MOMENT PROBLEMS FOR OPERATOR POLYNOMIALS

JAKA CIMPRIČ AND ALJAŽ ZALAR

Abstract. Haviland's theorem states, that given a closed subset K in $\mathbb{R}^n$ each functional $L\colon \mathbb{R}[\underline{x}] \to \mathbb{R}$ positive on $\operatorname{Pos}(K) := \{p \in \mathbb{R}[\underline{x}] \mid p|_K \geq 0\}$ admits an integral representation by a positive Borel measure. Schmüdgen proved, that in the case of compact semialgebraic set K it suffices to check positivity of L on a preordering T, having K as the non-negativity set. Further he showed, that the compactness of K is equivalent to the archimedianity of T. The aim of this paper is to extend these results from functionals on the usual real polynomials to operators mapping from the real matrix or operator polynomials into $\mathbb{R}, M_n(\mathbb{R})$ or $B(\mathcal{K})$.

1. Introduction

Let K be a closed subset of $\mathbb{R}^d$, $d \geq 1$. The *K-moment problem* asks for which multisequences $c\colon \mathbb{N}^d \to \mathbb{R}$ there exists a positive Borel measure μ on K such that $c_\alpha = \int_K x^\alpha \, d\mu := \int_K x_1^{\alpha_1} \cdots x_d^{\alpha_d} \, d\mu$ for every $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$. A solution to this problem is given by the following result, see [26, Theorem 3.1.2]:

Theorem 1 (Haviland, 1935). *For a linear functional $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ and a closed set K in $\mathbb{R}^d$ the following statements are equivalent:*

(1) *There exists a positive Borel measure μ on K such that $L(p) = \int_K p \, d\mu$ for every $p \in \mathbb{R}[\underline{x}]$.*
(2) *$L(p) \geq 0$ holds for all $p \in \mathbb{R}[\underline{x}]$ satisfying $p \geq 0$ on K.*

Remark 1. If K is compact, then the measure μ is unique, see [26, Corollary 3.3.1]. For noncompact K, the question of uniqueness is highly nontrivial and will not be discussed here, see [30] and [31].

Theorem 1 is not considered entirely satisfactory, because the set

$$\operatorname{Pos}(K) := \{p \in \mathbb{R}[\underline{x}] \mid p \geq 0 \text{ on } K\}$$

is very big. If the set K is defined by finitely many polynomial inequalities, then the condition $L(\operatorname{Pos}(K)) \geq 0$ is equivalent to $L(T) \geq 0$ for some set T which is much smaller that $\operatorname{Pos}(K)$. This is the contents of Theorem 2.

For a finite set $S = \{g_1, \ldots, g_k\}$ in $\mathbb{R}[\underline{x}]$ write

$$K_S := \{\underline{x} \in \mathbb{R}^d \mid g_1(\underline{x}) \geq 0, g_2(\underline{x}) \geq 0, \ldots, g_k(\underline{x}) \geq 0\}$$

and

$$M_S := \{\sigma_0 + \sigma_1 g_1 + \ldots + \sigma_k g_k \mid \sigma_0, \sigma_1, \ldots, \sigma_k \in \sum \mathbb{R}[\underline{x}]^2\}.$$

Theorem 2. *Let S be a finite subset of $\mathbb{R}[\underline{x}]$ such that K_S be compact. Then there exists a finite subset S_1 of $\mathbb{R}[\underline{x}]$ containing S such that $K_{S_1} = K_S$ and*

(1) *every $p \in \mathbb{R}[\underline{x}]$ such that $p|_{K_S} > 0$ belongs to M_{S_1},*

Date: February 20, 2023.

(2) *for every linear functionals* L *on* $\mathbb{R}[\underline{x}]$ *such that* $L(M_{S_1}) \geq 0$ *there exists a positive Borel measure* μ *on* K_S *such that* $L(p) = \int_{K_S} p\, d\mu$ *for all* $p \in \mathbb{R}[\underline{x}]$.

More precisely, we can take S_1 *to be either the set* $\prod S$ *of all square-free products of elements from* S *(Schmüdgen 1991, see* [38]*, a nice refinement is* [18]*) or the set* $S \cup \{l^2 - \sum_{i=1}^d x_i^2\}$ *for some* $l \in \mathbb{N}$ *(Putinar 1993, see* [29]*).*

Note that claim (2) of Theorem 2 is a consequence of claim (1) and Theorem 1.

The aim of this paper is to extend Theorems 1 and 2 to matrix polynomials. We also have some partial results (both positive and negative) for operator polynomials.

In most of the current literature, the term *operator moment problem* refers to the question of existence of integral representations for linear mappings $L\colon \mathbb{R}[\underline{x}] \to B(\mathcal{K})_h$ where $B(\mathcal{K})_h$ is the real vector space of all bounded self-adjoint operators on a Hilbert space $\mathcal{K}$. The univariate case is well-understood, see e.g. [22] and [21]. In the multivariate case, see [42, Theorem I.4.3] for a result related to our Theorem 2. A different kind of a moment problem is considered in [1] where the authors study the question of existence of integral representations for linear functionals $L\colon \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h \to \mathbb{R}$. Here, $\mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h = B(\mathcal{H})_h[\underline{x}]$ is the real vector space of all polynomials with coefficients from $B(\mathcal{H})_h$. For the unit cube in $\mathbb{R}^d$, their Theorem 3 extends our Theorem 2.

In this paper, we unify both approaches by studying integral representations of linear mappings $L\colon \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h \to B(\mathcal{K})_h$. The relevant measure and integration theory was developed in [14]. It is recalled and slightly modified in Section 2. In Section 3 we prove a generalization of Theorem 1 to arbitrary $\mathcal{H}$ and $\mathcal{K}$, see Theorem 4 and its special case Theorem 3 for $\mathcal{K} = \mathbb{R}$. In Section 4, we prove a generalization of Putinar's part of Theorem 2 to arbitrary $\mathcal{H}$ and $\mathcal{K}$ and a generalization of Schmüdgen's part of Theorem 2 to finite-dimensional $\mathcal{H}$ and arbitrary $\mathcal{K}$, see Theorems 5 and 6. Finally, in Section 5, we show that the main step in the proof of Theorem 6 fails for infinite dimensional $\mathcal{H}$ even if $\mathcal{K} = \mathbb{R}$.

2. Operator-valued measures

Let $\mathcal{P}$ be a ring of sets and let $\mathcal{H}$ and $\mathcal{K}$ be real Hilbert spaces. We denote by $\mathcal{L}(B(\mathcal{H})_h, B(\mathcal{K})_h)$ the Banach space of all bounded linear operators from $B(\mathcal{H})_h$ to $B(\mathcal{K})_h$, where $B(\mathcal{H})_h$ and $B(\mathcal{K})_h$ are the Banach spaces of all bounded self-adjoint linear operators on $\mathcal{H}$ and $\mathcal{K}$, respectively. A set function

$$m\colon \mathcal{P} \to \mathcal{L}(B(\mathcal{H})_h, B(\mathcal{K})_h)$$

is a *non-negative operator-valued measure* if for every $A \in B(\mathcal{H})_+$ the set function

$$m_A\colon \mathcal{P} \to B(\mathcal{K})_h, \quad m_A(\Delta) = m(\Delta)(A),$$

is a positive operator-valued measure.

Remark 2. Recall from [6, Definition 1] that a set function

$$E\colon \mathcal{P} \to B(\mathcal{K})_h$$

is a *positive operator-valued measure*, if it satisfies the following conditions:

(a) $E(\Delta) \succeq 0$ for all $\Delta \in \mathcal{P}$.
(b) $E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2)$ if Δ_1 and Δ_2 are disjoint subsets in $\mathcal{P}$.
(c) If Δ_i is an increasing sequence in $\mathcal{P}$ and $\Delta = \bigcup_i \Delta_i$ belongs to $\mathcal{P}$ then $E(\Delta) = \sup_i E(\Delta_i)$.

When $\mathcal{H} = \mathbb{R}$, we can identify $\mathcal{L}(B(\mathcal{H})_h, B(\mathcal{K})_h)$ with $B(\mathcal{K})_h$. In this identification the non-negative operator-valued measure m corresponds to the positive operator-valued measure m_1. Therefore, positive operator-valued measures are special cases of non-negative operator-valued measures.

Remark 3. Our definition of a non-negative operator-valued measure is similar to the following definition from [14, p. 511]: A set function $m\colon \mathcal{P} \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$, where $\mathcal{P}$ is a δ-ring of sets and $\mathcal{X}$, $\mathcal{Y}$ are Banach spaces, is called an *operator-valued measure countably additive in the strong operator topology* if for every $x \in X$ the set function $m_x\colon \mathcal{P} \to \mathcal{Y}, \Delta \mapsto m(\Delta)x$, is a countably additive vector measure.

These definitions coincide if $\mathcal{X} = B(\mathcal{H})_h$ for some Hilbert space $\mathcal{H}$, $\mathcal{Y} = B(\mathcal{K})_h$ for some finite-dimensional Hilbert space $\mathcal{K}$, and $m_x(\Delta) \in B(\mathcal{K})_+$ for every $x \in B(\mathcal{H})_+$ and every $\Delta \in \mathcal{P}$. The problem with infinite-dimensional $\mathcal{K}$ is that the definitions of convergence of $m_x(\Delta_i)$ to $m_x(\bigcup_i \Delta_i)$ do not coincide.

Let X be a set, $\mathcal{P}$ a σ-algebra of subsets of X and $m\colon \mathcal{P} \to \mathcal{L}(B(\mathcal{H})_h, B(\mathcal{K})_h)$ a non-negative operator-valued measure. Let $\mathcal{I}$ denote the set of all $\mathcal{P}$-measurable real-valued functions on X which are m_A-integrable for every $A \in B(\mathcal{H})_+$. It is a real vector space and it consists at least of all bounded measurable functions. In particular, if $\mathcal{P} = \operatorname{Bor}(X)$ (the Borel σ-algebra of X) then $C_c(X, \mathbb{R}) \subset \mathcal{I}$.

Remark 4. Let $E\colon \mathcal{P} \to B(\mathcal{K})_h$ be a positive operator-valued measure. For every $x \in \mathcal{K}$ we define a positive measure $E_x\colon \mathcal{P} \to \mathbb{R}^{\geq 0}$ by $E_x(\Delta) = \langle E(\Delta)x, x\rangle$. We say that a $\mathcal{P}$-measurable function $f\colon X \to \mathbb{R}$ is *E-integrable* if there exists a constant $K_f \in \mathbb{R}$ such that $\int |f| \, dE_x \leq K_f \|x\|^2$ for every $x \in \mathcal{K}$. (If $\|f\|_\infty < \infty$ then $K_f = \|E(X)\| \, \|f\|_\infty$ works.) The mapping $(x, y) \mapsto \frac{1}{4}(\int f \, dE_{x+y} - \int f \, dE_{x-y})$ is then a bounded bilinear form; see [6, Section 5]. Therefore, there exists a bounded operator $\int f \, dE \in B(\mathcal{K})_h$ such that $\int f \, dE_x = \langle (\int f \, dE)x, x\rangle$ for every $x \in \mathcal{K}$.

For every $f \in \mathcal{I}$ and every operator $A \in B(\mathcal{H})_h$, we define

$$\int f \, dm_A := \int f \, dm_{A_+} - \int f \, dm_{A_-}$$

where $A_+, A_- \in B(\mathcal{H})_+$ are the positive and the negative part of A. Namely, $A = A_+ - A_-$, $A_+A_- = A_-A_+ = 0$ and hence $\|A_\pm\| \leq \|A\|$ (see [24, Proposition 5.2.2(4)]).

Let $\mathcal{I} \otimes B(\mathcal{H})_h$ be the algebraic tensor product of $\mathcal{I}$ and $B(\mathcal{H})_h$ over $\mathbb{R}$. By the universal property of tensor products, the bilinear form

$$\mathcal{I} \times B(\mathcal{H})_h \to B(\mathcal{K})_h, \quad (f, A) \mapsto \int f \, dm_A$$

extends to a linear map

$$\mathcal{I} \otimes B(\mathcal{H})_h \to B(\mathcal{K})_h, \quad F = \sum_{i=1}^n f_i \otimes A_i \mapsto \int F \, dm := \sum_{i=1}^n \int f_i \, dm_{A_i}.$$

We first recall the following operator-valued version of the F. Riesz representation theorem for positive functionals, see [6, Theorem 19]. A positive operator-valued measure with $\mathcal{P} = \operatorname{Bor}(X)$ will be called a *Borel positive operator-valued measure.*

Proposition 1. *Let X be a locally compact and σ-compact metrizable space, $\mathcal{K}$ a Hilbert space and $T\colon C_c(X, \mathbb{R}) \to B(\mathcal{K})_h$ a positive bounded linear map. Then there exists one and only one Borel positive operator-valued measure E on X such that $T(f) = \int f \, dE$ for every $f \in C_c(X, \mathbb{R})$.*

Proposition 2 extends Proposition 1 from $C_c(X,\mathbb{R})$ to $C_c(X,\mathbb{R}) \otimes B(\mathcal{H})_h$. It is similar to [13, Theorem 2]. The vector space $C_c(X,\mathbb{R}) \otimes B(\mathcal{H})_h$ can be identified with a subspace of $C_c(X, B(\mathcal{H})_h)$ from where it inherits the supremum norm and the positive cone $C_c(X, B(\mathcal{H})_+)$. Unlike [14] we will never integrate functions from $C_c(X, B(\mathcal{H})_h)$ that do not belong to $C_c(X,\mathbb{R}) \otimes B(\mathcal{H})_h$.

Proposition 2. *Let X be a locally compact and σ-compact metrizable space, $\mathcal{H}$ and $\mathcal{K}$ Hilbert spaces and $L : C_c(X,\mathbb{R}) \otimes B(\mathcal{H})_h \to B(\mathcal{K})_h$ a positive bounded linear map. Then there exists a unique non-negative operator-valued measure*

$$m : \mathrm{Bor}(X) \to \mathcal{L}(B(\mathcal{H})_h, B(\mathcal{K})_h)$$

such that

$$L(F) = \int F\, dm$$

holds for all $F \in C_c(X,\mathbb{R}) \otimes B(\mathcal{H})_h$.

Proof. For every $A \in B(\mathcal{H})_+$ we define an operator $L_A \colon C_c(X,\mathbb{R}) \to B(\mathcal{K})_h$ by $L_A(f) = L(f \otimes A)$. Since L is positive, it follows that $L_A(C_c(X,\mathbb{R})_+) \succeq 0$. By Proposition 1 there exists a unique Borel positive operator-valued measure E_A such that $L_A(f) = \int f\, dE_A$ for all $f \in C_c(X,\mathbb{R})$. Let us define a map

$$m\colon \mathrm{Bor}(X) \to \mathcal{L}(B(\mathcal{H})_h, B(\mathcal{K})_h), \quad m(\Delta)(A) = E_{A_+}(\Delta) - E_{A_-}(\Delta).$$

For every $f \in C_c(X,\mathbb{R})$ and $A, B \in B(\mathcal{H})_+$ we have

$$\begin{aligned}\int f\, dE_{A+B} &= L_{A+B}(f) = L(f \otimes (A+B)) = \\ &= L(f \otimes A) + L(f \otimes B) = L_A(f) + L_B(f) = \\ &= \int f\, dE_A + \int f\, dE_B = \int f\, d(E_A + E_B).\end{aligned}$$

It follows that $E_{A+B} = E_A + E_B$ by the uniqueness part of Proposition 1. For general $A, B \in B(\mathcal{H})_h$ we deduce that $m(\Delta)(A+B) - m(\Delta)(A) - m(\Delta)(B) = (E_{(A+B)_+}(\Delta) - E_{(A+B)_-}(\Delta)) - (E_{A_+}(\Delta) - E_{A_-}(\Delta)) - (E_{B_+}(\Delta) - E_{B_-}(\Delta)) = E_{(A+B)_+ + A_- + B_-}(\Delta) - E_{(A+B)_- + A_+ + B_+}(\Delta) = 0$. Therefore, $m(\Delta)$ is additive for every $\Delta \in \mathrm{Bor}(X)$. Similarly we show that it is also homogeneous.

We claim that $m(\Delta)$ is bounded for every $\Delta \in \mathrm{Bor}(X)$. Pick an increasing sequence of compact $\Delta_i \in \mathrm{Bor}(X)$ such that $X = \bigcup_i \Delta_i$. By Urysohn's Lemma there exist functions $u_i \in C_c(X, [0,1])$ such that $u_i|_{\Delta_i} \equiv 1$. For every $A \in B(\mathcal{H})_+$ we have that $E_A(\Delta_i) = \int \chi_{\Delta_i}\, dE_A \le \int u_i\, dE_A = L_A(u_i) = L(u_i \otimes A)$ which implies that $\|E_A(\Delta_i)\| \le \|L\|\, \|u_i \otimes A\| = \|L\|\, \|A\|$. Furthermore, $(E_A)_x(\Delta) \le (E_A)_x(X) = \sup_i (E_A)_x(\Delta_i)$ for every $x \in \mathcal{K}$, which implies that

$$\|E_A(\Delta)\| \le \sup_i \|E_A(\Delta_i)\| \le \|L\|\, \|A\|.$$

For non-positive $A \in B(\mathcal{H})$ we need an additional factor 2 because

$$\|m(\Delta)(A)\| \le \|E_{A_+}(\Delta)\| + \|E_{A_-}(\Delta)\| \le 2\|L\|\|A\|.$$

Therefore, the set function m is a non-negative operator-valued measure.

To prove that m is a representing measure for L, it suffices by linearity to prove that $L(f \otimes A) = \int (f \otimes A)\, dm$ for all $f \otimes A \in C_c(X,\mathbb{R}) \otimes B(\mathcal{H})_+$. This follows from

$$L(f \otimes A) = L_A(f) = \int f\, dE_A = \int f\, dm_A = \int (f \otimes A)\, dm.$$

The uniqueness of m follows from the uniqueness of the measures E_A for every $A \in B(\mathcal{H})_+$. □

3. Haviland's Theorem

Theorem 3 extends Theorem 1 to operator polynomials. Here we will restrict ourselves to $\mathcal{K} = \mathbb{R}$.

Theorem 3. *For a linear map $L\colon \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h \to \mathbb{R}$ and a closed set X in $\mathbb{R}^d$, the following are equivalent:*

(1) *There exists a non-negative Borel measure $m\colon \operatorname{Bor}(X) \to \mathcal{L}(B(\mathcal{H})_h, \mathbb{R})$ such that $L(F) = \int F\,dm$ for every $F \in \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h$.*
(2) *$L(F) \geq 0$ for every $F \in \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h$ such that $F \succeq 0$ on X.*

For $\mathcal{H} = \mathbb{R}$, this is [26, Theorems 3.1.2 and 3.2.2].

Proof. The nontrivial direction is that (2) implies (1). Let A_0 be the range of the natural mapping $\hat{}\colon \mathbb{R}[\underline{x}] \to C(X, \mathbb{R})$. By (2), $\bar{L}(\hat{p} \otimes B) := L(p \otimes B)$ is a well-defined positive linear functional on $A_0 \otimes B(\mathcal{H})_h$. The set

$$C'(X, \mathbb{R}) := \{f \in C(X, \mathbb{R}) \mid \exists p \in \mathbb{R}[\underline{x}] : |f| \leq |\hat{p}| \text{ on } X\}$$

is clearly a vector space which contains $C_c(X, \mathbb{R})$. Since A_0 is cofinal in $C'(X, \mathbb{R})$, also $A_0 \otimes B(\mathcal{H})_h$ is cofinal in $C'(X, \mathbb{R}) \otimes B(\mathcal{H})_h$. By the M. Riesz extension theorem, $\bar{L}$ extends (non-uniquely) to a positive linear functional on $C'(X, \mathbb{R}) \otimes B(\mathcal{H})_h$ which will also be denoted by $\bar{L}$. Note, that $\bar{L}|_{C_c(X,\mathbb{R}) \otimes B(\mathcal{H})_h}$ is bounded, since for every $F \in C_c(X, \mathbb{R}) \otimes B(\mathcal{H})_h$ we have $F \preceq \|F\|_\infty \otimes \mathrm{Id}$ and hence $\bar{L}(F) \leq \bar{L}(\|F\|_\infty \otimes \mathrm{Id}) = \bar{L}(1 \otimes \mathrm{Id}) \|F\|_\infty$. By Proposition 2, there exists a non-negative operator-valued Borel measure $m\colon \operatorname{Bor}(X) \to \mathcal{L}(B(\mathcal{H})_h, \mathbb{R})$ such that

$$\bar{L}(F) = \int F\,dm \tag{*}$$

for all $F \in C_c(X, \mathbb{R}) \otimes B(\mathcal{H})_h$. We have to show that (*) holds for all $F \in C'(X, \mathbb{R}) \otimes B(\mathcal{H})_h$ (and hence for all $F \in A_0 \otimes B(\mathcal{H})_h$). Clearly, it suffices to show that (*) holds for every $F = f \otimes B$ where $f \in C'(X, \mathbb{R})_+$ and $B \in B(\mathcal{H})_+$.

Write $p = x_1^2 + \ldots + x_n^2$. By the proof of Claim 3 of [26, Theorem 3.2.2] there exists an increasing sequence $f_i \in C_c(X, \mathbb{R})_+$ such that $0 \leq f - f_i \leq \frac{1}{i}(f + \hat{p})^2$ for every i. Thus,

$$\bar{L}(f \otimes B) = \bar{L}_B(f) = \lim_{i \to \infty} \bar{L}_B(f_i) = \lim_{i \to \infty} \int f_i\,dE_B \overbrace{=}^{(*)} \int f\,dE_B = \int f \otimes B\,dm.$$

Note that in this case E_B are the usual positive Borel measures. Therefore, the existence of $\int f\,dE_B$ and $(*)$ follow from the monotone convergence theorem and the fact that the sequence $\int f_i\,dE_B$ is bounded above by $\bar{L}(f \otimes B)$. □

Remark 5. If the Hilbert space $\mathcal{H}$ in Theorem 3 is finite-dimensional, then we can identify $\mathcal{L}(B(\mathcal{H})_h, \mathbb{R})$ with $B(\mathcal{H})_h$ via the trace map tr. The representation $L(F) = \int F\,dm$ then reads as $L(F) = \int \operatorname{tr}(F\,dE)$ where $E\colon \operatorname{Bor}(X) \to B(\mathcal{H})_h$ is the positive operator-valued measure that corresponds to m in the above identification.

To obtain versions of Hamburger, Stieltjes and Hausdorff moment problems for operator polynomials, we combine Theorem 3 with the following:

Proposition 3. *For every operator polynomial $F \in \mathbb{R}[x] \otimes B(\mathcal{H})_h$ we have the following equivalences:*

(1) *$F(a) \succeq 0$ for every $a \in \mathbb{R}$ iff F is a sum of hermitian squares of polynomials from $\mathbb{R}[x] \otimes B(\mathcal{H})$.*
(2) *$F(a) \succeq 0$ for every $a \in [0, \infty)$ iff $F = \sigma_0 + x\sigma_1$ where σ_0, σ_1 are sums of hermitian squares of polynomials from $\mathbb{R}[x] \otimes B(\mathcal{H})$.*
(3) *$F(a) \succeq 0$ for every $a \in [0, 1]$ iff $F = \sigma_0 + x\sigma_1 + (1-x)\sigma_2 + x(1-x)\sigma_3$ where σ_i are sums of hermitian squares of polynomials from $\mathbb{R}[x] \otimes B(\mathcal{H})$.*

In the proof we use the operator version of the Fejér-Riesz theorem, see [33] in the matrix case, [34] in the operator case and [15, Theorem 2.1] for a survey. Since $\mathcal{H}$ is a real Hilbert space, while the Fejér-Riesz theorem works only for complex Hilbert spaces, we have to complexify our $\mathcal{H}$ to $\mathcal{H}_{\mathbb{C}}$. From the proof it will also follow, that F in (1) and σ_0, σ_1 in (2) can be chosen as a sum of at most two hermitian squares.

Proof. (1) By assumption, $\deg F = 2n$ for some n. Replacing $x = \tan t$, we get

$$F(x) = (\cos t)^{-2n} \tilde{F}(\cos t, \sin t)$$

where $\tilde{F}(u, v) := F\left(\frac{v}{u}\right) u^{2n}$ is homogeneous and $\tilde{F} \succeq 0$ on $\mathbb{R}^2$. Clearly,

$$\tilde{F}(\cos t, \sin t) = u(e^{2it})$$

for some operator Laurent polynomial u, i.e., $u(z) = \sum_{k=-n}^{n} A_k z^k$ and $A_k \in B(\mathcal{H}_{\mathbb{C}}) = B(\mathcal{H})_{\mathbb{C}}$. Since $u(e^{it}) \succeq 0$ for $t \in \mathbb{R}$, it follows by the Fejér-Riesz theorem that $u(e^{it}) = P(e^{it})P^*(e^{-it})$, where P is a usual operator polynomial, i.e., $P(z) = \sum_{k=0}^{n} B_k z^k$ and $B_k \in B(\mathcal{H})_{\mathbb{C}}$. Hence

$$\tilde{F}(\cos t, \sin t) = G(\cos t, \sin t) G^*(\cos t, \sin t),$$

where

$$\begin{aligned} G(\cos t, \sin t) &= P(e^{2it})e^{-itn} = \sum_{k=0}^{n} B_k e^{2itk - itn} = \sum_{k=0}^{n} B_k (e^{it})^k (e^{-it})^{n-k} = \\ &= \sum_{k=0}^{n} (B'_k + iB''_k)(\cos t + i \sin t)^k (\cos t - i \sin t)^{n-k} = \\ &= H(\cos t, \sin t) + iK(\cos t, \sin t), \end{aligned}$$

with $B'_k, B''_k \in B(\mathcal{H})$ and $H, K \in \mathbb{R}[u, v] \otimes B(\mathcal{H})$ are homogeneous polynomials of degree n. It follows that

$$\tilde{F}(\cos t, \sin t) = H(\cos t, \sin t)H^*(\cos t, \sin t) + K(\cos t, \sin t)K^*(\cos t, \sin t).$$

Note that $i(-H(\cos t, \sin t)K^*(\cos t, \sin t) + K(\cos t, \sin t)H^*(\cos t, \sin t)) = 0$ since the coefficients of $\tilde{F}$ are "real", i.e., they belong to $B(\mathcal{H})$. Therefore,

$$F(x) = H(1, x)H^*(1, x) + K(1, x)K^*(1, x).$$

(2) From $F|_{\mathbb{R}_+} \succeq 0$ it follows $G(a) := F(a^2) \succeq 0$ on $\mathbb{R}$. By (1)

$$G(a) = \sum_i P_i(a)P_i^*(a) = \sum_i (R_i(a^2) + aQ_i(a^2))(R_i^*(a^2) + aQ_i^*(a^2)) =$$

$$\sum_i R_i(a^2)R_i^*(a^2) + a\sum_i (Q_i(a^2)R_i^*(a^2) + R_i(a^2)Q_i^*(a^2)) + a^2 \sum_i Q_i(a^2)Q_i^*(a^2)$$

Since $G(a) = G(-a)$ we get

$$F(a^2) = G(a) = \frac{1}{2}\left(\sum_i R_i(a^2)R_i^*(a^2) + a^2\sum_i Q_i(a^2)Q_i^*(a^2)\right)$$

and with substitution $t = a^2$ the result follows.

(3) The proof is the same as in the matrix case, see [12, Theorem 2.5] or [40, Section 7] .

□

Now we can explicitly formulate Hamburger's, Stieltjes' and Hausdorff's theorems for matrix polynomials.

Corollary 1. *Let L be a linear functional on $\mathbb{R}[x] \otimes S_n(\mathbb{R})$. For each $p \in \mathbb{N}_0$ write $S_p := [L(x^p E_{k,l})]_{k,l=1,\ldots,n}$ where $E_{k,l}$ are coordinate matrices. Then*

(1) *L has an integral representation (in the sense of Remark 5) with a positive operator-valued measure E whose support is contained in $\mathbb{R}$ iff $[S_{i+j}]_{i,j=0,\ldots,m}$ is positive semidefinite for every $m \in \mathbb{N}_0$,*
(2) *L has an integral representation with a positive operator-valued measure E whose support is contained in $[0,\infty)$ iff $[S_{i+j}]_{i,j=0,\ldots,m}$ and $[S_{i+j+1}]_{i,j=0,\ldots,m}$ are positive semidefinite for every $m \in \mathbb{N}_0$,*
(3) *L has an integral representation with a positive operator-valued measure E whose support is contained in $[0,1]$ iff $[S_{i+j}]_{i,j=0,\ldots,m}$, $[S_{i+j+1}]_{i,j=0,\ldots,m}$, $[S_{i+j} - S_{i+j+1}]_{i,j=0,\ldots,m}$ and $[S_{i+j+1} - S_{i+j+2}]_{i,j=0,\ldots,m}$ are positive semidefinite for every $m \in \mathbb{N}_0$.*

The operator version of Corollary 1 is less straightforward. For the Hamburger's theorem one has to require that for every $m \in \mathbb{N}_0$ and every tuple of operators $(A_0, \ldots, A_m) \in B(\mathcal{H})$, the matrix

$$\left[L\left(x^{i+j}A_i^*A_j\right)\right]_{i,j=0,\ldots,m}$$

is positive semidefinite. For the Stieltjes' theorem we require that for every $m \in \mathbb{N}_0$ and every tuple of operators $(A_0, \ldots, A_m) \in B(\mathcal{H})$, the matrices

$$\left[L\left(x^{i+j}A_i^*A_j\right)\right]_{i,j=0,\ldots,m} \text{ and } \left[L\left(x^{i+j+1}A_i^*A_j\right)\right]_{i,j=0,\ldots,m}$$

are positive semidefinite, while for Hausdorff's theorems we additionaly require that

$$\left[L\left(\left(x^{i+j} - x^{i+j+1}\right)A_i^*A_j\right)\right]_{i,j=0,\ldots,m} \text{ and } \left[L\left(\left(x^{i+j+1} - x^{i+j+2}\right)A_i^*A_j\right)\right]_{i,j=0,\ldots,m}$$

are positive semidefinite.

The problem with the extension of Theorem 3 to $\mathcal{K} \neq \mathbb{R}$ is that M. Riesz extension theorem is known to fail in general. However, if the mapping L is completely positive then we can use the following version of Arveson's extension theorem.

Proposition 4. *Suppose $(E, K_1(E), K_2(E), \ldots)$ is a real matrix ordered vector space. Let E_0 be a cofinal subspace of E. Let $\mathcal{K}$ be a real Hilbert space and $L\colon E_0 \to B(\mathcal{K})_h$ a completely positive map from the matrix ordered space E_0 to $B(\mathcal{K})_h$. Then there exists a completely positive map $L'\colon E \to B(\mathcal{K})_h$ such that $L'|_{E_0} = L$.*

Proposition 4 is very similar to [28, Theorem 3.7.]. The differences are that our E and E_0 are real vector spaces with trivial involution instead of complex vector spaces with general involution and that the codomain of our L is bounded operators instead of (not necessarily bounded) sesquilinear forms. We advice the reader to consult [39, Section 11.1] before continuing.

Proof. If $L = 0$, put $L' = 0$. Assume that $L \neq 0$. By Zorn's Lemma we may assume that $E = \mathbb{R}x_0 \oplus E_0$ for some $x_0 \in E \setminus E_0$. We consider the real $*$-vector space $\mathcal{K} \otimes \mathcal{K}$ with involution $(k_1 \otimes k_2)^* = k_2 \otimes k_1$. Let G be the real vector space $(\mathcal{K} \otimes \mathcal{K})_h \oplus \mathbb{R}$ and let C be the convex hull of elements

$$\left(\sum_{j,l=1}^{n} \alpha_{jl} k_l \otimes k_j, \sum_{j,l=1}^{n} \langle L(x_{jl}) k_l, k_j \rangle \right) \in (\mathcal{K} \otimes \mathcal{K}) \oplus \mathbb{R}$$

where $\alpha_{jl} \in \mathbb{R}$, $x_{jl} \in E_0$ and $k_j \in \mathcal{K}$ are such that $[\alpha_{jl} x_0 + x_{jl}]_{jl} \in K_n(E)$. It follows that $\alpha_{lj} = \alpha_{jl}$ and $x_{lj} = x_{jl}$ for every $j, l = 1, \ldots, n$, hence $C \subseteq G$.

Next, we show that $(0, 1)$ is an algebraic interior point of C - i.e., for every $(y, \lambda) \in G$ we will find $\delta > 0$ such that $\gamma(y, \lambda) + (0, 1) \in C$ for every $\gamma \in (0, \delta)$. Since $L \neq 0$ and E_0 is cofinal in E, there exist $x \in K_1(E_0)$, $k \in \mathcal{K}$, such that $\langle L(x)k, k\rangle > 0$. Hence $(0, \langle L(x)k, k\rangle) \in C$ and with scaling we conclude $(0, \alpha) \in C$ for every $\alpha > 0$. Suppose that $y = \sum_{j,l=1}^{n} \alpha_{jl} k_l \otimes k_j$ where $[\alpha_{jl}]_{jl} \in M_n(\mathbb{R})_h$ and $k_1, k_2, \ldots, k_n \in \mathcal{K}$. Since E_0 is cofinal in E, there exist $z_{jl} \in K_1(E_0)$, $j, l = 1, \ldots, n$, such that $z_{jl} \pm \alpha_{jl} x_0 \in K_1(E)$. Set $[x_{jl}]_{jl} := \sum_j E_{jj}^T z_{jj} E_{jj} + \sum_{j<l} (E_{jj} + E_{jl})^T z_{jl} (E_{jj} + E_{jl}) + \sum_{j<l} (E_{jj} + E_{ll})^T z_{jl} (E_{jj} + E_{ll}) \in K_n(E_0)$ where E_{jl} are coordinate matrices. Clearly, $[\alpha_{jl} x_0 + x_{jl}]_{jl} = [\alpha_{jl}]_{jl} x_0 + [x_{jl}]_{jl} = \sum_j E_{jj}^T (z_{jj} + \alpha_{jj} x_0) E_{jj} + \sum_{j<l} (E_{jj} + E_{jl})^T (z_{jl} + \alpha_{jl} x_0)(E_{jj} + E_{jl}) + \sum_{j<l} (E_{jj} + E_{ll})^T (z_{jl} - \alpha_{jl} x_0)(E_{jj} + E_{ll}) \in K_n(E)$. Write $\lambda_1 := \sum_{j,l=1}^{n} \langle L(x_{jl}) k_l, k_j \rangle \geq 0$ and note that $(y, \lambda_1) \in C$. For every $0 < \gamma < \min\left\{ \frac{1}{|\lambda - \lambda_1|}, 1 \right\} =: \delta$ we have $\gamma(y, \lambda) + (0, 1) = \gamma(y, \lambda_1) + (1 - \gamma) \left(0, \frac{\gamma(\lambda - \lambda_1) + 1}{1 - \gamma} \right) \in C$.

On the other hand, $(0, 0)$ is not an algebraic interior point in C. The proof is the same as in the complex case, see [39, Theorem 11.1.5]. (Namely, if $(0, -\epsilon) \in C$ for some $\epsilon > 0$ then we get a contradiction after a short computation.)

Now the separation theorem for convex sets, see e.g. [11, Ch. IV, Theorem 3.3], gives us a linear functional $f \colon G \to \mathbb{R}$ such that $f(C) \geq 0$. Since $(0, 1)$ is in the interior of C, we have that $f((0, 1)) > 0$, so we may assume that $f((0, 1)) = 1$. We claim that the bilinear form $M(k_1, k_2) := \frac{1}{2} f((k_1 \otimes k_2 + k_2 \otimes k_1, 0))$ is bounded. Namely, since E_0 is cofinal in E we can pick $z \in K_1(E_0)$ such that $z \pm x_0 \in K_1(E)$. By the definition of C, it follows that $(\pm k \otimes k, \langle L(z)k, k\rangle) \in C$ for every $k \in \mathcal{K}$, which implies that $\pm M(k, k) + \langle L(z)k, k\rangle = \pm f((k \otimes k, 0)) + \langle L(z)k, k\rangle f((0, 1)) \geq 0$ for every $k \in \mathcal{K}$. Since $L(z)$ is bounded, the polarization identity implies that M is also bounded. By [11, Ch. II, Theorem 2.2], there exists $L_0(x_0) \in B(\mathcal{K})_h$ such that $\langle L_0(x_0) k_1, k_2 \rangle = M(k_1, k_2)$ for every $k_1, k_2 \in \mathcal{K}$.

The mapping $L' \colon \mathbb{R}x_0 + E_0 \to B(\mathcal{K})_h$, $L'(\alpha x_0 + z) := \alpha L_0(x_0) + L(z)$ clearly extends L. To show that L' is completely positive, pick any $n \in \mathbb{N}$, $X \in K_n(E)$ and $k_1, \ldots, k_n \in \mathcal{K}$. Clearly, $X = [\alpha_{jl} x_0 + x_{jl}]_{jl}$ for some $[\alpha_{jl}]_{jl} \in M_n(\mathbb{R})_h$ and $[x_{jl}]_{jl} \in M_n(E)_h$. If $y = \sum_{j,l=1}^{n} \alpha_{jl} k_l \otimes k_j$ and $\lambda = \sum_{j,l=1}^{n} \langle L(x_{jl}) k_l, k_j \rangle$ then

$$\sum_{j,l=1}^{n} \langle (L' \otimes \mathrm{Id}_{M_n(\mathbb{R})})(X) k_l, k_j \rangle = \sum_{j,l=1}^{n} \langle L'(\alpha_{jl} x_0 + x_{jl}) k_l, k_j \rangle =$$

$$= \sum_{j,l=1}^{n} \alpha_{jl} \langle L(x_0) k_l, k_j \rangle + \sum_{j,l=1}^{n} \langle L(x_{jl}) k_l, k_j \rangle = f((y, 0)) + \lambda = f((y, \lambda)).$$

Since $(y, \lambda) \in C$, we have that $f((y, \lambda)) \geq 0$ which implies the claim. □

Theorem 4 is a generalization of Theorem 3. It is also a generalization of [37, Proposition 2.1], where the author studies the case $\mathcal{H} = \mathbb{C}$.

Theorem 4. *If $\mathcal{H}, \mathcal{K}$ are Hilbert spaces, X is a closed set in $\mathbb{R}^d$ and*

$$L\colon \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h \to B(\mathcal{K})_h$$

is a linear map such that

$$L \otimes \mathrm{Id}_{M_n(\mathbb{R})}(G) \succeq 0$$

for every integer $n \in \mathbb{N}$ and every symmetric polynomial $G \in \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h \otimes M_n(\mathbb{R})$ such that $G(a) \succeq 0$ for every $a \in X$, then there exists a non-negative Borel measure

$$m\colon \operatorname{Bor}(X) \to \mathcal{L}(B(\mathcal{H})_h, B(\mathcal{K})_h)$$

such that for every $F \in \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})$

$$L(F) = \int F\, dm.$$

Proof. With the notation from the proof of Theorem 3, we have that $E_0 = A_0 \otimes B(\mathcal{H})_h$ is cofinal in $E = C'(X, \mathbb{R}) \otimes B(\mathcal{H})_h$ where $K_n(E)$ consists of all elements of $M_n(E)$ which are positive semidefinite in every point of X. Furthermore, the mapping $\bar{L}\colon E_0 \to B(\mathcal{K})_h$ defined by $\bar{L}(\hat{p} \otimes B) := L(p \otimes B)$ is completely positive by assumption. By Proposition 4, there exists a completely positive extension of $\bar{L}$ to E. As in the proof of Theorem 3, the restriction of $\bar{L}$ from E to $C_c(X, \mathbb{R}) \otimes B(\mathcal{H})_h$ is bounded. By Proposition 2, it has the desired integral representation.

It remains to show that this integral representation also works on E. By linearity, it suffices to take $F = f \otimes B$ where $f \in C'(X, \mathbb{R})_+$ and $B \in B(\mathcal{H})_+$ are arbitrary. Let p and f_i be as in the proof of Theorem 3 and let $x \in \mathcal{K}$ be arbitrary. Then

$$\langle \bar{L}(F)x, x\rangle = \langle \bar{L}_B(f)x, x\rangle = \lim_{i\to\infty} \langle \bar{L}_B(f_i)x, x\rangle.$$

Since $\bar{L}_B(f_i) = \int f_i\, dE_B$, it follows by the monotone convergence theorem that

$$\lim_{i\to\infty} \langle \bar{L}_B(f_i)x, x\rangle = \lim_{i\to\infty} \int f_i\, d(E_B)_x = \int f\, d(E_B)_x.$$

It follows that f is E_B-integrable (with $K_f = \|\bar{L}(F)\|$; see Remark 4). Therefore,

$$\int f\, d(E_B)_x = \langle (\int f\, dE_B)x, x\rangle = \langle (\int F\, dm)x, x\rangle.$$

Since x was arbitrary, we have that $\bar{L}(F) = \int F\, dm$ as claimed. □

Remark 6. If X is compact, we can replace the complete positivity assumption in Theorem 4 with the weaker positivity assumption, see Theorem 5 below. This can also be done if $\mathcal{H} = \mathbb{R}$ and $\dim \mathcal{K} < \infty$ and X is either $\mathbb{R}$ or $[0, \infty)$, see [43, 44].

4. Schmüdgen's theorem

Let $\mathcal{H}$ be a Hilbert space. A subset $\mathcal{M} \subseteq \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h$ is a *quadratic module* if $\mathrm{Id}_{\mathcal{H}} \in \mathcal{M}$, $\mathcal{M} + \mathcal{M} \subseteq \mathcal{M}$ and $A^*\mathcal{M}A \subseteq \mathcal{M}$ for every $A \in \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})$. The smallest quadratic module which contains a given subset $\mathcal{G}$ of $\mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h$ will be denoted by $\mathcal{M}_{\mathcal{G}}$. For $\mathcal{H} = \mathbb{R}$ we get the definition of a quadratic module in $\mathbb{R}[\underline{x}]$.

A quadratic module $\mathcal{M}$ in $\mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h$ is *archimedean* if for every operator polynomial $F \in \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h$ there exists a number $n \in \mathbb{N}$ such that $n \cdot \mathrm{Id}_{\mathcal{H}} \pm F \in$

$\mathcal{M}$. If M is an archimedean quadratic module in $\mathbb{R}[\underline{x}]$ then the set M' which consists of all finite sums of elements of the form mA^TA where $m \in M$ and $A \in \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})$ is clearly an archimedean quadratic module in $\mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h$.

Theorem 5 is an operator version of the Putinar's part of Theorem 2.

Theorem 5. *Let $L : \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h \to B(\mathcal{K})_h$ be a linear operator, $M \subseteq \mathbb{R}[\underline{x}]$ an archimedean quadratic module and $K_M := \{\underline{x} \in \mathbb{R}^d \mid p(\underline{x}) \succeq 0$ for all $p \in M\}$. Then the following statements are equivalent:*

(1) *There exists a unique non-negative operator-valued measure*

$$m : \mathrm{Bor}(K_M) \to \mathcal{L}(B(\mathcal{H})_h, B(\mathcal{K})_h),$$

such that

$$L(F) = \int_{K_M} F\, dm$$

holds for all $F \in \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h$.

(2) *$L(mA^TA) \succeq 0$ for every $m \in M$ and $A \in \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})$ (i.e., $L(M') \succeq 0$).*

For an archimedean quadratic module M in $\mathbb{R}[\underline{x}]$ we define a set $\overline{M'} = \{F \in R[\underline{x}] \otimes B(\mathcal{H})_h \mid \epsilon + F \in M'$ for all $\epsilon > 0\}$. In the sequel, we will need the following version of the Scherer-Hol theorem, which is a special case of [10, Theorem 12].

Proposition 5. *Let M be an archimedean quadratic module in $\mathbb{R}[\underline{x}]$ and $\mathcal{H}$ a Hilbert space. For every element $F \in \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h$, the following are equivalent:*

(1) *$F \in \epsilon + M'$ for some real $\epsilon > 0$.*
(2) *For every $a \in K_M$ we have that $F(a) \succ 0$.*

Proof of Theorem 5. Clearly, (1) implies (2). Suppose now that (2) is true. Our plan is to extend L to a positive bounded linear map from $C(K_M, \mathbb{R}) \otimes B(\mathcal{H})_h$ to $B(\mathcal{K})_h$ and then apply Proposition 2. This will prove that (1) is true. Recall that the norm and the positive cone of $C(K_M, \mathbb{R}) \otimes B(\mathcal{H})_h$ are inherited from $C(K_M, B(\mathcal{H})_h)$, i.e., $\|F\| = \sup_{a \in K_M} \|F(a)\|$ and $F \geq 0$ iff $F(a) \succeq 0$ for every $a \in K_M$.

Let A_0 be the range of the natural mapping $\hat{}: \mathbb{R}[\underline{x}] \to C(K_M, \mathbb{R})$. For every $F = \sum_i p_i \otimes A_i \in \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h$ we will write $\hat{F} := \sum_i \hat{p}_i \otimes A_i \in C(K_M, \mathbb{R}) \otimes B(\mathcal{H})_h$. We define a linear map $\bar{L}\colon A_0 \otimes B(\mathcal{H})_h$ by $\bar{L}(\hat{F}) := L(F)$. To see that $\bar{L}$ is well-defined and positive, note that if $\hat{F} \succeq 0$ on K_M, then $F \in \overline{M'}$ by Proposition 5. Now, (2) implies that $L(F) \succeq 0$.

Next, we show that $\bar{L}$ is bounded. For every $v \in \mathcal{K}$, where $\|v\| = 1$, we define a functional $\bar{L}_v : A_0 \to \mathbb{R}$ by $\bar{L}_v(\hat{F}) = \langle L(F)v, v\rangle$. Since $\bar{L}_v(M') \geq 0$, it follows that

$$|\bar{L}_v(\hat{F})| \leq n_{M'}(F)\ \bar{L}_v(1),$$

where

$$n_{M'}(F) = \inf\left\{q \in \mathbb{Q}^+ \mid q \cdot \mathrm{Id} \pm F \in M'\right\}.$$

It follows that

$$\|\bar{L}(\hat{F})\| = \max_{\|v\|=1} |\bar{L}_v(\hat{F})| \leq n_{M'}(F) \max_{\|v\|=1} \bar{L}_v(\hat{\mathrm{Id}}) = n_{M'}(F)\, \|\bar{L}(\hat{\mathrm{Id}})\|.$$

By Proposition 5, $n_{M'}(F) = \|\hat{F}\|$. Hence $\|\bar{L}(\hat{F})\| \leq \|\hat{F}\|\|\bar{L}(\hat{\mathrm{Id}})\|$ for every $\hat{F}$. Therefore $\bar{L}$ is bounded.

By the Stone-Weierstrass theorem, A_0 is dense in $C(K_M, \mathbb{R})$. It follows that $A_0 \otimes B(\mathcal{H})_h$ is dense in $C(K_M, \mathbb{R}) \otimes B(\mathcal{H})_h$. Therefore, $\bar{L}$ has a unique extension to a positive bounded map from $C(K_M, \mathbb{R}) \otimes B(\mathcal{H})_h$ to $B(\mathcal{K})_h$ by continuity. □

Let us recall from [9] that a quadratic module $\mathcal{M}$ in $S_n(\mathbb{R}[\underline{x}])$ is a *preordering* if the set $E_{11}\mathcal{M}E_{11}$ (or equivalently the set $\mathcal{M} \cap \mathbb{R}[\underline{x}] \cdot \mathrm{I}_n$) is closed under multiplication. The smallest preordering which contains a given set $\mathcal{G} \subseteq S_n(\mathbb{R}[\underline{x}])$ will be denoted by $\mathcal{T}_\mathcal{G}$. We will prove the following matrix version of the Schmüdgen's part of Theorem 2.

Theorem 6. *Suppose that* $\mathcal{G} = \{G_1, G_2, \ldots, G_k\} \subseteq S_n(\mathbb{R}[\underline{x}])$ *are such that the set* $K_\mathcal{G} := \left\{\underline{x} \in \mathbb{R}^d \mid G_1(\underline{x}) \succeq 0, G_2(\underline{x}) \succeq 0, \ldots, G_k(\underline{x}) \succeq 0\right\}$ *is compact. Then:*

1. *The preordering* $\mathcal{T}_\mathcal{G}$ *is an archimedean quadratic module.*
2. *Every* $F \in S_n(\mathbb{R}[\underline{x}])$ *which satisfies* $F(\underline{x}) \succ 0$ *on* $K_\mathcal{G}$ *belongs to* $\mathcal{T}_\mathcal{G}$.
3. *For every Hilbert space* $\mathcal{K}$ *and every linear map* $L\colon S_n(\mathbb{R}[\underline{x}]) \to B(\mathcal{K})_h$ *such that* $L(\mathcal{T}_\mathcal{G}) \succeq 0$ *there exists a unique non-negative measure* $m : \mathrm{Bor}(K_\mathcal{G}) \to \mathcal{L}(S_n(\mathbb{R}), B(\mathcal{K})_h)$ *such that* $L(F) = \int_{K_\mathcal{G}} F\, dm$ *for every* $F \in S_n(\mathbb{R}[\underline{x}])$.

The following special case of [9, Proposition 5] will be used in the proof:

Proposition 6. *For every subset* $\mathcal{G} \subseteq S_n(\mathbb{R}[\underline{x}])$ *there exists a subset* $\tilde{\mathcal{G}} \subseteq \mathcal{M}_\mathcal{G} \cap \mathbb{R}[\underline{x}] \cdot \mathrm{I}_n$ *such that* $K_\mathcal{G} = K_{\tilde{\mathcal{G}}}$. *If* $\mathcal{G}$ *is finite, then* $\tilde{\mathcal{G}}$ *can also be chosen finite.*

Proof of Theorem 6. By Proposition 6, there exist $g_1, g_2, \ldots, g_k \in \mathbb{R}[\underline{x}]$ such that $K_\mathcal{G} = K_{\{g_1 \cdot \mathrm{I}_n, g_2 \cdot \mathrm{I}_n, \ldots, g_k \cdot \mathrm{I}_n\}} = K_{\{g_1, g_2, \ldots, g_k\}}$ and $g_1 \cdot \mathrm{I}_n, g_2 \cdot \mathrm{I}_n, \ldots, g_k \cdot \mathrm{I}_n \in \mathcal{M}_\mathcal{G}$. Since $K_\mathcal{G}$ is compact, it follows by Theorem 2 that $T_{\{g_1, g_2, \ldots, g_k\}}$ is an archimedean preordering in $\mathbb{R}[\underline{x}]$. Now $\mathcal{T}_\mathcal{G}$ is an archimedean because it contains the archimedean quadratic module $\left(T_{\{g_1, g_2, \ldots, g_k\}}\right)'$. This proves claim (1). Claim (2) follows from claim (1) and Proposition 5. Claim (3) follows from claim (1) and Theorem 5. □

5. An example

Let $\mathcal{H}$ be a Hilbert space. A quadratic module $\mathcal{T} \subseteq \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h$ is a *preordering* if for some (and hence every) rank one projector $P \in B(\mathcal{H})_h$ the set $P\mathcal{T}P$ is closed under multiplication. Recall that P is the form $P_u : x \to \langle x, u\rangle\, u$ for some $u \in \mathcal{H}$ of norm 1. Moreover, $P_{Su} = SP_uS^*$ and $P_uSP_u = \langle Su, u\rangle\; P_u$ for all $S \in \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})$. For a subset $\mathcal{G}$ of $\mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h$ write $\mathcal{T}_\mathcal{G}$ for the smallest preordering containing $\mathcal{G}$.

Lemma 1. *Let* $\mathcal{G}$ *be a subset of* $\mathbb{R}[\underline{x}] \otimes B(\mathcal{H})_h$ *and* u *an element of* $\mathcal{H}$ *of norm* 1. *Write* $\mathcal{G}_u$ *for the set of all finite products of elements of the form*

$$P_u S^* G S P_u = \langle GSu, Su\rangle P_u$$

where $G \in \mathcal{G} \cup \{\mathrm{Id}\}$ *and* $S \in \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})$. *Then*

$$\mathcal{T}_\mathcal{G} = \mathcal{M}_{\mathcal{G} \cup \mathcal{G}_u}.$$

Proof. The inclusion $\mathcal{M}_{\mathcal{G}\cup\mathcal{G}_u} \subseteq \mathcal{T}_\mathcal{G}$ is clear. To prove the opposite inclusion, it suffices to show that the quadratic module $\mathcal{M}_{\mathcal{G}\cup\mathcal{G}_u}$ is a preordering. Every element $F \in \mathcal{M}_{\mathcal{G}\cup\mathcal{G}_u}$ is of the form $F = \sum_i R_i^* G_i R_i + \sum_j S_j^* H_i S_j$ where $G_i \in \mathcal{G} \cup \mathrm{Id}$, $H_j \in \mathcal{G}_u$, $R_i, S_j \in \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})$ and both sums are finite. It follows that $P_u F P_u = \sum_i P_u R_i^* G_i R_i P_u + \sum_j P_u S_j^* P_u H_i P_u S_j P_u = \sum_i P_u R_i^* G_i R_i P_u + \sum_j H_i P_u S_j^* P_u^2 S_j P_u$ is a finite sum of elements from $\mathcal{G}_u$. Therefore, the set $P_u \mathcal{M}_{\mathcal{G}_u} P_u = \sum_{\text{finite}} \mathcal{G}_u$ is closed under multiplication. □

Note that for every $f \in \mathbb{R}[\underline{x}] \otimes \mathcal{H}$ and every $u \in \mathcal{H}$ of norm 1 there exists an element $F \in \mathbb{R}[\underline{x}] \otimes B(\mathcal{H})$ such that $f = Fu$. It follows that the set $\mathcal{G}_u$ consists of all finite products of elements of the form $\langle Gf, f\rangle P_u$ where $G \in \mathcal{G} \cup \{\mathrm{Id}\}$ and $f \in \mathbb{R}[\underline{x}] \otimes \mathcal{H}$.

5.1. **Construction of a compact non-archimedean preordering.** We define polynomials $p_i(x) = \frac{x^3}{i} - x^2$, $i \in \mathbb{N}$. We have $K_{\{p_i\}} = \{0\} \cup [i, \infty)$. Let us define operator polynomial $G(x) \in \mathbb{R}[x] \otimes B(\ell^2)$ as

$$G(x) = \mathrm{diag}(p_1(x), p_2(x), \ldots),$$

which is equivalent to

$$G = x^3 \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} - x^2 \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We have $K_{\{G\}} = \{0\}$. Let $u = (1, 0, 0, \ldots)$. Clearly, the leading coefficient of G as well as the leading coefficients of all elements from $\{G\}_u$ are positive semidefinite operators. It follows that the leading coefficient of every element from $\mathcal{T}_{\{G\}} = \mathcal{M}_{\{G\} \cup \{G\}_u}$ is a positive semidefinite operator. Therefore, $T_{\{G\}}$ does not contain $(K^2 - x^2)\,\mathrm{Id}$ for any real K. It follows that the preordering $T_{\{G\}}$ is not archimedean. Moreover, the operator polynomial $(1 - x^2)\,\mathrm{Id}$ is positive definite on $K_{\{G\}} = \{0\}$ but it does not belong to $T_{\{G\}}$.

This proves that assertions (1) and (2) of Theorem 6 do not extend from matrix polynomials to operator polynomials. It is still an open question whether assertion (3) of Theorem 6 extends from matrix polynomials to operator polynomials.

We claim that in our example, every functional L on $\mathbb{R}[x] \otimes B(\ell^2)$ such that $L(\mathcal{T}_{\{G\}}) \geq 0$ has an integral representation. Let $S\colon (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots)$ be the shift operator. Note that for every $n \in \mathbb{N}$, $S^n G (S^*)^n = A_n x^3 - \mathrm{Id}\, x^2$ where

$$A_n = \begin{pmatrix} \frac{1}{n+1} & 0 & 0 & \cdots \\ 0 & \frac{1}{n+2} & 0 & \cdots \\ 0 & 0 & \frac{1}{n+3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \preceq \frac{1}{n+1}\,\mathrm{Id}$$

Since $L(\mathcal{T}_{\{G\}}) \geq 0$, it follows that $L(A_n x^3) - L(\mathrm{Id}\, x^2) = L(S^n G (S^*)^n) \geq 0$ for every n. By the Cauchy-Schwartz inequality, it follows that $0 \leq L(\mathrm{Id}\, x^2) \leq L(A_n x^3) \leq L(A_n^2)^{1/2} L(\mathrm{Id}\, x^6)^{1/2} \leq \frac{1}{(n+1)} L(\mathrm{Id})^{1/2} L(\mathrm{Id}\, x^6)^{1/2}$. In the limit, we get that $L(\mathrm{Id}\, x^2) = 0$. Using Cauchy-Schwartz again, we deduce that $L(x^k B_k) = 0$ for every $k \in \mathbb{N}$ and $B_k \in B(\ell^2)$. Therefore, for every $F = \sum_{k=0}^m x^k B_k$, we have that $L(F) = L(B_0) = L|_{B(\ell^2)}(F(0))$. Therefore L has a representing measure which assigns to the set $\{0\}$ the functional $L|_{B(\ell^2)}$.

References

[1] C.-G. Ambrozie, F.-H. Vasilescu, Operator-theoretic Positivstellensätze, Z. Anal. Anwend. 22 (2003), No. 2, 299–314.

[2] N.I. Akhiezer, The Classical Moment Problem, Oliver and Boyd, New York, 1965.

[3] N.I. Akhiezer, I. Glazman, Theory of Linear Operators in Hilbert Space, Dover Publications, New York, 1993.

[4] A. Athavale, Holomorphic kernels and commuting operators, Trans. Amer. Math. Soc. 304 (1987) 101–110.
[5] R. G. Bartle, N. Dunford, J. Schwartz. Weak compactness and vector measures, Canad. J. Math. 7 (1955) 289–305.
[6] S. K. Berberian, Notes on Spectral Theory, D. van Nostrand Company, Princeton, 1966.
[7] J. Cimprič, A representation theorem for archimedean quadratic modules on *-rings, Can. math. bull. 52 (2009) 39–52.
[8] J. Cimprič, Strict Positivstellensätze for matrix polynomials with scalar constraints, Linear algebra appl., 434 (2011), iss. 8, 1879–1883.
[9] J. Cimprič, Real algebraic geometry for matrices over commutative rings, J. Algebra 359 (2012), 89–103.
[10] J. Cimprič, Archimedean operator-theoretic Positivstellensätze, J. Funct. Anal. 260 (2011), no. 10, 3132—3145.
[11] J. Conway, A Course in Functional Analysis, Springer-Verlag, New York, 1990.
[12] H. Dette, W. J. Studden, Matrix measures, moment spaces and Favard's theorem for the interval $[0, 1]$ and $[0, \infty)$, Linear Algebra Appl. 345 (2002), 169—193.
[13] I. Dobrakov, On representation of linear operators on $C_0(T, X)$, Czechoslovak Math. J. 21 (1971) 13–30.
[14] I. Dobrakov, On integration in Banach spaces I, Czechoslovak Math. J. 20 (1970) 511–536.
[15] M. A. Dritschel, J. Rovnyak, The operator Fejér-Riesz theorem, Oper. Theory Adv. Appl. 207 (2010) 223–254.
[16] W. Forrest Stinespring, Positive functions on C^*-algebras, Proc. Amer. Math. Soc. 6 (1955) 211–216.
[17] C.W.J. Hol, C.W. Scherer, Matrix sum-of-squares relaxations for robust semi-definite programs, Math. Programming 107 (2006) 189–211.
[18] T. Jacobi, A. Prestel, Distinguished representations of strictly positive polynomials, J. Reine Angew. Math. 532 (2001) 223—235.
[19] G. W. Johnson, The dual of C(S,F), Math. Ann. 187 (1970) 1–8.
[20] I. Klep, M. Schweighofer, Pure states, positive matrix polynomials and sums of hermitian squares, Indiana Univ. Math. J. 59 (2010),No. 3, 857–874.
[21] I. V. Kovalishina, Analytic theory of a class of interpolation problems, Izv. Akad. Nauk SSSR Ser. Mat. 47 (1983), no. 3, 455—497.
[22] M. Krein, Infinite J-matrices and a matrix-moment problem, Doklady Akad. Nauk SSSR (N.S.) 69 (1949) 125—128.
[23] P.D. Lax, Functional Analysis, John Wiley & Sons, New York, 2002.
[24] B. Li, Real Operator Algebras, World Scientific Publishing, Singapore, 2003.
[25] A.J. Durán, P. López-Rodríguez, The matrix moment problem, Margarita mathematica (2001) 333—348.
[26] M. Marshall, Positive Polynomials and Sums of Squares, American Mathematical Society, Providence, 2008.
[27] R. Phelps, Lectures on Choquet's Theorem, Springer-Verlag, Berlin, 2001.
[28] R.T. Powers, Selfadjoint algebras of unbounded operators. II, Trans. Amer. Math. Soc. 187 (1974), 261–293.
[29] M. Putinar, Positive polynomials on compact semi-algebraic sets, Indiana Univ. Math. J. 42 (1993), no. 3, 969—984.
[30] M. Putinar, C. Scheiderer, Multivariate moment problems: Geometry and indeterminateness, Ann. Sc. Norm. Super. Pisa Cl. Sci. 5 (2006), no. 2, 137—157.
[31] M. Putinar, K. Schmüdgen, Multivariate determinateness, Indiana Univ. Math. J. 57 (2008), no. 6, 2931—2968.
[32] F. Riesz, B.Sz. Nagy, Functional Analysis, Blackie & Son Limited, London, 1956.
[33] M. Rosenblatt, A multi-dimensional prediction problem, Ark. Mat. 3 (1958), 407—424.
[34] M. Rosenblum, Vectorial Toeplitz operators and the Fejér-Riesz theorem, J. Math. Anal. Appl. 23 (1968) 139-–147.
[35] W. Rudin, Functional Analysis, McGraw-Hill, New York, 1973.
[36] S. K. Mitter, S. K. Young, Integration with respect to operator-valued measures with applications to quantum estimation theory, `http://dspace.mit.edu/handle/1721.1/2845`.
[37] K. Schmüdgen, On a generalization of the classical moment problem, J. Math. Anal. Appl. 125 (1987) 461–470.

[38] K. Schmüdgen, The K-moment problem for compact semi-algebraic sets. Math. Ann. 289 (1991), no. 2, 203—206.
[39] K. Schmüdgen. *Unbounded operator algebras and representation theory.* Operator Theory: Advances and Applications, 37. Basel etc.: Birkhäuser Verlag. 1989.
[40] Y. Savchuk, K. Schmüdgen, Positivstellensätze for Algebras of Matrices, arXiv:1004.1529 (Preprint, April 2010).
[41] F.-H. Vasilescu, Subnormality and moment problems, Extracta Math. 24 (2009) 167–186.
[42] F.-H. Vasilescu, Spectral measures and moment problems, in the volume *Spectral theory and its applications* (2003) 173–215.
[43] S.M. Zagorodnyuk, The matrix Stieltjes moment problem: a description of all solutions, `http://arxiv.org/pdf/1002.4511.pdf`, 24.8.2012.
[44] S.M. Zagorodnyuk, A description of all solutions of the matrix Hamburger moment problem in a general case, `http://arxiv.org/pdf/1002.4511.pdf`, 24.8.2012.

Jaka Cimprič, University of Ljubljana, Faculty of Math. and Phys., Dept. of Math., Jadranska 19, SI-1000 Ljubljana, Slovenija. www page: http://www.fmf.uni-lj.si/~cimpric.
Email address: `cimpric@fmf.uni-lj.si`

Aljaž Zalar, University of Ljubljana, Faculty of Math. and Phys., Dept. of Math.
Email address: `aljaz.zalar@student.fmf.uni-lj.si`