THE FINAL SOLUTION OF THE HITCHIKER'S PROBLEM #5

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ABSTRACT. The recent survey [2] nicknamed "Hitchhiker's Guide" has raised the rating of quasi-copula problems in the dependence modeling community in spite of the lack of statistical interpretation of quasi-copulas. In our previous work we addressed the question of extreme values of the mass distribution associated with a mutidimensional quasi–copulas. Using linear programming approach we were able to settle [2, Open Problem 5] up to $d = 17$ and disprove a recent conjecture from [25] on solution to that problem. In this note we use an analytical approach to provide a complete answer to the original question.

1. Introduction

Copula is simply a multivariate distribution with uniform margins, but when we insert arbitrary univariate distributions as margins into it, we can get any multivariate distribution. This seminal 1959 result of Sklar [23] has made them the most important tool of dependence modeling [9, 16]. On the other hand, the statistical interpretation of quasi-copulas is problematic since they may have negative volumes locally. They come naturally as (pointwise) lower and upper bounds of sets of copulas. The lower Fréchet-Hoeffding bound W, say, the pointwise infimum of all d-variate copulas, $d \geq 2$, is in general a quasi-copula, unless $d = 2$, while the upper bound M, the pointwise supremum of all d-variate copulas, is always a copula. Quasi-copulas were first introduced in 1993 by Alsina, Nelsen, and Schweizer [1], but the dependence modeling community may have been somehow withheld from wider use of them perhaps due to their deficiency described above.

The new era for this notion starts in 2020 with the seminal paper [2] by Arias-García, Mesiar, and De Baets, listing the most important results and open problems in the area. Some vivid activity followed this paper. The six open problems listed there are sometimes nicknamed "hitchhiker's problems". The first one solved seems to have been hitchhiker's problem #6, Omladič and Stopar [20] in 2020; the second one was hitchhiker's problem #3, Omladič and Stopar [21] in 2022; the next solution appeared in 2023, it was hitchhiker's problem #2, Klemment et al. [14]; and then, a possible solution for the bivariate case of hitchhiker's problem #4 was given by Stopar [24] in 2024. The hitchhiker's problem #5 appears to be more involved. It asks to find the maximal negative and positive mass that a quasi-copula can have as well as to characterize the type of boxes that have the maximal mass. This problem started originally by Nelsen et al. in 2002 [17], where an analytical solution for the bivariate case is given. In 2007 de Baets et al. [5] propose a linear programming approach to give the trivariate case. In 2023 Úbeda-Flores [25] extends the linear

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programming approach up to dimension four and conjectures the growth for higher dimensions. In our very recent work [6] we extended this approach further, up to dimension 17 and disproved conjecture of [25]. In this paper we give the final solution of the problem for all dimensions. It turns out that there is no closed formula that applies simultaneously in all dimensions. However, there is an explicit formula for each dimension that requires performing a simple algorithm to determine the smallest positive integer that satisfies a particular inequality. This smallest integer then determines the maximal positive (resp. maximal negative) volume of a box over all quasi-copulas and also a realization of a box and a quasi-copula. The overall growth of the maximal volumes is exponential. For the maximal positive value, the boxes $\mathcal{B}$ are always of the form $[a,1]^d$, while for maximal negative values this applies to $d \geq 7$.

2. Statements of the main theorems

In this section we introduce the notation and state our main results that solve [2, Open Problem 5], i.e., Theorem 1 solves the maximal negative volume problem, while Theorem 2 solves the maximal positive volume problem. Concrete examples are also presented (see Examples 1 and 2).

Let $\mathcal{D} \subseteq [0,1]^d$ be a set. We say that a function $Q : \mathcal{D} \to [0,1]$ satisfies:

(1) *Boundary condition:* If for $\underline{u} := (u_1, \ldots, u_d) \in \mathcal{D}$, the following holds for any index $i = 1, \ldots, d$:

$$\begin{aligned} &\text{(a)} \quad Q(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_d) = 0 \quad \text{and} \\ &\text{(b)} \quad Q(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i. \end{aligned}$$

(2) *Monotonicity condition:* Q is nondecreasing in every variable, i.e., for each $i = 1, \ldots, d$ and each pair of d-tuples

$$\begin{aligned} \underline{u} &:= (u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_d) \in \mathcal{D}, \\ \underline{\widetilde{u}} &:= (u_1, \ldots, u_{i-1}, \widetilde{u}_i, u_{i+1}, \ldots, u_d) \in \mathcal{D}, \end{aligned}$$

such that $u_i \leq \widetilde{u}_i$, it follows that $Q(\underline{u}) \leq Q(\underline{\widetilde{u}})$.

(3) *Lipschitz condition*: Given d–tuples $(u_1, \ldots, u_d)$ and $(v_1, \ldots, v_d)$ in $\mathcal{D}$ it holds that

$$|Q(u_1, \ldots, u_d) - Q(v_1, \ldots, v_d)| \leq \sum_{i=1}^{d} |u_i - v_i|.$$

If $\mathcal{D} = [0,1]^d$ and Q satisfies (1), (2), (3), then Q is called a *d-variate quasi-copula* (or *d-quasi-copula*). We will omit the dimension d when it is clear from the context and write quasi-copula for short.

Let Q be a quasi-copula and $\mathcal{B} = \prod_{i=1}^{d} [a_i, b_i] \subseteq [0,1]$ a d-box with $a_i < b_i$ for each i. We will use multi-indices of the form $\mathbb{I} := (\mathbb{I}_1, \mathbb{I}_2, \ldots, \mathbb{I}_d) \in \{0,1\}^d$ to index 2^d vertices $\prod_{i=1}^{d} \{a_i, b_i\}$ of $\mathcal{B}$. We write

$$x_{\mathbb{I}} := ((x_{\mathbb{I}})_1, \ldots, (x_{\mathbb{I}})_d), \quad \text{where} \quad (x_{\mathbb{I}})_k = \begin{cases} a_k, & \text{if } \mathbb{I}_k = 0, \\ b_k, & \text{if } \mathbb{I}_k = 1. \end{cases}$$

Let us denote the value of Q at the point $x_{\mathbb{I}}$ by

$$q_{\mathbb{I}} := Q(x_{\mathbb{I}}).$$

Let $\|\mathbb{I}\|_1 := \sum_{i=1}^{d} \mathbb{I}_i$ be the 1-norm of the multi-index $\mathbb{I}$ and $\operatorname{sign}(\mathbb{I}) := (-1)^{d-\|\mathbb{I}\|_1}$ its sign. The **Q-volume** of $\mathcal{B}$ is defined by:

$$V_Q(\mathcal{B}) = \sum_{\mathbb{I}\in\{0,1\}^d} \operatorname{sign}(\mathbb{I}) q_{\mathbb{I}}. \tag{2.1}$$

If the quasi-copula Q is understood, we call it simply the volume of $\mathcal{B}$ and denote it $V(\mathcal{B})$.

The complete solution to [2, Open Problem 5] for the maximal negative volume is the following.

Theorem 1. *Assume the notation above. Let $d \in \mathbb{N}$ and $d \geq 7$. Define*

$$(c_1, c_2, \ldots, c_{\lfloor \frac{d}{2}\rfloor}) :=$$
$$= \begin{cases} \left(\binom{d-1}{0}, \binom{d-1}{1}, \binom{d-1}{2}, \ldots, \binom{d-1}{\frac{d}{2}-1}\right), & \text{if } d \text{ is even,} \\ \left(\binom{d-1}{1}, \binom{d-1}{1}, \binom{d-1}{3}, \binom{d-1}{3}, \ldots, \binom{d-1}{\lfloor\frac{d}{2}\rfloor-1}, \binom{d-1}{\lfloor\frac{d}{2}\rfloor-1}, \binom{d-1}{\lfloor\frac{d}{2}\rfloor}\right), & \text{if } d \text{ is odd.} \end{cases}$$

Define $c_0 = 0$ and

$$w_{i,-} := \frac{1}{i+1}\Big(\sum_{j=0}^{i-1} c_{\lfloor\frac{d}{2}\rfloor - j} - 1\Big) \quad \text{for } i = 1, \ldots, \lfloor\frac{d}{2}\rfloor.$$

Let i_0 be the smallest integer in $\{1, \ldots, \lfloor\frac{d}{2}\rfloor\}$ such that

$$w_{i_0,-} \geq c_{\lfloor\frac{d}{2}\rfloor - i_0}.$$

Then:

1. *The maximal negative volume $V_Q(\mathcal{B})$ of some box $\mathcal{B}$ over all d-quasi-copulas Q is equal to $-w_{i_0,-}$.*
2. *One of the realizations of $\mathcal{B}$ and Q is*

$$\mathcal{B} = \left[\frac{i_0}{i_0+1}, 1\right]^d, \quad q_{(0,\ldots,0)} = 0, \quad q_{\mathbb{I}} = \sum_{i=1}^{\|\mathbb{I}\|_1} \delta_i \quad \text{for all } \mathbb{I} \in \{0,1\}^d \setminus \{0\}^d,$$

where:

(a) *If d is even, then for $j = 1, \ldots, d$ we have*

$$\delta_j = \begin{cases} \frac{1}{i_0+1}, & \text{if } j = d \text{ or } (j \text{ is odd and } \binom{d-1}{j-1} > c_{\frac{d}{2}-i_0}), \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

(b) *If d is odd, then for $j = 1, \ldots, d$ we have*

$$\delta_j = \begin{cases} \frac{1}{i_0+1}, & \text{if } j = d \text{ or } (j \text{ is even and } \binom{d-1}{j-1} > c_{\lfloor\frac{d}{2}\rfloor-i_0}), \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

This realization of Q on $\left\{\frac{i_0}{i_0+1}, 1\right\}^d$ indeed extends to a quasi-copula $Q : [0,1]^d \to [0,1]$ by [6, Theorem 2.1].

Let us demonstrate the statement of Theorem 1 in the smallest dimension $d = 7$.

Example 1. Assume the notation from Theorem 1. Let $d = 7$. Define

$$(c_1, c_2, c_3) := \left(\binom{6}{1}, \binom{6}{1}, \binom{6}{3}\right) = (6, 6, 20)$$

and $c_0 = 0$. Let

$$\begin{aligned} w_{1,-} &:= \frac{1}{2}(c_3 - 1) = \frac{19}{2}, \\ w_{2,-} &:= \frac{1}{3}(c_3 + c_2 - 1) = \frac{25}{3}, \\ w_{3,-} &:= \frac{1}{4}(c_3 + c_2 + c_1 - 1) = \frac{31}{4}. \end{aligned}$$

Note that the smallest $i \in \{1, 2, 3\}$ such that $w_{i,-} \geq c_{3-i}$, is $i_0 = 1$, i.e., $w_{1,-} \geq c_2$. By Theorem 1, the maximal negative volume of some box $\mathcal{B}$ over all 7-quasi-copulas Q is equal to

$$-w_{1,-} = -\frac{19}{2}.$$

Further on,

$$\delta_j = \begin{cases} \frac{1}{2}, & \text{if } j \in \{4, 7\}, \\ 0, & \text{if } j \in \{1, 2, 3, 5, 6\}. \end{cases}$$

Hence, one of the realizations of $\mathcal{B}$ and Q is

$$\mathcal{B} = \left[\frac{1}{2}, 1\right]^7, \quad q_{(0,\ldots,0)} = 0, \quad q_{\mathbb{I}} = \begin{cases} 0, & \text{if } \|\mathbb{I}\|_1 \in \{0, 1, 2, 3\}, \\ \frac{1}{2}, & \text{if } \|\mathbb{I}\|_1 \in \{4, 5, 6\}, \\ 1, & \text{if } \mathbb{I} = (1, \ldots, 1). \end{cases}$$

Note that this realization agrees with the one obtained by computer software approach in [6, Table 1].

The complete solution to [2, Open Problem 5] for the maximal positive volume is the following.

Theorem 2. *Assume the notation above. Let $d \in \mathbb{N}$ and $d \geq 2$. Define*

$$(c_1, c_2, \ldots, c_{\lfloor \frac{d+1}{2} \rfloor - 1}) :=$$
$$= \begin{cases} \left(\binom{d-1}{1}, \binom{d-1}{2}, \binom{d-1}{3}, \ldots, \binom{d-1}{\lfloor \frac{d+1}{2} \rfloor - 1}\right), & \text{if } d \text{ is even}, \\ \left(\binom{d-1}{0}\binom{d-1}{2}, \binom{d-1}{2}, \binom{d-1}{4}, \binom{d-1}{4}, \ldots, \binom{d-1}{\lfloor \frac{d}{2} \rfloor - 2}, \binom{d-1}{\lfloor \frac{d}{2} \rfloor - 2}, \binom{d-1}{\lfloor \frac{d}{2} \rfloor - 1}\right), & \text{if } d \text{ is odd}. \end{cases}$$

Define $c_0 = 0$ and

$$w_{i,+} := \frac{1}{i+1}\Big(\sum_{j=0}^{i-1} c_{\lfloor \frac{d+1}{2} \rfloor - 1 - j} + 1\Big) \quad \text{for } i = 1, \ldots, \lfloor \frac{d+1}{2} \rfloor - 1.$$

Let i_0 be the smallest integer in $\{1, \ldots, \lfloor \frac{d+1}{2} \rfloor - 1\}$ such that

$$w_{i_0,+} \geq c_{\lfloor \frac{d+1}{2} \rfloor - 1 - i_0}.$$

Then:

(1) The maximal positive volume $V_Q(\mathcal{B})$ of some box $\mathcal{B}$ over all d-quasi-copulas Q is equal to $w_{i_0,+}$.

(2) One of the realizations of $\mathcal{B}$ and Q is

$$\mathcal{B} = \left[\frac{i_0}{i_0+1}, 1\right]^d, \quad q_{(0,\ldots,0)} = 0, \quad q_{\mathbb{I}} = \sum_{i=1}^{\|\mathbb{I}\|_1} \delta_i \quad \textit{for all } \mathbb{I} \in \{0,1\}^d \setminus \{0\}^d,$$

where:

(a) If d is even, then for $j = 1, \ldots, d$ we have

$$\delta_j = \begin{cases} \frac{1}{i_0+1}, & \textit{if } j = d \textit{ or } (j \textit{ is even and } \binom{d-1}{j-1} > c_{\frac{d}{2}-1-i_0}), \\ 0, & \textit{otherwise.} \end{cases} \tag{2.4}$$

(b) If d is odd, then for $j = 1, \ldots, d$ we have

$$\delta_j = \begin{cases} \frac{1}{i_0+1}, & \textit{if } j = d \textit{ or } (j \textit{ is odd and } \binom{d-1}{j-1} > c_{\lfloor\frac{d+1}{2}\rfloor-1-i_0}), \\ 0, & \textit{otherwise.} \end{cases} \tag{2.5}$$

This realization of Q on $\left\{\frac{i_0}{i_0+1}, 1\right\}^d$ indeed extends to a quasi-copula $Q : [0,1]^d \to [0,1]$ by [6, Theorem 2.1].

Let us demonstrate the statement of Theorem 1 in dimension $d = 8$.

Example 2. Assume the notation from Theorem 2. Let $d = 8$. Define

$$(c_1, c_2, c_3) := \left(\binom{7}{1}, \binom{7}{2}, \binom{7}{3}\right) = (7, 21, 35)$$

and $c_0 = 0$. Let

$$\begin{aligned} w_{1,+} &:= \frac{1}{2}(c_3 + 1) = 18, \\ w_{2,+} &:= \frac{1}{3}(c_3 + c_2 + 1) = 19, \\ w_{3,+} &:= \frac{1}{4}(c_3 + c_2 + c_1 + 1) = 16. \end{aligned}$$

Note that the smallest $i \in \{1,2,3\}$ such that $w_{i,+} \geq c_{3-i}$ is $i_0 = 2$, i.e., $w_{2,+} \geq c_1$. By Theorem 2, the maximal positive volume of some box $\mathcal{B}$ over all 8-quasi-copulas Q is equal to

$$w_{2,+} = 19.$$

Further on,

$$\delta_j = \begin{cases} \frac{1}{3}, & \text{if } j \in \{4,6,8\}, \\ 0, & \text{if } j \in \{1,2,3,5,7\}. \end{cases}$$

Hence, one of the realizations of $\mathcal{B}$ and Q is

$$\mathcal{B} = \left[\frac{2}{3}, 1\right]^8, \quad q_{(0,\ldots,0)} = 0, \quad q_{\mathbb{I}} = \begin{cases} 0, & \text{if } \|\mathbb{I}\|_1 \in \{0,1,2,3\}, \\ \frac{1}{3}, & \text{if } \|\mathbb{I}\|_1 \in \{4,5\}, \\ \frac{2}{3}, & \text{if } \|\mathbb{I}\|_1 \in \{6,7\}, \\ 1, & \text{if } \mathbb{I} = (1,\ldots,1). \end{cases}$$

Note that this realization agrees with the one obtained by computer software approach in [6, Table 2].

3. Towards the proof of Theorems 1 and 2

In this section we first recall the formulation of the maximal volume problems from [6] in terms of the linear programs (see Proposition 1). Then we simplify the linear programs by using symmetries in the variables (see Proposition 2 and Corollary 1). Next, the simplified linear programs are further reduced to smaller ones observing redundancy of some conditions (see Proposition 3). Finally, the dual linear programs, which need to be solved, are obtained (see Proposition 4). We use them to prove that the proposed solution is indeed optimal.

3.1. **Notation and preliminaries.** Fix $d \in \mathbb{N}$. For multi-indices

$$\mathbb{I} = (\mathbb{I}_1, \ldots, \mathbb{I}_d) \in \{0,1\}^d \quad \text{and} \quad \mathbb{J} = (\mathbb{J}_1, \ldots, \mathbb{J}_d) \in \{0,1\}^d$$

let

$$\mathbb{J} - \mathbb{I} = (\mathbb{J}_1 - \mathbb{I}_1, \ldots, \mathbb{J}_d - \mathbb{I}_d) \in \{-1,0,1\}^d$$

stand for their usual coordinate-wise difference. Let $\mathbb{E}^{(\ell)}$ stand for the multi-index with the only non-zero coordinate the ℓ–th one, which is equal to 1. For each $\ell = 1, \ldots, \ell$ we define a relation on $\{0,1\}^d$ by

$$\mathbb{I} \prec_\ell \mathbb{J} \quad \Leftrightarrow \quad \mathbb{J} - \mathbb{I} = \mathbb{E}^{(\ell)}.$$

For a point $\underline{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we define the functions

$$G_d : \mathbb{R}^d \to \mathbb{R}, \quad G_d(\underline{x}) := \sum_{i=1}^d x_i - d + 1,$$
$$H_d : \mathbb{R}^d \to \mathbb{R}, \quad H_d(\underline{x}) := \min\{x_1, x_2, \ldots, x_d\}.$$

In our previous work we proved the following proposition.

Proposition 1 ([6, Propositions 3.1 and 3.2]). *Define the following linear program*

$$\begin{aligned}
&\min_{\substack{a_1,\ldots,a_d,\\ b_1,\ldots,b_d,\\ q_\mathbb{I} \text{ for } \mathbb{I}\in\{0,1\}^d}} \quad \sum_{\mathbb{I}\in[1]^d} \operatorname{sign}(\mathbb{I}) q_\mathbb{I}, \\
&\quad \textit{subject to} \quad 0 \le a_i < b_i \le 1 \quad i = 1, \ldots, d, \\
&\qquad\qquad\qquad 0 \le q_\mathbb{J} - q_\mathbb{I} \le b_\ell - a_\ell \quad \textit{for all } \ell = 1, \ldots, d \textit{ and all } \mathbb{I} \prec_\ell \mathbb{J}, \\
&\qquad\qquad\qquad \max\{0, G_d(x_\mathbb{I})\} \le q_\mathbb{I} \le H_d(x_\mathbb{I}) \quad \textit{for all } \mathbb{I} \in \{0,1\}^d.
\end{aligned} \tag{3.1}$$

Let $\mathbb{I}^{(1)}, \ldots, \mathbb{I}^{(2^d)}$ be some order of all multi-indices $\mathbb{I} \in \{0,1\}^d$. If there exists an optimal solution $(a_1^*, \ldots, a_d^*, b_1^*, \ldots, b_d^*, q_{\mathbb{I}^{(1)}}, \ldots, q_{\mathbb{I}^{(2^d)}})$ to (3.1)*, which satisfies $b_1^* = \ldots = b_d^* = 1$, then the optimal value of* (3.1) *is the maximal negative volume of some box over all d-quasi-copulas.*

Moreover, if there exists an optimal solution to (3.1) *where* min *is replaced with* max*, which satisfies $b_1^* = \ldots = b_d^* = 1$, then the optimal value of* (3.1) *is the maximal positive volume of some box over all d-quasi-copulas.*

Remark 3.1. In [6, Section 3] we presented numerical solutions to (3.1) up to $d = 18$ for min and up to $d = 17$ for max. In the case of min and $7 \le d \le 18$ we obtained optimal solutions with $b_1^* = \ldots = b_d^* = 1$ and so we solved the maximal negative volume problem. Cases $d < 7$ have to be considered separately by extending Proposition 1 from the 2^d-element grid $\prod_{i=1}^d \{a_i, b_i\}$ to the 3^d-element grid $\prod_{i=1}^d \{a_i, b_i, 1\}$. In the case of max and $2 \le d \le 17$ we obtained optimal solutions with $b_1^* = \ldots = b_d^* = 1$ and so we solved the maximal positive volume problem. ■

3.2. **Symmetrization of the linear program** (3.1) **and the dual.** By the following proposition it suffices to consider symmetric solutions to the linear program (3.1).

Proposition 2. *Assume the notation from Proposition 1. There exists an optimal solution to the linear program* (3.1) *of the form*

$$\big(\underbrace{a^*,\dots,a^*}_{d},\underbrace{b^*,\dots,b^*}_{d},q^*_{\|\mathbb{I}^{(1)}\|_1},\dots,q^*_{\|\mathbb{I}^{(2^d)}\|_1}\big)$$

for some $a,b,q_1,\dots,q_d\in[0,1]$.

Analogously, replacing min *with* max *in* (3.1) *above, the same statement holds.*

Proof. Let S_d be the set of all permutations of a d-element set $\{1,\dots,d\}$. For $\Phi\in S_d$ and $\mathbb{I}^{(j)}:=(\mathbb{I}^{(j)}_1,\dots,\mathbb{I}^{(j)}_d)\in\{0,1\}^d$, let $\Phi(\mathbb{I}^{(j)}):=\big(\mathbb{I}^{(j)}_{\Phi(1)},\dots,\mathbb{I}^{(j)}_{\Phi(d)}\big)$. If

$$\big(a^*_1,\dots,a^*_d,b^*_1,\dots,b^*_d,q^*_{\mathbb{I}^{(1)}},\dots,q^*_{\mathbb{I}^{(2^d)}}\big)$$

is an optimal solution to (3.1), then

$$\big(a^*_{\Phi(1)},\dots,a^*_{\Phi(d)},b^*_{\Phi(1)},\dots,b^*_{\Phi(d)},q^*_{\Phi(\mathbb{I}^{(1)})},\dots,q^*_{\Phi(\mathbb{I}^{(2^d)})}\big)$$

is also an optimal solution to (3.1). Hence, an optimal solution as stated in the proposition is equal to

$$\frac{1}{n!}\sum_{\Phi\in S_n}\big(a^*_{\Phi(1)},\dots,a^*_{\Phi(d)},b^*_{\Phi(1)},\dots,b^*_{\Phi(d)},q^*_{\Phi(\mathbb{I}^{(1)})},\dots,q^*_{\Phi(\mathbb{I}^{(2^d)})}\big),$$

where the summation is the coordinate-wise one. The proof for max instead of min is the same. □

An immediate corollary to Proposition 2 is the following.

Corollary 1. *The optimal value of the linear program* (3.1) *is equal to the optimal value of the linear program*

$$
\begin{aligned}
&\min_{a,b,q_0,q_1,\dots,q_d} && \sum_{i=0}^{d}(-1)^{d-i}\binom{d}{i}q_i,\\
&\textit{subject to} && 0\le a<b\le 1,\\
&&& 0\le q_i-q_{i-1}\le b-a \quad \textit{for } i=1,\dots,d,\\
&&& \max\{0,(d-i)a+ib-d+1\}\le q_i\le a\\
&&& \qquad\qquad \textit{for } i=0,\dots,d-1,\\
&&& \max\{0,db-d+1\}\le q_d\le b.
\end{aligned}
\tag{3.2}
$$

Analogously, replacing min *with* max *in* (3.1) *above, the same statement holds.*

It turns out that some of the constraints in (3.2) are redundant, while introducing new variables $\delta_i=q_i-q_{i-1}$ further decreases the number of constraints.

Proposition 3. *The optimal value of the linear program* (3.2) *is equal to the optimal value of the linear program*

$$
\begin{aligned}
&\min_{a,b,q_0,\delta_1,\ldots,\delta_d} && \sum_{j=1}^{d}(-1)^{d+j}\binom{d-1}{j-1}\delta_j,\\
&\textit{subject to} && b\le 1,\\
& && \delta_i \le b-a \quad \textit{for } i=1,\ldots,d,\\
& && q_0+\sum_{i=1}^{d-1}\delta_i \le a,\\
& && db-d+1\le q_0+\sum_{i=1}^{d}\delta_i,\\
& && a\ge 0,\ b\ge 0,\ q_0\ge 0,\ \delta_i\ge 0 \quad \textit{for } i=1,\ldots,d.
\end{aligned}
\tag{3.3}
$$

Analogously, replacing min *with* max *in* (3.3) *above, the same statement holds.*

Proof. First let us prove that in (3.2) the conditions

$$
\begin{aligned}
&(d-i)a+ib-d+1\le q_i \quad \text{for } i=0,\ldots,d-1,\\
&q_i\le a \quad \text{for } i=0,\ldots,d-2,\\
&q_d\le b
\end{aligned}
\tag{3.4}
$$

follow from the other conditions. We have that

$$q_{d-1}=q_d+(q_{d-1}-q_d)\ge db-d+1+a-b=a+(d-1)b-d+1,$$

where we used the second and the fourth conditions from (3.2) for the inequality. This is the condition in the first line of (3.4) for $i=d-1$. Inductively, using

$$q_{d-j}=q_{d-j+1}+(q_{d-j}-q_{d+j+1}),$$

the conditions in the first line of (3.4) for $i=d-2,\ldots,1$ follow.

The conditions in the second line of (3.4) follow easily from

$$a\ge q_{d-1}\ge q_{d-2}\ge\ldots\ge q_0.$$

Finally, the condition in the last line of (3.4) follows from

$$q_d=q_{d-1}+(q_d-q_{d-1})\le a+(b-a)=b,$$

where we used the second and the third conditions from (3.2) for the inequality.

Now we introduce new variables $\delta_i=q_i-q_{i-1}$, $i=1,\ldots,d$ and the constraints of the reduced linear program (3.2), i.e., (3.2) without the constraints from (3.4), become as in (3.3). The only non-trivial thing to verify is the form of the objective function. But this follows by a simple computation:

$$
\begin{aligned}
\sum_{i=0}^{d}(-1)^{d-i}\binom{d}{i}q_i &= \sum_{i=0}^{d}(-1)^{d-i}\binom{d}{i}\Big(q_0+\sum_{j=1}^{i}\delta_j\Big)\\
&= q_0\underbrace{\sum_{k=0}^{d}(-1)^{d-i}\binom{d}{i}}_{0}+\sum_{j=1}^{d}\delta_j\underbrace{\Big(\sum_{k=j}^{d}(-1)^{d-k}\binom{d}{k}\Big)}_{(-1)^{d+j}\binom{d-1}{j-1}},
\end{aligned}
$$

where we used $\sum_{k\le n}(-1)^k\binom{r}{k}=(-1)^n\binom{r-1}{n}$ in the last equality. □

3.3. **The dual linear program of** (3.3)**.** Recall that writing the linear program in the form $\max_x\{c^T x\colon Ax \le b, x \ge 0\}$ where $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are real vectors, $A \in \mathbb{R}^{m\times n}$ is a matrix, x is a vector of variables and the inequalities are the coordinate-wise ones, its dual program is given by $\min_y\{b^T y\colon A^T y \ge c, y \ge 0\}$, where y is a vector of dual variables. Namely, for each constraint of the primal program, there is a variable for the dual program, while for each variable of the primal program, there is a constraint for the dual program.

Rewriting the objective function $\min_x c^T x$ of (3.3) in the form $-(\max_x(-c)^T x)$ the dual linear program of (3.3) is the following:

$$\begin{aligned}
-\Big(\min_{y_1,l_1,\ldots,l_d,y_2,y_3} \quad & y_1 + (d-1)y_3\Big),\\
\text{subject to} \quad & \sum_{i=1}^{d} l_i - y_2 \ge 0,\\
& y_1 - \sum_{i=1}^{d} l_i + dy_3 \ge 0,\\
& y_2 - y_3 \ge 0,\\
& l_j + y_2 - y_3 \ge (-1)^{d+j+1}\binom{d-1}{j-1} \quad \text{for } j = 1,\ldots,d-1,\\
& l_d - y_3 \ge -1,\\
& y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0,\ l_i \ge 0 \quad \text{for } i = 1,\ldots,d.
\end{aligned} \tag{3.5}$$

Analogously, the dual linear program of (3.3) with the objective function $\max_x c^T x$ is the following:

$$\begin{aligned}
\min_{y_1,l_1,\ldots,l_d,y_2,y_3} \quad & y_1 + (d-1)y_3,\\
\text{subject to} \quad & \sum_{i=1}^{d} l_i - y_2 \ge 0,\\
& y_1 - \sum_{i=1}^{d} l_i + dy_3 \ge 0,\\
& y_2 - y_3 \ge 0,\\
& l_j + y_2 - y_3 \ge (-1)^{d+j}\binom{d-1}{j-1} \quad \text{for } j = 1,\ldots,d-1,\\
& l_d - y_3 \ge 1,\\
& y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0,\ l_i \ge 0 \quad \text{for } i = 1,\ldots,d.
\end{aligned} \tag{3.6}$$

It turns out that both linear programs can be simplified.

Proposition 4. *The optimal solutions to* (3.5) *and* (3.6) *are equal to the optimal solutions to*

$$
\begin{aligned}
&-\Big(\min_{l_1,\ldots,l_d,y_3} && (d-1)y_3\Big),\\
&\textit{subject to} && -\sum_{i=1}^{d} l_i + dy_3 = 0,\\
&&& l_j + (d-1)y_3 \geq (-1)^{d+j+1}\binom{d-1}{j-1} \quad \textit{for } j = 1,\ldots,d-1,\\
&&& l_d = \max(0, -1+y_3),\\
&&& y_3 \geq 0,\ l_i \geq 0 \quad \textit{for } i = 1,\ldots,d,
\end{aligned}
\tag{3.7}
$$

and

$$
\begin{aligned}
&\min_{l_1,\ldots,l_d,y_3} && (d-1)y_3,\\
&\textit{subject to} && -\sum_{i=1}^{d} l_i + dy_3 = 0,\\
&&& l_j + (d-1)y_3 \geq (-1)^{d+j}\binom{d-1}{j-1} \quad \textit{for } j = 1,\ldots,d-1,\\
&&& l_d - y_3 = 1,\\
&&& y_3 \geq 0,\ l_i \geq 0 \quad \textit{for } i = 1,\ldots,d,
\end{aligned}
\tag{3.8}
$$

respectively.

Proof. We will prove the proposition by first establishing a few claims.

Claim 1. Their exists an optimal solution $(y_1^*, l_1^*, \ldots, l_d^*, y_2^*, y_3^*)$ to the linear program (3.5) (resp. (3.6)) which satisfies $y_2^* = y_1^* + dy_3^*$.

Proof of Claim 1. Let $(y_1^*, l_1^*, \ldots, l_d^*, y_2^*, y_3^*)$ be an optimal solution to (3.5). From the first two constraints in (3.5) it follows that

$$y_1^* + dy_3^* \geq \sum_{i=1}^{d} l_i \geq y_2^*. \tag{3.9}$$

If the left inequality in (3.9) is strict and $y_1^* > 0$, then there exists $\epsilon > 0$ such that $(y_1^* - \epsilon, l_1^*, \ldots, l_d^*, y_2^*, y_3^*)$ is a feasible solution to (3.5) such that

$$(y_1^* - \epsilon) + (d-1)y_3^* < y_1^* + (d-1)y_3^*.$$

But this contradicts to the optimality of $(y_1^*, l_1^*, \ldots, l_d^*, y_2^*, y_3^*)$.

If the left inequality in (3.9) is strict, $y_1^* = 0$ and $y_3^* > 0$, then there exists $\epsilon > 0$ such that $(y_1^*, l_1^*, \ldots, l_d^*, y_2^* - \epsilon, y_3^* - \epsilon)$ is a feasible solution to (3.5) such that

$$y_1^* + (d-1)(y_3^* - \epsilon) < y_1^* + (d-1)y_3^*.$$

But this contradicts to the optimality of $(y_1^*, l_1^*, \ldots, l_d^*, y_2^*, y_3^*)$.

Hence, the left inequality in (3.9) is an equality. If the right inequality is strict, we just increase y_2^* to $\sum_{i=1}^{d} l_i$ and we get another optimal solution to (3.5). This proves the lemma for (3.5).

The proof for (3.6) is the same. ■

Using Claim 1, the linear programs (3.5) and (3.6) simplify to:

$$
\begin{aligned}
&-\Big(\min_{y_1,l_1,\dots,l_d,y_3} && y_1+(d-1)y_3\Big),\\
&\text{subject to} && y_1-\sum_{i=1}^{d} l_i + dy_3 = 0,\\
& && y_1+(d-1)y_3 \ge 0,\\
& && l_j + y_1 + (d-1)y_3 \ge (-1)^{d+j+1}\binom{d-1}{j-1}\\
& && \qquad\qquad \text{for } j=1,\dots,d-1,\\
& && l_d - y_3 \ge -1,\\
& && y_1\ge 0,\ \ y_3\ge 0,\ l_i \ge 0 \quad \text{for } i=1,\dots,d,
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
&\min_{y_1,l_1,\dots,l_d,y_3} && y_1+(d-1)y_3,\\
&\text{subject to} && y_1-\sum_{i=1}^{d} l_i + dy_3 = 0,\\
& && y_1+(d-1)y_3 \ge 0,\\
& && l_j + y_1 + (d-1)y_3 \ge (-1)^{d+j}\binom{d-1}{j-1}\\
& && \qquad\qquad \text{for } j=1,\dots,d-1,\\
& && l_d - y_3 \ge 1,\\
& && y_1\ge 0,\ \ y_3\ge 0,\ l_i \ge 0 \quad \text{for } i=1,\dots,d.
\end{aligned}
\tag{3.11}
$$

Observation 1. Since $y_1 \ge 0, y_3 \ge 0$, the inequality $y_1 + (d-1)y_3 \ge 0$ in (3.10) and (3.11) is redundant.

Claim 2. Their exists an optimal solution $(y_1^*, l_1^*, \dots, l_d^*, y_3^*)$ to the linear program (3.10) (resp. (3.11)) which satisfies $y_1^* = 0$.

Proof of Claim 2. Let $(y_1^*, l_1^*, \dots, l_d^*, y_2^*, y_3^*)$ be an optimal solution to (3.10). Assume that $y_1^* > 0$. Then $\big(0, l_1^*, \dots, l_{d-1}^*, l_d^* + \frac{y_1^*}{d-1}, y_3^* + \frac{y_1^*}{d-1}\big)$ is another optimal solution to (3.10).

The proof for (3.11) is the same. ■

Using Observation 1 and Claim 2, the linear programs (3.10) and (3.11) simplify to

$$
\begin{aligned}
&-\Big(\min_{l_1,\dots,l_d,y_3} && (d-1)y_3\Big),\\
&\text{subject to} && -\sum_{i=1}^{d} l_i + dy_3 = 0,\\
& && l_j + (d-1)y_3 \ge (-1)^{d+j+1}\binom{d-1}{j-1} \quad \text{for } j=1,\dots,d-1,\\
& && l_d - y_3 \ge -1,\\
& && y_3\ge 0,\ l_i \ge 0 \quad \text{for } i=1,\dots,d,
\end{aligned}
\tag{3.12}
$$

and

$$(3.13)\quad \begin{aligned} \min_{l_1,\ldots,l_d,y_3} \quad & (d-1)y_3, \\ \text{subject to} \quad & -\sum_{i=1}^{d} l_i + dy_3 = 0, \\ & l_j + (d-1)y_3 \geq (-1)^{d+j}\binom{d-1}{j-1} \quad \text{for } j = 1,\ldots,d-1, \\ & l_d - y_3 \geq 1, \\ & y_3 \geq 0,\ l_i \geq 0 \quad \text{for } i = 1,\ldots,d, \end{aligned}$$

respectively.

The following claim shows that in the optimal solution to the linear programs (3.12) and (3.13), one of constraints $l_d \geq 0$ and $l_d - y_3 \geq \pm 1$ is an equality.

Claim 3. If $(y_1^*, l_1^*, \ldots, l_d^*, y_3^*)$ is an optimal solution to the linear program (3.12) (resp. (3.13)), then $l_d^* = \max(0, -1 + y_3^*)$ (resp. $l_d^* = 1 + y_3^*$).

Moreover, $l_d^* = 0$ can only occur for $d \leq 6$.

Proof of Claim 3. Let $(y_1^*, l_1^*, \ldots, l_d^*, y_3^*)$ be an optimal solution to (3.12). Assume that $c := l_d^* - \max(0, -1 + y_3^*) > 0$. Then there exists $\epsilon > 0$ such that

$$(y_1^*, l_1^* + (d-1)\epsilon, \ldots, l_{d-1}^* + (d-1)\epsilon, l_d - (d^2 - d + 1)\epsilon, y_3^* - \epsilon)$$

is still a feasible solution to (3.12) with a smaller objective function, which is a contradiction to the optimality of $(y_1^*, l_1^*, \ldots, l_d^*, y_3^*)$.

The proof for (3.13) is the same, using also that $l_d^* = \max(0, 1 + y_3^*) = 1 + y_3^*$.

Let us establish the moreover part. In this case $y_3^* \leq 1$. Since every y_3 large enough extends to a feasible solution to (3.12) (e.g., $l_1 = \ldots = l_{d-1} = 0$ and $l_d = dy_3$), convexity of the feasible region implies that there is a feasible solution with $y_3 = 1$. But then for this solution the first constraint in (3.12) implies that

$$(3.14)\qquad \sum_{i=1}^{d} l_i = d.$$

We separate two cases according to the parity of d.

Case 1: $d = 2d'$ for some $d' \in \mathbb{N}$. Summing up the inequalities with positive right hand side in (3.12) we get

$$(3.15)\qquad \sum_{\substack{j \text{ odd}, \\ j<d}} l_j + (d-1)\Big(\frac{d}{2} - 1\Big) \geq \sum_{\substack{j \text{ odd}, \\ j<d}} \binom{d-1}{j-1}.$$

Using (3.14) in (3.15), it follows that

$$d + (d-1)\Big(\frac{d}{2} - 1\Big) \geq \sum_{\substack{j \text{ odd}, \\ j<d}} \binom{d-1}{j-1} = \binom{d-1}{0} + \binom{d-1}{2} + \ldots + \binom{d-1}{d-2}.$$

For $d \geq 8$ this is a contradiction.

Case 2: $d = 2d' - 1$ for some $d' \in \mathbb{N}$. Summing up the inequalities with positive right hand side in (3.12) above we get

$$\sum_{\substack{j \text{ even}, \\ j<d}} l_j + \frac{(d-1)^2}{2} \geq \sum_{\substack{j \text{ even}, \\ j<d}} \binom{d-1}{j-1}. \tag{3.16}$$

Using (3.14) in (3.16), it follows that

$$d + \frac{(d-1)^2}{2} \geq \sum_{\substack{j \text{ even}, \\ j<d}} \binom{d-1}{j-1} = \binom{d-1}{1} + \binom{d-1}{3} + \ldots + \binom{d-1}{d-2}.$$

For $d \geq 7$ this is a contradiction.

This proves Claim 3. ■

Using Claim 3, the linear programs (3.12) and (3.13) simplify to the linear programs (3.7) and (3.8), respectively. □

4. Basic technical result

The following proposition is the basic result, which will be used to prove Theorems 1 and 2.

Proposition 5. *Let*

$$0 < c_1 \leq c_2 \leq \ldots \leq c_k$$

be given positive real numbers with $c_1 < c_k$*,*

$$e_1 < 0, \ldots, e_r < 0$$

given negative real numbers, $\alpha \in \mathbb{R}$ *such that*

$$-c_k < \alpha \leq c_k$$

and a linear program

$$\begin{aligned}
\min_{\substack{w,y_1,y_2,\ldots,y_k, \\ z_1,z_2,\ldots,z_r}} \quad & w \\
\textit{subject to} \quad & y_1 + \ldots + y_k + z_1 + \ldots + z_r = w + \alpha, \\
& y_i + w \geq c_i \quad \textit{for } i = 1, \ldots, k, \\
& z_i + w \geq e_i \quad \textit{for } i = 1, \ldots, r, \\
& w \geq 0, \\
& y_i \geq 0 \quad \textit{for } i = 1, \ldots, k, \\
& z_i \geq 0 \quad \textit{for } i = 1, \ldots, r.
\end{aligned} \tag{4.1}$$

Define

$$w_i := \frac{1}{i+1} \Big(\sum_{j=0}^{i-1} c_{k-j} - \alpha \Big) \quad \textit{for } i = 1, \ldots, k.$$

and $c_0 := 0$*. Let*

(4.2) i_0 *be the smallest integer in* $\{1, \ldots, k\}$ *such that* $w_{i_0} \geq c_{k-i_0}$*.*

Then the optimal solution $(w^*, y_1^*, \ldots, y_k^*, z_1^*, \ldots, z_r^*)$ *to* (4.1) *is*

$$(w^*, y_1^*, \ldots, y_k^*, z_1^*, \ldots, z_r^*) = (w_{i_0}, \underbrace{0, \ldots, 0}_{k-i_0}, c_{k-i_0+1} - w_{i_0}, \ldots, c_k - w_{i_0}, \underbrace{0, \ldots, 0}_{r}).$$

Proof. First we establish two claims.

Claim 1. There is at most one $i \in \{1, \ldots, k\}$ such that $w_i \in [c_{k-i}, c_{k-i+1})$.

Proof of Claim 1. Let $\widetilde{i}$ be such that $w_{\widetilde{i}} \in [c_{k-\widetilde{i}}, c_{k-\widetilde{i}+1})$. Note that $w_{\widetilde{i}} \geq c_{k-\widetilde{i}}$ is equivalent to

$$\sum_{j=0}^{\widetilde{i}-1} c_{k-j} - \alpha \geq (\widetilde{i}+1)c_{k-\widetilde{i}}. \tag{4.3}$$

Hence,

$$w_{\widetilde{i}+1} = \frac{1}{\widetilde{i}+2}\Big(\sum_{j=0}^{\widetilde{i}} c_{k-j} - \alpha\Big) \underbrace{\geq}_{(4.3)} c_{k-\widetilde{i}},$$

which in particular implies that $w_{\widetilde{i}+1} \notin [c_{k-\widetilde{i}-1}, c_{k-\widetilde{i}})$. Inductively we can prove that $w_{\widetilde{i}+l} \geq c_{k-\widetilde{i}}$ for $l = 1, 2, \ldots, k - \widetilde{i}$, whence $w_{\widetilde{i}+l} \notin [c_{k-\widetilde{i}-l}, c_{k-\widetilde{i}-l+1})$. This proves Claim 1. ■

Claim 2. Let i_0 be as in (4.2). Then $w_{i_0} \in [c_{k-i_0}, c_{k-i_0+1})$.

Proof of Claim 2. First note that i_0 is well-defined, since there exists $i \in \{1, \ldots, k\}$ such that $w_i \geq c_{k-i}$, e.g., $i = k$ due to $w_k = \frac{1}{k}\big(\sum_{j=1}^{k} c_k - \alpha\big) \geq 0 = c_0$. Also note that $i = i_0$ is the unique candidate for the containment $w_i \in [c_{k-i}, c_{k-i+1})$. This follows from the following two observations:

- By definition of i_0, we have that $w_i < c_{k-i}$ for $i = 1, \ldots, i_0 - 1$.
- As in the proof of Claim 1, $w_i \geq c_{k-i_0} \geq c_{k-i+1}$ for $i = i_0 + 1, \ldots, k$.

So it only remains to prove that $w_{i_0} < c_{k-i_0+1}$. Assume on the contrary that $w_{i_0} \geq c_{k-i_0+1}$. First notice that in this case $i_0 \neq 1$, since $w_1 = \frac{1}{2}(c_k - \alpha) < \frac{1}{2}(c_k + c_k) = c_k$. Then

$$\sum_{j=0}^{i_0-1} c_{k-j} - \alpha \geq (i_0+1)c_{k-i_0+1},$$

which implies that

$$\sum_{j=0}^{i_0-2} c_{k-j} - \alpha \geq i_0 \cdot c_{k-i_0+1}.$$

Further on,

$$w_{i_0-1} = \frac{1}{i_0}\Big(\sum_{j=0}^{i_0-2} c_{k-j} - \alpha\Big) \geq c_{k-i_0+1}.$$

But this is a contradiction with the minimiality of i_0. ■

Let now $(w^*, y_1^*, \ldots, y_k^*, z_1^*, \ldots, z_r^*)$ be an optimal solution to (4.1).

Claim 3. $z_i^* = 0$ for $i = 1, \ldots, r$.

Proof of Claim 3. If there is i such that $z_i > 0$, then there exists $\epsilon > 0$, such that

$$(w^* - \epsilon, y_1^* + \epsilon, \ldots, y_k^* + \epsilon, z_1^*, \ldots, z_{i-1}^*, z_i^* - (k+1)\epsilon, z_{i+1}^*, \ldots, z_r^*) \tag{4.4}$$

is still a feasible solution to (4.1). This contradicts to the optimality of w^*. ■

Clearly $w = c_k$ extends to a feasible solution of (4.1), e.g., $y_1 = c_k + \alpha$, $y_i = 0$ for $i = 2, \ldots, k$ and $z_i = 0$ for $i = 1, \ldots, r$. So $w^* \leq c_k$. We separate two cases for the value of w^*.

Case 1. $w^* = c_k$. Note that there is i such that $y_i > 0$. But then there exists $\epsilon > 0$, such that

$$(w^* - \epsilon, y_1^* + \epsilon, \ldots, y_{i-1}^* + \epsilon, y_i^* - k\epsilon, y_{i+1}^* + \epsilon, \ldots, y_k^* + \epsilon, \underbrace{0, \ldots, 0}_{r}) \tag{4.5}$$

is still a feasible solution of (4.1), which contradicts to the optimality of w^*.

Case 2. $w^* < c_k$. Clearly $y_i^* \geq \max(c_i - w^*, 0)$. If there exists i such that $y_i^* > \max(c_i - w^*, 0)$, then there exists $\epsilon > 0$, such that (4.5) is still a feasible solution to (4.1). This contradicts to the optimality of w^*. So $y_i^* = \max(c_i - w^*, 0)$. Let $\widehat{i}_0$ be such that $w^* \in [c_{k-\widehat{i}_0}, c_{k-\widehat{i}_0+1})$. Then

$$y_1^* = \ldots = y_{k-\widehat{i}_0}^* = 0 \quad \text{and} \quad y_i^* = c_i - w^* \text{ for } i > k - \widehat{i}_0.$$

Further on,

$$\sum_{i=1}^{k} y_i^* = \sum_{i=k-\widehat{i}_0+1}^{k} (c_i - w^*) = w^* + \alpha,$$

whence

$$w^* = \frac{1}{\widehat{i}_0 + 1}\Big(\sum_{i=k-\widehat{i}_0+1}^{k} c_i - \alpha\Big) = \frac{1}{\widehat{i}_0 + 1}\Big(\sum_{j=0}^{\widehat{i}_0 - 1} c_{k-j} - \alpha\Big) = w_{\widehat{i}_0}.$$

Since by Claims 1 and 2, the only i such that $w_i \in [c_{k-i_0}, c_{k-i_0+1})$ is i_0 defined by (4.2), it follows that $\widehat{i}_0 = i_0$ and the optimal solution to (4.1) is as stated in the proposition. □

5. Final step in the proof of Theorem 1

Assume the notation as in the statement of Theorem 1 and Section 3.2. In this section we will solve the linear programs (3.7) and (3.2), which by Proposition 1 also solves the maximal negative volume quasi-copula problem and proves Theorem 1.

Let $(l_1^*, \ldots, l_d^*, y_3^*)$ be an optimal solution to (3.7). By the moreover part of Claim 3 in the proof of Proposition 4, under the assumption $d \geq 7$, we have that $l_d^* = -1 + y_3^*$. Writing $w := (d-1)y_3$, (3.7) becomes

$$\begin{aligned} & -\Big(\min_{l_1,\ldots,l_{d-1},w} \quad w\Big), \\ & \text{subject to} \quad -\sum_{i=1}^{d-1} l_i + w + 1 = 0, \\ & \qquad l_j + w \geq (-1)^{d+j+1}\binom{d-1}{j-1} \quad \text{for } j = 1, \ldots, d-1, \\ & \qquad w \geq 0,\ l_i \geq 0 \quad \text{for } i = 1, \ldots, d-1. \end{aligned} \tag{5.1}$$

The solution to (5.1) will lean on the use of Proposition 5. Let c_i, $i = 1, \ldots, \lfloor \frac{d}{2} \rfloor$ be as in Theorem 1. Let $r := d - 1 - \lfloor \frac{d}{2} \rfloor$ and let $e_1, \ldots, e_r$ be the right hand sides $-\binom{d-1}{j-1}$ in (5.1) in some order. By Proposition 5 with $\alpha = 1$, the optimal solution to (5.1) is $w^* = w_{i_0,-}$ and:

(1) If d is even, then

$$l_j^* = \begin{cases} \binom{d-1}{j-1} - w_{i_0,-}, & j \text{ is odd and } \binom{d-1}{j-1} > c_{\frac{d}{2}-i_0}, \\ 0, & \text{otherwise}. \end{cases} \tag{5.2}$$

(2) If d is odd, then

$$l_j^* = \begin{cases} \binom{d-1}{j-1} - w_{i_0,-}, & j \text{ is even and } \binom{d-1}{j-1} > c_{\lfloor\frac{d}{2}\rfloor-i_0}, \\ 0, & \text{otherwise}. \end{cases} \tag{5.3}$$

Hence, in an optimal solution to (3.5) except (5.2) or (5.3), also the following hold: $y_1^* = 0$, $l_d^* = -1 + \frac{1}{d-1}w_{i_0,-}$, $y_2^* = \frac{d}{d-1}w_{i_0,-}$, $y_3^* = \frac{1}{d-1}w_{i_0,-}$. Using the complementary slackness, in an optimal solution to the dual (3.3) of (3.5), we have $a^* = \frac{i_0}{i_0+1}$, $b^* = 1$, $q_0^* = 0$ and δ_j^* are as in (2.2) if d is even, while δ_j^* are as in (2.3) if d is odd. This give the desired solution to (3.2) and proves Theorem 1.

6. Final step in the proof of Theorem 2

Assume the notation as in the statement of Theorem 2 and Section 3.2. In this section we will solve the linear program (3.8) and (3.2), where min is replaced by max, This solves the maximal positive volume quasi-copula problem and proves Theorem 2.

Let $(l_1^*, \ldots, l_d^*, y_3^*)$ be an optimal solution to (3.8). By the moreover part of Claim 3 in the proof of Proposition 4, we have that $l_d^* = 1 + y_3^*$. Writing $w := (d-1)y_3$, (3.13) becomes

$$\begin{aligned} &\min_{l_1,\ldots,l_{d-1},w} && w, \\ &\text{subject to} && -\sum_{i=1}^{d-1} l_i + w - 1 = 0, \\ & && l_j + w \geq (-1)^{d+j}\binom{d-1}{j-1} \quad \text{for } j = 1, \ldots, d-1, \\ & && w \geq 0,\ l_i \geq 0 \quad \text{for } i = 1, \ldots, d-1. \end{aligned} \tag{6.1}$$

The solution to (6.1) will lean on the use of Proposition 5. Let c_i, $i = 1, \ldots, \lfloor\frac{d+1}{2}\rfloor - 1$ be as in Theorem 2. Let $r := d - \lfloor\frac{d+1}{2}\rfloor$ and let $e_1, \ldots, e_r$ be the right hand sides $-\binom{d-1}{j-1}$ in (6.1) in some order. By Proposition 5 with $\alpha = -1$, the optimal solution to (6.1) is $w^* = w_{i_0,+}$ and:

(1) If d is even, then

$$l_j^* = \begin{cases} \binom{d-1}{j-1} - w_{i_0,-}, & j \text{ is even and } \binom{d-1}{j-1} > c_{\frac{d}{2}-1-i_0}, \\ 0, & \text{otherwise}. \end{cases} \tag{6.2}$$

(2) If d is odd, then

$$l_j^* = \begin{cases} \binom{d-1}{j-1} - w_{i_0,-}, & j \text{ is odd and } \binom{d-1}{j-1} > c_{\lfloor\frac{d+1}{2}\rfloor-1-i_0}, \\ 0, & \text{otherwise}. \end{cases} \tag{6.3}$$

Hence, in an optimal solution to (3.6) except (6.2) or (6.3), also the following hold: $y_1^* = 0$, $l_d^* = 1 + \frac{1}{d-1}w_{i_0,+}$, $y_2^* = \frac{d}{d-1}w_{i_0,+}$, $y_3^* = \frac{1}{d-1}w_{i_0,+}$. Using the complementary slackness, in an optimal solution to the dual (3.3), with max instead of min, of (3.6), we have $a^* = \frac{i_0}{i_0+1}$, $b^* = 1$, $q_0^* = 0$ and δ_j^* are as in (2.4) if d is even, while δ_j^* are as in (2.5) if d is odd. This give the desired solution to (3.2) with max instead of min, and proves Theorem 2.

7. MAXIMAL NEGATIVE VOLUME PROBLEM IN DIMENSIONS $d \le 6$ ANALYTICALLY

The solutions to the maximal negative volume quasi-copula problem in dimensions $2 \le d \le 6$ are already known, i.e., see [17] for $d = 2$, [5] for $d = 3$, [25] for $d = 4$ and [6] for $d = 5, 6$. However, the solutions for $d \ge 3$ are numerical, based on the solution to the corresponding linear programs using computer software. In this section we will present analytical solutions.

Assume the notation as in Section 3.2. We will solve the linear program (3.7) and consequently (3.2) analytically for $3 \le d \le 6$.

Let $(l_1^*, \dots, l_d^*, y_3^*)$ be an optimal solution to (3.7). By the moreover part of Claim 3 in the proof of Proposition 4, we have $l_d^* = 0$ for $d \le 6$.

7.1. **Case $d = 3$.** The linear program (3.7) for $d = 3$ is

$$
\begin{aligned}
-\Big(\min_{l_1,l_2,y_3} \quad & 2y_3\Big),\\
\text{subject to} \quad & 3y_3 = l_1 + l_2,\\
& l_1 + 2y_3 \ge -\binom{2}{0} = -1\\
& l_2 + 2y_3 \ge \binom{2}{1} = 2\\
& 1 \ge y_3 \ge 0,\ l_1 \ge 0,\ l_2 \ge 0.
\end{aligned}
$$

Repeating the arguments from the proof of Proposition 5, in the optimal solution (l_1^*, l_2^*, y_3^*) we have that $l_1^* = 0$ and $l_2^* + 2y_3^* = 2$. Using this in $3y_3^* = l_1^* + l_2^*$ we get $y_3^* = \frac{2}{5}$. So the minimal volume box of some 3-quasi-copula has value $-\frac{4}{5}$. Hence, an optimal solution $(y_1^*, l_1^*, l_2^*, l_3^*, y_2^*, y_3^*)$ to (3.5) for $d = 3$ is $\big(0, 0, \frac{2}{5}, 0, \frac{6}{5}, \frac{2}{5}\big)$. Using the complementary slackness, an optimal solution $(a^*, b^*, q_0^*, \delta_1^*, \delta_2^*, \delta_3^*)$ to its dual (3.3) is equal to $\big(\frac{2}{5}, \frac{4}{5}, 0, 0, \frac{2}{5}, 0\big)$. Finally, an optimal solution to (3.2) is

$$
(a^*, b^*, q_0^*, q_1^*, q_2^*, q_3^*) = \Big(\frac{2}{5}, \frac{4}{5}, 0, 0, \frac{2}{5}, \frac{2}{5}\Big).
$$

7.2. **Case $d = 4$.** The linear program (3.7) for $d = 4$ is

$$
\begin{aligned}
-\Big(\min_{l_1,l_2,l_3,y_3} \quad & 3y_3\Big),\\
\text{subject to} \quad & 4y_3 = l_1 + l_2 + l_3,\\
& l_1 + 3y_3 \ge \binom{3}{0} = 1\\
& l_2 + 3y_3 \ge -\binom{3}{1} = -3\\
& l_3 + 3y_3 \ge \binom{3}{2} = 3\\
& 1 \ge y_3 \ge 0,\ l_1 \ge 0,\ l_2 \ge 0,\ l_3 \ge 0.
\end{aligned}
$$

Repeating the arguments from the proof of Proposition 5, in the optimal solution (l_1^*, l_2^*, y_3^*) we have that $l_2^* = 0$ and one of the cases:

- $l_1^* + 3y_3^* = 1$ and $l_3^* + 3y_3^* = 3$, or
- $l_1^* = 0$ and $l_3^* + 3y_3^* = 3$,

Using these in $4y_3^* = l_1^* + l_2^*$ we get $y_3^* = \frac{2}{5}$ in the first case and $y_3^* = \frac{3}{7}$ in the second case. In the first case $l_1^* = -\frac{1}{5} < 0$, which is not in the feasible region. So the second case applies and the minimal volume box of some 4-quasi-copula has value $-\frac{9}{7}$. Hence, an optimal solution $(y_1^*, l_1^*, l_2^*, l_3^*, y_2^*, y_3^*)$ to (3.5) for $d = 4$ is $\big(0, 0, 0, \frac{12}{7}, \frac{12}{7}, \frac{3}{7}\big)$. Using the complementary slackness, an optimal solution $(a^*, b^*, q_0^*, \delta_1^*, \delta_2^*, \delta_3^*, \delta_4^*)$ to its dual (3.3) is equal to $\big(\frac{3}{7}, \frac{6}{7}, 0, 0, 0, \frac{3}{7}, 0\big)$. Finally, an optimal solution to (3.2) is

$$(a^*, b^*, q_0^*, q_1^*, q_2^*, q_3^*, q_4^*) = \Big(\frac{3}{7}, \frac{6}{7}, 0, 0, 0, \frac{3}{7}, \frac{3}{7}\Big).$$

7.3. **Case $d = 5$.** The linear program (3.7) for $d = 5$ is

$$\begin{aligned}
-\Big(\min_{l_1,l_2,l_3,l_4,y_3} \quad & 4y_3\Big),\\
\text{subject to} \quad & 5y_3 = l_1 + l_2 + l_3 + l_4,\\
& l_1 + 4y_3 \geq -\binom{4}{0} = -1\\
& l_2 + 4y_3 \geq \binom{4}{1} = 4\\
& l_3 + 4y_3 \geq -\binom{4}{2} = -6\\
& l_4 + 4y_3 \geq \binom{4}{3} = 4\\
& 1 \geq y_3 \geq 0,\ l_1 \geq 0,\ l_2 \geq 0,\ l_3 \geq 0,\ l_4 \geq 0.
\end{aligned}$$

Repeating the arguments from the proof of Proposition 5, in the optimal solution $(l_1^*, l_2^*, l_3^*, l_4^*, y_3^*)$ we have that $l_1^* = l_3^* = 0$ and $l_2^* + 4y_3^* = l_4^* + 4y_3^* = 4$. Using these in $5y_3^* = l_2^* + l_4^*$ we get $y_3^* = \frac{8}{13}$ So the minimal volume box of some 5-quasi-copula has value $-\frac{32}{13}$. Hence, an optimal solution $(y_1^*, l_1^*, l_2^*, l_3^*, l_4^*.y_2^*, y_3^*)$ to (3.5) for $d = 5$ is $\big(0, 0, \frac{20}{13}, 0, \frac{20}{13}, \frac{30}{13}, \frac{8}{13}\big)$. Using the complementary slackness, an optimal solution $(a^*, b^*, q_0^*, \delta_1^*, \delta_2^*, \delta_3^*, \delta_4^*, \delta_5^*)$ to its dual (3.3) is equal to $\big(\frac{8}{13}, \frac{12}{13}, 0, 0, \frac{4}{13}, 0, \frac{4}{13}, 0\big)$. Finally, an optimal solution to (3.2) is

$$(a^*, b^*, q_0^*, q_1^*, q_2^*, q_3^*, q_4^*, q_5^*) = \Big(\frac{8}{13}, \frac{12}{13}, 0, 0, \frac{4}{13}, \frac{4}{13}, \frac{8}{13}, \frac{8}{13}\Big).$$

7.4. **Case $d = 6$.** The linear program (3.7) for $d = 6$ is

$$
\begin{aligned}
-\Big(\min_{l_1,l_2,l_3,l_4,l_5,y_3} \quad & 5y_3\Big),\\
\text{subject to} \quad & 6y_3 = l_1+l_2+l_3+l_4+l_5+l_6,\\
& l_1+5y_3 \geq \binom{5}{0} = 1\\
& l_2+5y_3 \geq -\binom{5}{1} = -5\\
& l_3+5y_3 \geq \binom{5}{2} = 10\\
& l_4+5y_3 \geq -\binom{5}{3} = -10\\
& l_5+5y_3 \geq \binom{5}{4} = 5\\
& 1 \geq y_3 \geq 0,\ l_1 \geq 0,\ l_2 \geq 0,\ l_3 \geq 0,\ l_4 \geq 0,\ l_5 \geq 0.
\end{aligned}
$$

Repeating the arguments from the proof of Proposition 5, in the optimal solution $(l_1^*, l_2^*, l_3^*, l_4^*, l_5^*, y_3^*)$ we have that $l_2^* = l_4^* = 0$ and and one of the cases:

- $l_1^* + 5y_3^* = 1$, $l_3^* + 5y_3^* = 10$ and $l_5^* + 5y_3^* = 5$, or
- $l_1^* = 0$, $l_3^* + 5y_3^* = 10$ and $l_5^* + 5y_3^* = 5$, or
- $l_1^* = 0$, $l_3^* + 5y_3^* = 10$ and $l_5^* = 0$.

Using these in $6y_3^* = l_1^* + l_3^* + l_5^*$ we get $y_3^* = \frac{16}{21}$ in the first case, $y_3^* = \frac{15}{16}$ in the second case and $y_3^* = \frac{10}{11}$ in the third case. In the smallest candidate for the solution, i.e., $\frac{16}{21}$, we have that $l_1^* = -\frac{59}{21} < 0$, which is not in the feasible region. The second smallest candidate for the solution, i.e., $\frac{15}{16}$ indeed comes from the feasible (and hence optimal) solution, i.e., $(l_1^*, l_2^*, l_3^*, l_4^*, l_5^*, y_3^*) = \big(0, 0, \frac{85}{16}, 0, \frac{5}{16}, \frac{15}{16}\big)$. So the minimal volume box of some 6-quasi-copula has value $-\frac{75}{16}$. Hence, an optimal solution $(y_1^*, l_1^*, l_2^*, l_3^*, l_4^*, l_5^*, l_6^*, y_2^*, y_3^*)$ to (3.5) for $d = 6$ is $\big(0, 0, 0, \frac{85}{16}, 0, \frac{5}{16}, 0, \frac{90}{16}, \frac{15}{16}\big)$. Using the complementary slackness, an optimal solution $(a^*, b^*, q_0^*, \delta_1^*, \delta_2^*, \delta_3^*, \delta_4^*, \delta_5^*, \delta_6^*)$ to its dual (3.3) is equal to $\big(\frac{5}{8}, \frac{15}{16}, 0, 0, 0, \frac{5}{16}, 0, \frac{5}{16}, 0\big)$. Finally, an optimal solution to (3.2) is

$$
(a^*, b^*, q_0^*, q_1^*, q_2^*, q_3^*, q_4^*, q_5^*, q_6^*) = \Big(\frac{5}{8}, \frac{15}{16}, 0, 0, 0, \frac{5}{16}, \frac{5}{16}, \frac{5}{8}, \frac{5}{8}\Big).
$$

References

[1] C. Alsina, R.B. Nelsen and B. Schweizer, On the characterization of a class of binary operations on distribution functions, *Statistics and Probability Letters* 17 (1993) 85–89.

[2] J.J. Arias-García, R. Mesiar and B. De Baets, A hitchhiker's guide to quasi-copulas, *Fuzzy Sets and Systems* 393 (2020), 1–28.

[3] B. De Baets, H. De Meyer, B. De Schuymer and S. Jenei Cyclic evaluation of transitivity of reciprocal relations, *Social Choice and Welfare*, 26 (2006), 217–238.

[4] B. De Baets, S. Janssens and H. De Meyer, Meta-theorems on inequalities for scalar fuzzy set cardinalities, *Fuzzy Sets and Systems*, 157 (2006) 1463–1476.

[5] B. De Baets, H. De Meyer and M. Úbeda-Flores, Extremes of the mass distribution associated with a trivariate quasi-copula, *Comptes Rendus Mathematique*, 344 (2007), 587–590.

[6] M. Belšak, M. Omladič, M. Vuk, A. Zalar, Extreme values of the mass distribution associated with d-quasi-copulas via linear programming, arXiv:2410.19339, to appear in *Fuzzy Sets and Systems*.

[7] J. Bezanson, A. Edelman, S. KarpinskI V.B Shah, Julia: A Fresh Approach to Numerical Computing, *SIAM Rev.*, 59 (2017), 65–98.

[8] I. Cuculescu and R. Theodorescu, Copulas: Diagonals and tracks, *Rev. Roumaine Math. Pures Appl.*, 46 (2001), 731–742.
[9] F. Durante and C. Sempi, *Principles of Copula Theory*, CRC Press, 2015.
[10] J. Fernández-Sánchez, R.B. Nelsen and M. Úbeda-Flores, Multivariate copulas, quasi-copulas and lattices, *Statistics and Probability Letters*, 81 (2011), 1365–1369.
[11] J. Fernández-Sánche and M. Úbeda-Flores, A note on quasi-copulas and signed measures, *Fuzzy Sets and Systems*, 234 (2014) 109–112.
[12] C. Genest, J.J. Quesada Molina, J.A. Rodríguez Lallena and C. Sempi, A Characterization of Quasi-copulas, *Journal of Multivariate Analysis* 69 (1999), 193–205.
[13] Q. Huangfu and J.A.J. Hall, Parallelizing the dual revised simplex method, *Mathematical Programming Computation* 10 (2018) 119–142.
[14] E.P. Klement, D. Kokol Bukovšek, M. Omladič, S. Saminger-Platz, N. Stopar, Multivariate copulas with given values at two arbitrary points, *Stat. Papers* **64** (2023), 2015–2055.
[15] A. Kolesárová, A. Stupňanová and J. Beganová, Aggregation-based extensions of fuzzy measures, *Fuzzy Sets and Systems* 194 (2012) 1–14.
[16] R.B. Nelsen and M. Úbeda-Flores, The lattice-theoretic structure of sets of bivariate copulas and quasi-copulas, *Comptes Rendus. Mathématique* 341 (2005) 583–586.
[17] R.B. Nelsen, J.J. Quesada-Molina, J.A. Rodríguez-Lallena and M. Úbeda-Flores, Some New Properties of Quasi-Copulas, *Distributions With Given Marginals and Statistical Modelling*, Springer Netherlands, (2002), 187–194.
[18] R.B. Nelsen, J.J. Quesada-Molina, J.A. Rodríguez-Lallena and M. Úbeda-Flores, Some new properties of quasi-copulas, *Distributions with given marginals and Statistical Modelling* (2002), 187–194.
[19] R.B. Nelsen and M. Úbeda Flores, The lattice-theoretic structure of sets of bivariate copulas and quasi-copulas, *Comptes Rendus Mathematique*, 341 (2005) 583–586.
[20] M. Omladič, N. Stopar, Final solution to the problem of relating a true copula to an imprecise copula, *Fuzzy Sets and Systems* **393** (2020), 96–112.
[21] M. Omladič, N. Stopar, Dedekind-MacNeille completion of multivariate copulas via ALGEN method, *Fuzzy Sets and Systems* **441** (2022), 321–334.
[22] J.A. Rodríguez-Lallena and M. Úbeda-Flores, Some new characterizations and properties of quasi-copulas, *Fuzzy Sets and Systems*, 160 (2009), 717–725.
[23] A. Sklar, Fonctions de répartition à ndimensions et leurs marges, *Publ. Inst. Stat. Univ. Paris*, 8 (1959), 229–231.
[24] N. Stopar, Bivariate measure-inducing quasi-copulas, *Fuzzy Sets and Systems* **492** (2024), 109068.
[25] M. Úbeda-Flores, Extreme values of the mass distribution associated with a tetravariate quasi-copula, *Fuzzy Sets and Systems* 473 (2023).
[26] M. Vuk, *QuasiCopula.jl*, `https://gitlab.com/mrcinv/QuasiCopula.jl`, Accessed: 15 March 2025.

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