THE TRUNCATED HAMBURGER MOMENT PROBLEMS WITH GAPS IN THE INDEX SET

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ABSTRACT. In this article we solve four special cases of the truncated Hamburger moment problem (THMP) of degree $2k$ with one or two missing moments in the sequence. As corollaries we obtain, by using appropriate substitutions, the solutions to bivariate truncated moment problems of degree $2k$ for special curves. Namely, for the curves $y = x^3$ (first solved by Fialkow [Fia11]), $y^2 = x^3$, $y = x^4$ where a certain moment of degree $2k + 1$ is known and $y^3 = x^4$ with a certain moment given. The main technique is the completion of the partial positive semidefinite matrix (ppsd) such that the conditions of Curto and Fialkow's solution of the THMP are satisfied. The main tools are the use of the properties of positive semidefinite Hankel matrices and a result on all completions of a ppsd matrix with one unknown entry, proved by the use of the Schur complements for 2×2 and 3×3 block matrices.

1. INTRODUCTION

For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $i = (i_1, \ldots, i_d) \in \mathbb{Z}_+^d$, we set $|i| = i_1 + \ldots + i_d$ and $x^i = x_1^{i_1} \cdots x_d^{i_d}$. Given a real d-dimensional multisequence $\beta = \beta^{(2k)} = \{\beta_i\}_{i \in \mathbb{Z}_+^d, |i| \leq 2k}$ of degree $2k$ and a closed subset K of $\mathbb{R}^d$, the **truncated moment problem (TMP)** supported on K for β asks to characterize the existence of a positive Borel measure μ on $\mathbb{R}$ with support in K, such that

$$\beta_i = \int_K x^i d\mu(x) \quad \text{for} \quad i \in \mathbb{Z}_+^d, |i| \leq 2k. \tag{1.1}$$

If such measure exists, we say that β has a representing measure supported on K and μ is its K-**representing measure**.

We denote by $M(k) = M(k)(\beta) = (\beta_{i,j})_{i,j=0}^k$ the moment matrix associated with β, where the rows and columns are indexed by X^i, $|i| \leq k$, in degree-lexicographic order. Let $\mathbb{R}[x]_k := \{p \in \mathbb{R}[x] \colon \deg p \leq k\}$ stand for the set of polynomials in d variables of degree at most k. To every $p := \sum_{i \in \mathbb{Z}_+^d, |i| \leq k} a_i x^i \in \mathbb{R}[x]_k$, we denote by $p(X) = \sum_{i \in \mathbb{Z}_+^d, |i| \leq k} a_i X^i$ the vector from the column space $\mathcal{C}(M(k))$ of the matrix $M(k)$. Recall from [CF96], that β has a representing measure μ with the support $\operatorname{supp} \mu$ being a subset of $\mathcal{Z}_p := \{x \in \mathbb{R}^d \colon p(x) = 0\}$ if and only if $p(X) = 0$. We say that the matrix $M(k)$ is **recursively generated (rg)** if for $p, q, pq \in \mathbb{R}[x]_k$ such that $p(X) = 0$, it follows that $(pq)(X) = 0$.

The full moment problem (MP), where β_i is given for every $i \in \mathbb{Z}_+^d$, being the classical question in analysis and also due to its relation with real algebraic geometry via the duality with positive

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Date: July 21, 2020.

2010 *Mathematics Subject Classification.* Primary 47A57, 47A20, 44A60; Secondary 15A04, 47N40.

Key words and phrases. Hamburger moment problem, truncated moment problems, representing measure, moment matrix.

Supported by the Slovenian Research Agency grants J1-2453, P1-0288.

polynomials given by Haviland's theorem [Hav35], has been widely studied, see e.g., [Akh65, AhK62, KN77, Las09, Lau05, Lau09, Mar08, PS06, PS08, Put93, PV99, Sch91, Sch03, Sch17]. The TMP, which is more general than the full MP [Sto01], has been intensively studied in a series of papers by Curto and Fialkow [CF91, CF96, CF98a, CF98b, CF02, CF04, CF05, CF08] with the celeberated flat extension theorem they established as a core tool in the field. There are also various generalizations of the TMP (e.g., [**?**, Bol96, **?**, DU18], to matrix moments, [BK10, **?**] to tracial moments, [IKLS17] to infinitely many variables). Recently, Fialkow's core variety [Fia17] approach led to many new results on the TMP; see also [BF20, DS18]. A **concrete solution** to the TMP is a set of necessary and sufficient conditions for the existence of a K-representing measure. Among necessary conditions, $M(k)$ must be psd and rg [CF91, CF98b], which also suffice in some cases. Concrete solutions to the TMP are known in the following cases:

(1) (Truncated Hamburger moment problem (THMP)) $d = 1$ and $K = \mathbb{R}$. See [AhK62, Theorem I.3] or [Ioh82, Theorem A.II.1] for the special case of even k with an invertible moment matrix and [CF91, Section 3] for the general case.
(2) (Truncated Hausdorff moment problem) $d = 1$ and $K = [0, \infty)$. See [KN77, p. 175] for the special case of an invertible moment matrix and [CF91, Section 5] for the general case.
(3) (Truncated Stieltjes moment problem) $d = 1$ and $K = [a, b]$, $a < b$. See [KN77, Theorems III.2.4 and II.2.3] and [CF91, Section 4] for the general case.
(4) $d = 2$ and K is a curve $p(x, y) = 0$ with $\deg p \leq 2$. See [CF02, CF04, CF05, FN10, Fia14, CS16].
(5) $d = 2$ and K is a curve $y = x^3$. See [Fia11].
(6) $d = 2$ and the moment matrix has a special feature called *recursive determinateness*. See [CF13] for details.
(7) (Extremal case) The rank of the moment matrix is the same as the cardinality of the corresponding variety; see [CFM08].
(8) Some special cases are solved in [CS15, Fia17, Ble15, BF20].

In (5), β must satisfy certain numerical conditions, which are equivalent to the conditions from Corollary 3.3 below. The proof is by separating the nonsingular case from the singular one. In the nonsingular case the existence of a flat extension is established by a detailed and technically demanding analysis, while the singular case is done by the use of additional features of the moment matrix such as recursive determinateness and known results for such matrices.

In this article we present concrete solutions to the four cases of the THMP of degree $2k$ with some unknown moments $\beta_{i_1}, \ldots, \beta_{i_j}$, $1 \leq i_1 \leq \cdots \leq i_j \leq 2k-1$, in the sequence, which we call the **THMP with gaps** $(\beta_{i_1}, \ldots, \beta_{i_j})$. Namely, we solve the THMP with gaps (β_{2k-1}), $(\beta_{2k-2}, \beta_{2k-1})$, (β_1) and (β_1, β_2). The motivation to solve this cases of the THMP with gaps is to obtain the solutions to the special cases of the 2-dimensional TMP. Namely, the solution of the THMP with gaps:

(1) (β_{2k-1}) gives an alternative solution to the TMP with $d = 2$ and K being the curve $y = x^3$ (see (5) above). The advantage of our approach is that the proof is short and we also do not need to separate three subcases, i.e., $k = 1$, $k = 2$ and $k \geq 3$.
(2) $(\beta_{2k-2}, \beta_{2k-1})$ solves the TMP with $d = 2$, K being the curve $y = x^4$ and in addition the moment $\beta_{3,2k-2}$ of degree $2k+1$ is known. To solve the TMP for the curve $y = x^4$ without this additional moment, one needs to solve the THMP with gaps $(\beta_{2k-5}, \beta_{2k-2}, \beta_{2k-1})$ which is a possible topic of future research.
(3) (β_1) solves the TMP with $d = 2$ and K being the curve $y^2 = x^3$.

(4) (β_1, β_2) solves the TMP with $d = 2$, K being the curve $y^3 = x^4$ and known $\beta_{\frac{5}{3},0}$. By $\beta_{\frac{5}{3},0}$ we mean the moment of $x_1^{\frac{5}{3}}$, i.e., $\int_K x_1^{\frac{5}{3}} d\mu$. To solve the TMP for the curve $y^3 = x^4$ without this additional information, one needs to solve the THMP with gaps $(\beta_1, \beta_2, \beta_5)$, which is another open question for future research.

1.1. Readers Guide.

The paper is organized as follows. In Section 2 we present the tools used in the proofs of our main results:

- Generalized Schur complements and verification of positive semidefiniteness of block matrices (Subsection 2.1).
- Properties of psd Hankel matrices (Subsection 2.2).
- The solution to the THMP (Subsection 2.3).
- A result about psd completions of partial psd matrices with one unknown entry (Subsetion 2.4).
- An extension principle for psd matrices (Subsection 2.5).
- A result about subsequences of moment sequences (Subsection 2.6).

In Section 3 we solve the THMP of degree $2k$ with gaps (β_{2k-1}) (see Theorem 3.1) and $(\beta_{2k-2}, \beta_{2k-1})$ (see Theorem 3.5). Corollary 3.3, being a special case of the (β_{2k-1})-case, is the solution to the TMP with $d = 2$ and the curve $y = x^3$ as K, while Corollary 3.6, being a special case of the $(\beta_{2k-2}, \beta_{2k-1})$-case, is the solution to the TMP with $d = 2$, the curve $y = x^4$ as K and an additional moment $\beta_{3,2k-2}$ known.

In Section 4 we solve the THMP of degree $2k$ with gaps (β_1) (see Theorem 4.1) and (β_1, β_2) (see Theorem 4.5). Corollary 4.4, being a special case of the (β_1)-case, is the solution to the TMP with $d = 2$ and the curve $y^2 = x^3$ as K, while Corollary 4.7, being a special case of the (β_1, β_2)-case, is the solution to the TMP with $d = 2$, the curve $y^3 = x^4$ as K and an additional moment $\beta_{\frac{5}{3},0}$ known.

Acknowledgement. I would like to thank Jaka Cimprič and Abhishek Bhardwaj for useful suggestions on the preliminary versions of this article.

2. Preliminaries

In this section we present some tools which will be needed in the proofs of our main results in Sections 3 and 4.

We write $M_{n,m}$ (resp. M_n) for the set of $n \times m$ (resp. $n \times n$) real matrices. For a matrix M we denote by $\mathcal{C}(M)$ its column space. The set of real symmetric matrices of size n will be denoted by S_n. For a matrix $A \in S_n$ the notation $A \succ 0$ (resp. $A \succeq 0$) means A is positive definite (pd) (resp. positive semidefinite (psd)).

2.1. Generalized Schur complements.

Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S_{n+m} \tag{2.1}$$

be a real matrix where $A \in M_n$, $B \in M_{n,m}$, $C \in M_{m,n}$ and $D \in M_m$. The **generalized Schur complement** [Zha05] of A (resp. D) in M is defined by

$$M/A = D - CA^+B \quad \text{(resp. } M/D = A - BD^+C\text{)},$$

where A^+ (resp. D^+) stands for the Moore-Penrose inverse of A (resp. D).

Remark 2.1. (1) If A (resp. D) is invertible, then M/A (resp. M/D) is the usual Schur complement of A (resp. D) in M.
(2) Note that $M/A = \begin{pmatrix} D & C \\ B & A \end{pmatrix} /A$.

The following theorem gives conditions for verifying positive semidefiniteness of a block matrix of size 2.

Theorem 2.2. [Alb69] *Let*

$$M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \in S_{n+m} \tag{2.2}$$

be a real symmetric matrix where $A \in S_n$, $B \in M_{n,m}$ and $C \in S_m$. Then the following conditions are equivalent:

(1) $M \succeq 0$.
(2) $C \succeq 0$, $\mathcal{C}(B^T) \subseteq \mathcal{C}(C)$ and $M/C \succeq 0$.
(3) $A \succeq 0$, $\mathcal{C}(B) \subseteq \mathcal{C}(A)$ and $M/A \succeq 0$.

If $m = 1$ in (2.2), then $\operatorname{rank} M \in \{\operatorname{rank} A, \operatorname{rank} A+1\}$. The following proposition characterizes w.r.t. the value of M/A when each of the possibilities occurs in the case M is psd.

Proposition 2.3. *Let*

$$M = \begin{pmatrix} A & b \\ b^T & c \end{pmatrix} \in S_{n+1}$$

be a real symmetric matrix where $A \in S_n$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then $\operatorname{rank} M = \operatorname{rank} A$ *if and only if $M/A = 0$. Otherwise* $\operatorname{rank} M = \operatorname{rank} A + 1$.

Proof. By Theorem 2.2, the psd assumption implies that $b \in \mathcal{C}(A)$. By the properties of the Moore-Penrose inverse $\{A^+b + w \colon w \in \ker A\}$ is the set of solutions z of the system $Az = b$. Therefore,

$$\mathcal{C}(M) = \mathcal{C}\Big(\begin{pmatrix} A & 0 \\ b^T & c - b^T(A^+b + w) \end{pmatrix} \Big) = \mathcal{C}\Big(\begin{pmatrix} A & 0 \\ b^T & M/A \end{pmatrix} \Big), \tag{2.3}$$

where the second equality follows from the fact that A is symmetric, $b \in \mathcal{C}(A)$ and $w \in \ker A$. Now, the statement of the proposition follows from (2.3). □

The following proposition gives an explicit formula, called the *quotient formula* [CH69], for expressing the Schur complement of a 2×2 upper left-hand or a 2×2 lower right-hand block in a 3×3 block matrix using 2×2 block submatrices.

Proposition 2.4. *Let*

$$K = \begin{pmatrix} A & B & D \\ B^T & C & E \\ D^T & E^T & F \end{pmatrix} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{M} & \begin{matrix} D \\ E \end{matrix} \\ \hline D^T & E^T & F \end{array}\right) = \left(\begin{array}{c|cc} A & B & D \\ \hline \begin{matrix} B^T \\ D^T \end{matrix} & \multicolumn{2}{c}{N} \end{array}\right) \in S_{n_1+n_2+n_3}$$

be a 3×3 block real matrix, where $A \in S_{n_1}, C \in S_{n_2}, F \in S_{n_3}$ are real symmetric matrices and $B \in M_{n_1,n_2}, D_{n_1,n_3}, E_{n_2,n_3}$ are rectangular matrices. If M and A are nonsingular, then

$$K/M = \begin{pmatrix} A & D \\ D^T & F \end{pmatrix} /A - \left[\begin{pmatrix} A & B \\ D^T & E^T \end{pmatrix} \Big/ A \right] (M/A)^{-1} \left[\begin{pmatrix} A & D \\ B^T & E \end{pmatrix} \Big/ A \right]. \tag{2.4}$$

If N and C are nonsingular, then

$$K/N = \begin{pmatrix} C & B^T \\ B & A \end{pmatrix} \Big/ C - \left[\begin{pmatrix} C & E \\ B & D \end{pmatrix} \Big/ C \right] (N/C)^{-1} \left[\begin{pmatrix} C & B^T \\ E^T & D^T \end{pmatrix} \Big/ C \right]. \tag{2.5}$$

Proof. By an easy calculation we have that

$$K/A = \begin{pmatrix} M/A & \begin{pmatrix} A & D \\ B^T & E \end{pmatrix}/A \\ \begin{pmatrix} A & B \\ D^T & E^T \end{pmatrix}/A & \begin{pmatrix} A & D \\ D^T & F \end{pmatrix}/A \end{pmatrix}.$$

Now the quotient formula [CH69] $K/M = (K/A)/(M/A)$ yields (2.4).

By Remark 2.1 (2), it is true that $K/N = L/N$ where

$$L = \left(\begin{array}{c|c} N & \begin{matrix} B^T \\ D^T \end{matrix} \\ \hline \begin{matrix} B & D \end{matrix} & A \end{array}\right).$$

Now (2.5) follows from (2.4). □

2.2. **Hankel matrices.** Let $k \in \mathbb{N}$. For

$$\beta = (\beta_0, \ldots, \beta_{2k}) \in \mathbb{R}^{2k+1},$$

we denote by

$$A_\beta := (\beta_{i+j})_{i,j=0}^{k} = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_k \\ \beta_1 & \beta_2 & \iddots & \iddots & \beta_{k+1} \\ \beta_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \beta_{2k-1} \\ \beta_k & \beta_{k+1} & \cdots & \beta_{2k-1} & \beta_{2k} \end{pmatrix} \in S_{k+1}$$

the corresponding Hankel matrix. We denote by $\mathbf{v_j} := (\beta_{j+\ell})_{\ell=0}^{k}$ the $(j+1)$-th column of A_β, $0 \leq j \leq k$, i.e.,

$$A_\beta = \begin{pmatrix} \mathbf{v_0} & \cdots & \mathbf{v_k} \end{pmatrix}.$$

As in [CF91], the **rank** of β, denoted by $\operatorname{rank} \beta$, is defined by

$$\operatorname{rank} \beta = \begin{cases} k+1, & \text{if } A_\beta \text{ is nonsingular}, \\ \min\{i \colon \mathbf{v_i} \in \operatorname{span}\{\mathbf{v_0}, \ldots, \mathbf{v_{i-1}}\}\}, & \text{if } A_\beta \text{ is singular}. \end{cases}$$

We denote the upper left-hand corner of A_β of size $m+1$ by

$$A_\beta(m) = (\beta_{i+j})_{i,j=0}^{m} \in S_{m+1}.$$

The following proposition is the alternative description of $\operatorname{rank} \beta$ if A_β is singular.

Proposition 2.5. [CF91, Proposition 2.2] *Let $k \in \mathbb{N}$, $\beta = (\beta_0, \ldots, \beta_{2k})$, and assume that A is positive semidefinite and singular. Then*

$$\operatorname{rank} \beta = \min\{j \colon 0 \leq j \leq k \textit{ such that } A_\beta(j) \textit{ is singular}\}.$$

Important property of psd Hankel matrices is the following rank principle.

Theorem 2.6. [CF91, Corollary 2.5] *Let $k \in \mathbb{N}$, $\beta = (\beta_0, \ldots, \beta_{2k})$, $\widetilde{\beta} = (\beta_0, \ldots, \beta_{2k-2})$, $A_\beta \succeq 0$ and $r = \operatorname{rank} \widetilde{\beta}$. Then:*

(1) $\operatorname{rank} A_{\widetilde{\beta}} = r$.

(2) $r \leq \operatorname{rank} A_\beta \leq r+1$.

(3) $\operatorname{rank} A_\beta = r+1$ *if and only if*

$$\beta_{2k} > \varphi_0\beta_{2k-r} + \ldots + \varphi_{r-1}\beta_{2k-1},$$

where $(\varphi_0,\ldots,\varphi_{r-1}) := A_\beta(r-1)^{-1}(\beta_r,\ldots,\beta_{2r-1})^T$

We will use the following corollary of Proposition 2.5 and Theorem 2.6 in the sequel.

Corollary 2.7. *In the notation of Theorem 2.6, under the assumptions* $A_\beta \succeq 0$*,* A_β *is singular, and* $r = \operatorname{rank}\widetilde{\beta}$*, then*

$$r = \operatorname{rank}\beta = \operatorname{rank} A_\beta(r-1) = \operatorname{rank} A_\beta(r) = \ldots = \operatorname{rank} A_\beta(k-1) = \operatorname{rank} A_{\widetilde{\beta}}.$$

We denote the lower right-hand corner of A_β of size $m+1$ by

$$A_\beta[m] = (\beta_{i+j})_{i,j=m-k}^{k} = \begin{pmatrix} \beta_{2(k-m)} & \beta_{2(k-m)+1} & \beta_{2(k-m+1)} & \cdots & \beta_{2k-m} \\ \beta_{2(k-m)+1} & \beta_{2(k-m+1)} & \iddots & \iddots & \beta_{2k-m+1} \\ \beta_{2(k-m+1)} & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \beta_{2k-1} \\ \beta_{2k-m} & \beta_{2.k-m+1} & \cdots & \beta_{2k-1} & \beta_{2k} \end{pmatrix} \in S_{m+1}$$

Let

$$\beta^{(\mathrm{rev})} := (\beta_{2k}, \beta_{2k-1}, \ldots, \beta_0)$$

be the sequence obtained from β by reversing the order of numbers. Using Corollary 2.7 for a reversed sequence implies the following corollary.

Corollary 2.8. *In the notation of Theorem 2.6, under the assumption* $A_\beta \succeq 0$*,* A_β *is singular and* $r = \operatorname{rank}\widetilde{\beta}^{(rev)}$*, where* $\widetilde{\beta}^{(rev)} := (\beta_{2k},\ldots,\beta_2)$*, it holds that*

$$r = \operatorname{rank}\beta^{(rev)} = \operatorname{rank} A_\beta[r-1] = \operatorname{rank} A_\beta[r] = \ldots = \operatorname{rank} A_\beta[k-1] = \operatorname{rank} A_{\widetilde{\beta}^{(rev)}}.$$

Proof. Corollary 2.7 used for $\beta^{(\mathrm{rev})}$ implies that

$$r = \operatorname{rank}\beta^{(\mathrm{rev})} = \operatorname{rank} A_{\beta^{(\mathrm{rev})}}(r-1) = \operatorname{rank} A_{\beta^{(\mathrm{rev})}}(r) = \ldots = A_{\beta^{(\mathrm{rev})}}(k-1) = \operatorname{rank} A_{\widetilde{\beta}^{(\mathrm{rev})}}. \tag{2.6}$$

For $\ell = 0,\ldots,k$ define the permutation matrices $P_\ell : \mathbb{R}^{\ell+1} \to \mathbb{R}^{\ell+1}$ by $e_i^{(\ell)} \mapsto e_{\ell+2-i}^{(\ell)}$, $i = 1,\ldots,\ell+1$, where $e_1^{(\ell)},\ldots,e_{\ell+1}^{(\ell)}$ is the standard basis for $\mathbb{R}^{\ell+1}$. Note that $A_{\beta^{(\mathrm{rev})}}(\ell) = P_\ell^T A_\beta[\ell] P_\ell$ and hence $\operatorname{rank} A_{\beta^{(\mathrm{rev})}}(\ell) = \operatorname{rank} A_\beta[\ell]$, which together with (2.6) implies the statement of the corollary. $\square$

A sequence $\beta = (\beta_0,\ldots,\beta_{2k})$ with $r := \operatorname{rank}\beta$ is **positively recursively generated** if $A_\beta(r-1) \succ 0$ and denoting $(\varphi_0,\ldots,\varphi_{r-1}) := A_\beta(r-1)^{-1}(\beta_r,\ldots,\beta_{2r-1})^T$, it is true that

$$\beta_j = \varphi_0\beta_{j-r} + \cdots + \varphi_{r-1}\beta_{j-1} \quad \text{for} \quad j = r,\ldots,2k. \tag{2.7}$$

Note that (2.7) is equivalent to

$$\mathbf{v_j} = \varphi_0\mathbf{v_{j-r}} + \cdots + \varphi_{r-1}\mathbf{v_{j-1}} \quad \text{for} \quad j = r,\ldots,k. \tag{2.8}$$

2.3. **Solution of the truncated Hamburger moment problem.**

Theorem 2.9. [CF91, Theorem 3.9] *For $k \in \mathbb{N}$ and $\beta = (\beta_0, \ldots, \beta_{2k})$ with $\beta_0 > 0$, the following statements are equivalent:*

(1) There exists a representing measure for β supported on $K = \mathbb{R}$.
(2) There exists a $(\operatorname{rank} \beta)$-atomic representing measure for β.
(3) β is positively recursively generated.
(4) $A_\beta \succeq 0$ and $\operatorname{rank} A_\beta = \operatorname{rank} \beta$.

A straightforward corollary of Theorem 2.9 and Corollary 2.7 is the following.

Corollary 2.10. *Let $k \in \mathbb{N}$ and $\beta = (\beta_0, \ldots, \beta_{2k})$ with $\beta_0 > 0$. Suppose that A_β is singular. The following statements are equivalent:*

(1) There exists a representing measure for β supported on $K = \mathbb{R}$.
(2) There exists a $(\operatorname{rank} \beta)$-atomic representing measure for β.
(3) β is positively recursively generated.
(4) $A_\beta \succeq 0$ and $\operatorname{rank} A_\beta = \operatorname{rank} A_\beta(k-1)$.

2.4. **Partially positive semidefinite matrices and their completions.** A **partial matrix** $A = (a_{ij})_{i,j=1}^n$ is a matrix of real numbers $a_{ij} \in \mathbb{R}$, where some of the entries are not specified.

A partial symmetric matrix $A = (a_{ij})_{i,j=1}^n$ is **partially positive semidefinite (ppsd)** (resp. **partially positive definite (ppd)**) if the following two conditions hold:

(1) a_{ij} is specified if and only if a_{ji} is specified and $a_{ij} = a_{ji}$.
(2) All fully specified principal minors of A are psd (resp. pd).

It is well-known that a ppsd matrix $A(x)$ of the form as in Lemma 2.11 below admits a psd completion. (This follows from the fact that the corresponding graph is chordal, see e.g. [GJSW84, Dan92, BW11].) In the notation of Lemma 2.11, if $A(x_0)$, $x_0 \in \mathbb{R}$, is a psd Hankel matrix, then Corollary 2.7 implies that (2.9) below holds. Since we will need an additional information about the rank of the completion $A(x_0)$ and the explicit interval of all possible x_0 for our results, we give a proof of Lemma 2.11 based on the use of generalized Schur complements assuming (2.9) holds.

Lemma 2.11. *Let*

$$A(x) := \begin{pmatrix} A_1 & a & b \\ a^T & \alpha & x \\ b^T & x & \beta \end{pmatrix} \in S_n$$

be a partially positive semidefinite symmetric matrix, where $A_1 \in S_{n-2}$, $a, b \in \mathbb{R}^{n-2}$, $\alpha, \beta \in \mathbb{R}$ and x is a variable. Let

$$A_2 := \begin{pmatrix} A_1 & a \\ a^T & \alpha \end{pmatrix} \in S_{n-1}, \qquad A_3 := \begin{pmatrix} A_1 & b \\ b^T & \beta \end{pmatrix} \in S_{n-1},$$

and

$$x_\pm := b^T A_1^+ a \pm \sqrt{(A_2/A_1)(A_3/A_1)} \in \mathbb{R}.$$

Suppose the following holds:

$$A_1 \text{ is invertible} \quad \text{or} \quad \operatorname{rank} A_1 = \operatorname{rank} A_2. \tag{2.9}$$

Then:

(1) $A(x_0)$ is positive semidefinite if and only if $x_0 \in [x_-, x_+]$.

(2)

$$\operatorname{rank} A(x_0) = \begin{cases} \max\{\operatorname{rank} A_2, \operatorname{rank} A_3\}, & \textit{for } x_0 \in \{x_-, x_+\}, \\ \max\{\operatorname{rank} A_2, \operatorname{rank} A_3\} + 1, & \textit{for } x_0 \in (x_-, x_+). \end{cases}$$

(3) If $A(x)$ is partially positive definite, then $A(x')$ is positive definite for $x' \in (x_-, x_+)$.

Proof. By Theorem 2.2, $A(x) \succeq 0$ if and only if

$$A_2 \succeq 0, \qquad \begin{pmatrix} b \\ x \end{pmatrix} \in \mathcal{C}(A_2) \qquad \text{and} \qquad f(x) := A(x)/A_2 \geq 0, \tag{2.10}$$

The first condition of (2.10) is true by the ppsd assumption.

Since $A_2 \succeq 0$, it follows by Theorem 2.2 that $a \in \mathcal{C}(A_1)$ and hence by the properties of the Moore-Penrose inverse we have that $A_1(A_1^+ a) = a$. Thus,

$$\mathcal{C}(A_2) = \mathcal{C}\Big(\begin{pmatrix} A_1 & 0 \\ a^T & \alpha - a^T A_1^+ a \end{pmatrix}\Big) = \mathcal{C}\Big(\begin{pmatrix} A_1 & 0 \\ a^T & A_2/A_1 \end{pmatrix}\Big). \tag{2.11}$$

Now we separate two cases according to A_2/A_1.

Case 1: $A_2/A_1 > 0$.

(2.11) and the assumption of Case 1 imply that $\mathcal{C}(A_2) = \mathcal{C}(A_1 \oplus 1)$. Since $A_3 \succeq 0$, it follows by Theorem 2.2 that $b \in \mathcal{C}(A_1)$. Therefore $\begin{pmatrix} b & x \end{pmatrix}^T \in \mathcal{C}(A_1 \oplus 1)$ for every $x \in \mathbb{R}$. Thus the second condition of (2.10) is true for every $x \in \mathbb{R}$.

Note that the assumption of Case 1 and Proposition 2.3 imply that $\operatorname{rank} A_2 > \operatorname{rank} A_1$ and hence the assumption (2.9) implies invertibility of A_1 and A_2. By Proposition 2.4, used for $A(x)$ as K, A_2 as M and A_1 as A, we have that

$$f(x) = A_3/A_1 - (A_2/A_1)^{-1}(x - b^T A_1^+ a)^2. \tag{2.12}$$

Therefore $f(x_0) \geq 0$ if and only if $x_0 \in [x_-, x_+]$, which is the third condition of (2.10). Now by Proposition 2.3 we know that $\operatorname{rank} A(x) > \operatorname{rank} A_2$ if and only if $f(x_0) > 0$, which establishes (1),(2) in the case $A_2/A_1 > 0$.

Case 2: $A_2/A_1 = 0$.

(2.11) and the assumption of Case 2 imply that

$$\mathcal{C}(A_2) = \mathcal{C}\Big(\begin{pmatrix} A_1 \\ a^T \end{pmatrix}\Big). \tag{2.13}$$

Therefore, using (2.13), it is true that

$$\begin{pmatrix} b \\ x \end{pmatrix} \in \mathcal{C}(A_2) \;\Leftrightarrow\; \begin{pmatrix} b \\ x \end{pmatrix} = \begin{pmatrix} A_1 \\ a^T \end{pmatrix} z = \begin{pmatrix} A_1 z \\ a^T z \end{pmatrix} \quad \text{for some } z \in \mathbb{R}^{n-2}. \tag{2.14}$$

Since $A_3 \succeq 0$, it follows by Theorem 2.2 that $b \in \mathcal{C}(A_1)$ and hence by the properties of the Moore-Penrose inverse $\{A_1^+ b + w \colon w \in \ker A_1\}$ is the set of all solutions z of the system $A_1 z = b$. Therefore, using (2.14), it follows that

$$\begin{pmatrix} b \\ x \end{pmatrix} \in \mathcal{C}(A_2) \;\Leftrightarrow\; x \in \{a^T A_1^+ b + a^T w \colon w \in \ker A_1\} = \{a^T A_1^+ b\},$$

where we used the fact that A_1 is symmetric, $a \in \mathcal{C}(A_1)$ and $w \in \ker A_1$ for the last equality. So only $x_0 = a^T A_1^+ b$ satisfies the second condition of (2.10).

Now by definition of the generalized Schur complement, we have

$$f(x) = \beta - \begin{pmatrix} b^T & x \end{pmatrix} A_2^+ \begin{pmatrix} b \\ x \end{pmatrix}.$$

By the properties of the Moore-Penrose inverse

$$A_2^+ \begin{pmatrix} b \\ x_0 \end{pmatrix} = \begin{pmatrix} A_1^+ b \\ 0 \end{pmatrix} + v \quad \text{for some } v \in \ker A_2.$$

Hence,

$$f(x_0) = \beta - \begin{pmatrix} b^T & x_0 \end{pmatrix} \left(\begin{pmatrix} A_1^+ b \\ 0 \end{pmatrix} + v \right) = \beta - b^T A_1^+ b = A_3/A_1 \geq 0,$$

where the second equality follows from the fact that A_2 is symmetric, $\begin{pmatrix} b^T & x_0 \end{pmatrix}^T \in \mathcal{C}(A_2)$ and $v \in \ker A_2$, and the last inequality follows by the ppsd assumption. Note that $x_0 = x_+ = x_-$ and by Proposition 2.3, $\operatorname{rank} A(x_0) = \operatorname{rank} A_2$ if and only if $A_3/A_1 = 0$, in which case also $\operatorname{rank} A_3 = \operatorname{rank} A_2$. Otherwise we have $f(x_0) = A_3/A_1 > 0$, which implies by Proposition 2.3 that $\operatorname{rank} A(x_0) = \operatorname{rank} A_3 = \operatorname{rank} A_1 + 1$. Thus (1),(2) are true in the case $A_2/A_1 = 0$.

(3) follows from (2) by noticing that $A_2/A_1 > 0$, $A_3/A_1 > 0$ and $\operatorname{rank} A_2 = \operatorname{rank} A_3 = n - 1$. □

2.5. **Extension principle.** The extension principle for psd matrices is the following.

Lemma 2.12. *Let $A \in S_n$ be a positive semidefinite matrix, $Q \subseteq \{1, \ldots, n\}$ a subset and A_Q be the restriction of A to rows and columns from the set Q. If $v \in \ker A_Q$ is a nonzero vector from the kernel of A_Q, then the vector $\widehat{v}$ with the only nonzero entries in rows from Q and such that the restriction $\widehat{v}|_Q$ to the rows from Q equals to v, belongs to* $\ker A$.

Proof. By permuting rows and columns we may assume that A is of the form $A = \begin{pmatrix} A_Q & B \\ B^T & C \end{pmatrix}$. We have to prove that

$$A\begin{pmatrix} v \\ 0 \end{pmatrix} = 0. \tag{2.15}$$

Since A is psd, for every $w := \begin{pmatrix} v^T & u^T \end{pmatrix} \in \mathbb{R}^n$ we have that

$$0 \leq wAw^T = 2u^T B^T v + u^T C u. \tag{2.16}$$

If $B^T v \neq 0$, then we define $u := -\alpha B^T v$ where $\alpha > 0$ is an arbitrary positive real number, and plug into (2.16) to get

$$0 \leq -2\alpha \left\| B^T v \right\|^2 + \alpha^2 v^T BCB^T v = \alpha(\alpha v^T BCB^T v - 2 \left\| B^T v \right\|^2) =: \alpha S(\alpha). \tag{2.17}$$

Since $\lim_{\alpha \to 0} S(\alpha) = -2 \left\| B^T v \right\|^2 < 0$, (2.17) cannot be true for α small enough. Hence $B^T v = 0$, which proves (2.15). □

2.6. **Subsequences of one-dimensional moment sequences.**

Proposition 2.13. *Let $k \in \mathbb{N}$ and $\beta = (\beta_0, \ldots, \beta_{2k})$ with $\beta_0 > 0$ be a sequence which admits a representing measure supported on $K = \mathbb{R}$. Then for every $i, j \in \mathbb{N}$, where $0 \le i \le j \le k$, a subsequence $\beta^{(i,j)} := (\beta_{2i}, \ldots, \beta_{2j})$ also admits a representing measure supported on $K = \mathbb{R}$.*

Proof. Note that A_β is of the form

$$A_\beta = \begin{pmatrix} A_{\beta(0,i-1)} & * & * \\ * & A_{\beta(i,j)} & * \\ * & * & A_{\beta(j+1,k)} \end{pmatrix}.$$

By Theorem 2.9, $A_\beta \succeq 0$ and hence $A_{\beta(i,j)} \succeq 0$. For $i = j$ the statement is clear, i.e., the representing atom is β_{2i} with density 1. Assume that $i < j$. We separate two cases according to the invertibility of $A_{\beta(i,j)}$.

(1) If $A_{\beta(i,j)} \succ 0$, then $\operatorname{rank} A_{\beta(i,j)} = \operatorname{rank} \beta^{(i,j)} = j - i + 1$ and by Theorem 2.9, $\beta^{(i,j)}$ admits a measure.

(2) Else

$$A_{\beta(i,j)} = \begin{pmatrix} A_{\beta(i,j-1)} & v^T \\ v & \beta_{2j} \end{pmatrix}$$

is singular, where $v = \begin{pmatrix} \beta_j & \cdots & \beta_{2j-1} \end{pmatrix}$. We separate two cases according to the invertibility of $A_{\beta(i,j-1)}$.

- If $A_{\beta(i,j-1)}$ is invertible, then $\operatorname{rank} A_{\beta(i,j-1)} = \operatorname{rank} A_{\beta(i,j)}$.
- Else $A_{\beta(i,j-1)}$ is singular and by Corollary 2.7 used for $\beta^{(i,j)}$ as β, we get $\operatorname{rank} A_{\beta(i,j-2)} = \operatorname{rank} A_{\beta(i,j-1)}$. This implies that the last column of $A_{\beta(i,j-1)}$ is in the span of the other columns of $A_{\beta(i,j-1)}$. By Lemma 2.12, the j-th column of A_β is in the span of the columns $i+1, \ldots, j-1$. Since β is positively recursively generated, the $(j+1)$-th column of A_β is in the span of the columns $i+2, \ldots, j$ and in particular the last column of $A_{\beta(i,j)}$ is in the span of the other columns of $A_{\beta(i,j)}$. Hence $\operatorname{rank} A_{\beta(i,j-1)} = \operatorname{rank} A_{\beta(i,j)}$.

In both subcases of (2), $\operatorname{rank} A_{\beta(i,j-1)} = \operatorname{rank} A_{\beta(i,j)}$ and Corollary 2.10 implies that $\beta^{(i,j)}$ admits a measure.

□

3. Truncated Hamburger moment problem of degree $2k$ with gap (β_{2k-1}) and $(\beta_{2k-2}, \beta_{2k-1})$

In this section we solve the THMP of degree $2k$ with gaps (β_{2k-1}) (see Theorem 3.1) and $(\beta_{2k-2}, \beta_{2k-1})$ (see Theorem 3.5). As a corollary of Theorem 3.1 we obtain the solution to the TMP for the curve $y = x^3$ (see Corollary 3.3), while as a corollary of Theorem 3.5 we get the solution to the TMP for the curve $y = x^4$ and an additional moment $\beta_{3,2k-2}$ given (see Corollary 3.6).

3.1. **Truncated Hamburger moment problem of degree $2k$ with gap (β_{2k-1}).**

Theorem 3.1. *Let $k \in \mathbb{N}$ and*

$$\beta(x) := (\beta_0, \beta_1, \ldots, \beta_{2k-2}, x, \beta_{2k})$$

be a sequence where each β_i is a real number, $\beta_0 > 0$ and x is a variable. Let

$$\widehat{\beta} := (\beta_0, \ldots, \beta_{2k-4}) \quad \text{and} \quad \widetilde{\beta} := (\beta_0, \ldots, \beta_{2k-2})$$

be subsequences of $\beta(x)$, $v := \begin{pmatrix} \beta_k & \cdots & \beta_{2k-2} \end{pmatrix}$ *a vector and*

$$\widetilde{A} := \begin{pmatrix} A_{\widehat{\beta}} & v^T \\ v & \beta_{2k} \end{pmatrix}$$

a matrix. Then the following statements are equivalent:

(1) *There exists* $x_0 \in \mathbb{R}$ *and a representing measure for* $\beta(x_0)$ *supported on* $K = \mathbb{R}$.
(2) *There exists* $x_0 \in \mathbb{R}$ *and a* $(\operatorname{rank} \widetilde{\beta})$*-atomic representing measure for* $\beta(x_0)$.
(3) $A_{\beta(x)}$ *is partially positive semidefinite and one of the following conditions is true:*
   (a) $k = 1$.
   (b) $k > 1$ *and one of the following conditions is true:*
      (i) $A_{\widetilde{\beta}} \succ 0$.
      (ii) $\operatorname{rank} A_{\widehat{\beta}} = \operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} \widetilde{A}$.

Proof. First we prove the implication (1) ⇒ (3). By Theorem 2.9, $A_{\beta(x_0)} \succeq 0$ and $\operatorname{rank} A_{\beta(x_0)} = \operatorname{rank} \beta(x_0)$. $A_{\beta(x_0)} \succeq 0$ in particular implies that $A_{\beta(x)}$ is ppsd. If $k = 1$, then (3a) holds. Otherwise $k > 1$. If $A_{\widetilde{\beta}} \succ 0$, then (3(b)i) holds. Else $A_{\widetilde{\beta}}$ is singular and hence

$$\operatorname{rank} A_{\widehat{\beta}} = \operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} \beta(x_0) = A_{\beta(x_0)}, \tag{3.1}$$

where the first two equalities follow by Corollary 2.7 used for $\beta(x_0)$ as β and the last by Theorem 2.9. $A_{\widehat{\beta}}$ being a principal submatrix of $\widetilde{A}$ and $\widetilde{A}$ being a principal submatrix of

$$A_{\beta(x_0)} = \begin{pmatrix} A_{\widehat{\beta}} & u^T & v^T \\ u & \beta_{2k-2} & x_0 \\ v & x_0 & \beta_{2k} \end{pmatrix},$$

where $u = \begin{pmatrix} \beta_{k-1} & \cdots & \beta_{2k-3} \end{pmatrix}$, imply together with (3.1) that (3(b)ii) holds and concludes the proof of the implication (1) ⇒ (3).

Second we prove the implication (3) ⇒ (2). We separate two cases according to k.

- $k = 1$. We have that $A_{\beta(x)} = \begin{pmatrix} \beta_0 & x \\ x & \beta_2 \end{pmatrix}$. For $x_0 = \sqrt{\beta_0\beta_2}$, $A_{\beta(x_0)}$ is of rank 1 and the second column is the multiple of the first. Hence, by Corollary 2.10, a 1-atomic measure exists, proving the implication (3) ⇒ (2) in this case.
- $k > 1$. Notice that $A_{\beta(x)}$ is of the same form as $A(x)$ from Lemma 2.11, where $A_{\widehat{\beta}}$, $A_{\widetilde{\beta}}$, $\widetilde{A}$ correspond to A_1, A_2, A_3, respectively. Since both cases (3(b)i) and (3(b)ii) satisfy the assumption (2.9), it follows by Lemma 2.11 that there exists x_0 such that $A_{\beta(x_0)} \succeq 0$ and

$$\operatorname{rank} A_{\beta(x_0)} = \max \left\{ \operatorname{rank} A_{\widetilde{\beta}}, \operatorname{rank} \widetilde{A} \right\}. \tag{3.2}$$

  Since in the case (3(b)i), it holds that $\operatorname{rank} \widetilde{A} \leq \operatorname{rank} A_{\widetilde{\beta}}$, while in the case (3(b)ii), $\operatorname{rank} \widetilde{A} = \operatorname{rank} A_{\widetilde{\beta}}$, we obtain from (3.2) that $\operatorname{rank} A_{\beta(x_0)} = \operatorname{rank} A_{\widetilde{\beta}}$. By Corollary 2.10, $(\operatorname{rank} \widetilde{\beta})$-representing measure for $\beta(x_0)$ exists, which proves (2).

The implication (2) ⇒ (1) is trivial. □

Example 3.2. For $k=9$, let

$$\beta^{(1)}(x)=(1,0,1,0,2,0,5,0,14,0,42,0,132,0,429,0,2000,x,338881),$$

$$\beta^{(2)}(x)=\Big(14,\frac{7}{2},\frac{79}{4},-\frac{67}{8},\frac{1055}{16},-\frac{1935}{32},\frac{18195}{64},-\frac{43115}{128},\frac{336151}{256},-\frac{926695}{512},\frac{6407195}{1024},-\frac{19736547}{2048},\frac{124731423}{4096},-\frac{419176415}{8192},\frac{2469281827}{16384},-\frac{8894873563}{32768},\frac{49568350247}{65536},x,\frac{1006568996907}{262144}\Big),$$

$$\beta^{(3)}(x)=(8,0,78,0,1446,0,32838,0,794886,0,19651398,0,489352326,0,12216629958,0,305262005766,x,7630169896518).$$

Let $\widetilde{A}^{(i)}$, $i=1,2,3$, denote $\widetilde{A}$ from Theorem 3.1 corresponding to $\beta^{(i)}(x)$. Using *Mathematica* [Wol] one can check that:

- $\widetilde{A}^{(i)}\succeq 0$ for $i=1,2,3$.
- $A_{\widetilde{\beta}^{(1)}}\succ 0$, $A_{\widetilde{\beta}^{(2)}}\not\succeq 0$, $A_{\widetilde{\beta}^{(3)}}\succeq 0$ and $\dim\Big(\ker A_{\widetilde{\beta}^{(3)}}\Big)=1$.
- $\operatorname{rank} A_{\widehat{\beta}^{(3)}}=\operatorname{rank}\widetilde{A}^{(3)}=\operatorname{rank} A_{\widetilde{\beta}^{(3)}}=8$.

Therefore:

- $A_{\beta^{(1)}(x)}$ is ppsd and $\widetilde{\beta}^{(1)}$ satisfies (3(b)i) of Theorem 3.1, implying that a 9-atomic measure for $\beta^{(1)}(x)$ exists.
- $A_{\beta^{(2)}(x)}$ is not ppsd and by Theorem 3.1, there is no representing measure for $\beta^{(2)}(x)$.
- $A_{\beta^{(3)}(x)}$ is ppsd and $\widetilde{\beta}^{(3)}$ satisfies (3(b)ii) of Theorem 3.1, implying that an 8-atomic measure for $\beta^{(3)}(x)$ exists.

The following corollary is a consequence of Theorem 3.1 and is an alternative solution of the bivariate TMP for the curve $y=x^3$, first solved by Fialkow in [Fia11].

Corollary 3.3. *Let $k\in\mathbb{N}$ and $\beta=(\beta_{i,j})_{i,j\in\mathbb{Z}_+^2,i+j\leq 2k}$ be a 2-dimensional real multisequence of degree $2k$. Suppose $M(k)$ is positive semidefinite and recursively generated. Let*

$$u^{(i)}:=(\beta_{0,i},\beta_{1,i},\beta_{2,i})\quad \text{for } i=0,\ldots,2k-2,$$

$$\widehat{\beta}:=(u^{(0)},\ldots,u^{(2k-2)})\qquad \text{and}\qquad \widetilde{\beta}:=(u^{(0)},\ldots,u^{(2k-2)},\beta_{0,2k-1},\beta_{1,2k-1})$$

be subsequences of β. Then β has a representing measure supported on $y=x^3$ if and only if the following statements hold:

(1) One of the following holds:
- *If $k\geq 3$, then $Y=X^3$ is a column relation of $M(k)$.*
- *If $k=2$, then the equalities $\beta_{0,1}=\beta_{3,0}$, $\beta_{1,1}=\beta_{4,0}$, $\beta_{0,2}=\beta_{3,1}$ hold.*

(2) One of the following holds:
- *(a) $A_{\widetilde{\beta}}\succ 0$.*
- *(b) $A_{\widetilde{\beta}}\succeq 0$ and $\operatorname{rank} A_{\widehat{\beta}}=\operatorname{rank} A_{\widetilde{\beta}}=\operatorname{rank} M(k)$.*

Moreover, if the representing measure exists, then:

- *If $A_{\widetilde{\beta}}$ is nonsingular, there exists a $(3k)$-atomic measure.*
- *If $A_{\widetilde{\beta}}$ is singular, then the measure is $(\operatorname{rank} M(k))$-atomic.*

Proof. For $m\in\{0,1\ldots,6k-2,6k\}$ we define the numbers $\widetilde{\beta}_m$ by the following rule

$$\widetilde{\beta}_m:=\beta_{m \pmod 3),\lfloor\frac{m}{3}\rfloor}.$$

Claim 1. Every number $\widetilde{\beta}_m$ is well-defined.

We have to prove that $m \pmod 3 + \lfloor \frac{m}{3} \rfloor \leq 2k$. We separate three cases according to m.

- $m \leq 6k-4$: $\lfloor \frac{m}{3} \rfloor + m \pmod 3 \leq (2k-2)+2 = 2k$.
- $m \in \{6k-3, 6k-2\}$: $\lfloor \frac{m}{3} \rfloor + m \pmod 3 \leq (2k-1)+1 = 2k$.
- $m = 6k$: $\lfloor \frac{m}{3} \rfloor + m \pmod 3 = 2k+0 = 2k$.

Claim 2. Let $t \in \mathbb{N}$. The atoms $(x_1, x_1^3), \ldots (x_t, x_t^3)$ with densities $\lambda_1, \ldots, \lambda_t$ are the $(y-x^3)$-representing measure for β if and only if the atoms $x_1, \ldots, x_t$ with densities $\lambda_1, \ldots, \lambda_t$ are the $\mathbb{R}$-representing measure for $\widetilde{\beta}(x) = (\widetilde{\beta}_0, \ldots, \widetilde{\beta}_{2k-2}, x, \widetilde{\beta}_{2k})$.

The if part follows from the following calculation:

$$\widetilde{\beta}_m = \beta_{m \pmod 3, \lfloor \frac{m}{3} \rfloor} = \sum_{\ell=1}^{t} \lambda_\ell x_\ell^{m \pmod 3} x_\ell^{3\lfloor \frac{m}{3} \rfloor} = \sum_{\ell=1}^{t} \lambda_\ell x_\ell^{m \pmod 3 + 3\lfloor \frac{m}{3} \rfloor} = \sum_{\ell=1}^{t} \lambda_\ell x_\ell^m,$$

where $m = 0, 1, \ldots, 6k-2, 6k$.

The only if part follows from the following calculation:

$$\begin{aligned}\beta_{i,j} &= \beta_{i-3,j+1} = \cdots = \beta_{i \pmod 3, j+\lfloor \frac{i}{3} \rfloor} \\ &= \widetilde{\beta}_{i \pmod 3 + 3(j+\lfloor \frac{i}{3} \rfloor)} = \sum_{\ell=1}^{t} \lambda_\ell x_\ell^{i \pmod 3 + 3(j+\lfloor \frac{i}{3} \rfloor)} = \sum_{\ell=1}^{t} \lambda_\ell x_\ell^{i \pmod 3 + 3\lfloor \frac{i}{3} \rfloor} x_\ell^{3j} = \sum_{\ell=1}^{t} \lambda_\ell x_\ell^i (x_\ell^3)^j,\end{aligned}$$

where the equalities in the first line follow by $M(k)$ being rg.

Using Claim 2 and a theorem of Bayer and Teichmann [BT06], implying that if a finite sequence has a K-representing measure, then it has a finitely atomic K-representing measure, the statement of the Corollary follows by Theorem 3.1. □

Remark 3.4.

(1) Corollary 3.3 in case $k=1$ is an improvement of [Fia11, Proposition 5.6.ii)] by decreasing the number of atoms from 6 to 3.

(2) For $M(1) \succ 0$ and $A_{\widetilde{\beta}} \not\succ 0$, (2) of Corollary 3.3 is not satisfied and hence the measure does not exist. Since this is the case under the assumptions of [Fia11, Proposition 5.6.iii)], the additional conditions in [Fia11, Proposition 5.6.iii)] are never satisfied.

(3) Examples in the Example 3.2 above are derived from [Fia11, Example 5.2], [Fia08, Example 4.18], [Fia08, Example 3.3], which demonstrate the solution of the moment problem for the curve $y = x^3$.

3.2. Truncated Hamburger moment problem of degree $2k$ with gaps $(\beta_{2k-2}, \beta_{2k-1})$.

Theorem 3.5. *Let $k \in \mathbb{N}$, $k > 1$, and*

$$\beta(x,y) := (\beta_0, \beta_1, \ldots, \beta_{2k-3}, y, x, \beta_{2k})$$

be a sequence, where each β_i is a real number, $\beta_0 > 0$ and x, y are variables. Let

$$\widehat{\beta} := (\beta_0, \ldots, \beta_{2k-6}) \quad \textit{and} \quad \widetilde{\beta} := (\beta_0, \ldots, \beta_{2k-4})$$

be subsequences of $\beta(x,y)$,

$$u := \begin{pmatrix} \beta_k & \cdots & \beta_{2k-3} \end{pmatrix}, \quad s := \begin{pmatrix} \beta_{k-1} & \cdots & \beta_{2k-3} \end{pmatrix} \quad \textit{and} \quad w := \begin{pmatrix} \beta_{k-2} & \cdots & \beta_{2k-5} \end{pmatrix}$$

vectors and

$$\widetilde{A} := \begin{pmatrix} A_{\widehat{\beta}} & u^T \\ u & \beta_{2k} \end{pmatrix}$$

a matrix. Then the following statements are equivalent:

(1) *There exist* $x_0, y_0 \in \mathbb{R}$ *and a representing measure for* $\beta(x_0, y_0)$ *supported on* $K = \mathbb{R}$.
(2) *There exist* $x_0, y_0 \in \mathbb{R}$ *and a* $(\operatorname{rank} \widetilde{\beta})$ *or* $(\operatorname{rank} \widetilde{\beta} + 1)$*-atomic representing measure for* $\beta(x_0, y_0)$.
(3) $A_{\beta(x,y)}$ *is partially positive semidefinite and one of the following conditions holds:*
   (a) $k = 2$ *and* $\frac{\beta_1^2}{\beta_0} \leq \sqrt{\beta_0 \beta_4}$.
   (b) $k > 2$*, the inequality*

$$sA_{\widetilde{\beta}}^{+} s^T \leq uA_{\widehat{\beta}}^{+} w^T + \sqrt{(A_{\widetilde{\beta}}/A_{\widehat{\beta}})(\widetilde{A}/A_{\widehat{\beta}})}. \tag{3.3}$$

   holds and one of the following conditions is true:
      (i) $A_{\widetilde{\beta}} \succ 0$.
      (ii) $\operatorname{rank} A_{\widehat{\beta}} = \operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} \begin{pmatrix} A_{\widetilde{\beta}} & s^T \end{pmatrix} = \operatorname{rank} \widetilde{A}$.

Moreover, if the representing measure for β *exists, then:*

- *If* $k = 2$*, then there is a* 1*-atomic measure if* $\frac{\beta_1^2}{\beta_0} = \sqrt{\beta_0 \beta_4}$*. Otherwise there is a 2-atomic measure.*
- *If* $k > 2$*, there exists a* $(\operatorname{rank} \widetilde{\beta})$*-atomic if and only if one of the equalities*

$$sA_{\widetilde{\beta}}^{+} s^T = uA_{\widehat{\beta}}^{+} w^T - \sqrt{(A_{\widetilde{\beta}}/A_{\widehat{\beta}})(\widetilde{A}/A_{\widehat{\beta}})} \quad \text{or} \quad sA_{\widetilde{\beta}}^{+} s^T = uA_{\widehat{\beta}}^{+} w^T + \sqrt{(A_{\widetilde{\beta}}/A_{\widehat{\beta}})(\widetilde{A}/A_{\widehat{\beta}})} \tag{3.4}$$

  holds.

Proof. Note that $\beta(x, y)$ admits a measure if and only if there exist $y_0 \in \mathbb{R}$ such that $\beta(x, y_0)$ admits a measure. Theorem 3.1 implies that the following claim holds.

Claim 1. $\beta(x, y_0)$ admits a measure if and only if the following conditions hold:

(1) $A_{\beta(x,y_0)}$ is ppsd.
(2) One of the following is true:
   (a) $A_{(\widetilde{\beta}, \beta_{2k-3}, y_0)} \succ 0$, where

$$A_{(\widetilde{\beta}, \beta_{2k-3}, y)} = \begin{cases} \begin{pmatrix} \beta_0 & \beta_1 \\ \beta_1 & y \end{pmatrix}, & \text{if } k = 2, \\ \begin{pmatrix} A_{\widetilde{\beta}} & s^T \\ s & y_0 \end{pmatrix} = \begin{pmatrix} A_{\widehat{\beta}} & w^T & s_1^T \\ w & \beta_{2k-4} & \beta_{2k-3} \\ s_1 & \beta_{2k-3} & y_0 \end{pmatrix} \quad \text{where} \quad s_1^T = \begin{pmatrix} \beta_{k-1} \\ \vdots \\ \beta_{2k-4} \end{pmatrix}, & \text{otherwise.} \end{cases}$$

   (b) $\operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} \widehat{A}(y_0)$, where

$$\widehat{A}(y) := \begin{cases} \begin{pmatrix} \beta_0 & y \\ y & \beta_4 \end{pmatrix}, & \text{if } k = 2, \\ \begin{pmatrix} A_{\widetilde{\beta}} & u(y)^T \\ u(y) & \beta_{2k} \end{pmatrix} = \begin{pmatrix} A_{\widehat{\beta}} & w^T & u^T \\ w & \beta_{2k-4} & y \\ u & y & \beta_{2k} \end{pmatrix} \quad \text{and} \quad u(y) := \begin{pmatrix} u & y \end{pmatrix}, & \text{otherwise.} \end{cases}$$

Claim 2. Let $k>2$. Assume $A_{\widehat{\beta}} \succ 0$ or $\operatorname{rank} A_{\widehat{\beta}} = \operatorname{rank} A_{\widetilde{\beta}}$. Then $\widehat{A}(y_0) \succeq 0$ if and only if

$$\widehat{A}(y) \text{ is ppsd} \quad \text{and} \quad y_0 \in \Big[uA_{\widehat{\beta}}^{+}w^T - \sqrt{(A_{\widetilde{\beta}}/A_{\widehat{\beta}})(\widetilde{A}/A_{\widehat{\beta}})}, uA_{\widehat{\beta}}^{+}w^T + \sqrt{(A_{\widetilde{\beta}}/A_{\widehat{\beta}})(\widetilde{A}/A_{\widehat{\beta}})}\Big] =: [y_-, y_+]. \tag{3.5}$$

Moreover,

$$\operatorname{rank} \widehat{A}(y_0) = \begin{cases} \max\big\{\operatorname{rank} A_{\widetilde{\beta}}, \operatorname{rank} \widetilde{A}\big\}, & \text{for } y_0 \in \{y_-, y_+\}, \\ \max\big\{\operatorname{rank} A_{\widetilde{\beta}}, \operatorname{rank} \widetilde{A}\big\} + 1, & \text{for } y_0 \in (y_-, y_+). \end{cases} \tag{3.6}$$

The assumption (2.9) of Lemma 2.11 used for $\widehat{A}(y)$, $A_{\widehat{\beta}}$, $A_{\widetilde{\beta}}$, $\widetilde{A}$ as $A(x)$, A_1, A_2, A_3, respectively, are by the assumption of Claim 2 satisfied and hence Claim 2 follows by Lemma 2.11.

Claim 3. Let $k>2$. Assume $A_{\widehat{\beta}} \succ 0$ or $\operatorname{rank} A_{\widehat{\beta}} = \operatorname{rank} A_{\widetilde{\beta}}$. Then $A_{\beta(x,y_0)}$ is ppsd for some $y_0 \in \mathbb{R}$ if and only if $A_{\beta(x,y)}$ is ppsd, $s^T \in \mathcal{C}(A_{\widetilde{\beta}})$ and (3.3) holds.

Note that $A_{\beta(x,y_0)}$ is ppsd if and only if $A_{(\widetilde{\beta},\beta_{2k-3},y_0)} \succeq 0$ and $\widehat{A}(y_0) \succeq 0$. By Theorem 2.2, $A_{(\widetilde{\beta},\beta_{2k-3},y_0)} \succeq 0$ if and only if

$$A_{\widetilde{\beta}} \succeq 0, \qquad s^T \in \mathcal{C}(A_{\widetilde{\beta}}) \qquad \text{and} \qquad A_{(\widetilde{\beta},\beta_{2k-3},y_0)}/A_{\widetilde{\beta}} = y_0 - sA_{\widetilde{\beta}}^{+}s^T \geq 0, \tag{3.7}$$

By Claim 2, $\widehat{A}(y_0)$ is psd if and only if (3.5) holds. Now note that the first condition of (3.5) (which also includes the first condition of (3.7)) is equivalent to $A_{\beta(x,y)}$ being ppsd and that y_0 satisfying the third condition of (3.7) and the second condition of (3.5) exists if and only if (3.3) holds. This proves Claim 3.

First we prove the implication (1) $\Rightarrow$ (3). By Claim 1, in particular $A_{\beta(x,y_0)}$ is ppsd.

If $k=2$, then $A_{(\widetilde{\beta},\beta_1,y)} \succeq 0$, which implies that $y_0 \geq \frac{\beta_1^2}{\beta_0}$, and $\widehat{A}(y_0) \succeq 0$, which implies that $y_0 \leq \sqrt{\beta_0\beta_4}$. Hence, $\frac{\beta_1^2}{\beta_0} \leq \sqrt{\beta_0\beta_4}$, which is (3a). Since $A_{\beta(x,y_0)}$ being ppsd implies that also $A_{\beta(x,y)}$ is ppsd, this proves the implication (1) $\Rightarrow$ (3) in this case.

It remains to prove (1) $\Rightarrow$ (3) in the case $k>2$. We separate two cases according to the invertibility of $A_{\widetilde{\beta}}$.

- $A_{\widetilde{\beta}} \succ 0$: Using Claim 3, $A_{\beta(x,y)}$ is ppsd, (3.3) and (3(b)i) holds, which proves the implication (1) $\Rightarrow$ (3) in this case.
- $A_{\widetilde{\beta}} \not\succ 0$: It follows that $A_{(\widetilde{\beta},\beta_{2k-3},y_0)} \not\succ 0$ and hence (2b) of Claim 1 must hold. Corollary 2.7 used for $(\widetilde{\beta},\beta_{2k-3},y_0)$ as β implies that

$$\operatorname{rank} A_{\widehat{\beta}} = \operatorname{rank} A_{\widetilde{\beta}}. \tag{3.8}$$

By Proposition 2.13, $(\widetilde{\beta},\beta_{2k-3},y_0)$ also admits a measure and Corollary 2.10 used for $(\widetilde{\beta},\beta_{2k-3},y_0)$ as β implies that

$$\operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} A_{(\widetilde{\beta},\beta_{2k-3},y_0)}. \tag{3.9}$$

(2b) of Claim 1 together with (3.8) implies that all the inequalities in the estimate $\operatorname{rank} A_{\widehat{\beta}} \leq \operatorname{rank} \widetilde{A} \leq \operatorname{rank} \widehat{A}(y_0)$ are equalities and in particular,

$$\operatorname{rank} A_{\widehat{\beta}} = \operatorname{rank} \widetilde{A}. \tag{3.10}$$

(3.8), (3.9), (3.10) and Claim 3 imply that $A_{\beta(x,y)}$ is ppsd, (3.3) and (3(b)ii) holds, which proves the implication (1) $\Rightarrow$ (3) in this case.

Second we prove the implication (3) $\Rightarrow$ (1). We separate two cases according to k.

If $k=2$, then we are in the case (3a). For $y_0=\sqrt{\beta_0\beta_4}$, $\beta(x,y_0)$ is ppsd and satisfies (2a) of Claim 1 if $\frac{\beta_1^2}{\beta_0}<\sqrt{\beta_0\beta_4}$ and (2b) if $\frac{\beta_1^2}{\beta_0}=\sqrt{\beta_0\beta_4}$. In both cases Claim 1 implies the implication (3) $\Rightarrow$ (1) is true in this case.

Else $k>2$. If (3(b)i) holds, then in particular $A_{\widehat{\beta}}\succ 0$. Otherwise (3(b)ii) holds and in particular $\operatorname{rank} A_{\widehat{\beta}}=\operatorname{rank} A_{\widetilde{\beta}}$. In both cases the assumptions of Claims 2 and 3 are fulfilled. By Claim 3, the matrix $A_{\beta(x,y_+)}$ is ppsd and by (3.6) of Claim 2, $\operatorname{rank}\widehat{A}(y_+)=\max\{\operatorname{rank} A_{\widetilde{\beta}},\operatorname{rank}\widetilde{A}\}$. If (3(b)i) holds, then $\operatorname{rank}\widehat{A}(y_+)=\operatorname{rank} A_{\widetilde{\beta}}=k-1$. Else (3(b)ii) holds and $\operatorname{rank}\widehat{A}(y_+)=\operatorname{rank} A_{\widetilde{\beta}}=\operatorname{rank}\widetilde{A}$. In both cases, $\beta(x,y_+)$ satisfies (1) and (2b) of Claim 1 above and thus the measure exists which proves the implication (3) $\Rightarrow$ (1).

The implication (2) $\Rightarrow$ (1) is trivial.

Now we prove the implication (1) $\Rightarrow$ (2). If $\beta(x,y_0)$ has a representing measure, then:

- By Theorem 3.1 it has a $(\operatorname{rank}(\widetilde{\beta},\beta_{2k-3},y_0))$-atomic representing measure.
- By Proposition 2.13, $\widetilde{\beta}$ and $(\widetilde{\beta},\beta_{2k-3},y_0)$ also have measures and hence by Theorem 2.9, $\operatorname{rank} A_{\widetilde{\beta}}=\operatorname{rank}\widetilde{\beta}$ and $\operatorname{rank}(\widetilde{\beta},\beta_{2k-3},y_0)=\operatorname{rank} A_{(\widetilde{\beta},\beta_{2k-3},y_0)}$.

Since $\operatorname{rank} A_{(\widetilde{\beta},\beta_{2k-3},y_0)}\in\{\operatorname{rank} A_{\widetilde{\beta}},\operatorname{rank} A_{\widetilde{\beta}}+1\}$, the implication (1) $\Rightarrow$ (2) is true.

It remains to prove the moreover part. We separate two cases according to k.

- If $k=2$, then $\operatorname{rank} A_{\widetilde{\beta}}=\operatorname{rank}(\beta_0)=1$. So 1-atomic measure exists if and only if $\operatorname{rank} A_{(\beta_0,\beta_1,y_0)}=\operatorname{rank}\widehat{A}(y_0)=1$ for some y_0. But from the form of $A_{(\beta_0,\beta_1,y)}$ and $\widehat{A}(y)$ this is possible only if $y_0=\frac{\beta_1^2}{\beta_0}=\sqrt{\beta_0\beta_4}$. Otherwise there is a 2-atomic measure.
- Else $k>2$. By Proposition 2.3 and (3.7) above, $\operatorname{rank} A_{(\widetilde{\beta},\beta_{2k-3},y_0)}=\operatorname{rank} A_{\widetilde{\beta}}$ if and only if $y_0=sA_{\widetilde{\beta}}^{+}s^T$. In the proof of the implication (3) $\Rightarrow$ (1) we see that $\operatorname{rank} A_{\widetilde{\beta}}\geq\operatorname{rank}\widetilde{A}$. Using this in (3.6) above, it follows that $sA_{\widetilde{\beta}}^{+}s^T$ must be equal to y_- or y_+, which is exactly (3.4).

This concludes the proof of the theorem. □

The following corollary is a consequence of Theorem 3.5 and solves the bivariate TMP for the curve $y=x^4$ where also $\beta_{3,2k-2}$ is given.

Corollary 3.6. *Let $\beta=(\beta_{i,j})_{i,j\in\mathbb{Z}_+^2,i+j\leq 2k}$ be a 2-dimensional real multisequence of degree $2k$ and let $\beta_{3,2k-2}$ be also given. Suppose $M(k)$ is positive semidefinite and recursively generated. Let*

$$u^{(i)}=(\beta_{0,i},\beta_{1,i},\beta_{2,i},\beta_{3,i})\quad \textit{for } i=0,\ldots,2k-1,$$

$$\widehat{\beta}:=(u^{(0)},\ldots,u^{(2k-3)},\beta_{0,2k-2},\beta_{1,2k-2},\beta_{2,2k-2})\quad\textit{and}\quad \widetilde{\beta}:=(\widehat{\beta},\beta_{3,2k-2},\beta_{0,2k-1})$$

be subsequences of β,

$$u:=\begin{pmatrix} u^{(k)} & \cdots & u^{(2k-1)} & \beta_{1,2k-1}\end{pmatrix},\quad s:=\begin{pmatrix}\beta_{3,k-1} & u^{(k)} & \cdots & u^{(2k-1)} & \beta_{1,2k-1}\end{pmatrix},$$

$$w:=\begin{pmatrix}\beta_{2,k-1} & \beta_{3,k-1} & u^{(k)} & \cdots & u^{(2k-2)} & \beta_{1,2k-2} & \beta_{2,2k-2} & \beta_{3,2k-2}\end{pmatrix}$$

vectors and

$$\widetilde{A}:=\begin{pmatrix} A_{\widehat{\beta}} & u^T\\ u & \beta_{0,2k}\end{pmatrix}$$

a matrix. Then β has a representing measure supported on $y = x^4$ if and only if

$$sA_{\widetilde{\beta}}^{+}s^T \leq uA_{\widehat{\beta}}^{+}w^T + \sqrt{(A_{\widetilde{\beta}}/A_{\widehat{\beta}})(\widetilde{A}/A_{\widehat{\beta}})}.$$

one of the following statements hold:

(1) One of the following holds:
- *If $k \geq 4$, then $Y = X^4$ is a column relation of $M(k)$.*
- *If $k = 3$, then the equalities $\beta_{0,1} = \beta_{4,0}$, $\beta_{1,1} = \beta_{5,0}$, $\beta_{2,1} = \beta_{6,0}$ hold.*
- *If $k = 2$, then the equality $\beta_{0,1} = \beta_{4,0}$ holds.*
- *$k = 1$.*

(2) One of the following conditions holds:
(a) $A_{\widetilde{\beta}} \succ 0$.
(b) $A_{\widetilde{\beta}} \succeq 0$ and $\operatorname{rank} A_{\widehat{\beta}} = \operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} \begin{pmatrix} A_{\widetilde{\beta}} & s^T \end{pmatrix} = \operatorname{rank} \widetilde{A}$.

Moreover, if the representing measure exists, then there is a $(\operatorname{rank} \widetilde{\beta})$-atomic measure if

$$sA_{\widetilde{\beta}}^{+}s^T \in \left\{ uA_{\widehat{\beta}}^{+}w^T - \sqrt{(A_{\widetilde{\beta}}/A_{\widehat{\beta}})(\widetilde{A}/A_{\widehat{\beta}})}, uA_{\widehat{\beta}}^{+}w^T + \sqrt{(A_{\widetilde{\beta}}/A_{\widehat{\beta}})(\widetilde{A}/A_{\widehat{\beta}})} \right\}.$$

and $(\operatorname{rank} \widetilde{\beta} + 1)$-atomic otherwise.

Proof. For $m \in \{0, 1 \ldots, 8k-3, 8k\}$ we define the numbers $\widetilde{\beta}_m$ by the following rule

$$\widetilde{\beta}_m := \beta_{m \pmod 4, \lfloor \frac{m}{4} \rfloor}.$$

Claim 1. Every number $\widetilde{\beta}_m$ is well-defined.

We will prove that $m \pmod 4 + \lfloor \frac{m}{4} \rfloor \leq 2k$ if $m \neq 8k-5$, while for $m = 8k-5$ we have $\widetilde{\beta}_{8k-5} = \beta_{3,2k-2}$. We separate three cases according to m.
- $m < 8k-8$: $\lfloor \frac{m}{4} \rfloor + m \pmod 4 \leq (2k-3) + 3 = 2k$.
- $m \in \{8k-8, 8k-7, 8k-6\}$: $\lfloor \frac{m}{4} \rfloor + m \pmod 4 \leq (2k-2) + 2 = 2k$.
- $m \in \{8k-4, 8k-3\}$: $\lfloor \frac{m}{4} \rfloor + m \pmod 4 \leq (2k-1) + 1 = 2k$.
- $m = 8k$: $\lfloor \frac{m}{4} \rfloor + m \pmod 3 = 2k + 0 = 2k$.

Claim 2. Let $t \in \mathbb{N}$. The atoms $(x_1, x_1^4), \ldots (x_t, x_t^4)$ with densities $\lambda_1, \ldots, \lambda_t$ are the $(y - x^4)$-representing measure for β and $\beta_{3,2k-2}$ if and only if the atoms $x_1, \ldots, x_t$ with densities $\lambda_1, \ldots, \lambda_t$ are the $\mathbb{R}$-representing measure for $\widetilde{\beta}(x, y) = (\widetilde{\beta}_0, \ldots, \widetilde{\beta}_{2k-2}, y, x, \widetilde{\beta}_{2k})$.

The if part follows from the following calculation:

$$\widetilde{\beta}_m = \beta_{m \pmod 4, \lfloor \frac{m}{4} \rfloor} = \sum_{\ell=1}^{t} \lambda_\ell x_\ell^{m \pmod 4} x_\ell^{4\lfloor \frac{m}{4} \rfloor} = \sum_{\ell=1}^{t} \lambda_\ell x_\ell^{m \pmod 4 + 4\lfloor \frac{m}{4} \rfloor} = \sum_{\ell=1}^{t} \lambda_\ell x_\ell^m,$$

where $m = 0, \ldots, 8k-3, 8k$.

The only if part follows from the following calculation for $i + j \leq 2k$:

$$\begin{aligned} \beta_{i,j} &= \beta_{i-4,j+1} = \cdots = \beta_{i \pmod 4, j + \lfloor \frac{i}{4} \rfloor} \\ &= \widetilde{\beta}_{i \pmod 4 + 4(j + \lfloor \frac{i}{4} \rfloor)} = \sum_{\ell=1}^{t} \lambda_\ell x_\ell^{i \pmod 4 + 4(j + \lfloor \frac{i}{4} \rfloor)} = \sum_{\ell=1}^{t} \lambda_\ell x_\ell^{i \pmod 4 + 4\lfloor \frac{i}{4} \rfloor} x_\ell^{4j} = \sum_{\ell=1}^{t} \lambda_\ell x_\ell^i (x_\ell^4)^j, \end{aligned}$$

where the equalities in the first line follow by $M(k)$ being rg, and

$$\beta_{3,2k-2} = \widetilde{\beta}_{8k-5} = \sum_{\ell=1}^{t} \lambda_\ell x_\ell^{8k-5} = \sum_{\ell=1}^{t} \lambda_\ell x_\ell^3 (x_\ell^4)^{2k-2}.$$

Using Claim 2 and a theorem of Bayer and Teichmann [BT06], implying that if a finite sequence has a K-representing measure, then it has a finitely atomic K-representing measure, the statement of the Corollary follows by Theorem 3.5. □

4. Truncated Hamburger moment problem of degree $2k$ with gap(s) (β_1), (β_1, β_2)

In this section we solve the THMP of degree $2k$ with gaps (β_1) (see Theorem 4.1) and (β_1, β_2) (see Theorem 4.5). As a corollary of Theorem 4.1 we obtain the solution to the TMP for the curve $y^2 = x^3$ (see Corollary 4.4), while as a corollary of Theorem 4.5 we get the solution to the TMP for the curve $y^3 = x^4$ and an additional moment $\beta_{\frac{5}{3},0}$ given (see Corollary 4.7).

4.1. **Truncated Hamburger moment problem of degree $2k$ with gaps (β_1).**

Theorem 4.1. *Let $k \in \mathbb{N}$, $k > 1$, and*

$$\beta(x) := (\beta_0, x, \beta_2, \ldots, \beta_{2k})$$

be a sequence where each β_i is a real number, $\beta_0 > 0$ and x is a variable. Let

$$\widehat{\beta} := (\beta_2, \ldots, \beta_{2k-2}), \quad \widetilde{\beta} := (\beta_2, \ldots, \beta_{2k}), \quad \overline{\beta} := (\beta_4, \ldots, \beta_{2k-2}) \quad \text{and} \quad \breve{\beta} := (\beta_4, \ldots, \beta_{2k})$$

be subsequences of $\beta(x)$,

$$v := \begin{pmatrix} \beta_2 & \cdots & \beta_{k-1} \end{pmatrix} \quad \text{and} \quad u := \begin{pmatrix} \beta_2 & \ldots & \beta_k \end{pmatrix}$$

vectors, and

$$\widetilde{A} := \begin{pmatrix} \beta_0 & v \\ v^T & A_{\overline{\beta}} \end{pmatrix} \quad \text{and} \quad \widehat{A} := \begin{pmatrix} \beta_0 & u \\ u^T & A_{\breve{\beta}} \end{pmatrix}$$

matrices. Then the following statements are equivalent:

1. *There exists $x_0 \in \mathbb{R}$ and a representing measure for $\beta(x_0)$ supported on $K = \mathbb{R}$.*
2. *There exists $x_0 \in \mathbb{R}$ and a $(\operatorname{rank} \widetilde{\beta})$ or a $(\operatorname{rank} \widetilde{\beta} + 1)$-atomic representing measure for $\beta(x_0)$.*
3. *$A_{\beta(x)}$ is partially positive semidefinite and one of the following conditions is true:*
   - *(a) (i) $k = 2$ and $A_{\widetilde{\beta}} \succ 0$.*
     (ii) $k > 2$, $A_{\widetilde{\beta}} \succ 0$ and $\widetilde{A} \succ 0$.
   - *(b) $\operatorname{rank} A_{\widehat{\beta}} = \operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} A_{\breve{\beta}}$.*

Moreover, if the representing measure exists, then there does not exist a $(\operatorname{rank} \widetilde{\beta})$-atomic measure if and only if (3b) holds and $\operatorname{rank} A_{\widehat{\beta}} < \operatorname{rank} \widehat{A}$.

Proof. First we prove the implication (1) ⇒ (3). By Theorem 2.9, $A_{\beta(x_0)} \succeq 0$ and $\operatorname{rank} A_{\beta(x_0)} = \operatorname{rank} \beta(x_0)$. The condition $A_{\beta(x_0)} \succeq 0$ implies that $A_{\beta(x)}$ is ppsd. We separate two cases according to the invertibility of $A_{\widetilde{\beta}}$.

- $A_{\widetilde{\beta}} \succ 0$: Since $A_{\widetilde{\beta}}$ is a principal submatrix of $A_{\beta(x_0)}$, we conclude that $\operatorname{rank} A_{\beta(x_0)} \geq \operatorname{rank} A_{\widetilde{\beta}} = k$, and hence $A_{\beta(x_0)}$ is either invertible or $\operatorname{rank} A_{\beta(x_0)}$ is singular and by Corollary 2.10 used for $\beta(x_0)$ as β, $\operatorname{rank} A_{\beta(x_0)} = \operatorname{rank} A_{(\beta_0,x_0,\widehat{\beta})}$. In both cases

$$A_{(\beta_0,x_0,\widehat{\beta})} = \begin{cases} \begin{pmatrix} \beta_0 & x_0 \\ x_0 & \beta_2 \end{pmatrix}, & \text{if } k=2, \\ \begin{pmatrix} \beta_0 & x_0 & v \\ x_0 & \beta_2 & v_1 \\ v^T & v_1^T & A_{\overline{\beta}} \end{pmatrix} \quad \text{where} \quad v_1 = \begin{pmatrix} \beta_3 & \cdots & \beta_k \end{pmatrix}, & \text{if } k>2, \end{cases},$$

is invertible. If $k>2$, $\widetilde{A}$ is a principal submatrix of $A_{(\beta_0,x_0,\widehat{\beta})}$ and it follows that $\widetilde{A} \succ 0$. Hence, (3a) holds. Together with $A_{\beta(x)}$ being ppsd, proves the implication (1) $\Rightarrow$ (3) in this case.
- $A_{\widetilde{\beta}}$ is singular: Since $\widetilde{\beta}$ is a subsequence of $\beta(x_0)$ of the form from Proposition 2.13 with $i=1, j=k$, it admits a measure. By Corollary 2.10 used for $\widetilde{\beta}$ as β, it follows that

$$\operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} A_{\widehat{\beta}}. \tag{4.1}$$

By Corollary 2.8 used for $\beta(x_0)$ as β, it follows that

$$\operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} A_{\breve{\beta}}. \tag{4.2}$$

Hence, (4.1) and (4.2) imply that (3b) holds. Together with $A_{\beta(x)}$ being ppsd, proves the implication (1) $\Rightarrow$ (3) in this case.

Second we prove the implication (3) $\Rightarrow$ (2). Let $P_1 : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ be the following permutation matrix

$$P_1 = \begin{pmatrix} \mathbf{0} & 0 & 1 \\ \mathbf{0} & 1 & 0 \\ I_{k-1} & 0 & 0 \end{pmatrix},$$

where $\mathbf{0}$ stands for the row of $k-2$ zeros and I_{k-1} is the identity matrix of size $k-1$. Then $P_1^T A_{\beta(x)} P_1$ is of the form

$$P_1^T A_{\beta(x)} P_1 = \begin{pmatrix} A_{\breve{\beta}} & w^T & u^T \\ w & \beta_2 & x \\ u & x & \beta_0 \end{pmatrix},$$

where $w = \begin{pmatrix} \beta_3 & \cdots & \beta_{k+1} \end{pmatrix}$ is a vector.

Claim. $A_{\beta(x_0)}$ is psd if and only if

$$x_0 \in \left[uA_{\breve{\beta}}^{+} w^T - \sqrt{(A_{\widetilde{\beta}}/A_{\breve{\beta}})(\widehat{A}/A_{\breve{\beta}})}, uA_{\breve{\beta}}^{+} w^T + \sqrt{(A_{\widetilde{\beta}}/A_{\breve{\beta}})(\widehat{A}/A_{\breve{\beta}})} \right] =: [x_-, x_+].$$

Moreover,

$$\operatorname{rank} A_{\beta(x_0)} := \begin{cases} \max\{\operatorname{rank} A_{\widetilde{\beta}}, \operatorname{rank} \widehat{A}\}, & \text{if } x_0 \in \{x_-, x_+\}, \\ \max\{\operatorname{rank} A_{\widetilde{\beta}}, \operatorname{rank} \widehat{A}\} + 1, & \text{if } x_0 \in (x_-, x_+). \end{cases} \tag{4.3}$$

Denoting the matrices

$$\mathcal{A} := \begin{pmatrix} A_{\breve{\beta}} & w^T \\ w & \beta_2 \end{pmatrix} \quad \text{and} \quad \mathcal{B} := \begin{pmatrix} A_{\breve{\beta}} & u^T \\ u & \beta_0 \end{pmatrix},$$

and the permutation matrix $P_2 : \mathbb{R}^k \to \mathbb{R}^k$ by

$$P_2 = \begin{pmatrix} \mathbf{0} & 1 \\ I_{k-1} & 0 \end{pmatrix},$$

where $\mathbf{0}$ stands for the row of $k-1$ zeroes and I_{k-1} the identity matrix of size $k-1$, we have that

$$\mathcal{A} = P_2^T A_{\widetilde{\beta}} P_2 \quad \text{and} \quad \mathcal{B} = P_2^T \widehat{A} P_2.$$

In particular,

$$\operatorname{rank} \mathcal{A} = \operatorname{rank} A_{\widetilde{\beta}} \quad \text{and} \quad \operatorname{rank} \mathcal{B} = \operatorname{rank} \widehat{A}. \tag{4.4}$$

If (3a) holds, then $A_{\widetilde{\beta}} \succ 0$ implies that $A_{\breve{\beta}} \succ 0$. If (3b) holds, then in particular $\operatorname{rank} A_{\breve{\beta}} = \operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} \mathcal{A}$. Hence, the assumption (2.9) of Lemma 2.11 used for $P_1^T A_{\beta(x)} P_1, A_{\breve{\beta}}, \mathcal{A}, \mathcal{B}$ as $A(x), A_1, A_2, A_3$, respectively, is satisfied and using also $\mathcal{A}/A_{\breve{\beta}} = A_{\widetilde{\beta}}/A_{\breve{\beta}}$ and $\mathcal{B}/A_{\breve{\beta}} = \widehat{A}/A_{\breve{\beta}}$, Claim follows.

First assume that (3a) holds. We separate two cases according to the inverbility of $\widehat{A}$.

- $\widehat{A} \succ 0$: From $A_{\widetilde{\beta}} \succ 0$ and $\widehat{A} \succ 0$ it follows, using Proposition 2.3, that $A_{\widetilde{\beta}}/A_{\breve{\beta}} > 0$ and $\widehat{A}/A_{\breve{\beta}} > 0$. Hence by the definition of $x_\pm$, we have $x_- < x_+$ and by Claim, $A_{\beta(x_0)} \succ 0$ for $x_0 \in (x_-, x_+)$. By Theorem 2.9, $(\operatorname{rank} \beta(x_0)) = (\operatorname{rank} \widetilde{\beta}+1)$-atomic representing measure for $\beta(x_0)$ exists, which proves the implication (3) $\Rightarrow$ (2) in this case.
- $\widehat{A}$ is singular: From $A_{\widetilde{\beta}} \succ 0$ it follows that $A_{\breve{\beta}} \succ 0$. Since $\widehat{A}$ is singular, Proposition 2.3 implies that $\widehat{A}/A_{\breve{\beta}} = 0$, and hence by the definition of $x_\pm$, we have $x_- = x_+$. By Claim, $A_{\beta(x_\pm)} \succeq 0$ with $\operatorname{rank} A_{\beta(x_\pm)} = \operatorname{rank} A_{\widetilde{\beta}}$. We separate two cases according to k.
  - $k = 2$: Since $\widehat{A} = \begin{pmatrix} \beta_0 & \beta_2 \\ \beta_2 & \beta_4 \end{pmatrix}$ and $\beta_0 > 0$, it follows that the second (also the last) column of $\widehat{A}$ is in the span of the first (also the others) one.
  - $k > 2$: By assumptions $\widetilde{A} \succ 0$ and $\widehat{A} = \begin{pmatrix} \widetilde{A} & u_1^T \\ u_1 & \beta_{2k} \end{pmatrix}$ being singular, where the u_1 is equal to $u_1 = \begin{pmatrix} \beta_k & \beta_{k+2} & \cdots & \beta_{2k-1} \end{pmatrix}$, it follows that the last column of $\widehat{A}$ is in the span of the others.

  By Lemma 2.12, the last column of $A_{\beta(x_\pm)}$ is also in the span of the others and by Corollary 2.10, we have that $(\operatorname{rank} \beta(x_\pm)) = (\operatorname{rank} \widetilde{\beta})$-atomic representing measure for $\beta(x_\pm)$ exists, which proves the implication (3) $\Rightarrow$ (2) in this case.

Otherwise (3b) holds. Proposition 2.3 implies that $A_{\widetilde{\beta}}/A_{\widehat{\beta}} = 0$, and hence by the definition of $x_\pm$, we have $x_- = x_+$. By Claim, $A_{\beta(x_\pm)} \succeq 0$. The assumption $\operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} A_{\widehat{\beta}}$, also implies that the last column of $A_{\widetilde{\beta}} = \begin{pmatrix} A_{\widehat{\beta}} & u_2^T \\ u_2 & \beta_{2k} \end{pmatrix}$, where $u_2 = \begin{pmatrix} \beta_k & \cdots & \beta_{2k-1} \end{pmatrix}$, is in the span of the others. By Lemma 2.12, the last column of $A_{\beta(x_\pm)}$ is in the span of the others. Hence, by Corollary 2.10, $(\operatorname{rank} \beta(x_\pm))$-atomic measure for $\beta(x_\pm)$ exists. Since $\widetilde{\beta}$ is a subsequence of $\beta(x_0)$ of the form from Proposition 2.13 with $i = 1, j = k$, it admits a measure and hence Theorem 2.9 implies that $\operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} \widetilde{\beta}$. From (4.3), it follows that:

- If $\operatorname{rank} \widehat{A} \leq \operatorname{rank} A_{\widetilde{\beta}}$, then $\operatorname{rank} \beta(x_\pm) = \operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} \widetilde{\beta}$.
- Else $\operatorname{rank} \widehat{A} = \operatorname{rank} A_{\widetilde{\beta}} + 1$ and $\operatorname{rank} \beta(x_\pm) = \operatorname{rank} \widehat{A} = \operatorname{rank} \widetilde{\beta} + 1$.

This proves the implication (3) $\Rightarrow$ (2) in this case.

The implication (2) $\Rightarrow$ (1) is trivial.

It remains to prove the moreover part. Observe that in the proof of the implication (3) $\Rightarrow$ (2), $(\operatorname{rank}\widetilde{\beta})$-atomic measure might not exist if (3a) holds with $\widehat{A} \succ 0$ and does not exist if (3b) holds with $\operatorname{rank} A_{\widetilde{\beta}} < \operatorname{rank} \widehat{A}$. We will prove that in the first case there always exists a $(\operatorname{rank}\widetilde{\beta})$-atomic measure. Assume that $A_{\widetilde{\beta}} \succ 0$ and $\widehat{A} \succ 0$. We will prove that one of $A_{\beta(x_{\pm})}$ or $A_{\beta(x_+)}$ satisfies

$$\operatorname{rank} A_{\beta(x_{\pm})} = \operatorname{rank} A_{\beta(x_{\pm})}(k-1), \tag{4.5}$$

and hence by Corollary 2.10, a $(\operatorname{rank}\beta(x_{\pm})) = (\operatorname{rank}\widetilde{\beta})$-atomic measure exists. Using Proposition 2.4 with for $A_{\beta(x)}, A_{\widetilde{\beta}}, A_{\widehat{\beta}}$ as K, N, C, respectively, and denoting $u := A_{\widetilde{\beta}}/A_{\widehat{\beta}}$, we have that

$$f(x) := A_{\beta(x)}/A_{\widetilde{\beta}} = \left(\beta_0 - e(x)A_{\widehat{\beta}}^{-1}e(x)^T\right) - \frac{1}{u}\left(\beta_k - e(x)A_{\widehat{\beta}}^{-1}z^T\right)^2 =: g(x) - \frac{1}{u}h(x)^2,$$

where $e(x) := \begin{pmatrix} x & \beta_2 & \cdots & \beta_{k-1} \end{pmatrix}$ and $z := \begin{pmatrix} \beta_{k+1} & \cdots & \beta_{2k-1} \end{pmatrix}$. From the proof of the implication (3) $\Rightarrow$ (2), we know that $x_- < x_+$ and

$$f(x_-) = f(x_+) = 0. \tag{4.6}$$

Note that $g(x) = A_{(\beta_0, x, \widehat{\beta})}/A_{\widehat{\beta}}$. If

$$g(x_-) = g(x_+) = 0, \tag{4.7}$$

then $h(x_-) = h(x_+) = 0$. But $h(x)$ is a linear function in x, so this is possible only if $h(x) = 0$ for every $x \in \mathbb{R}$. This is possible only if

$$A_{\widehat{\beta}}^{-1}z^T = \begin{pmatrix} 0 & b_2 & \cdots & b_{k-1} \end{pmatrix}^T \quad \text{for some } b_2, \ldots, b_k \in \mathbb{R} \qquad \text{and} \qquad \beta_k = \sum_{i=2}^{k-1} \beta_i b_i. \tag{4.8}$$

We write $(A_{\beta(x)})|_{S_1,S_2}$ for the restriction of $A_{\beta(x)}$ to rows from S_1 and columns from S_2. Since $A_{\beta(x)}$ is a Hankel matrix, we have

$$(A_{\beta(x)})_{\{1,\ldots,X^{k-1}\},\{X,\ldots,X^k\}} = (A_{\beta(x)})_{\{X,\ldots,X^k\},\{1,\ldots,X^{k-1}\}},$$

which is equal to

$$\begin{pmatrix} e(x) & \beta_k \\ A_{\widehat{\beta}} & z^T \end{pmatrix} = \begin{pmatrix} e(x)^T & A_{\widehat{\beta}} \\ \beta_k & z \end{pmatrix}.$$

(4.8) implies that the last column of $(A_{\beta(x)})_{\{1,\ldots,X^{k-1}\},\{X,\ldots,X^k\}}$ is in the span of the columns $2, \ldots, k-1$. From $(A_{\beta(x)})_{\{X,\ldots,X^k\},\{1,\ldots,X^{k-1}\}}$ this in particular implies that the last column of $A_{\widehat{\beta}}$ is in the span of the others and $A_{\widehat{\beta}}$ is singular, which is a contradiction with the assumption $A_{\widetilde{\beta}} \succ 0$. Therefore (4.7) cannot be true and one of $g(x_-)$ and $g(x_+)$ is positive. By Proposition 2.3, this means that $A_{(\beta_0,x_+,\widehat{\beta})} \succ 0$ or $A_{(\beta_0,x_-,\widehat{\beta})} \succ 0$ and hence $\operatorname{rank} A_{(\beta_0,x_-,\widehat{\beta})} = k$ or $\operatorname{rank} A_{(\beta_0,x_+,\widehat{\beta})} = k$. By Proposition 2.3 and (4.6), $\operatorname{rank} A_{\beta(x_-)} = \operatorname{rank} A_{\beta(x_+)} = \operatorname{rank} A_{\widetilde{\beta}} = k$. Therefore $\operatorname{rank} A_{\beta(x_-)} = \operatorname{rank} A_{(\beta_0,x_-,\widehat{\beta})}$ or $\operatorname{rank} A_{\beta(x_+)} = \operatorname{rank} A_{(\beta_0,x_+,\widehat{\beta})}$. Noticing that $A_{(\beta_0,x_\pm,\widehat{\beta})} = A_{\beta(x_\pm)}(k-1)$, it follows that one of $x_\pm$ satisfies (4.5). This concludes the proof of the moreover part. $\square$

Remark 4.2. For $k = 1$, the THMP with gaps (β_1) coincides with the THMP with gaps (β_{2k-1}) and hence the case $k = 1$ is already covered by Theorem 3.1.

Example 4.3. For $k = 9$, let

$$\beta^{(1)}(x) = \Big(1, x, 11, 0, \frac{979}{5}, 0, 4103, 0, \frac{462979}{5}, 0, 2174855, 0, \frac{261453379}{5}, 0, 1275350087, 0, \frac{156925970179}{5}, 0, 776760884999\Big),$$

$$\beta^{(2)}(x) = \Big(1, x, \frac{15}{2}, 0, \frac{177}{2}, 0, \frac{2445}{2}, 0, \frac{36177}{2}, 0, \frac{554325}{2}, 0, \frac{8656377}{2}, 0, \frac{136617405}{2}, 0, \frac{2169039777}{2}, 0, \frac{138214318741}{8}\Big),$$

$$\beta^{(3)}(x) = \Big(1, x, \frac{15}{2}, 0, \frac{177}{2}, 0, \frac{2445}{2}, 0, \frac{36177}{2}, 0, \frac{554325}{2}, 0, \frac{8656377}{2}, 0, \frac{136617405}{2}, 0, \frac{2169039777}{2}, 0, \frac{34553579685}{2}\Big),$$

$$\beta^{(4)}(x) = \frac{1}{9}(9, x, 133, -235, 3157, -7987, 86893, -281995, 2598757, -10096867, 82154653, -362972155, 2699153557, -13062280147, 91112865613, -470199300715, 3134918735557, -16926788453827, 109327177835773),$$

Let $\widetilde{A}^{(i)}$ and $\widehat{A}^{(i)}$, $i = 1, 2, 3$, denote $\widetilde{A}$, $\widehat{A}$, respectively, from Theorem 4.1 corresponding to $\beta^{(i)}(x)$. Using *Mathematica* [Wol] one can check that:

- $\widehat{A}^{(1)} \succ 0$, while for $i = 2, 3, 4$ it holds that $\widehat{A}^{(i)} \succeq 0$ and $\dim\big(\ker \widehat{A}^{(i)}\big) = 1$.
- For $i = 1, 4$ we have $\widetilde{A}^{(i)} \succ 0$ for $i = 1, 4$, while for $i = 2, 3$ it holds that $\widetilde{A}^{(i)} \succeq 0$ and $\dim\big(\ker \widetilde{A}^{(i)}\big) = 1$.
- $A_{\widetilde{\beta}^{(i)}} \succ 0$ for $i = 1, 2, 4$, $A_{\widetilde{\beta}^{(3)}} \succeq 0$ and $\dim\big(\ker A_{\widetilde{\beta}^{(3)}}\big) = 1$.
- $A_{\widehat{\beta}^{(3)}} \succ 0$ and $A_{\breve{\beta}^{(3)}} \succ 0$.

Therefore:

- $A_{\beta^{(1)}(x)}$ is ppsd and (3a) of Theorem 4.1 is true, implying that a 9-atomic measure for $\beta^{(1)}(x)$ exists.
- $\beta^{(2)}(x)$ does not satisfy (3a) neither (3b) of Theorem 4.1, implying there is no representing measure for $\beta^{(2)}(x)$.
- $A_{\beta^{(3)}(x)}$ is ppsd and $\widetilde{\beta}^{(3)}$ satisfies (3b) of Theorem 4.1 together with $\operatorname{rank} A_{\widehat{\beta}^{(3)}} = \operatorname{rank} \widehat{A}^{(3)}$, implying that an 8-atomic measure for $\beta^{(3)}(x)$ exists.
- $A_{\beta^{(4)}(x)}$ is ppsd and $\widetilde{\beta}^{(4)}$ satisfies (3a) of Theorem 4.1, implying that a 9-atomic measure for $\beta^{(4)}(x)$ exists.

The following corollary is a consequence of Theorem 4.1 and gives the solution of the bivariate TMP for the curve $y^2 = x^3$.

Corollary 4.4. *Let $\beta = (\beta_{i,j})_{i,j\in\mathbb{Z}_+^2, i+j\leq 2k}$ be a 2-dimensional real multisequence of degree $2k$. Suppose $M(k)$ is positive semidefinite and recursively generated. Let*

$$u^{(i)} := (\beta_{1,i}, \beta_{0,i+1}, \beta_{2,i}) \quad \textit{for } i = 0, \ldots, 2k-2,$$

$$\widehat{\beta} := (u^{(0)}, \ldots, u^{(2k-2)}), \quad \widetilde{\beta} := (\widehat{\beta}, \beta_{1,2k-1}, \beta_{0,2k}), \quad \overline{\beta} := (\beta_{2,0}, u^{(1)}, \ldots, u^{(2k-2)})$$

$$\textit{and} \quad \breve{\beta} := (\overline{\beta}, \beta_{1,2k-1}, \beta_{0,2k})$$

be subsequences of β*,*

$$v := \begin{pmatrix} u^{(0)} & \cdots & u^{(k-2)} & \beta_{1,k-1} \end{pmatrix}$$

a vector and

$$\widetilde{A} := \begin{pmatrix} \beta_0 & v \\ v^T & A_{\overline{\beta}} \end{pmatrix}$$

a matrix. Then β *has a representing measure supported on* $y^2 = x^3$ *if and only if the following statements hold:*

(1) *One of the following holds:*
- *If* $k \geq 3$*, then* $Y^2 = X^3$ *is a column relation of* $M(k)$*.*
- *If* $k = 2$*, then the equalities* $\beta_{0,2} = \beta_{3,0}$*,* $\beta_{1,2} = \beta_{4,0}$*,* $\beta_{0,3} = \beta_{3,1}$ *hold.*
- $k = 1$*.*

(2) *One of the following holds:*
  (a) $A_{\widetilde{\beta}} \succ 0$ *and* $\widetilde{A} \succ 0$*.*
  (b) $A_{\widetilde{\beta}} \succeq 0$ *and* $\operatorname{rank} A_{\widehat{\beta}} = \operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} A_{\breve{\beta}}$*.*

Moreover, if the representing measure exists, then there exists a $(\operatorname{rank} \widetilde{\beta})$*-atomic measure if* (2a) *is true or* (2b) *holds with* $\operatorname{rank} A_{\widehat{\beta}} = \operatorname{rank} M(k)$*. Otherwise there is a* $(\operatorname{rank} \widetilde{\beta} + 1)$*-atomic measure.*

Proof. For $m \in \{0, 2 \ldots, 6k\}$ we define the numbers $\widetilde{\beta}_m$ by the following rule

$$\widetilde{\beta}_m := \begin{cases} \beta_{0,\frac{m}{3}}, & \text{if } m \pmod 3 = 0, \\ \beta_{2,\lfloor \frac{m}{3} \rfloor - 1}, & \text{if } m \pmod 3 = 1, \\ \beta_{1,\lfloor \frac{m}{3} \rfloor}, & \text{if } m \pmod 3 = 2. \end{cases}$$

Claim 1. Every number $\widetilde{\beta}_m$ is well-defined.

We have to prove that $i + j \leq 2k$, where i, j are indices of $\beta_{i,j}$ used in the definition of $\widetilde{\beta}_m$. We separate three cases according to m:

- $m \pmod 3 = 0$: $\frac{m}{3} \leq 2k$.
- $m \pmod 3 = 1$: $2 + (\lfloor \frac{m}{3} \rfloor - 1) \leq 2 + (2k - 2) = 2k$.
- $m \pmod 3 = 2$: $1 + \lfloor \frac{m}{3} \rfloor \leq 1 + (2k - 1) = 2k$.

Claim 2. Let $t \in \mathbb{N}$. The atoms $(x_1^2, x_1^3), \ldots (x_t^2, x_t^3)$ with densities $\lambda_1, \ldots, \lambda_t$ are the $(y^2 - x^3)$-representing measure for β if and only if the atoms $x_1, \ldots, x_t$ with densities $\lambda_1, \ldots, \lambda_t$ are the $\mathbb{R}$-representing measure for $\widetilde{\beta}(x) = (\widetilde{\beta}_0, x, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_{2k})$.

The if part follows from the following calculation:

$$\widetilde{\beta}_m = \begin{cases} \beta_{0,\frac{m}{3}}, & \text{if } m \pmod 3 = 0, \\ \beta_{2,\lfloor \frac{m}{3} \rfloor - 1}, & \text{if } m \pmod 3 = 1, \\ \beta_{1,\lfloor \frac{m}{3} \rfloor}, & \text{if } m \pmod 3 = 2, \end{cases} = \begin{cases} \sum_{\ell=1}^t \lambda_\ell (x_\ell^3)^{\frac{m}{3}}, & \text{if } m \pmod 3 = 0, \\ \sum_{\ell=1}^t \lambda_\ell (x_\ell^2)^2 (x_\ell^3)^{\lfloor \frac{m}{3} \rfloor - 1}, & \text{if } m \pmod 3 = 1, \\ \sum_{\ell=1}^t \lambda_\ell x_\ell^2 (x_\ell^3)^{\lfloor \frac{m}{3} \rfloor}, & \text{if } m \pmod 3 = 2, \end{cases} = \sum_{\ell=1}^t \lambda_\ell x_\ell^m,$$

where $m = 0, 2, \ldots, 6k$.

The only if part follows from the following calculation:

$$\begin{aligned}\beta_{i,j} &= \beta_{i-3,j+2} = \cdots = \beta_{i \pmod 3, j+2\lfloor \frac{i}{3}\rfloor} = \widetilde{\beta}_{2(i \pmod 3)+3(j+2\lfloor \frac{i}{3}\rfloor)}\\ &= \sum_{\ell=1}^{t} \lambda_\ell x_\ell^{2(i \pmod 3)+3(j+2\lfloor \frac{i}{3}\rfloor)} = \sum_{\ell=1}^{t} \lambda_\ell x_\ell^{2(i \pmod 3+3\lfloor \frac{i}{3}\rfloor)} x_\ell^{3j} = \sum_{\ell=1}^{t} \lambda_\ell (x_\ell^2)^i (x_\ell^3)^j,\end{aligned}$$

where the first three equalities in the first line follow by $M(k)$ being rg.

Using Claim 2 and a theorem of Bayer and Teichmann [BT06], implying that if a finite sequence has a K-representing measure, then it has a finitely atomic K-representing measure, the statement of the Corollary follows by Theorem 4.1. □

4.2. **Truncated Hamburger moment problem of degree $2k$ with gaps (β_1, β_2).**

Theorem 4.5. *Let $k \in \mathbb{N}$, $k > 2$, and*

$$\beta(x,y) := (\beta_0, x, y, \beta_3, \ldots, \beta_{2k})$$

be a sequence, where each β_i is a real number, $\beta_0 > 0$ and x, y are variables. Let

$$\widetilde{\beta} := (\beta_4, \ldots, \beta_{2k-2}), \quad \overline{\beta} := (\beta_4, \ldots, \beta_{2k}), \quad \breve{\beta} := (\beta_6, \ldots, \beta_{2k-2}), \quad \overline{\overline{\beta}} := (\beta_6, \ldots, \beta_{2k})$$

be subseqeunces of $\beta(x,y)$,

$$v := \begin{pmatrix} \beta_3 & \ldots & \beta_{k-1} \end{pmatrix}, \quad u := \begin{pmatrix} \beta_3 & \cdots & \beta_k \end{pmatrix}, \quad s := \begin{pmatrix} \beta_3 & \cdots & \beta_{k+1} \end{pmatrix},$$
$$w := \begin{pmatrix} \beta_5 & \cdots & \beta_{k+2} \end{pmatrix},$$

vectors, and

$$\overline{A} := \begin{pmatrix} \beta_0 & v \\ v^T & A_{\breve{\beta}} \end{pmatrix} \quad \text{and} \quad \widetilde{A} := \begin{pmatrix} \beta_0 & u \\ u^T & A_{\overline{\overline{\beta}}} \end{pmatrix}$$

matrices. Then the following statements are equivalent:

(1) There exist $x_0, y_0 \in \mathbb{R}$ and a representing measure for $\beta(x_0, y_0)$ supported on $K = \mathbb{R}$.
(2) There exist $x_0, y_0 \in \mathbb{R}$ and a $(\operatorname{rank} \overline{\beta})$ or $(\operatorname{rank} \overline{\beta} + 1)$-atomic representing measure for $\beta(x_0, y_0)$.
(3) $A_{\beta(x,y)}$ is partially positive semidefinite,

$$sA_{\overline{\beta}}^{\pm} s^T \leq uA_{\overline{\beta}}^{\pm} w^T + \sqrt{(A_{\overline{\beta}}/A_{\overline{\overline{\beta}}})(\widetilde{A}/A_{\overline{\overline{\beta}}})} \tag{4.9}$$

and one of the following statements is true:
  (a) $A_{\overline{\beta}} \succ 0$ and one of the following holds:
    (i) (A) $k = 3$ and the inequality in (4.9) is strict..
      (B) $k > 3$, $\overline{A} \succ 0$ and the inequality in (4.9) is strict.
    (ii) The following inequalities holds:

$$uA_{\widetilde{\beta}}^{\pm} u^T < sA_{\overline{\beta}}^{\pm} s^T \quad \text{and} \quad uA_{\overline{\overline{\beta}}}^{\pm} w^T - \sqrt{(A_{\overline{\beta}}/A_{\overline{\overline{\beta}}})(\widetilde{A}/A_{\overline{\overline{\beta}}})} \leq sA_{\overline{\beta}}^{\pm} s^T.$$

  (b) $\operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} A_{\overline{\beta}} = \operatorname{rank} \begin{pmatrix} s^T & A_{\overline{\beta}} \end{pmatrix}$.

Moreover, if the representing measure exists, then there is a $(\operatorname{rank} \overline{\beta})$-atomic if and only if (3(a)ii) or (3b) holds.

Proof. Note that $\beta(x,y)$ admits a measure if and only if there exist $y_0\in\mathbb{R}$ such that $\beta(x,y_0)$ admits a measure. Theorem 4.1 implies the following claim holds.

Claim 1. $\beta(x,y_0)$ admits a measure if and only if the following conditions hold:

(1) $A_{\beta(x,y_0)}$ is ppsd.

(2) Denoting

$$\widetilde{\beta}(y_0) := (y_0,\beta_3,\ldots,\beta_{2k-2}) \quad \text{and} \quad \overline{\beta}(y_0) := (y_0,\beta_3,\ldots,\beta_{2k}),$$

one of the following is true:

(a) $A_{\overline{\beta}(y_0)} = \begin{pmatrix} y_0 & s \\ s^T & A_{\overline{\beta}} \end{pmatrix} \succ 0$ and $\overline{A}(y_0)\succ 0$, where

$$\overline{A}(y) := \begin{cases} \left(\begin{array}{c|cc} \beta_0 & y & \beta_3 \\ \hline y & & \\ \beta_3 & & A_{\widetilde{\beta}} \end{array}\right), & \text{if } k=3, \\ \begin{pmatrix} \beta_0 & v(y) \\ v(y)^T & A_{\widetilde{\beta}} \end{pmatrix} = \begin{pmatrix} \beta_0 & y & v \\ y & \beta_4 & w_1 \\ v^T & w_1^T & A_{\breve{\beta}} \end{pmatrix}, \quad v(y)^T = \begin{pmatrix} y \\ v \end{pmatrix} \text{ and } w_1^T = \begin{pmatrix} \beta_5 \\ \vdots \\ \beta_{k+2} \end{pmatrix}, & \text{otherwise.} \end{cases}$$

(b) $\operatorname{rank} A_{\widetilde{\beta}(y_0)} = \operatorname{rank} A_{\overline{\beta}(y_0)} = \operatorname{rank} A_{\overline{\beta}}$.

We denote by

$$\widetilde{A}(y) := \begin{pmatrix} \beta_0 & u(y) \\ u(y)^T & A_{\overline{\beta}} \end{pmatrix} \quad \text{where} \quad u(y) = \begin{pmatrix} y & u \end{pmatrix}.$$

Claim 2. Assume $A_{\overline{\overline{\beta}}}\succ 0$ or $\operatorname{rank} A_{\overline{\overline{\beta}}} = \operatorname{rank} A_{\overline{\beta}}$. Then $\widetilde{A}(y_0)\succeq 0$ if and only if

(4.10)
$$\widetilde{A}(y) \text{ is ppsd} \quad \text{and} \quad y_0\in\Big[uA_{\overline{\overline{\beta}}}^{\pm}w^T-\sqrt{(A_{\overline{\beta}}/A_{\overline{\overline{\beta}}})(\widetilde{A}/A_{\overline{\overline{\beta}}})}, uA_{\overline{\overline{\beta}}}^{\pm}w^T+\sqrt{(A_{\overline{\beta}}/A_{\overline{\overline{\beta}}})(\widetilde{A}/A_{\overline{\overline{\beta}}})}\Big] =: [y_-,y_+].$$

Moreover,

$$\operatorname{rank}\widetilde{A}(y_0) = \begin{cases} \max\{\operatorname{rank} A_{\overline{\beta}}, \operatorname{rank}\widetilde{A}\}, & y\in\{y_-,y_+\}, \\ \max\{\operatorname{rank} A_{\overline{\beta}}, \operatorname{rank}\widetilde{A}\}+1, & y\in(y_-,y_+). \end{cases} \tag{4.11}$$

Let P_2 be the permutation matrix as in the proof of Theorem 4.1. We have that $P_2^T\widetilde{A}(y)P_2$ is of the form

$$P_2^T\widetilde{A}(y)P_2 = \begin{pmatrix} A_{\overline{\overline{\beta}}} & w^T & u^T \\ w & \beta_4 & y \\ u & y & \beta_0 \end{pmatrix}, \tag{4.12}$$

and denoting the matrices

$$\mathcal{A} := \begin{pmatrix} A_{\overline{\overline{\beta}}} & w^T \\ w & \beta_4 \end{pmatrix} \quad \text{and} \quad \mathcal{B} := \begin{pmatrix} A_{\overline{\overline{\beta}}} & u^T \\ u & \beta_0 \end{pmatrix},$$

and the permuation matrix $P_3:\mathbb{R}^{k-1}\to\mathbb{R}^{k-1}$ by

$$P_3 = \begin{pmatrix} \mathbf{0} & 1 \\ I_{k-2} & 0 \end{pmatrix},$$

where $\mathbf{0}$ stands for the row of $k-2$ zeros and I_{k-2} is the identity matrix of size $k-2$, we have that

$$\mathcal{A} = P_3^T A_{\overline{\beta}} P_3 \quad \text{and} \quad \mathcal{B} = P_3^T \widetilde{A} P_3. \tag{4.13}$$

By the assumptions in Claim 2 and (4.13), $A_{\overline{\overline{\beta}}} \succ 0$ or $\operatorname{rank} A_{\overline{\overline{\beta}}} = \operatorname{rank} \mathcal{A}$. Hence, the assumption (2.9) of Lemma 2.11 used for $P_2^T \widetilde{A}(y) P_2, A_{\overline{\overline{\beta}}}, \mathcal{A}, \mathcal{B}$ as $A(x), A_1, A_2, A_3$, respectively, is satisfied and using also $\mathcal{A}/A_{\overline{\overline{\beta}}} = A_{\overline{\beta}}/A_{\overline{\overline{\beta}}}$, $\mathcal{B}/A_{\overline{\overline{\beta}}} = \widetilde{A}/A_{\overline{\overline{\beta}}}$, Claim 2 follows.

Theorem 2.2 implies the following claim.

Claim 3. It is true that:

(1) $A_{\overline{\beta}(y_0)} \succeq 0$ if and only if

$$A_{\overline{\beta}} \succeq 0, \qquad s^T \in \mathcal{C}(A_{\overline{\beta}}) \qquad \text{and} \qquad A_{\overline{\beta}(y_0)}/A_{\overline{\beta}} = y_0 - sA_{\overline{\beta}}^{+} s^T \geq 0. \tag{4.14}$$

(2) $A_{\widetilde{\beta}(y_0)} \succeq 0$ if and only if

$$A_{\widetilde{\beta}} \succeq 0, \qquad u^T \in \mathcal{C}(A_{\widetilde{\beta}}) \qquad \text{and} \qquad A_{\widetilde{\beta}(y_0)}/A_{\widetilde{\beta}} = y_0 - uA_{\widetilde{\beta}}^{+} u^T \geq 0. \tag{4.15}$$

Claim 4. Assume $A_{\overline{\overline{\beta}}} \succ 0$ or $\operatorname{rank} A_{\overline{\overline{\beta}}} = \operatorname{rank} A_{\overline{\beta}}$. Then $A_{\beta(x,y_0)}$ is ppsd for some $y_0 \in \mathbb{R}$ if and only if $A_{\beta(x,y)}$ is ppsd, $s^T \in \mathcal{C}(A_{\overline{\beta}})$ and (4.9) holds.

Note that $A_{\beta(x,y_0)}$ is ppsd if and only if $A_{\overline{\beta}(y_0)} \succeq 0$ and $\widetilde{A}(y_0) \succeq 0$. The first condition of (4.10) (which also includes the first condition of (4.14)) is equivalent to $A_{\beta(x,y)}$ being ppsd. Further on, y_0 satisfying the third condition of (4.14) and the second condition of (4.10) exists if and only if (4.9) holds. This proves Claim 4.

First we prove the implication (1) $\Rightarrow$ (3). By Claim 1, in particular $A_{\beta(x,y_0)}$ (and hence also $A_{\beta(x,y)}$) is ppsd. Since $\widetilde{\beta}(y_0)$ also admits a measure by Proposition 2.13, we either have $A_{\widetilde{\beta}(y_0)} \succ 0$ and in particular $A_{\overline{\overline{\beta}}} \succ 0$, or $A_{\widetilde{\beta}(y_0)}$ is singular and it follows by Corollary 2.8 that $A_{\overline{\overline{\beta}}} \succ 0$ or $\operatorname{rank} A_{\overline{\overline{\beta}}} = \operatorname{rank} A_{\overline{\beta}}$.

If (2a) of Claim 1 holds, then in particular $A_{\overline{\beta}} \succ 0$ and if $k > 3$ also $\overline{A} \succ 0$. Since $A_{\overline{\beta}}(y_0) \succ 0$, it follows using Proposition 2.3 that $A_{\overline{\beta}}(y_0)/A_{\overline{\beta}} > 0$ or equivalently $y_0 > sA_{\overline{\beta}^+} s^T$. Since by Claim 2, $y_0 \in [y_-, y_+]$, this implies that $sA_{\overline{\beta}^+} s^T < y_+$ which means that the inequality in (4.9) is strict. Hence, $A_{\beta(x,y)}$ is ppsd, $A_{\overline{\beta}} \succ 0$ and (3(a)i) holds. This proves the implication (1) $\Rightarrow$ (3) in this case.

Assume now that (2b) of Claim 1 holds. There are two cases to consider:

- $A_{\overline{\beta}} \succ 0$: It follows that $\operatorname{rank} A_{\widetilde{\beta}(y_0)} = \operatorname{rank} A_{\overline{\beta}(y_0)} = k-1$, which implies that:
  - $A_{\widetilde{\beta}(y_0)} \succ 0$ since $A_{\widetilde{\beta}(y_0)}$ is of size $k-1$.
  - By Proposition 2.3, $y_0 = sA_{\overline{\beta}}^{+} s^T$ since $A_{\overline{\beta}(y_0)} = \begin{pmatrix} y_0 & s \\ s^T & A_{\overline{\beta}} \end{pmatrix}$ is singular.
  - $k-1 \leq \operatorname{rank} A_{\beta(x_0,y_0)} \leq k$ for some $x_0 \in \mathbb{R}$ such that $A_{\beta(x_0,y_0)} \succeq 0$, since

$$A_{\beta(x_0,y_0)} = \begin{pmatrix} \beta_0 & u(x_0,y_0) \\ u(x_0,y_0)^T & A_{\overline{\beta}(y_0)} \end{pmatrix} \quad \text{where} \quad u(x_0,y_0) = \begin{pmatrix} x_0 & y_0 & u \end{pmatrix}.$$

From $A_{\widetilde{\beta}(y_0)} \succ 0$ and $y_0 = sA_{\overline{\beta}}^{+} s^T$, it follows by Proposition 2.3 and (4.15) that $uA_{\overline{\beta}}^{+} u^T < sA_{\overline{\beta}}^{+} s^T$. Further on, $y_- \le sA_{\overline{\beta}}^{+} s^T$ since by Claim 2, $\widetilde{A}(y_0) \succeq 0$ implies that $y_0 \in [y_-, y_+]$. Hence, $A_{\beta(x,y)}$ is ppsd, $A_{\overline{\beta}} \succ 0$ and (3(a)ii) holds. This proves the implication (1) $\Rightarrow$ (3) in this case.

- $A_{\overline{\beta}} \not\succ 0$: By Lemma 2.13, $\overline{\beta}$ also admits a measure and hence by Corollary 2.10 used for $\overline{\beta}$ as β, rank $A_{\widetilde{\beta}} =$ rank $A_{\overline{\beta}}$. Together with the second condition in (4.14), this implies that (3b) holds. Since $A_{\beta(x,y)}$ is ppsd and (4.9) holds, this proves the implication (1) $\Rightarrow$ (3) in this case.

Second we prove the implication (3) $\Rightarrow$ (1). If (3a) holds, then $A_{\overline{\beta}} \succ 0$ and in particular $A_{\overline{\overline{\beta}}} \succ 0$. Else (3b) holds and in particular $A_{\overline{\beta}}$ is singular. By Claim 3, $A_{\overline{\beta}(sA_{\overline{\beta}}^{+} s^T)} \succeq 0$ and hence by Corollary 2.8 used for $\overline{\beta}(sA_{\overline{\beta}}^{+} s^T)$ as β we conclude that rank $A_{\overline{\overline{\beta}}} =$ rank $A_{\overline{\beta}}$. Hence the assumption of Claims 2 and 4 is satisfied and $A_{\beta(x,y_0)}$ is ppsd for every y_0 from the interval $[\max\{y_-, sA_{\overline{\beta}}^{+} s^T\}, y_+]$. We separate cases three cases according to the assumptions:

- Case (3(a)i): We separate two cases according to the invertibility of $\widetilde{A}$.
  - $\widetilde{A} \succ 0$: Since $A_{\overline{\beta}} \succ 0$ and $\widetilde{A} \succ 0$, it follows that $A_{\overline{\beta}}/A_{\overline{\overline{\beta}}} > 0$ and $\widetilde{A}/A_{\overline{\overline{\beta}}} > 0$. By the form of $y_\pm$ given in Claim 2, we have that $y_- < y_+$. Since by assumption also the inequality (4.9) is strict, the interval $(\max\{y_-, sA_{\overline{\beta}}^{+} s^T\}, y_+)$ is not empty and hence for every $y_0 \in (\max\{y_-, sA_{\overline{\beta}}^{+} s^T\}, y_+)$, $A_{\beta(x,y_0)}$ satisfies (2a) above by Claims 2 and 3. This proves the implication (3) $\Rightarrow$ (1) in this case.
  - $\widetilde{A}$ in singular: First we show that the last column of $\widetilde{A}$ is in the span of others. We separate two cases according to k.
    * $k = 3$: Since $\widetilde{A} = \begin{pmatrix} \beta_0 & \beta_3 \\ \beta_3 & \beta_6 \end{pmatrix}$ and $\beta_0 > 0$, it follows that the second (also the last) column of $\widetilde{A}$ is a multiple of the first (also it the span of the others).
    * $k > 3$: Since $\overline{A} \succ 0$, the last column of $\widetilde{A} = \begin{pmatrix} \overline{A} & r^T \\ r & \beta_{2k} \end{pmatrix}$ is in the span of the others, where $r = \begin{pmatrix} \beta_k & \beta_{k+2} & \cdots & \beta_{2k-1} \end{pmatrix}$.

    Since $A_{\overline{\beta}} \succ 0$, it follows that $A_{\overline{\overline{\beta}}} \succ 0$ and $\widetilde{A}/A_{\overline{\overline{\beta}}} = 0$. By the form of $y_\pm$ given in Claim 2, we have that $y_- = y_+$. By Claim 2, $\widetilde{A}(y_+) = \begin{pmatrix} \overline{A}(y_+) & r_1^T \\ r_1 & \beta_{2k} \end{pmatrix} \succeq 0$, where $r_1 = \begin{pmatrix} \beta_k & \cdots & \beta_{2k-1} \end{pmatrix}$, and rank $\widetilde{A}(y_+) =$ rank $A_{\overline{\beta}} = k-1$. By Lemma 2.12, the last column of $\widetilde{A}(y_+)$ is in the span of the others and hence $\overline{A}(y_+) \succ 0$. Since by assumption also the inequality (4.9) is strict, $A_{\overline{\beta}(y_+)} \succ 0$ by Claim 3. Hence, (2a) of Claim 1 holds for $y_0 = y_+$, which proves the implication (3) $\Rightarrow$ (1) in this case.
- Case (3(a)ii): $\beta(x, sA_{\overline{\beta}}^{+} s^T)$ is ppsd. Since

$$A_{\overline{\beta}(sA_{\overline{\beta}}^{+} s^T)} = \begin{pmatrix} A_{\widetilde{\beta}(sA_{\overline{\beta}}^{+} s^T)} & r_2^T \\ r_2 & \beta_{\beta_{2k}} \end{pmatrix},$$

where $r_2 = \begin{pmatrix} \beta_{k+1} & \cdots & \beta_{2k-1} \end{pmatrix}$, is singular, the assumption $uA_{\overline{\beta}}^{+}u^T < sA_{\overline{\beta}}^{+}s^T$ and Claim 3 imply that $A_{\widetilde{\beta}(sA_{\overline{\beta}}^{+}s^T)} \succ 0$, hence $\operatorname{rank} A_{\widetilde{\beta}(sA_{\overline{\beta}}^{+}s^T)} = \operatorname{rank} A_{\overline{\beta}(sA_{\overline{\beta}}^{+}s^T)} = \operatorname{rank} A_{\overline{\beta}}$. Hence, (2b) of Claim 1 for $y_0 = sA_{\overline{\beta}}^{+}s^T$ holds, which proves the implication (3) $\Rightarrow$ (1) in this case.

- Case (3b): By assumption $\operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} A_{\overline{\beta}}$, it follows that the last column of $A_{\overline{\beta}}$ is in the span of the others. There exists $x_0 \in \mathbb{R}$ such that $A_{\beta(x_0,y_+)}$ is psd and by Lemma 2.12, the last column of $A_{\overline{\beta}(y_+)}$ is in the span of the others and hence $\operatorname{rank} A_{\widetilde{\beta}(y_+)} = \operatorname{rank} A_{\overline{\beta}(y_+)}$. Since $A_{\overline{\beta}(y_+)}$ is singular, using Corollary 2.8 with β equal to $\beta(x_0, y_+)$, we get $\operatorname{rank} A_{\overline{\beta}(y_+)} = \operatorname{rank} A_{\overline{\beta}}$, which in particular implies that $y_+ = sA_{\overline{\beta}}^{+}s^T$. Hence, $\operatorname{rank} A_{\widetilde{\beta}(y_+)} = \operatorname{rank} A_{\overline{\beta}(y_+)} = \operatorname{rank} A_{\overline{\beta}}$, which is (2b) of Claim 1. This proves the implication (3) $\Rightarrow$ (1) in this case.

It remains to prove the implication (1) $\Rightarrow$ (2). By Theorem 4.1, if $\beta(x, y_0)$ has a representing measure, then there is a $(\operatorname{rank} \overline{\beta}(y_0))$ or $(\operatorname{rank} \overline{\beta}(y_0) + 1)$-atomic representing measure. By Corollary 2.8, $\operatorname{rank} \overline{\beta}(y_0) = \operatorname{rank} A_{\overline{\beta}(y_0)} = \operatorname{rank} A_{\overline{\beta}}$ if $A_{\overline{\beta}(y_0)}$ is singular and $\operatorname{rank} \overline{\beta}(y_0) = \operatorname{rank} A_{\overline{\beta}} + 1 = \operatorname{rank} \overline{\beta} + 1$ otherwise.

For the moreover part, note from the previous paragraph that $(\operatorname{rank} \overline{\beta})$-atomic measure exists if and only if $A_{\overline{\beta}(y_0)} = \operatorname{rank} A_{\overline{\beta}}$ for some y_0 such the $\beta(x, y_0)$ admits a measure. The only $y_0 \in \mathbb{R}$ satisfying $\operatorname{rank} A_{\overline{\beta}(y_0)} = \operatorname{rank} A_{\overline{\beta}}$ is $sA_{\overline{\beta}}^{+}s^T$ and hence a $(\operatorname{rank} \overline{\beta})$-atomic measure exists if and only if $\beta(x, sA_{\overline{\beta}}^{+}s^T)$ admits a measure. From the proof of the implication (3) $\Rightarrow$ (1) we see that this is true in the cases (3(a)ii) and (3b). Finally, if (3(a)i) holds, then we see that:

- If $\widetilde{A} \succ 0$, then we must have $y_- \leq sA_{\overline{\beta}}^{+}s^T$ and $uA_{\overline{\beta}}^{+}u^T < sA_{\overline{\beta}}^{+}s^T$ (see the proof of (3(a)ii)), which means that (3(a)ii) holds.
- If $\widetilde{A}$ is singular, then $sA_{\overline{\beta}}^{+}s^T < y_- = y_+$ and $\beta(x, sA_{\overline{\beta}}^{+}s^T)$ does not admit a $(\operatorname{rank} \overline{\beta})$-atomic measure.

This establishes the proof of the moreover part. □

Remark 4.6. For $k = 2$, the THMP with gaps (β_1, β_2) coincides with the THMP with gaps $(\beta_{2k-2}, \beta_{2k-1})$ and hence the case $k = 2$ is already covered by Theorem 3.5 .

The following corollary is a consequence of Theorem 4.5 and solves the bivariate TMP for the curve $y^3 = x^4$ where also $\beta_{\frac{5}{3},0}$ is given. Here $\beta_{\frac{5}{3},0}$ stands for the integral of $x^{\frac{5}{3}}$ w.r.t. μ, i.e., $\int_K x^{\frac{5}{3}} d\mu$.

Corollary 4.7. *Let $\beta = (\beta_{i,j})_{i,j \in \mathbb{Z}_+^2, i+j \leq 2k}$ β be a 2-dimensional real multisequence of degree $2k$ and let $\beta_{\frac{5}{3},0}$ be also given. Suppose $M(k)$ is positive semidefinite and recursively generated. Let*

$$u^{(1)} = (\beta_{0,1}, \beta_{\frac{5}{3},0}, \beta_{2,0}, \beta_{1,2}), \quad u^{(i)} = (\beta_{0,i}, \beta_{3,i-2}, \beta_{2,i-1}, \beta_{1,i}) \quad \textit{for} \quad i = 2, \ldots, 2k-1,$$

$$\widetilde{\beta} := (u^{(1)}, \ldots, u^{(2k-2)}, \beta_{0,2k-1}, \beta_{3,2k-3}, \beta_{2,2k-2}), \quad \overline{\beta} := (\widetilde{\beta}, \beta_{1,2k-1}, \beta_{0,2k}),$$

$$\breve{\beta} := (\widehat{\beta}, \beta_{3,2k-3}, \beta_{2,2k-2}) \quad \textit{and} \quad \overline{\overline{\beta}} := (\breve{\beta}, \beta_{3,2k-1}, \beta_{0,2k})$$

be subsequences of β,

$$v := \begin{pmatrix} \beta_{1,0} & u^{(1)} & \cdots & u^{(k-2)} & \beta_{0,k-1} & \beta_{3,k-3} & \beta_{2,k-2} & \beta_{1,k-1} \end{pmatrix}, \quad u := \begin{pmatrix} v & \beta_{0,k} \end{pmatrix},$$

$$s := \begin{pmatrix} u & \beta_{3,k-2} \end{pmatrix}, \quad w := \begin{pmatrix} \beta_{\frac{5}{3},0} & \beta_{2,0} & \beta_{1,1} & u^{(2)} & \cdots & u^{(k-1)} & \beta_{0,k} & \beta_{3,k-2} & \beta_{2,k-1} \end{pmatrix}$$

vectors and

$$\overline{A} := \begin{pmatrix} \beta_0 & v \\ v^T & A_{\breve{\beta}} \end{pmatrix} \quad \textit{and} \quad \widetilde{A} := \begin{pmatrix} \beta_0 & u \\ u^T & A_{\overline{\overline{\beta}}} \end{pmatrix}$$

matrices. Then β has a representing measure supported on $y^3 = x^4$ if and only if

$$sA_{\overline{\beta}}^{+}s^T \leq uA_{\overline{\overline{\beta}}}^{+}w^T + \sqrt{(A_{\overline{\beta}}/A_{\overline{\overline{\beta}}})(\widetilde{A}/A_{\overline{\overline{\beta}}})} \tag{4.16}$$

one of the following statements hold:

(1) *One of the following holds:*
- *If $k \geq 4$, then $Y^3 = X^4$ is a column relation of $M(k)$.*
- *If $k = 3$, then the equalities $\beta_{0,3} = \beta_{4,0}$, $\beta_{1,3} = \beta_{5,0}$, $\beta_{2,3} = \beta_{6,0}$, $\beta_{0,4} = \beta_{4,1}$, $\beta_{0,5} = \beta_{4,2}$.*
- *If $k = 2$, then the equality $\beta_{0,3} = \beta_{4,0}$ holds.*
- *$k = 1$.*

(2) *One of the following holds:*
  (a) *$A_{\overline{\beta}} \succ 0$, $\overline{A} \succ 0$ and the inequality in (4.16) is strict.*
  (b) *$A_{\overline{\beta}} \succ 0$ and the following inequalities holds:*
$$uA_{\widetilde{\beta}}^{+}u^T < sA_{\overline{\beta}}^{+}s^T \quad \textit{and} \quad uA_{\overline{\overline{\beta}}}^{+}w^T - \sqrt{(A_{\overline{\beta}}/A_{\overline{\overline{\beta}}})(\widetilde{A}/A_{\overline{\overline{\beta}}})} \leq sA_{\overline{\beta}}^{+}s^T.$$
  (c) *$A_{\overline{\beta}} \succeq 0$ and $\operatorname{rank} A_{\widetilde{\beta}} = \operatorname{rank} A_{\overline{\beta}} = \operatorname{rank} \begin{pmatrix} s^T & A_{\overline{\beta}} \end{pmatrix}$.*

Moreover, if the representing measure exists, then there exists a $(\operatorname{rank} \overline{\beta})$-atomic measure if and only if (2b) or (2c) holds. Otherwise there is a $(\operatorname{rank} \overline{\beta} + 1)$-atomic measure

Proof. For $\{0, 3, 4, 6, \ldots, 8k\}$ we define the numbers $\widetilde{\beta}_m$ by the following rule

$$\widetilde{\beta}_m := \begin{cases} \beta_{0,\frac{m}{4}}, & \text{if } m \pmod 4 = 0, \\ \beta_{3,\lfloor\frac{m}{4}\rfloor-2}, & \text{if } m \pmod 4 = 1, \\ \beta_{2,\lfloor\frac{m}{4}\rfloor-1}, & \text{if } m \pmod 4 = 2, \\ \beta_{1,\lfloor\frac{m}{4}\rfloor}, & \text{if } m \pmod 4 = 3. \end{cases}$$

Claim 1. Every number $\widetilde{\beta}_m$ is well-defined.

We have to prove that $i + j \leq 2k$, where i, j are indices of $\beta_{i,j}$ used in the definition of $\widetilde{\beta}_m$. We separate four cases according to m:

- $m \pmod 4 = 0$: $\frac{m}{4} \leq 2k$.
- $m \pmod 4 = 1$: $\lfloor\frac{m}{4}\rfloor - 2 + 3 \leq (2k-1) + 1 = 2k$.
- $m \pmod 4 = 2$: $\lfloor\frac{m}{4}\rfloor - 1 + 2 \leq (2k-1) + 1 = 2k$.
- $m \pmod 4 = 3$: $\lfloor\frac{m}{4}\rfloor + 1 \leq (2k-1) + 1 = 2k$.

We also define $\widetilde{\beta}_5 := \beta_{\frac{5}{3},0}$.

Claim 2. Let $t \in \mathbb{N}$. The atoms $(x_1^3, x_1^4), \ldots (x_t^3, x_t^4)$ with densities $\lambda_1, \ldots, \lambda_t$ are the $(y^3 - x^4)$-representing measure for β with $\beta_{\frac{5}{3},0}$ known if and only if the atoms $x_1, \ldots, x_t$ with densities $\lambda_1, \ldots, \lambda_t$ are the $\mathbb{R}$-representing measure for $\widetilde{\beta}(x, y) = (\widetilde{\beta}_0, x, y, \widetilde{\beta}_3, \ldots, \widetilde{\beta}_{2k})$.

The if part follows from the following calculation:

$$\widetilde{\beta}_m = \begin{cases} \beta_{0,\frac{m}{4}}, & \text{if } m \pmod 4 = 0, \\ \beta_{3,\lfloor\frac{m}{4}\rfloor-2}, & \text{if } m \pmod 4 = 1, \\ \beta_{2,\lfloor\frac{m}{4}\rfloor-1}, & \text{if } m \pmod 4 = 2, \\ \beta_{1,\lfloor\frac{m}{4}\rfloor}, & \text{if } m \pmod 4 = 3, \end{cases} = \begin{cases} \sum_{\ell=1}^t \lambda_\ell (x_\ell^4)^{\frac{m}{4}}, & \text{if } m \pmod 4 = 0, \\ \sum_{\ell=1}^t \lambda_\ell (x_\ell^3)^3 (x_\ell^4)^{\lfloor\frac{m}{4}\rfloor-2}, & \text{if } m \pmod 4 = 1, \\ \sum_{\ell=1}^t \lambda_\ell (x_\ell^3)^2 (x_\ell^4)^{\lfloor\frac{m}{4}\rfloor-1}, & \text{if } m \pmod 4 = 2, \\ \sum_{\ell=1}^t \lambda_\ell x_\ell^3 (x_\ell^4)^{\lfloor\frac{m}{4}\rfloor}, & \text{if } m \pmod 4 = 3, \end{cases} = \sum_{\ell=1}^t \lambda_\ell x_\ell^m,$$

where $m = 0, 3, 4, 6, \ldots, 8k$ and

$$\widetilde{\beta}_5 = \beta_{\frac{5}{3},0} = \sum_{\ell=1}^t \lambda_\ell (x_\ell^3)^{\frac{5}{3}} = \sum_{\ell=1}^t \lambda_\ell x_\ell^5.$$

The only if part follows from the following calculation:

$$\begin{aligned} \beta_{i,j} &= \beta_{i-4,j+3} = \cdots = \beta_{i \pmod 4, j+3\lfloor\frac{i}{4}\rfloor} = \widetilde{\beta}_{3(i \pmod 4)+4(j+3\lfloor\frac{i}{4}\rfloor)} \\ &= \sum_{\ell=1}^t \lambda_\ell x_\ell^{3(i \pmod 4))+4(j+3\lfloor\frac{i}{4}\rfloor)} = \sum_{\ell=1}^t \lambda_\ell x_\ell^{3(i \pmod 4)+4\lfloor\frac{i}{4}\rfloor)} x_\ell^{4j} = \sum_{\ell=1}^t \lambda_\ell (x_\ell^3)^i (x_\ell^4)^j, \end{aligned}$$

where the first three equalities in the first line follow by $M(k)$ being rg and

$$\beta_{\frac{5}{3},0} = \widetilde{\beta}_5 = \sum_{\ell=1}^t \lambda_\ell x_\ell^5 = \sum_{\ell=1}^t \lambda_\ell (x_\ell^3)^{\frac{5}{3}}.$$

Using Claim 2 and a theorem of Bayer and Teichmann [BT06], implying that if a finite sequence has a K-representing measure, then it has a finitely atomic K-representing measure, the statement of the Corollary follows by Theorem 4.5. □

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