. Let $\beta_{2k+1} \in \mathbb{R}$ be arbitrary and $\widetilde{\beta}$ be defined as in Theorem 2.4.(ii). By [17, Lemma 2.3] we have that $\operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} M(0,k)$ and hence $\widetilde{\beta}$ is singular. By Theorem 2.4.(i), $\widetilde{\beta}$ has a unique $(k+1)$–atomic representing measure supported on the set of roots $\mathcal{Z}(p_{\widetilde{\beta}})$ of the generating polynomial $p_{\widetilde{\beta}}$ of $\widetilde{\beta}$. To establish (1) it remains to prove that for all but one β_{2k+1}, $\mathcal{Z}(p_{\widetilde{\beta}})$ does not contain 0 and $\widetilde{\beta}$ is rg. We write $A_{\widetilde{\beta}} = \begin{pmatrix} \mathbf{u_0} & \mathbf{u_1} & \cdots & \mathbf{u_k} & \mathbf{u_{k+1}} \end{pmatrix}$. Assume that $\mathcal{Z}(p_{\widetilde{\beta}})$ contains 0. Then $p_{\widetilde{\beta}}(x) = x^{k+1} - \sum_{i=1}^k \varphi_i x^i$ for some $\varphi_i \in \mathbb{R}$ or equivalently $\mathbf{u_{k+1}} = \sum_{i=1}^k \varphi_i \mathbf{u_i}$. In particular,

$$\mathbf{u_{k+1}}(0:k-1) = \sum_{i=1}^k \varphi_i \mathbf{u_i}(0:k-1) \tag{2.7}$$

and $\beta_{2k+1} = \mathbf{u_{k+1}}(k) = \sum_{i=1}^k \varphi_i \mathbf{u_i}(k)$. If the vectors $\mathbf{u_1}(0:k-1), \ldots, \mathbf{u_k}(0:k-1)$ are linearly independent, then $\varphi_1, \ldots, \varphi_k$ satisfying (2.7) are uniquely determined and hence also β_{2k+1}, such that $\mathcal{Z}(p_{\widetilde{\beta}})$ contains 0, is unique. Otherwise $\mathbf{u_1}(0:k-1), \ldots, \mathbf{u_k}(0:k-1)$ are linearly dependent and thus

$$\begin{aligned} k &> \operatorname{rank} \begin{pmatrix} \mathbf{u_1}(0:k-1) & \mathbf{u_2}(0:k-1) & \cdots & \mathbf{u_{k+1}}(0:k-1) \end{pmatrix} \\ &= \operatorname{rank} \begin{pmatrix} \mathbf{u_1}(0:k-1) & \mathbf{u_2}(0:k-1) & \cdots & \mathbf{u_{k+1}}(0:k-1) \end{pmatrix}^T \\ &= \operatorname{rank} \begin{pmatrix} (\mathbf{u_1}(0:k-1))^T \\ M(1,k) \end{pmatrix} = k, \end{aligned} \tag{2.8}$$

where we used the Hankel structure of $A_{\widetilde{\beta}}$ in the second equality and $M(1,k) \succ 0$ in the third equality. (2.8) is a contradiction. Thus there is at most one β_{2k+1} such that $\mathcal{Z}(p_{\widetilde{\beta}})$ contains 0. If $\mathcal{Z}(p_{\widetilde{\beta}})$ does not contain 0, then $p_{\widetilde{\beta}}(x) = x^{k+1} - \sum_{i=0}^k \varphi_i x^i$ with $\varphi_0 \neq 0$. By Proposition 2.3, $\widetilde{\beta}$ is rg in this case. This proves (1). The statement (2) also follows from the proof of (1) above by noticing that $\mathbf{u_i}(0:k) = \mathbf{v_i}(0:k)$ for $i = 0, \ldots, k$ and $\mathbf{u_{k+1}}(0:k-1) = w$. □

3. THE STHMP AND THE TMP WITH VARIETY $xy = 1$

In this section we first solve the STHMP (see Theorem 3.1) and then as a corollary obtain the solution of the TMP for the curve $xy = 1$ (see Corollary 3.5).

Theorem 3.1. *For $k_1, k_2 \in \mathbb{N}$, let $\beta := \beta^{(-2k_1,2k_2)} = (\beta_{-2k_1}, \beta_{-2k_1+1}, \dots \beta_{2k_2})$ be a real sequence of degree $(-2k_1, 2k_2)$, such that $\beta_{-2k_1} > 0$, with the associated moment matrix $M(-k_1, k_2)$. The following statements are equivalent:*

(1) There exists a representing measure for β supported on $\mathbb{R} \setminus \{0\}$.
(2) There exists a $(\operatorname{rank}\beta)$–atomic representing measure for β supported on $\mathbb{R} \setminus \{0\}$.
(3) β is recursively generated.
(4) $M(-k_1, k_2) \succeq 0$ and one of the following statements holds:
  (a) $M(-k_1, k_2) \succ 0$.
  (b) $\operatorname{rank} M(-k_1, k_2) = \operatorname{rank} M(-k_1, k_2 - 1) = \operatorname{rank} M(-k_1 + 1, k_2)$.

Moreover, if β with $r = \operatorname{rank}\beta$ has a $(\mathbb{R} \setminus \{0\})$–representing measure and:

(i) $r \le k_1 + k_2$, then the representing measure is unique and of the form $\mu = \sum_{i=1}^{r} \rho_i \delta_{x_i}$, where $x_1, \dots, x_r$ are the roots of the generating polynomial of β and $\rho_1, \dots, \rho_r > 0$ the corresponding densities.

(ii) $r = k_1 + k_2 + 1$, then there are infinitely many $(k_1 + k_2 + 1)$–atomic representing measures for β. Denoting $M(-k_1, k_2) = \begin{pmatrix} \mathbf{v_0} & \mathbf{v_1} & \cdots & \mathbf{v_{k_1+k_2}} \end{pmatrix}$, they are obtained by choosing any $\beta_{2k_2+1} \in \mathbb{R}$, which is not equal to $v^T C^{-1} v$ if C is invertible, where

$$C = \begin{pmatrix} \mathbf{v_1}(0 : k_1 + k_2 - 1) & \cdots & \mathbf{v_k}(0 : k_1 + k_2 - 1) \end{pmatrix}$$

and $v = \begin{pmatrix} \beta_{-k_1+k_2+1} & \cdots & \beta_{2k_2} \end{pmatrix}^T$, defining

$$\beta_{2k_2+2} = u^T \left(M(-k_1, k_2)\right)^{-1} u,$$

where $u = \begin{pmatrix} \beta_{-k_1+k_2+1} & \cdots & \beta_{2k_2} & \beta_{2k_2+1} \end{pmatrix}^T$, and then use (i) for

$$\widetilde{\beta} = (\beta_{-2k_1}, \dots, \beta_{2k_2+1}, \beta_{2k_2+2}).$$

Remark 3.2. Before proving Theorem 3.1 let us mention that the matrix STHMP was already considered by Simonov in [60]. Let $N \in \mathbb{N}$ and $H_N(\mathbb{C})$ be the set of $N \times N$ complex hermitian matrices. The matrix STHMP of degree $(-2k_1, 2k_2)$, $k_1, k_2 \in \mathbb{Z}_+$ refers to the case when $\{S_i\}_{i=-2k_1}^{2k_2}$ is a sequence of hermitian $N \times N$ complex matrices and one wants to find all positive $H_N(\mathbb{C})$–valued Borel measure μ such that

$$S_i = \int_{\mathbb{R}} x^i d\mu \quad (i \in \mathbb{Z},\ -2k_1 \le i \le 2k_2). \tag{3.1}$$

holds. In [60], the author gave necessary and sufficient conditions for the solvability of the STHMP of degree $(-2m, 2m)$, $m \in \mathbb{N}$, and also described all solutions in terms of self-adjoint extensions of a certain, not necessarily everywhere defined, linear operator on the finite dimensional Hilbert space of N–vector Laurent polynomials. The operator techniques used in [60] are in fact not sensitive to the assumption that $k_1 = k_2 = m$ and can be verbatim extended to the general degree $(-2k_1, 2k_2)$ case, where $k_1, k_2 \in \mathbb{N}$. Moreover, using the same techniques one can also solve the matrix THMP, i.e., the sequence β is of degree $(0, 2m)$ or even of degree $(2m_1, 2m_2)$, where $m, m_1, m_2 \in \mathbb{N}$. Since except solvability we are also interested in the more concrete description of the minimal measures in the scalar STHMP case, where a **minimal measure** refers to the representing measure with the smallest possible number of atoms, we give a proof of Theorem 3.1 based on the application of

Theorem 2.4 in Subsection 3.1. Then, in Subsection 3.2, we explain the connection with Simonov's work.

3.1. **Proof of Theorem 3.1 using the solution of the THMP.** First we prove the implication (1) $\Rightarrow$ (4). If β is nonsingular, then we have $M(-k_1,k_2)\succ 0$, which is (a). Else β is singular and $M(-k_1,k_2)\not\succ 0$ holds. Since β admits a measure, it can be extended with

$$\beta_{-2k_1-2},\beta_{-2k_1-1},\beta_{2k_2+1},\beta_{2k_1+2}\in\mathbb{R}$$

to a sequence $\beta^{(-2k_1-2,2k_2+2)}$ which admits a measure. By (4) and (5) of Proposition 2.1, (b) holds.

Next we prove the implication (2) $\Leftarrow$ (4). We separate two cases:

Case 1. $M(-k_1,k_2)\succeq 0$ and $M(-k_1,k_2)\not\succ 0$: Since (b) holds, there exist by Proposition 2.1.(4) unique $\beta_{2k_2+1},\beta_{2k_1+2}\in\mathbb{R}$ such that $M(-k_1,k_2+1)\succeq 0$ and $\operatorname{rank} M(-k_1,k_2)=\operatorname{rank} M(-k_1,k_2+1)$. Inductively, for every $m\in\mathbb{N}$ there is a unique extension of $\beta^{(-2k_1,2k_2)}$ to $\beta^{(-2k_1,2(k_2+m))}$, such that $M(-k_1,k_2+m)\succeq 0$ and $\operatorname{rank} M(-k_1,k_2)=\operatorname{rank} M(-k_1,k_2+m)$. Write $r=\operatorname{rank}\beta$ and let $m\in\mathbb{N}$ be such that $k_2+m\geq r$. Let $p(x)=x^r-\sum_{i=0}^{r-1}\varphi_i x^i$ be the generating polynomial of $\beta^{(-2k_1,2k_2)}$. By Theorem 2.4, there exists a unique measure $\mu=\sum_{\ell=1}^{r}\rho_\ell\delta_{x_\ell}$ for $\beta^{(0,2(k_2+m))}$, where $x_1,\ldots,x_r\in\mathbb{R}$ are zeroes of p and $\rho_1,\ldots,\rho_r$ are the corresponding densities. First note by Proposition 2.3 that $\varphi_0\neq 0$ and hence all atoms x_ℓ are nonzero. We will prove that this is also the representing measure for $\beta^{(-2k_1,0)}$. Let us assume that μ represents $\beta_{j+1},\beta_{j+2},\ldots,\beta_{j+r}$ for some $-2k_1\leq j\leq -1$ and prove that it also represents β_j. Note that for $j=-1$ the assumption that μ represents $\beta_0,\beta_1,\ldots,\beta_{r-1}$ holds and the validity for $j<-1$ will hold by induction. We have:

$$\begin{aligned}\sum_{\ell=1}^{r}\rho_\ell x_\ell^j &= \sum_{\ell=1}^{r}\rho_\ell\left(\frac{1}{\varphi_0}x_\ell^{r+j}-\sum_{i=1}^{r-1}\frac{\varphi_i}{\varphi_0}x_\ell^{i+j}\right)=\sum_{\ell=1}^{r}\rho_\ell\left(\psi_{r-1}x_\ell^{r+j}+\sum_{i=1}^{r-1}\psi_{i-1}x_\ell^{i+j}\right)\\ &=\sum_{i=1}^{r}\psi_{i-1}\left(\sum_{\ell=1}^{r}\rho_\ell x_\ell^{i+j}\right)=\sum_{i=1}^{r}\psi_{i-1}\beta_{i+j}=\beta_j.\end{aligned}$$

where the first equality follows by expressing x_ℓ^j from $\frac{x_\ell^j}{\varphi_0}\cdot p(x_\ell)$ which is equal to 0, the second by Proposition 2.3.(4), the forth by the hypothesis that μ represents $\beta_{j+1},\ldots,\beta_{j+r}$, and the last by (2) and (5) of Proposition 2.1. Hence μ represents β_j and by induction also $\beta^{(-2k_1,0)}$.

Case 2. $M(-k_1,k_2)\succ 0$: By Proposition 2.5 there exist $\beta_{2k_2+1},\beta_{2k_2+2}\in\mathbb{R}$ such that $\widetilde{\beta}=(\beta,\beta_{2k_2+1},\beta_{2k_2+2})$ is singular and rg. By Proposition 2.1.(4),(5), $\widetilde{\beta}$ satisfies

$$\operatorname{rank} M(-k_1,k_2)=\operatorname{rank}\left(M(-k_1,k_2+1)(\widetilde{\beta})\right)=\operatorname{rank}\left(M(-k_1+1,k_2+1)(\widetilde{\beta})\right).$$

Now we use Case 1 for $\widetilde{\beta}$ to establish (4).

The implication (1) $\Leftarrow$ (2) is trivial. The equivalence (3) $\Leftrightarrow$ (4) follows from Theorem 2.4 used for $\beta^{(-2k_1,2k_2)}$ and its reversed sequence $(\beta^{(-2k_1,2k_2)})^{(\mathrm{rev})}=(\beta_{2k_2},\beta_{2k_2-1},\ldots,\beta_{-2k_1+1},\beta_{2k_1})$ as β to obtain the equivalences:

- $\beta^{(-2k_1,2k_2)}$ is prg if and only if $M(-k_1,k_2)\succ 0$ or $[M(-k_1,k_2)\succeq 0$ and $\operatorname{rank} M(-k_1,k_2)=\operatorname{rank} M(-k_1,k_2-1)]$.
- $(\beta^{(-2k_1,2k_2)})^{(\mathrm{rev})}$ is prg if and only if $\beta^{(-2k_1,2k_2)}$ is nrg if and only if it holds that $M(-k_1,k_2)\succ 0$ or $[M(-k_1,k_2)\succeq 0$ and the equality $\operatorname{rank} M(-k_1,k_2)=\operatorname{rank} M(-k_1+1,k_2)]$ is true.

Using both equivalences gives the equivalence (3) $\Leftrightarrow$ (4).

The moreover part can be read out of the proof of the implication (2) $\Leftarrow$ (4). In case β is a singular sequence, Case 1 applies, while if β is not singular, then Case 2 applies. In Case 1 the constructed representing measure is precisely the one stated in (i), while in Case 2 precisely singular, rg extensions $\widetilde{\beta} = (\beta, \beta_{2k_2+1}, \beta_{2k_2+2})$ have $(k_1 + k_2 + 1)$–atomic representing measures. By Proposition 2.5 these are precisely the ones stated in (ii). □

3.2. **Proof of (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) of Theorem 3.1 using the operator approach from [60].** Let

$$\mathbb{C}^N[x^{-1}, x]_{k_1,k_2} = \operatorname{span}\left\{ux^i \colon u \in \mathbb{C}^N, i = -k_1, -k_1+1, \ldots, k_2\right\}$$

be a linear space of N–vector Laurent polynomials of degree at most k_1 in x^{-1} and k_2 in x. Let $\{S_i\}_{i=-2k_1}^{2k_2}$ be a sequence of hermitian $N \times N$ complex matrices, which is **positive**, i.e., $\sum_{i,j=-k_1}^{k_2} v_j^* S_{i+j} v_i \geq 0$ for every sequence $\{v_i\}_{i=-k_1}^{k_2}$ where $v_i \in \mathbb{C}^N$. For a positive sequence $\{S_i\}_{i=-2k_1}^{2k_2}$, the Hermitian form

$$\langle u_1 x^i, u_2 x^j \rangle = u_2^* S_{i+j} u_2$$

on $\mathbb{C}^N[x^{-1}, x]_{k_1,k_2}$ is a semi-inner product. Quotienting out the vector subspace

$$\mathcal{N} = \left\{p \in \mathbb{C}^N[x^{-1}, x]_{k_1,k_2} \colon \langle p, p \rangle = 0\right\}$$

gives a finite dimensional Hilbert space $\mathcal{H}$. We denote by $[p] := p + \mathcal{N} \in \mathcal{H}$ the equivalence class of $p \in \mathbb{C}^N[x^{-1}, x]_{k_1,k_2}$.

We call the sequence $\{S_i\}_{i=-2k_1}^{2k_2}$:

- **matricially positively recursively generated (mat–prg)** if for any sequence $\{v_i\}_{i=-k_1}^{k_2-1}$ with $v_i \in \mathbb{C}^N$ the following holds:
$$\sum_{i,j=-k_1}^{k_2-1} v_j^* S_{i+j} v_i \geq 0 \quad \text{implies that} \quad \sum_{i,j=-k_1}^{k_2-1} v_j^* S_{i+j+2} v_i \geq 0.$$
- **matricially negatively recursively generated (mat–nrg)** if for any sequence $\{v_i\}_{i=-k_1}^{k_2-1}$ with $v_i \in \mathbb{C}^N$ the following holds:
$$\sum_{i,j=-k_1}^{k_2-1} v_j^* S_{i+j+2} v_i \geq 0 \quad \text{implies that} \quad \sum_{i,j=-k_1}^{k_2-1} v_j^* S_{i+j} v_i \geq 0.$$
- **matricially recursively generated (mat–rg)** if it is mat–prg and mat–nrg.

If the sequence $\{S_i\}_{i=-2k_1}^{2k_2}$ is mat–prg, the multiplication operator $A([p]) := [xp]$ on $\mathcal{H}$ with domain

$$\operatorname{dom} A := \operatorname{span}\left\{[ux^i] \colon u \in \mathbb{C}^N, i = -k_1, -k_1+1, \ldots, k_2-1\right\}$$

is well-defined. If moreover $\{S_i\}_{i=-2k_1}^{2k_2}$ is mat–nrg, it follows that $\ker A = \{0\}$.

Solution of the moment problem (3.1) from [60] is the following.

Theorem 3.3. [60, Theorems 3.3 and 3.4, Corollary 3.4.1]

(1) The moment problem (3.1) is solvable if and only if $\{S_i\}_{i=-2k_1}^{2k_2}$ is positive and matricially recursively generated.

(2) *There exists a one-to-one correspondence between the set of all solutions* μ *of* (3.1) *and the set of all equivalence classes for the relation of unitary equivalence of self-adjoint extensions* $\widetilde{A}$ *of* A *on some larger Hilbert space* $\widetilde{\mathcal{H}} \supseteq \mathcal{H}$*, satisfying* $\ker \widetilde{A} = \{0\}$ *and*

$$\widetilde{\mathcal{H}} = \overline{\operatorname{span}} \left\{ [u], (\widetilde{A} - \lambda)^{-1}[v] \colon u, v \in \mathbb{C}^N, \lambda \in \rho(\widetilde{A}) \right\},$$

where $\rho(\widetilde{A}) := \left\{ \lambda \in \mathbb{C} \mid \ker(\widetilde{A} - \lambda) = 0, \operatorname{Ran}(\widetilde{A} - \lambda) = \widetilde{\mathcal{H}} \right\}$ *is the resolvent set of* $\widetilde{A}$*. The correspondence is given by*

$$\langle \mu(t)u, v\rangle_{\mathcal{H}} = \langle E_{\widetilde{A}}(t)[u], [v]\rangle_{\widetilde{\mathcal{H}}}, \quad u, v \in \mathbb{C}^n, \tag{3.2}$$

where $E_{\widetilde{A}}$ *is the spectral measure of* $\widetilde{A}$*.*

(3) *The moment problem* (3.1) *has a unique solution if and only if* A *is self-adjoint.*

Using Theorem 3.3 the equivalence (1) $\Leftrightarrow$ (4) of Theorem 3.1 easily follows by noticing that being positive and mat–rg for $N = 1$ is equivalent to satisfying (4) of Theorem 3.1.

To prove the equivalence (1) $\Leftrightarrow$ (2) of Theorem 3.1 we have to argue in the following way: A is a symmetric operator on the finite dimensional Hilbert space. If $\operatorname{dom}(A) = \mathcal{H}$, then A is self-adjoint and by Theorem 3.3 its spectral measure, which is supported on the set of eigenvalues of A, gives the unique $(\operatorname{rank} A)$–atomic representing measure μ for β by the correspondence (3.2). Since

$$\begin{aligned}\operatorname{dom}(A) = \mathcal{H} \Leftrightarrow [x^{k_2}] = \left[\sum_{i=-k_1}^{k_2-1} \alpha_i x^i \right] \quad \text{for some } \alpha_i \in \mathbb{C} \\ \Leftrightarrow \operatorname{rank} M(-k_1, k_2) = \operatorname{rank} M(-k_1, k_2 - 1),\end{aligned}$$

this measure is also $(\operatorname{rank} \beta)$–atomic. Otherwise $\operatorname{dom}(A) \subset \mathcal{H}$ is a linear subspace of codimension 1 in $\mathcal{H}$ and A can be extended to a self-adjoint invertible operator $\widetilde{A}$ on $\mathcal{H}$. By Theorem 3.3, its spectral measure, which is $\dim \mathcal{H} = (k_1 + k_2 + 1)$–atomic, gives a $(k_1 + k_2 + 1)$–atomic representing measure μ for β by the correspondence (3.2).

Remark 3.4. (1) The moreover part in Theorem 3.1 does not directly follow from Theorem 3.3 since one would need to observe more carefully the minimal-rank self-adjoint extensions $\widetilde{A}$ of A from Theorem 3.3.(2) to describe precisely their spectral measures (or equivalently because of finite-dimensionality eigenpairs) in terms of the sequence β.

(2) Using the same technique as above one can give an alternative solution of the matrix THMP (see [1, 5, 6, 15, 30, 31]). Replacing N–vector Laurent polynomials with N–vector polynomials

$$\mathbb{C}^N[x]_{k_1,k_2} := \operatorname{span} \left\{ ux^i \colon u \in \mathbb{C}^N, i = k_1, \ldots, k_2 \right\},$$

following the proof of Theorem 3.3 in [60] one obtains the fact, that the sequence $\{S_i\}_{i=2k_1}^{2k_2}$ of hermitian $N \times N$ complex matrices admits a $H_N(\mathbb{C})$–valued Borel measure such that $S_i = \int_{\mathbb{R}} x^i d\mu$ for each i if and only if $\{S_i\}_{i=2k_1}^{2k_2}$ is positive and mat–prg, while all solutions are precisely those described in Theorem 3.3.(2) only that the condition $\ker \widetilde{A} = \{0\}$ is dropped. (This condition is needed only for the equality $S_{-2k_1} = \int x^{-2k_1} d\mu$ in the STHMP case.) The uniqueness part remains the same as in Theorem 3.3.(3).

3.3. **The TMP with variety $xy = 1$.** As a corollary of Theorem 3.1 we obtain a new proof of the TMP of degree $2k$ with variety $xy = 1$, solved in [22]. Moreover, our approach shows that in case the representing measure exists, there is always a $(\operatorname{rank} M(k))$–atomic one.

Let $M(k)$ be a moment matrix associated with a bivariate sequence $\beta^{(2k)}$. We write $(M(k))_{S_1,S_2}$ for the restriction of $M(k)$ to rows and columns indexed by the sets S_1 and S_2, respectively. We also write $(M(k))_S := (M(k))_{S,S}$ and $\mathcal{B} := \{Y^k, \ldots, Y, 1, X, \ldots, X^k\}$.

Corollary 3.5. *For $k \in \mathbb{N}$, let $\beta^{(2k)} = (\beta_{0,0}, \beta_{1,0}, \beta_{0,1}, \ldots, \beta_{1,2k-1}, \beta_{0,2k})$ be a 2–dimensional sequence of degree $2k$, such that $\beta_{0,0} > 0$, with the associated moment matrix $M(k)$. Then there exists a representing measure for $\beta^{(2k)}$ supported on $K := \{(x, y) \in \mathbb{R}^2 \colon xy = 1\}$ if and only if the following statements hold:*

1. *One of the following holds:*
   - *(a) $k \geq 2$ and $XY = 1$ is a column relation.*
   - *(b) $k = 1$ and $\beta_{1,1} = \beta_{0,0}$.*
2. *$M(k)$ is positive semidefinite, recursively generated and if $\operatorname{rank}(M(k))_{\mathcal{B}} = 2k$, then*

$$\operatorname{rank}(M(k))_{\mathcal{B}\setminus\{X^k\}} = \operatorname{rank}(M(k))_{\mathcal{B}\setminus\{Y^k\}} = 2k.$$

Moreover, let $r = \operatorname{rank} M(k)$ and β admits a K–representing measure. Let

$$\widetilde{\beta} := (\beta_{0,2k}, \beta_{0,2k-1}, \ldots, \beta_{0,1}, \beta_{0,0}, \beta_{1,0}, \ldots, \beta_{2k,0}).$$

Then:

- *(i) If $r \leq 2k$, then the representing measure is unique and of the form $\mu = \sum_{i=1}^{r} \rho_i \delta_{(x_i, x_i^{-1})}$, where $x_1, \ldots, x_r$ are the roots of the generating polynomial of $\widetilde{\beta}$ and $\rho_1, \ldots, \rho_r > 0$ the corresponding densities.*
- *(ii) If $r = 2k + 1$, then there are infinitely many $(2k + 1)$–atomic representing measures for β. Denoting $A_{\widetilde{\beta}} = \begin{pmatrix} \mathbf{v_0} & \mathbf{v_1} & \cdots & \mathbf{v_{2k}} \end{pmatrix}$, they are obtained by the following procedure:*
  - *Choose any $\beta_{2k+1,0} \in \mathbb{R}$, which is not equal to $v^T C^{-1} v$ if C is invertible, where $C = \begin{pmatrix} \mathbf{v_1}(0 : k_1 + k_2 - 1) & \cdots & \mathbf{v_k}(0 : k_1 + k_2 - 1) \end{pmatrix}$ and $v = \begin{pmatrix} \beta_{1,0} & \cdots & \beta_{2k,0} \end{pmatrix}^T$.*
  - *Define $\beta_{2k+2,0} = w^T (A_{\widetilde{\beta}})^{-1} w$, where $w = \begin{pmatrix} \beta_{1,0} & \cdots & \beta_{2k,0} & \beta_{2k+1,0} \end{pmatrix}^T$.*
  - *Use (i) for $\widehat{\beta} := (\widetilde{\beta}, \beta_{2k+1,0}, \beta_{2k+2,0})$.*

Proof. For $m \in \{-2k, -2k + 1, \ldots, 2k\}$ we define the numbers β_m by the following rule

$$\beta_m = \begin{cases} \beta_{m,0}, & m \geq 0, \\ \beta_{0,-m}, & m < 0. \end{cases}$$

Claim. Let $t \in \mathbb{N}$. The atoms $(x_1, x_1^{-1}), \ldots (x_t, x_t^{-1})$ with densities $\rho_1, \ldots, \rho_t$ are the $(xy - 1)$-representing measure for $\beta^{(2k)} = (\beta_{i,j})_{i,j \in \mathbb{Z}_+^2, i+j \leq 2k}$ if and only if the atoms $x_1, \ldots, x_t$ with densities $\rho_1, \ldots, \rho_t$ are the $(\mathbb{R} \setminus \{0\})$-representing measure for the 1–dimensional sequence $\beta := (\beta_{-2k}, \ldots, \beta_{-1}, \beta_0, \beta_1, \ldots, \beta_{2k})$.

The only if part follows from the following calculation:

$$\beta_{i,j} = \beta_{i-1,j-1} = \ldots = \begin{cases} \beta_{i-j,0}, & i \geq j, \\ \beta_{0,j-i}, & i < j. \end{cases} = \beta_{i-j} = \sum_{\ell=1}^{t} \rho_\ell x_\ell^{i-j} = \sum_{\ell=1}^{t} \rho_\ell x_\ell^{i} (x_\ell^{-1})^j,$$

where $i, j \in \mathbb{Z}_+^2$ such that $i + j \leq 2k$.

The if part follows from the following calculation:

$$\beta_m = \begin{cases} \beta_{m,0}, & m \geq 0, \\ \beta_{0,-m}, & m < 0. \end{cases} = \begin{cases} \sum_{\ell=1}^{t} \lambda_\ell x_\ell^m, & m \geq 0, \\ \sum_{\ell=1}^{t} \lambda_\ell (x_\ell^{-1})^{-m}, & m < 0. \end{cases} = \sum_{\ell=1}^{t} \lambda_\ell x_\ell^m,$$

where $m = -2k, -2k+1, \ldots, 2k$.

Using Claim, a result stating that if $\beta^{(2k)}$ has a K–representing measure, then it has a finitely atomic K–representing measure (see [57] or [7]), and Theorem 3.1, there exists a representing measure for $\beta^{(2k)}$ supported on K if and only if (1) and (A) are true, where

(A) $M(k)$ is psd, rg and one of the following conditions holds:
  (a) $(M(k))_{\mathcal{B}} \succ 0$.
  (b) $\operatorname{rank}(M(k))_{\mathcal{B}} = \operatorname{rank}(M(k))_{\mathcal{B}\setminus\{X^k\}} = \operatorname{rank}(M(k))_{\mathcal{B}\setminus\{Y^k\}}$.

It remains to prove the equivalence (A) $\Leftrightarrow$ (2). The nontrivial implication is (A) $\Leftarrow$ (2). If $\operatorname{rank}(M(k))_{\mathcal{B}} = 2k+1$, then (a) follows form the fact that $M(k)$ is psd. If $\operatorname{rank}(M(k))_{\mathcal{B}} = 2k$, then we are in case (b). It remains to prove that in case $\operatorname{rank}(M(k))_{\mathcal{B}} < 2k$, $(M(k))_{\mathcal{B}}$ being psd and rg implies (b). By symmetry it suffices to prove that $\operatorname{rank}(M(k))_{\mathcal{B}} = \operatorname{rank}(M(k))_{\mathcal{B}\setminus\{X^k\}}$. Let us assume on contrary that $\operatorname{rank}(M(k))_{\mathcal{B}} > \operatorname{rank}(M(k))_{\mathcal{B}\setminus\{X^k\}}$. This means that

$$\operatorname{rank}(M(k))_{\mathcal{B}\setminus\{X^k\}} \leq 2k-2.$$

Since $(M(k))_{\mathcal{B}}$ is a Hankel matrix in the order $Y^k, \ldots, Y, 1, X, \ldots, X^k$ of rows and columns, it follows that

$$X^{k-2} \in \operatorname{span}\{Y^k, \ldots, Y, 1, X, \ldots, X^{k-3}\}.$$

Proposition 2.1 implies that

$$X^{k-1} \in \operatorname{span}\{Y^{k-1}, \ldots, Y, 1, X, \ldots, X^{k-2}\}$$

or equivalently

$$X^{k-1} = \sum_{i=k-1}^{1} \alpha_i Y^i + \sum_{j=0}^{k-2} \beta_j X^j \quad \text{for some } \alpha_i, \beta_j \in \mathbb{R}. \tag{3.3}$$

Since $M(k)$ is rg, multiplying 3.3 with X and using $XY = 1$, implies that

$$X^k \in \operatorname{span}\{Y^{k-2}, \ldots, Y, 1, X, \ldots, X^{k-1}\},$$

which is a contradiction with $\operatorname{rank}(M(k))_{\mathcal{B}} > \operatorname{rank}(M(k))_{\mathcal{B}\setminus\{X^k\}}$. This proves (A) $\Leftarrow$ (2).

The moreover part of the corollary follows from the moreover part of Theorem 3.1 by also noticing that $\beta = \widetilde{\beta}$ and $\operatorname{rank} \widetilde{\beta} = \operatorname{rank}(M(k))_{\mathcal{B}} = \operatorname{rank} M(k)$. This concludes the proof of the corollary. □

Remark 3.6. [22, Proposition 2.14] states that in case $\operatorname{rank} M(k) = 2k+1$ there exists a $(\operatorname{rank} M(k))$ or $(\operatorname{rank} M(k)+1)$–atomic measure, depending on the choice of the moments $\beta_{2k+1,0}$ and $\beta_{0,2k+1}$ in the extension $M(k+1)$ (denoted by p and q in the proof of [22, Proposition 2.14]). By Corollary 3.5, p and q giving a $(\operatorname{rank} M(k))$–atomic measure, exist. Note also that this is not in contradiction with [22, Example 5.2] which only demonstrates the role of the choices of $\beta_{2k+1,0}$ and $\beta_{0,2k+1}$ on the rank of the extension of $M(k)$ to the moment matrix $M(k+1)$.

4. The STHMP with a gap β_{-2k_1+1} or β_{2k_2-1} and the TMP with variety $x^2y=1$

In this section we first solve the STHMP of degree $(-2k_1, 2k_2)$ with a missing moment β_{-2k+1} or β_{2k_2-1} and then as a corollary obtain the solution to the TMP for the curve $x^2y=1$.

A **partial matrix** $A=(a_{ij})_{i,j=1}^n$ is a matrix of real numbers $a_{ij}\in\mathbb{R}$, where some of the entries are not specified. A symmetric matrix $A=(a_{ij})_{i,j=1}^n$ is **partially positive semidefinite (ppsd)** if the following two conditions hold:

(1) a_{ij} is specified if and only if a_{ji} is specified and $a_{ij}=a_{ji}$.
(2) All fully specified principal minors of A are psd.

Let

$$M=\begin{bmatrix} A & B \\ C & D\end{bmatrix}\in M_{n+m} \tag{4.1}$$

be a real matrix where $A\in M_n$, $B\in M_{n,m}$, $C\in M_{m,n}$ and $D\in M_m$. The **generalized Schur complement** [64] of A (resp. D) in M is defined by

$$M/A=D-CA^{\dagger}B \quad (\text{resp. } M/D=A-BD^{\dagger}C),$$

where $A^{\dagger}$ (resp. $D^{\dagger}$) stands for the Moore-Penrose inverse of A (resp. D).

Theorem 4.1. *Let $k_1,k_2\in\mathbb{N}$, and*

$$\beta(x):=(\beta_{-2k_1},x,\beta_{-2k_1+2},\ldots,\beta_0,\ldots,\beta_{2k_2})$$

be a sequence where each β_i is a real number, $\beta_{-2k_1}>0$ and x is a variable. Let

$$v:=\begin{pmatrix}\beta_{-2k_1+2} & \cdots & \beta_{-k_1+k_2-1}\end{pmatrix} \quad \textit{and} \quad u:=\begin{pmatrix}\beta_{-2k_1+2} & \cdots & \beta_{-k_1+k_2}\end{pmatrix}$$

vectors, and

$$\widetilde{A}:=\begin{pmatrix}\beta_{-2k_1} & v \\ v^T & M(-k_1+2,k_2-1)\end{pmatrix} \quad \textit{and} \quad \widehat{A}:=\begin{pmatrix}\beta_{-2k_1} & u \\ u^T & M(-k_1+2,k_2)\end{pmatrix}$$

matrices. Then the following statements are equivalent:

(1) There exists $x_0\in\mathbb{R}$ and a representing measure for $\beta(x_0)$ supported on $K=\mathbb{R}\setminus\{0\}$.
(2) There exists $x_0\in\mathbb{R}$ such that $\beta(x_0)$ is recursively generated.
(3) There exists $x_0\in\mathbb{R}$ such that $\beta(x_0)$ is singular and recursively generated.
(4) There exists $x_0\in\mathbb{R}$ and a $(\operatorname{rank} M(-k_1+1,k_2))$–atomic representing measure for $\beta(x_0)$ supported on $K=\mathbb{R}\setminus\{0\}$.
(5) $A_{\beta(x)}$ is partially positive semidefinite and one of the following conditions is true:
   (a) $M(-k_1+1,k_2)\succ 0$ and $\widetilde{A}\succ 0$.
   (b) $\operatorname{rank} M(-k_1+1,k_2-1)=\operatorname{rank} M(-k_1+1,k_2)=\operatorname{rank} M(-k_1+2,k_2)=\operatorname{rank}\widehat{A}$.

Moreover, assume that there exists $x_0\in\mathbb{R}$ such that (4) holds. Let

$$s:=M(-k_1+1,k_2)/M(-k_1+2,k_2), \quad t:=\widehat{A}/M(-k_1+2,k_2)$$

and $w=\begin{pmatrix}\beta_{-2k_1+3} & \cdots & \beta_{-k_1+k_2+1}\end{pmatrix}$. Then:

(i) If $s=t=0$, then $x_0:=u(M(-k_1+2,k_2))^{\dagger}w^T$.
(ii) Else $s>0$, $t>0$ and there are two choices $x_{0,\pm}$ for x_0, i.e.,

$$x_{0,\pm}=u(M(-k_1+2,k_2))^{\dagger}w^T\pm\sqrt{s\cdot t}.$$

Once x_0 is fixed, the representing measure for $\beta(x_0)$ is unique and its support consists of the roots of the generating polynomial of $\beta(x_0)$.

Proof of Theorem 4.1. The equivalence (1) $\Leftrightarrow$ (2) follows from Theorem 3.1.

Now we prove the implication (2) $\Rightarrow$ (5). Since $\beta(x_0)$ is rg, the sequence $\beta(x_0)$ and the reversed sequence $\beta(x_0)^{(\text{rev})} := (\beta_{2k_2}, \beta_{2k_2-1}, \ldots, \beta_{-2k_1+2}, x_0, \beta_{-2k_1})$ are both prg. Regarding $\beta(x_0)$ and $\beta(x_0)^{(\text{rev})}$ as degree $(0, 2(k_1+k_2))$ sequences, the equivalence (1) $\Leftrightarrow$ (3) of Theorem 2.4 implies that they both admit representing measures on $\mathbb{R}$. Using [63, Theorem 4.1] for $\beta(x_0)$ and [63, Theorem 3.1] for $\beta(x_0)^{(\text{rev})}$, (5) holds.

The implication (5) $\Rightarrow$ (3) follows from [63, Theorem 4.1]. Indeed, the equivalence (ii) $\Leftrightarrow$ (iii) of [63, Theorem 4.1] implies that there exists $x_0 \in \mathbb{R}$ such that $\beta(x_0)$, regarded as a $(0, 2(k_1+k_2))$–degree sequence, admits a $(\operatorname{rank} M(-k_1+1, k_2))$–atomic $\mathbb{R}$–representing measure. So $\beta(x_0)$ is a singular sequence. By Theorem 2.4, $\beta(x_0)$ is prg and $\operatorname{rank} A_{\beta(x_0)} = \operatorname{rank} M(-k_1+1, k_2)$. It remains to prove that $\beta(x_0)$ is nrg. Let $p(x) := x^r - \sum_{j=0}^{r-1} \varphi_i x^i$ be the generating polynomial of $\beta(x_0)$. If $\varphi_0 = 0$, then the first column of $A_{\beta(x_0)}$ is not in the span of its other columns. But this is in contradiction with $\operatorname{rank} A_{\beta(x_0)} = \operatorname{rank} M(-k_1+1, k_2)$. Hence, $\varphi_0 \neq 0$ and by Proposition 2.3, $\beta(x_0)$ is rg.

The implication (3) $\Rightarrow$ (2) is trivial. So far we established the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5). The implication (4) $\Rightarrow$ (1) is trivial. It remains to prove the implication (3) $\Rightarrow$ (4). Since $\beta(x_0)$ is singular and rg, Proposition 2.1.(4) implies that $\operatorname{rank} \beta(x_0) = \operatorname{rank} A_{\beta(x_0)}$, while Proposition 2.1.(5) implies that $\operatorname{rank} A_{\beta(x_0)} = \operatorname{rank} M(-k_1+1, k_2)$. Hence, $\operatorname{rank} \beta(x_0) = \operatorname{rank} M(-k_1+1, k_2)$. Using Theorem 3.1 for $\beta(x_0)$ gives (4).

For the moreover part about possible choices of x_0 see the Claim in the proof of [63, Theorem 4.1]. The last sentence about the form of the representing measure for $\beta(x_0)$ follows from the moreover part of Theorem 2.4. □

Corollary 4.2. *Let $k_1, k_2 \in \mathbb{N}$, and*

$$\beta(x) := (\beta_{-2k_1}, \ldots, \beta_{2k_2-2}, x, \beta_{2k_2})$$

be a sequence where each β_i is a real number, $\beta_{-2k_1} > 0$ and x is a variable. Let

$$v := \begin{pmatrix} \beta_{-k_1+k_2+1} & \cdots & \beta_{2k_2-2} \end{pmatrix} \quad \text{and} \quad u := \begin{pmatrix} \beta_{-k_1+k_2} & \cdots & \beta_{2k_2-2} \end{pmatrix}$$

vectors, and

$$\widetilde{A} := \begin{pmatrix} M(-k_1+1, k_2-2) & v \\ v^T & \beta_{2k_2} \end{pmatrix} \quad \text{and} \quad \widehat{A} := \begin{pmatrix} M(-k_1, k_2-2) & u \\ u^T & \beta_{2k_2} \end{pmatrix}$$

matrices. Then the following statements are equivalent:

1. *There exists $x_0 \in \mathbb{R}$ and a representing measure for $\beta(x_0)$ supported on $K = \mathbb{R} \setminus \{0\}$.*
2. *There exists $x_0 \in \mathbb{R}$ such that $\beta(x_0)$ is recursively generated.*
3. *There exists $x_0 \in \mathbb{R}$ such that $\beta(x_0)$ is singular and recursively generated.*
4. *There exists $x_0 \in \mathbb{R}$ and a $(\operatorname{rank} M(-k_1, k_2-1))$–atomic representing measure for $\beta(x_0)$ supported on $K = \mathbb{R} \setminus \{0\}$.*
5. *$A_{\beta(x)}$ is partially positive semidefinite and one of the following conditions is true:*
   - (a) *$M(-k_1, k_2-1) \succ 0$ and $\widetilde{A} \succ 0$.*
   - (b) $\operatorname{rank} M(-k_1+1, k_2-1) = \operatorname{rank} M(-k_1, k_2-1) = \operatorname{rank} M(-k_1, k_2-2) = \operatorname{rank} \widehat{A}$.

Moreover, assume that there exists $x_0 \in \mathbb{R}$ such that (4) holds. Let

$$s := M(-k_1, k_2-1)/M(-k_1, k_2-2), \quad t := \widehat{A}/M(-k_1, k_2-2)$$

and $w = \begin{pmatrix} \beta_{-k_1+k_2-1} & \cdots & \beta_{2k_2-3} \end{pmatrix}$. Then:

- (i) *If $s = t = 0$, then $x_0 := u(M(-k_1, k_2-2))^{\dagger} w^T$.*

(ii) Else $s>0$, $t>0$ and there are two choices $x_{0,\pm}$ for x_0, i.e.,

$$x_{0,\pm}=u(M(-k_1,k_2-2))^{\dagger}w^T\pm\sqrt{s\cdot t}.$$

Once x_0 is fixed, the representing measure for $\beta(x_0)$ is unique and its support consists of the roots of the generating polynomial of $\beta(x_0)$.

Proof. Note that $\sum_{j=1}^{\ell}\rho_j\delta_{x_j}$, where $\rho_j>0$ are densities and $x_j\in\mathbb{R}\setminus\{0\}$ are atoms, is a $(\mathbb{R}\setminus\{0\})$–representing measure for $\beta(x)$ if and only if $\sum_{j=1}^{\ell}\rho_j\delta_{x_j^{-1}}$ is a $(\mathbb{R}\setminus\{0\})$–representing measure for

$$\widetilde{\beta}(x):=(\tilde{\beta}_{-2k_2},x,\tilde{\beta}_{-2k_2+2},\ldots,\tilde{\beta}_0,\ldots,\tilde{\beta}_{2k_1}),$$

where $\tilde{\beta}_i=\beta_{-i}$ for each i. Using Theorem 4.1, the corollary follows. □

The following corollary is a consequence of Theorem 4.1 and gives the solution of the bivariate TMP for the curve $x^2y=1$.

Corollary 4.3. *Let $\beta=(\beta_{i,j})_{i,j\in\mathbb{Z}_+^2,i+j\leq 2k}$ be a 2–dimensional real multisequence of degree $2k$. Suppose $\mathcal{M}(k)$ is positive semidefinite and recursively generated. Let*

$$u^{(i)}:=(\beta_{0,i},\beta_{1,i})\quad \textit{for } i=1,\ldots,2k-1,$$

$$\widehat{\beta}:=(u^{(2k-1)},u^{(2k-2)},\ldots,u^{(1)},\beta_{0,0},\beta_{1,0},\ldots,\beta_{2k-2,0}),\quad \widetilde{\beta}:=(\widehat{\beta},\beta_{2k-1,0},\beta_{2k,0}),$$
$$\overline{\beta}:=(u^{(2k-2)},u^{(2k-3)},\ldots,u^{(1)},\beta_{0,0},\beta_{1,0},\ldots,\beta_{2k-2,0}),\quad \breve{\beta}:=(\overline{\beta},\beta_{2k-1,0},\beta_{2k,0}),$$

be subsequences of β,

$$v:=\begin{cases}\begin{pmatrix}u^{(2k-1)} & u^{(2k-2)} & \cdots & u^{(\frac{k}{2}+1)}\end{pmatrix}, & \textit{if } k \textit{ is even},\\ \begin{pmatrix}u^{(2k-1)} & u^{(2k-2)} & \cdots & u^{(\lceil\frac{k}{2}\rceil+1)} & \beta_{0,\lceil\frac{k}{2}\rceil}\end{pmatrix}, & \textit{if } k \textit{ is odd},\end{cases}$$

a vector and

$$\widetilde{A}:=\begin{pmatrix}\beta_{0,2k} & v\\ v^T & A_{\overline{\beta}}\end{pmatrix}$$

a matrix. Then β has a representing measure supported on the variety $K:=\{(x,y)\in\mathbb{R}^2\colon x^2y=1\}$ if and only if the following statements hold:

(1) One of the following holds:
- *$k\geq 3$ and $X^2Y=1$ is a column relation of $M(k)$.*
- *$k=2$ and the equalities $\beta_{2,1}=\beta_{0,0}$, $\beta_{3,1}=\beta_{1,0}$ hold.*
- *$k=1$.*

(2) One of the following holds:
  - *(a) $A_{\widetilde{\beta}}\succ 0$ and $\widetilde{A}\succ 0$.*
  - *(b) $A_{\widetilde{\beta}}\succeq 0$ and $\operatorname{rank}A_{\widehat{\beta}}=\operatorname{rank}A_{\widetilde{\beta}}=\operatorname{rank}A_{\breve{\beta}}=\operatorname{rank}M(k)$.*

Moreover, let $r=\operatorname{rank}M(k)$ and β admits a K–representing measure. Let $\gamma(x):=(\beta_{0,2k},x,\widetilde{\beta})$ and $\widehat{A}:=\begin{pmatrix}\beta_{0,2k} & u\\ u^T & A_{\breve{\beta}}\end{pmatrix}$, where

$$u:=\begin{cases}\begin{pmatrix}v & \beta_{0,\frac{k}{2}}\end{pmatrix}, & \textit{if } k \textit{ is even},\\ \begin{pmatrix}v & \beta_{1,\lceil\frac{k}{2}\rceil}\end{pmatrix}, & \textit{if } k \textit{ is odd},\end{cases}$$

and

$$w := \begin{cases} \begin{pmatrix}\beta_{1,2k-1} & u^{(2k-2)} & \cdots & u^{(\frac{k}{2})}\end{pmatrix}, & \text{if } k \text{ is even},\\ \begin{pmatrix}\beta_{1,2k-1} & u^{(2k-2)} & \cdots & u^{(\lceil\frac{k}{2}\rceil)} & \beta_{0,\lceil\frac{k}{2}\rceil-1}\end{pmatrix}, & \text{if } k \text{ is odd}.\end{cases}$$

Then:

(i) If $r < 3k$, then the representing measure is unique and of the form $\mu = \sum_{i=1}^{\tilde{r}} \rho_i \delta_{(x_i, x_i^{-2})}$, where $\tilde{r} \in \{r, r+1\}$, $x_1, \ldots, x_{\tilde{r}}$ are the roots of the generating polynomial of $\gamma(x_0)$, $x_0 = u(A_{\breve{\beta}})^\dagger w^T$ and $\rho_1, \ldots, \rho_{\tilde{r}} > 0$ are the corresponding densities.

(ii) If $r = 3k$, then there are two $(3k)$–atomic representing measures. Let

$$x_\pm = u(A_{\breve{\beta}})^\dagger w^T \pm \sqrt{\Big(A_{\widetilde{\beta}}/A_{\breve{\beta}}\Big)\cdot\Big(\widehat{A}/A_{\breve{\beta}}\Big)}.$$

Then the two measures are of the form $\mu = \sum_{i=1}^{r} \rho_{i,\pm}\delta_{(x_{i,\pm}, x_{i,\pm}^{-2})}$, where $x_{1,\pm}, \ldots, x_{r,\pm}$ are the roots of the generating polynomial of $\gamma(x_\pm)$, and $\rho_{1,\pm}, \ldots, \rho_{r,\pm} > 0$ are the corresponding densities.

Proof. For $m \in \{-4k, -4k+2, -4k+3, \ldots, 2k\}$ we define the numbers $\widetilde{\beta}_m$ by the following rule

$$\widetilde{\beta}_m := \begin{cases} \beta_{0,\frac{|m|}{2}}, & \text{if } m \text{ is even and } m < 0,\\ \beta_{1,\lceil\frac{|m|}{2}\rceil}, & \text{if } m \text{ is odd and } m < 0,\\ \beta_{m,0}, & \text{if } m \geq 0.\end{cases}$$

Claim 1. Every number $\widetilde{\beta}_m$ is well-defined.

We have to prove that $i + j \leq 2k$, where i, j are indices of $\beta_{i,j}$ used in the definition of $\widetilde{\beta}_m$. We separate three cases according to m:

- m is even and $m < 0$: $\frac{|m|}{2} \leq \frac{4k}{2} = 2k$.
- m is odd and $m < 0$: $\lceil\frac{|m|}{2}\rceil + 1 \leq \lceil\frac{4k-3}{2}\rceil + 1 = 2k - 1 + 1 = 2k$.
- m is nonnegative: $m \leq 2k$.

Claim 2. Let $t \in \mathbb{N}$. The atoms $(x_1, x_1^{-2}), \ldots (x_t, x_t^{-2})$ with densities $\rho_1, \ldots, \rho_t$ are the $(x^2y - 1)$–representing measure for $(\beta_{i,j})_{i,j\in\mathbb{Z}_+^2, i+j\leq 2k}$ if and only if the atoms $x_1, \ldots, x_t$ with densities $\rho_1, \ldots, \rho_t$ are the $(\mathbb{R}\setminus\{0\})$–representing measure for $\widetilde{\beta}(x) = (\widetilde{\beta}_{-4k}, x, \widetilde{\beta}_{-4k+2}, \widetilde{\beta}_{-4k+3} \ldots, \widetilde{\beta}_{2k})$.

The if part follows from the following calculation:

$$\widetilde{\beta}_m = \begin{cases} \beta_{0,\frac{|m|}{2}}, & \text{if } m \text{ is even and } m < 0,\\ \beta_{1,\lceil\frac{|m|}{2}\rceil}, & \text{if } m \text{ is odd and } m < 0,\\ \beta_{m,0}, & \text{if } m \geq 0,\end{cases}$$
$$= \begin{cases} \sum_{\ell=1}^{t} \rho_\ell (x_\ell^{-2})^{\frac{|m|}{2}}, & \text{if } m \text{ is even and } m < 0,\\ \sum_{\ell=1}^{t} \rho_\ell x_\ell (x_\ell^{-2})^{\lceil\frac{|m|}{2}\rceil}, & \text{if } m \text{ is odd and } m < 0,\\ \sum_{\ell=1}^{t} \rho_\ell x_\ell^m, & \text{if } m \geq 0,\end{cases} \quad = \sum_{\ell=1}^{t} \rho_\ell x_\ell^m,$$

where $m = -4k, -4k+2, -4k+3, \ldots, 2k$.

The only if part follows from the following calculation:

$$\beta_{i,j} = \beta_{i-2,j-1} = \cdots = \begin{cases} \beta_{i-2j,0}, & \text{if } i-2j \geq 0, \\ \beta_{i \pmod 2, j-\lfloor \frac{i}{2} \rfloor}, & \text{if } i-2j < 0, \end{cases} = \widetilde{\beta}_{i-2j}$$
$$= \sum_{\ell=1}^{t} \rho_\ell x_\ell^{i-2j} = \sum_{\ell=1}^{t} \rho_\ell x_\ell^{i} (x_\ell^{-2})^j,$$

where the first three equalities in the first line follow by $M(k)$ being rg.

Using Claim 2 and a result stating that if $\beta^{(2k)}$ has a K–representing measure, then it has a finitely atomic K–representing measure (see [57] or [7]), the statements of the corollary follow by Theorem 4.1. □

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