OPERATOR FEJÉR–RIESZ THEOREMS WITH GAPS AND DEGREE BOUNDS

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ABSTRACT. Let K be a nonempty compact semialgebraic subset of the real line, and let S be a saturated scalar description of K. We prove that every hermitian polynomial with bounded-operator coefficients that is positive semidefinite on K belongs to the operator quadratic module generated by S. We also show that if such a polynomial has degree at most $\mathtt{d}$, then it admits a certificate of weighted degree at most $\mathtt{d} + c_S$, where c_S depends only on S, not on the polynomial or the coefficient Hilbert space.

In contrast to the scalar case, where natural preorderings admit the degree bound $\mathtt{d}$ independently of K, we show that no uniform additive constant exists for natural matrix preorderings. We construct quadratic 2×2 matrix polynomials positive semidefinite on unions of two intervals whose minimum certificate degrees in the corresponding natural preorderings diverge as one interval shrinks to a point, while the gap length remains bounded below. Conversely, we establish uniform bounds of the form $\mathtt{d} + C$ for families of interval unions contained in a fixed compact interval when interval lengths and gap lengths have uniform positive lower bounds.

Our proofs develop two techniques that may be of independent interest. First, a quadratic reduction principle represents arbitrary-degree operator polynomials as compressions of quadratic ones on enlarged Hilbert spaces. A perturbation argument and the closedness of truncated operator quadratic modules then extend uniform degree bounds from the quadratic case to arbitrary degrees with linear degree bounds of slope 1. Second, a rank-one symmetric matrix pencil transfers positivity on interval unions to $[0, 1]$, where the operator Fejér–Riesz theorem yields certificates.

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1. Introduction

1.1. **Background and motivation.** Positivity and sums of squares play a central role in real algebraic geometry and operator theory. The search for algebraic certificates of polynomial nonnegativity goes back to Hilbert's 17^{th} problem and is closely related with moment theory and the Positivstellensätze of Schmüdgen and Putinar [Sch91, Put93]; for classical results and modern treatment see [BCR98, Mar08, Sc24]. Two fundamental questions arise in this context: when does a nonnegative polynomial admit a sums-of-squares certificate, and how can one bound the degree of such a certificate?

The Fejér–Riesz theorem provides a sharp answer to both questions for trigonometric polynomials in one variable: nonnegativity is equivalent to a factorization $P = Q^*Q$, with Q analytic and $\deg Q \leq \deg P$. This extends to polynomials with bounded-operator coefficients on arbitrary Hilbert spaces [Ros68, DR10]. Fejér–Riesz factorization plays an important role in the operator-algebraic theory of Toeplitz operator systems [CvS21, Far21, DJ26]. Related developments include multivariable factorization [GW04, Dri04, DW05, DKMV26] and noncommutative positivity and factorization results [Hel02, McC01].

While the Fejér–Riesz theorem concerns positivity on the entire unit circle (or, in its real-line formulation, on the entire real line), applications often require positivity on prescribed subsets of $\mathbb{R}$. Such problems arise in systems and control theory, where finite-frequency generalizations of the Kalman–Yakubovich–Popov lemma address positivity on restricted frequency intervals [IH05, PIH14].

We study hermitian polynomials with bounded-operator coefficients that are positive semidefinite on arbitrary compact semialgebraic subsets of the real line, that is, unions of intervals and isolated points. We ask whether such polynomials admit exact positivity certificates in the corresponding operator quadratic modules, whether the certificate degrees can be bounded independently of the coefficient Hilbert space, and how these bounds depend on the underlying semialgebraic set.

1.2. **Notation.** Let $\mathcal{H}$ be a real or complex Hilbert space, and let $\mathcal{B}(\mathcal{H})[x]$ denote the polynomials in a real indeterminate x with coefficients in the bounded operators on $\mathcal{H}$. For a nonzero polynomial $F(x) = \sum_{k=0}^{\mathtt{d}} A_k x^k$, with $A_{\mathtt{d}} \neq 0$, we define $\deg F = \mathtt{d}$ and set $\deg 0 = -\infty$. We write

$$F^*(x) = \sum_{k=0}^{\mathtt{d}} A_k^* x^k$$

and call F *hermitian* if $F^* = F$, or equivalently, if $A_k^* = A_k$ for every k. Finally, we write $F \succeq 0$ on K if $F(x)$ is positive semidefinite for every $x \in K$.

For a complex Hilbert space $\mathcal{H}$, let $\mathrm{SOS}_{\mathcal{H}} = \{G^*G : G \in \mathcal{B}(\mathcal{H})[x]\}$. For a real Hilbert space, we use $\mathrm{SOS}_{\mathcal{H}} = \{G^*G + H^*H : G, H \in \mathcal{B}(\mathcal{H})[x]\}$. The fact that these definitions agree with the usual finite sums of hermitian squares follows from the factorization results recalled in Section 2 and the real descent argument in Appendix A. We also set $\mathrm{SOS}_{\mathbb{C}} = \{f^*f : f \in \mathbb{C}[x]\} \subset \mathbb{R}[x]$.

Let $S = \{g_1, \dots, g_s\} \subseteq \mathbb{R}[x]$ be finite, and put $g_0 = 1$. Define

$$K_S = \{x \in \mathbb{R} : \ g_j(x) \geq 0, \ 1 \leq j \leq s\},$$

$$M(S) = \left\{ \sum_{j=0}^{s} g_j \sigma_j \ : \ \sigma_j \in \mathrm{SOS}_{\mathbb{C}} \right\},$$

$$\mathcal{M}_{\mathcal{H}}(S) = \left\{ \sum_{j=0}^{s} g_j \Sigma_j \ : \ \Sigma_j \in \mathrm{SOS}_{\mathcal{H}} \right\}.$$

We call $M(S)$ the *quadratic module generated by* S and $\mathcal{M}_{\mathcal{H}}(S)$ the *operator quadratic module generated by* S. The module $M(S)$ is *saturated* if it contains every real polynomial nonnegative on K_S; in this case, S is called a *saturated scalar description* of K_S.

The *weighted degree* of an expression $\sum_j g_j \Sigma_j$ is the maximum of the degrees of its summands. We write $\mathcal{M}_{\mathcal{H}}(S)_{\leq n}$ for the polynomials admitting such a certificate of weighted degree at most n. The *operator preordering* generated by S is

$$\mathcal{T}_{\mathcal{H}}(S) = \left\{ \sum_{\alpha \in \{0,1\}^s} g^\alpha \Sigma_\alpha \ : \ \Sigma_\alpha \in \mathrm{SOS}_{\mathcal{H}} \right\}, \qquad g^\alpha = \prod_{j=1}^{s} g_j^{\alpha_j}.$$

The corresponding truncated preordering, with weighted degree at most n, is denoted by $\mathcal{T}_{\mathcal{H}}(S)_{\leq n}$.

These definitions also apply when $\mathcal{H} = \mathbb{C}^m$, in which case we speak of matrix quadratic modules and matrix preorderings. For $n \geq 0$, we write $\mathbb{R}[x]_{\leq n}$ for the polynomials of degree at most n, and use $\mathbb{N} = \{1, 2, \dots\}$.

1.3. **Main results.** We are now ready to state our first result, which is the main contribution of this paper.

Theorem A. Let $S = \{g_1, \dots, g_s\} \subseteq \mathbb{R}[x]$ be finite, with K_S nonempty and compact. Suppose that the scalar quadratic module $M(S)$ is saturated. Then, for every real or complex Hilbert space $\mathcal{H}$, every nonnegative integer $\mathtt{d}$, and every hermitian $F \in \mathcal{B}(\mathcal{H})[x]$ with $\deg F \leq \mathtt{d}$,

$$F \succeq 0 \text{ on } K_S \quad \Longleftrightarrow \quad F = \sum_{j=0}^{s} g_j \sum_{i=1}^{r} F_{j,i}^* F_{j,i},$$

where $g_0 = 1$, $F_{j,i} \in \mathcal{B}(\mathcal{H})[x]$, and $r = 1$ if $\mathcal{H}$ is complex and $r = 2$ if $\mathcal{H}$ is real. Moreover, there exists a nonnegative integer c_S, depending only on S, such that the factors can be chosen to satisfy

$$\deg(g_j F_{j,i}^* F_{j,i}) \leq \mathtt{d} + c_S \qquad (0 \leq j \leq s, \ 1 \leq i \leq r).$$

Remark 1.1. Several remarks related to Theorem A are in order.

(a) Every nonempty compact semialgebraic set $K \subseteq \mathbb{R}$ admits a saturated scalar description, and Scheiderer characterized all such descriptions [Sc03, Sc05b]; see Theorem 2.2. For natural descriptions, the scalar preorderings are saturated [KM02] and admit certificates of weighted degree at most $\deg f$ [KMS05].

(b) For a saturated scalar description S of a compact semialgebraic set, the second named author [Zal16] proved a non-quantitative version of Theorem A for matrix polynomials. Sun and the second named author [SZ26] obtained degree bounds, independent of matrix size, when the set consists of a compact interval and an isolated point.

(c) The degree bound in Theorem A is optimal up to a fixed additive constant. Indeed, fix $a < \min K_S$. For every $\mathtt{d} \geq 0$, the polynomial $(x-a)^{\mathtt{d}}$ is nonnegative on K_S, and any certificate for it must have weighted degree at least $\mathtt{d}$. Thus every uniform degree-bound function must be bounded below by $\mathtt{d}$, whereas Theorem A provides the upper bound $\mathtt{d} + c_S$.

(d) If the generators have real algebraic coefficients, the proof also gives a way to compute an upper bound on the constant c_S by exact semidefinite feasibility tests; see Subsection 5.4.

(e) For compact semialgebraic sets of dimension at least two, the weighted degree of positivity certificates generally cannot be bounded solely in terms of the polynomial degree and the fixed defining generators, even in the scalar case [Sc05].

(f) The classical Positivstellensätze of Schmüdgen and Putinar provide certificates for strictly positive polynomials under compactness or Archimedean assumptions [Sch91, Put93]. The degree bounds typically depend on the polynomial degree, its coefficients, and its minimum on the underlying set [Schw04, NS07, BM23, BS24]. Related matrix and operator Positivstellensätze in several variables have been established under additional hypotheses [AV03, SH06, KS10, Cim11a, Cim11b]. Other developments include Positivstellensätze for matrix algebras [SS12, Cim12] and quantitative degree bounds for matrix polynomials [Hua25].

Although Theorem A gives degree bounds independent of the coefficient Hilbert space, the additive constant depends on the scalar description and thus on the underlying semialgebraic set. Our second result shows that this dependence cannot be avoided even for natural matrix preorderings and quadratic 2×2 polynomials.

Fix $\ell \in \mathbb{N}$. For a finite union of nondegenerate compact intervals

$$K = \bigcup_{j=1}^{\ell} [a_j, b_j], \qquad a_1 < b_1 < \cdots < a_\ell < b_\ell,$$

we write

$$S_K = \{x - a_1, b_\ell - x\} \cup \{(x - b_j)(x - a_{j+1}) \ : \ 1 \leq j < \ell\}$$

for its natural scalar description.

Theorem B. For each $0 < \varepsilon < 1/2$, there exist sets $K_\varepsilon \subseteq [0,1]$ and hermitian matrix polynomials $F_\varepsilon \in \mathbb{C}[x]^{2\times 2}$ of degree at most 2 such that the following hold:

(i) Each K_ε is a union of two disjoint compact intervals, one of which shrinks to a point as $\varepsilon \to 0$, while the gap between intervals has length bounded below independently of ε.

(ii) $F_\varepsilon \succeq 0$ on K_ε.

(iii) If S_{K_ε} is the natural description of K_ε, then

$$\min\{n \ : \ F_\varepsilon \in \mathcal{T}_{\mathbb{C}^2}(S_{K_\varepsilon})_{\leq n}\} \longrightarrow \infty \qquad (\varepsilon \to 0).$$

Theorem B reveals a sharp contrast with the scalar case: natural scalar preorderings admit certificates of weighted degree at most $\deg f$, independently of the underlying set [KMS05].

Our third result shows that uniform additive degree bounds can nevertheless be established for operator-valued polynomials when interval lengths and gap lengths are bounded away from zero.

For the interval union K above, set

$$\Delta_K(x) = -\prod_{j=1}^{\ell} (x - a_j)(x - b_j).$$

Theorem C. Fix a compact interval $J \subset \mathbb{R}$ and $\mu, \omega > 0$. Let $\mathcal{K}(J, \mu, \omega)$ be the family of all sets

$$K = \bigcup_{j=1}^{\ell} [a_j, b_j] \subseteq J, \qquad \ell \geq 1,$$

where $a_1 < b_1 < \cdots < a_\ell < b_\ell$, each interval has length at least μ, and each gap between consecutive intervals has length at least ω. There exists a nonnegative integer C, depending only on J, μ, ω, such that for every $K \in \mathcal{K}(J, \mu, \omega)$, every real or complex Hilbert space $\mathcal{H}$, every nonnegative integer $\mathtt{d}$, and every hermitian $F \in \mathcal{B}(\mathcal{H})[x]$ with $\deg F \leq \mathtt{d}$,

$$F \succeq 0 \text{ on } K \quad \Longleftrightarrow \quad F \in \mathcal{M}_{\mathcal{H}}(\{\Delta_K\})_{\leq \mathtt{d}+C} \quad \Longleftrightarrow \quad F \in \mathcal{T}_{\mathcal{H}}(S_K)_{\leq \mathtt{d}+C}.$$

Theorem B shows that the lower bound on interval lengths in Theorem C is essential, even when the gap lengths remain bounded below.

1.4. **Proof overview and organization.** The starting point of our proof is the quadratic reduction principle (Theorem 3.4). It shows that any degree bound for quadratic operator polynomials that is independent of the coefficient Hilbert space extends to a bound of the form $\mathtt{d} + c_S$ for polynomials of degree $\mathtt{d}$. The idea is to represent an arbitrary-degree operator polynomial as a compression of a quadratic polynomial on an enlarged Hilbert space. A suitable perturbation makes the lifted polynomial positive semidefinite, and the closedness of truncated operator quadratic modules allows us to pass to the limit. Although we present the lifting construction in the one-variable setting, it does not rely on the one-dimensional nature of the underlying domain and may also be useful in related multivariable problems.

The next task is to establish degree bounds for quadratic operator polynomials that are independent of the coefficient Hilbert space. For unions of nondegenerate compact intervals, we construct a symmetric matrix pencil by a rank-one perturbation of a diagonal matrix. As the parameter varies over $[0, 1]$, the eigenvalues of this pencil cover the prescribed intervals. This allows us to transfer positivity to $[0, 1]$ and apply the operator Fejér–Riesz theorem. However, pulling the resulting certificate back to the original variable introduces a scalar multiplier that vanishes in the gaps between the intervals. To overcome this difficulty, we characterize the polynomial congruences that preserve the quadratic degree and use them to obtain a strictly positive multiplier. Scalar saturation then allows us to remove this multiplier. We handle isolated points using Douglas' factorization lemma, polynomial interpolation, and certificates for auxiliary matrix polynomials. Both the auxiliary polynomials and their certificates depend only on the underlying set. Combining these results with the quadratic reduction principle completes the proof of Theorem A.

The proof of Theorem B uses a compactness argument, building on examples studied by the second named author [Zal16]. We construct quadratic 2×2 matrix polynomials on unions of two intervals, one of which shrinks to a point while the separating gap remains bounded below. If the certificate degrees remained bounded, the sums-of-squares multipliers would admit coefficientwise convergent subsequences. The resulting limiting identity, together with the vanishing conditions at the collapsing point, leads to a contradiction.

For Theorem C, we reduce the uniformity problem to scalar sums-of-squares identities. We construct representations in the interiors of fixed-degree SOS cones, so that they persist under small perturbations of the interval endpoints. Compactness then yields a degree bound for quadratic operator polynomials that is uniform over the prescribed family of interval unions. By the quadratic reduction principle, this single quadratic bound gives the required degree bounds for polynomials of arbitrary degree.

In Section 2 we collect the preliminary results. Section 3 establishes the quadratic reduction principle and treats finite sets. In Section 4 we construct quadratic certificates on unions of

nondegenerate intervals, while Section 5 incorporates isolated points and completes the proof of Theorem A. In Sections 6 and 7 we prove Theorems B and C, respectively.

2. Preliminaries

By Lemma A.1, it suffices to prove Theorem A and Theorem C for complex Hilbert spaces. We therefore work with complex Hilbert spaces throughout Sections 2–7; the real case follows with the same weighted degree bounds.

The following two consequences of the operator Fejér–Riesz theorem [DR10, Ros68] are well known. Proofs for real Hilbert spaces can be found in [CZ13]; the arguments for complex Hilbert spaces are essentially the same.

Proposition 2.1. *Let $\mathtt{d}$ be a nonnegative integer, and let $F \in \mathcal{B}(\mathcal{H})[x]$ be hermitian with $\deg F \le 2\mathtt{d}$.*

(i) *If $F \succeq 0$ on $\mathbb{R}$, then $F = G^*G$ for some $G \in \mathcal{B}(\mathcal{H})[x]$ with $\deg G \le \mathtt{d}$.*

(ii) *If $F \succeq 0$ on $[0,1]$, then there exist $G, H \in \mathcal{B}(\mathcal{H})[x]$ such that*

$$F = G^*G + x(1-x)H^*H, \qquad \deg G \le \mathtt{d}, \quad \deg H \le \mathtt{d}-1, \tag{2.1}$$

where $H = 0$ if $\mathtt{d} = 0$.

Fix $\ell \in \mathbb{N}$. Recall that every nonempty compact semialgebraic subset $K \subseteq \mathbb{R}$ can be written as

$$K = \bigcup_{j=0}^{\ell} [a_j, b_j], \qquad a_j \le b_j, \quad b_{j-1} < a_j \quad (1 \le j \le \ell), \tag{2.2}$$

where $a_0 \le b_0$. In particular, an interval $[a_j, b_j]$ with $a_j = b_j$ represents an isolated point. The following characterization of saturation follows from [Sc03, Theorem 5.17] and [Sc05b, Corollary 4.4].

Theorem 2.2. *Let $S = \{g_1, \dots, g_s\} \subseteq \mathbb{R}[x]$ be finite such that K_S is nonempty and compact, written as in (2.2). Then $M(S)$ is saturated if and only if the following conditions hold:*

(i) *For each $j = 0, \dots, \ell$, there exists $i \in \{1, \dots, s\}$ such that $g_i(a_j) = 0$, $g_i'(a_j) > 0$.*

(ii) *For each $j = 0, \dots, \ell$, there exists $i \in \{1, \dots, s\}$ such that $g_i(b_j) = 0$, $g_i'(b_j) < 0$.*

Scalar saturation on a compact subset of $\mathbb{R}$ extends to matrices of every finite size. Theorem 2.3 establishes matrix saturation without providing degree bounds.

Theorem 2.3 ([Zal16, Theorem 2.1])**.** *Let $S \subset \mathbb{R}[x]$ be finite, with K_S nonempty and compact. Assume that $M(S)$ is saturated. For every integer $m \ge 1$ and every hermitian $F \in \mathbb{C}[x]^{m\times m}$,*

$$F \succeq 0 \text{ on } K_S \quad \Longleftrightarrow \quad F \in \mathcal{M}_{\mathbb{C}^m}(S).$$

Theorem 2.2 yields the following scalar identity, which will allow us to cancel powers of a fixed strictly positive polynomial in operator positivity certificates.

Lemma 2.4. *Let $\Delta \in \mathbb{R}[x]$ be such that $K := \{x \in \mathbb{R} : \Delta(x) \ge 0\}$ is nonempty and compact. Assume that Δ has a simple zero at every boundary point of K. Then, for every $h \in \mathbb{R}[x]$ strictly positive on $\mathbb{R}$, there exist $\sigma_0, \sigma_1 \in \mathrm{SOS}_{\mathbb{C}}$ such that*

$$\Delta = \sigma_0 + h\Delta\sigma_1. \tag{2.3}$$

Proof. The polynomial $h\Delta$ defines the same set K and has the same derivative signs at its boundary as Δ. By Theorem 2.2, $M(\{h\Delta\})$ is saturated. Apply this to the polynomial Δ, which is nonnegative on K. □

We use the following cancellation argument for a strictly positive multiplier.

Lemma 2.5. *Let $h, \Delta \in \mathbb{R}[x]$ satisfy $h > 0$ on $\mathbb{R}$ and*

$$\Delta = \sigma_0 + h\Delta\sigma_1, \qquad \sigma_0, \sigma_1 \in \mathrm{SOS}_{\mathbb{C}}\,.$$

If $F \in \mathcal{B}(\mathcal{H})[x]$ is hermitian and $hF \in \mathcal{M}_{\mathcal{H}}(\{\Delta\})_{\le n}$, then

$$F \in \mathcal{M}_{\mathcal{H}}(\{\Delta\})_{\le \max\{\deg F, n+\deg\sigma_1\}}.$$

Proof. Write $hF = A^*A + \Delta B^*B$, with $2\deg A \le n$ and $\deg\Delta + 2\deg B \le n$. Substituting the scalar identity gives

$$h\big(F - \Delta\sigma_1 B^*B\big) = A^*A + \sigma_0 B^*B \succeq 0 \quad \text{on } \mathbb{R}. \tag{2.4}$$

Since $h > 0$ on $\mathbb{R}$, the hermitian polynomial $G = F - \Delta\sigma_1 B^*B$ is positive semidefinite on $\mathbb{R}$. By Proposition 2.1, $G = H^*H$ for some $H \in \mathcal{B}(\mathcal{H})[x]$. Thus

$$F = H^*H + \Delta\sigma_1 B^*B \;\in\; \mathcal{M}_{\mathcal{H}}(\{\Delta\}).$$

The degree of the second summand is at most $n + \deg\sigma_1$, while $\deg G \le \max\{\deg F, n + \deg\sigma_1\}$. This gives the stated bound. □

Remark 2.1. The following remarks clarify the role of nonnegativity and strict positivity of the multiplier h.

(a) The hypothesis $h > 0$ on $\mathbb{R}$ in Lemma 2.4 cannot in general be replaced by $h \ge 0$. Indeed, if $h(c) = 0$ at a point where $\Delta(c) < 0$, then the identity $\Delta = \sigma_0 + h\Delta\sigma_1$, with $\sigma_0, \sigma_1 \in \mathrm{SOS}_{\mathbb{C}}$, would imply $\Delta(c) = \sigma_0(c) \ge 0$, a contradiction. This obstruction can occur even if h is strictly positive on K. A zero of h at a boundary point a of K also prevents this identity. Since $\Delta(a) = 0$ and $\Delta'(a) \ne 0$, the identity would imply $\sigma_0(a) = 0$ and $\sigma_0'(a) = \Delta'(a) \ne 0$, which is impossible for a nonnegative polynomial.

(b) For the cancellation of h in (2.4), however, it suffices to assume that $h \ge 0$ on $\mathbb{R}$ and $h \not\equiv 0$. Indeed, this identity implies $F - \Delta\sigma_1 B^*B \succeq 0$ wherever $h > 0$. Since a nonzero polynomial has only finitely many real zeros, the same inequality holds on $\mathbb{R}$ by continuity. Thus strict positivity is used to guarantee the scalar identity in Lemma 2.4, whereas the cancellation step itself only requires a nonzero nonnegative multiplier.

3. Reduction

3.1. **Finite sets.** Theorem A admits a direct proof when K_S is finite using the Lagrange interpolation polynomials.

Proposition 3.1. *Let $K = \{\xi_1, \dots, \xi_r\} \subset \mathbb{R}$, where $r \ge 1$ and the points are distinct. Let $S \subset \mathbb{R}[x]$ be finite with $K_S = K$ and $M(S)$ saturated. There is a nonnegative integer η_S such that, for every complex Hilbert space $\mathcal{H}$, every nonnegative integer $\mathtt{d}$, and every hermitian $F \in \mathcal{B}(\mathcal{H})[x]$,*

$$F \succeq 0 \text{ on } K, \qquad \deg F \le \mathtt{d} \quad\Longrightarrow\quad F \in \mathcal{M}_{\mathcal{H}}(S)_{\le \mathtt{d}+\eta_S}.$$

Proof. Let ℓ_i be the Lagrange interpolation polynomials

$$\ell_i(x) = \prod_{\substack{1\le j\le r\\ j\ne i}} \frac{x-\xi_j}{\xi_i-\xi_j} \qquad (1 \le i \le r), \tag{3.1}$$

and set

$$\varphi(x) = \prod_{i=1}^{r}(x-\xi_i), \qquad H(x) = \sum_{i=1}^{r} \ell_i(x)^2 F(\xi_i).$$

Since $F(\xi_i) \succeq 0$ for every i, we have $H \in \mathrm{SOS}_{\mathcal{H}}$. Moreover, $H(\xi_i) = F(\xi_i)$, so $F - H$ vanishes at every ξ_i and is therefore divisible by φ. Consequently, $F - H = \varphi G$ for some $G \in \mathcal{B}(\mathcal{H})[x]$. Since $F - H$ is hermitian and φ has real coefficients, G is also hermitian.

If $G = 0$, the assertion is immediate. Otherwise,

$$\deg G \le \max\{\mathtt{d}, 2r-2\} - r.$$

Set $k = \lceil \deg G/2 \rceil$. Since $\deg G \le 2k$, we can choose $N > 0$ sufficiently large that

$$G_+(x) = G(x) + N(1+x^2)^k I_{\mathcal{H}} \succeq 0, \qquad (x \in \mathbb{R}).$$

The polynomial

$$G_- = N(1+x^2)^k I_{\mathcal{H}}$$

is also positive semidefinite on $\mathbb{R}$. By Proposition 2.1, both G_+ and G_- belong to $\mathrm{SOS}_{\mathcal{H}}$, with

$$\deg G_\pm \le 2k \le \deg G + 1.$$

Now

$$F = H + \varphi G_+ + (-\varphi) G_-.$$

Since φ and $-\varphi$ vanish on K, saturation gives $\varphi, -\varphi \in M(S)$. Fix scalar certificates for both polynomials of weighted degree at most $\widetilde{\eta}_S$. Substituting these certificates yields $F \in \mathcal{M}_{\mathcal{H}}(S)_{\le n}$, where

$$n \le \max\big\{2r-2,\ \widetilde{\eta}_S + \max\{\mathtt{d}, 2r-2\} - r + 1\big\}.$$

Therefore, $n \le \mathtt{d} + \eta_S$, where $\eta_S = \max\{2r-2, \widetilde{\eta}_S + r - 1\}$. □

3.2. **Closedness of truncated operator quadratic modules.** The proof of Theorem 3.4 requires the closedness of the truncated operator quadratic module when the positivity set K_S has a nonempty interior. The following lemma establishes the required closedness.

Lemma 3.2. *Let $S = \{g_1, \dots, g_s\} \subseteq \mathbb{R}[x]$ be finite and suppose that K_S has a nonempty interior. Then, for every complex Hilbert space $\mathcal{H}$ and every nonnegative integer n, the cone $\mathcal{M}_{\mathcal{H}}(S)_{\le n}$ is closed under coefficientwise convergence in operator norm.*

Proof. Let $F_m \in \mathcal{M}_{\mathcal{H}}(S)_{\le n}$ converge coefficientwise in operator norm to F. Choose representations

$$F_m = \sum_{j=0}^{s} g_j Q_{j,m}, \qquad Q_{j,m} \in \mathrm{SOS}_{\mathcal{H}}, \quad g_0 = 1.$$

We may omit terms with $\deg g_j > n$. For the remaining indices, put

$$r_j = \left\lfloor \frac{n - \deg g_j}{2} \right\rfloor.$$

Then $\deg Q_{j,m} \le 2r_j$, and $Q_{j,m} \succeq 0$ on $\mathbb{R}$.

Choose a nondegenerate compact interval $J \subseteq K_S$ on which all the remaining generators g_j are strictly positive. Such an interval exists because a nonzero polynomial has only finitely many real zeros. Since F_m converges coefficientwise, there is $M > 0$ such that $0 \preceq F_m(x) \preceq M I_{\mathcal{H}}$ for $x \in J$ and all m. Each summand is positive semidefinite on J, so

$$0 \preceq Q_{j,m}(x) \preceq \frac{M}{g_j(x)} I_{\mathcal{H}} \qquad (x \in J).$$

Because g_j is bounded below by a positive constant on J, the values of $Q_{j,m}$ are uniformly bounded.

For each j, choose $2r_j + 1$ distinct points of J. Polynomial interpolation expresses every coefficient of $Q_{j,m}$ as a fixed linear combination of its values at these points. Hence all the coefficients of all the $Q_{j,m}$ are uniformly bounded in operator norm.

Closed bounded balls of $\mathcal{B}(\mathcal{H})$ are compact in the weak operator topology. As only finitely many coefficients occur, a common subnet converges coefficientwise in that topology to polynomials Q_j with $\deg Q_j \le 2r_j$. For every $x \in \mathbb{R}$, the values $Q_{j,m}(x)$ converge weakly to $Q_j(x)$. The positive cone is weak-operator closed, so $Q_j \succeq 0$ on $\mathbb{R}$. By Proposition 2.1, each Q_j belongs to $\mathrm{SOS}_{\mathcal{H}}$, with a factor

of degree at most r_j. Passing to the limit in the coefficient identities gives $F = \sum_j g_j Q_j$, a certificate of weighted degree at most n. □

3.3. **Reduction to quadratic operator polynomials.** A uniform degree bound for quadratic operator polynomials extends to polynomials of arbitrary degree, with degree bounds of Theorem A. The hypothesis must hold for all Hilbert spaces, since the reduction involves enlarging the coefficient Hilbert space.

The quadratic reduction requires an affine matrix polynomial whose kernel is spanned by the monomial column $(1, x, \dots, x^r)^{\mathsf{T}}$. The following lemma constructs all such polynomials.

Lemma 3.3. *Let $r \geq 1$ and set*

$$v(x) = \operatorname{col}(1, x, \dots, x^r) \in \mathbb{C}^{r+1}.$$

For every $M \in \mathbb{C}^{r\times r}$, the affine matrix polynomial

$$b_M(x) = b_0 + xb_1, \qquad b_0 = \begin{pmatrix} 0 & M \end{pmatrix}, \qquad b_1 = \begin{pmatrix} -M & 0 \end{pmatrix},$$

satisfies $b_M(x)v(x) = 0$ for all $x \in \mathbb{R}$. If M is invertible, then $b_M(x)$ has full row rank for every $x \in \mathbb{R}$ and

$$\ker b_M(x) = \operatorname{span}\{v(x)\}. \tag{3.2}$$

Conversely, every affine matrix polynomial $b(x) \in \mathbb{C}[x]^{r\times(r+1)}$ satisfying $b(x)v(x) = 0$ identically is of the above form for some $M \in \mathbb{C}^{r\times r}$.

Proof. Write $b(x) = b_0 + xb_1$, and let $c_0, \dots, c_r \in \mathbb{C}^r$ and $d_0, \dots, d_r \in \mathbb{C}^r$ denote the column vectors of b_0 and b_1, respectively. Thus, $b_0 = \begin{pmatrix} c_0 & \cdots & c_r \end{pmatrix}$, $b_1 = \begin{pmatrix} d_0 & \cdots & d_r \end{pmatrix}$. The identity $b(x)v(x) = 0$ gives

$$c_0 + \sum_{j=1}^{r} (c_j + d_{j-1})x^j + d_r x^{r+1} = 0.$$

Comparing coefficients, we obtain $c_0 = 0$, $d_r = 0$ and $c_j = -d_{j-1}$ for $1 \leq j \leq r$. Thus, with $M = (c_1, \dots, c_r)$, we have $b_0 = \begin{pmatrix} 0 & M \end{pmatrix}$, $b_1 = \begin{pmatrix} -M & 0 \end{pmatrix}$. Conversely, these formulas immediately imply $b_M(x)v(x) = 0$.

Suppose now that M is invertible. Then

$$b_M(x) = M \begin{pmatrix} -x & 1 & 0 & \cdots & 0 \\ 0 & -x & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -x & 1 \end{pmatrix}.$$

If $q = \operatorname{col}(q_0, \dots, q_r)$ belongs to $\ker b_M(x)$, then $q_j = xq_{j-1}$ for $1 \leq j \leq r$. Thus, $q = q_0 v(x)$, proving (3.2). Since $\dim(\ker b_M(x)) = 1$, $b_M(x)$ has rank r for every $x \in \mathbb{R}$. □

We now show that a uniform degree bound for quadratic operator polynomials extends to polynomials of arbitrary degree, with only an additive increase in the degree bound.

Theorem 3.4. *Let $S \subseteq \mathbb{R}[x]$ be finite, and suppose that K_S is compact and has a nonempty interior. Assume that there is an integer $N \geq 2$ such that for every complex Hilbert space $\mathcal{E}$ and every hermitian $L \in \mathcal{B}(\mathcal{E})[x]$ with $\deg L \leq 2$,*

$$L \succeq 0 \text{ on } K_S \quad \Longrightarrow \quad L \in \mathcal{M}_{\mathcal{E}}(S)_{\leq N}.$$

Then, for every complex Hilbert space $\mathcal{H}$, every integer $\mathtt{d} \geq 1$, and every hermitian $F \in \mathcal{B}(\mathcal{H})[x]$ with $\deg F \leq \mathtt{d}$,

$$F \succeq 0 \text{ on } K_S \quad \Longrightarrow \quad F \in \mathcal{M}_{\mathcal{H}}(S)_{\leq N+2\lceil \mathtt{d}/2\rceil-2} \subseteq \mathcal{M}_{\mathcal{H}}(S)_{\leq \mathtt{d}+N-1}.$$

For $\mathtt{d} = 0$, a certificate of degree zero suffices.

Proof. For $\mathtt{d} = 0$, positivity gives the degree-zero certificate $F = (F^{1/2})^* F^{1/2}$. If $1 \le \mathtt{d} \le 2$, the assertion follows directly from the quadratic hypothesis. We therefore assume $\mathtt{d} \ge 3$ and put $r = \lceil \frac{\mathtt{d}}{2} \rceil - 1 \ge 1$. Write

$$F(x) = \sum_{k=0}^{2r+2} F_k x^k,$$

with zero coefficients added if necessary. Set

$$\mathcal{E} = \mathbb{C}^{r+1} \otimes \mathcal{H} \cong \mathcal{H}^{\oplus(r+1)}, \qquad v(x) = \operatorname{col}(1, x, \dots, x^r),$$

and define the polynomial column

$$V(x) = v(x) \otimes I_{\mathcal{H}} : \mathcal{H} \longrightarrow \mathcal{E}.$$

Consider the block diagonal hermitian polynomial

$$A(x) = \operatorname{diag}(A_0(x), \dots, A_r(x)),$$

where

$$\begin{aligned} A_j(x) &= F_{2j} + xF_{2j+1} && (0 \le j < r), \\ A_r(x) &= F_{2r} + xF_{2r+1} + x^2 F_{2r+2}. \end{aligned}$$

Then $\deg A \le 2$, and direct multiplication gives

$$V(x)^* A(x) V(x) = F(x). \tag{3.3}$$

Although $F(x) \succeq 0$ on K_S, the identity (3.3) only gives positivity of $A(x)$ on the subspace $\operatorname{ran} V(x)$, not necessarily on the whole space $\mathcal{E}$. We therefore modify A by a quadratic positive semidefinite operator polynomial that vanishes under compression by V.

By Lemma 3.3, choosing $M = I_{\mathbb{C}^r}$ gives an affine matrix polynomial

$$b(x) = b_0 + xb_1, \qquad b_0 = \begin{pmatrix} 0 & I_r \end{pmatrix}, \qquad b_1 = \begin{pmatrix} -I_r & 0 \end{pmatrix},$$

such that

$$b(x)v(x) = 0, \qquad \ker b(x) = \operatorname{span}\{v(x)\}$$

for every $x \in \mathbb{R}$. Set

$$B(x) = b(x) \otimes I_{\mathcal{H}} : \mathcal{E} \longrightarrow \mathcal{H}^{\oplus r}.$$

It follows that

$$B(x)V(x) = 0, \qquad \ker B(x) = \operatorname{ran} V(x).$$

Since $b(x)$ has full row rank for every real x and K_S is compact, there exists $\kappa > 0$ such that

$$\|B(x)\eta\|^2 \ge \kappa \|\eta\|^2 \tag{3.4}$$

for every $x \in K_S$ and every $\eta \in (\ker B(x))^\perp$.

Fix $\delta > 0$ and put

$$A_\delta(x) = A(x) + \delta I_{\mathcal{E}}, \qquad \Lambda_\delta = \max_{x \in K_S} \|A_\delta(x)\|.$$

By (3.3),

$$V(x)^* A_\delta(x) V(x) = F(x) + \delta V(x)^* V(x) \succeq \delta V(x)^* V(x)$$

for every $x \in K_S$. Thus,

$$\langle A_\delta(x)\xi_0, \xi_0 \rangle \ge \delta \|\xi_0\|^2 \tag{3.5}$$

for every $\xi_0 \in \ker B(x)$.

We now show that a sufficiently large multiple of $B(x)^* B(x)$ makes $A_\delta(x)$ positive semidefinite on the whole space $\mathcal{E}$. Fix $x \in K_S$ and let $\xi \in \mathcal{E}$ be arbitrary. We write

$$\xi = \xi_0 + \xi_1, \qquad \xi_0 \in \ker B(x), \quad \xi_1 \in (\ker B(x))^\perp.$$

Using (3.5) and the definition of Λ_δ, we obtain

$$\langle A_\delta(x)\xi,\xi\rangle \geq \delta\|\xi_0\|^2 - 2\Lambda_\delta\|\xi_0\|\|\xi_1\| - \Lambda_\delta\|\xi_1\|^2 \geq \frac{\delta}{2}\|\xi_0\|^2 - \left(\Lambda_\delta + \frac{2\Lambda_\delta^2}{\delta}\right)\|\xi_1\|^2.$$

Choose

$$t_\delta \geq \kappa^{-1}\left(\Lambda_\delta + \frac{2\Lambda_\delta^2}{\delta}\right)$$

and define

$$L_\delta(x) = A_\delta(x) + t_\delta B(x)^* B(x).$$

Since $B(x)\xi_0 = 0$, the estimate (3.4) yields

$$\begin{aligned}\langle L_\delta(x)\xi,\xi\rangle &= \langle A_\delta(x)\xi,\xi\rangle + t_\delta\|B(x)\xi_1\|^2 \\ &\geq \frac{\delta}{2}\|\xi_0\|^2 + \left(t_\delta\kappa - \Lambda_\delta - \frac{2\Lambda_\delta^2}{\delta}\right)\|\xi_1\|^2 \\ &\geq 0.\end{aligned}$$

As $\xi \in \mathcal{E}$ and $x \in K_S$ were arbitrary, we have $L_\delta(x) \succeq 0$ for $x \in K_S$.

Moreover, L_δ is a hermitian operator polynomial of degree at most two. By the hypothesis,

$$L_\delta \in \mathcal{M}_{\mathcal{E}}(S)_{\leq N}.$$

Thus L_δ is of the form

$$L_\delta = \sum_{j=0}^{s} g_j H_{j,\delta}^* H_{j,\delta}, \qquad g_0 = 1,$$

where $H_{j,\delta} \in \mathcal{B}(\mathcal{E})[x]$ and $\deg(g_j H_{j,\delta}^* H_{j,\delta}) \leq N$. Since $B(x)V(x) = 0$, we have

$$V(x)^* L_\delta(x) V(x) = V(x)^* A_\delta(x) V(x) = F(x) + \delta V(x)^* V(x) = F(x) + \delta\left(\sum_{k=0}^{r} x^{2k}\right) I_{\mathcal{H}}.$$

On the other hand,

$$V(x)^* L_\delta(x) V(x) = \sum_{j=0}^{s} g_j(x) Q_{j,\delta}(x), \qquad \text{where} \quad Q_{j,\delta}(x) = V(x)^* H_{j,\delta}(x)^* H_{j,\delta}(x) V(x).$$

Each $Q_{j,\delta} \in B(\mathcal{H})[x]$ is positive semidefinite on $\mathbb{R}$, and $\deg(g_j Q_{j,\delta}) \leq N + 2r$. By Proposition 2.1, each $Q_{j,\delta}$ belongs to $\mathrm{SOS}_{\mathcal{H}}$, without increasing its degree. Consequently,

$$F + \delta\left(\sum_{k=0}^{r} x^{2k}\right) I_{\mathcal{H}} \in \mathcal{M}_{\mathcal{H}}(S)_{\leq N+2r}.$$

As $\delta \downarrow 0$, the left-hand side converges coefficientwise in operator norm to F. By Lemma 3.2, the truncated operator quadratic module $\mathcal{M}_{\mathcal{H}}(S)_{\leq N+2r}$ is closed. Therefore,

$$F \in \mathcal{M}_{\mathcal{H}}(S)_{\leq N+2r} = \mathcal{M}_{\mathcal{H}}(S)_{\leq N+2\lceil \mathrm{d}/2\rceil - 2}.$$

Finally, $N + 2\left\lceil \frac{\mathrm{d}}{2}\right\rceil - 2 \leq \mathrm{d} + N - 1$, which proves the assertion. □

4. Quadratic certificates on finite unions of intervals

Throughout this section, fix $\ell \in \mathbb{N}$,

$$K = \bigcup_{j=1}^{\ell} [a_j, b_j], \qquad a_1 < b_1 < \cdots < a_\ell < b_\ell, \tag{4.1}$$

and define

$$\begin{aligned} P(x) &= \prod_{j=1}^{\ell} (x - a_j), & Q(x) &= \prod_{j=1}^{\ell} (x - b_j), \\ R(x) &= Q(x) - P(x), & \Delta(x) &= -P(x)Q(x). \end{aligned} \tag{4.2}$$

The set K is $\{\Delta \geq 0\}$, with simple boundary zeros, so $M(\{\Delta\})$ is saturated by Theorem 2.2.

4.1. **A symmetric matrix pencil.** Our aim is to reduce polynomial positivity on K to positivity on $[0, 1]$, where the operator Fejér–Riesz theorem provides certificates with controlled degree. A natural first choice is the diagonal pencil

$$\widetilde{T}(y) = (1 - y)\,\mathrm{diag}(b_1, \ldots, b_\ell) + y\,\mathrm{diag}(a_1, \ldots, a_\ell).$$

Its eigenvalues cover the intervals $[a_j, b_j]$ as y runs over $[0, 1]$. However, recovering x from the jth eigenvalue requires the substitution $y = (b_j - x)/(b_j - a_j)$, which depends on j. Thus, this choice does not provide a common rational parameter for all the intervals.

We therefore seek a linear pencil $T(y)$ whose characteristic polynomial is affine in y, with Q and P as its characteristic polynomials at $y = 0$ and $y = 1$, respectively. Thus, we require

$$\det(xI - T(y)) = (1 - y)Q(x) + yP(x) = Q(x) - yR(x).$$

A rank-one perturbation has the required affine dependence on y. We therefore look for $v \in \mathbb{R}^\ell$ such that

$$T(y) = D - yvv^{\mathsf{T}}, \qquad D = \mathrm{diag}(b_1, \ldots, b_\ell), \tag{4.3}$$

has eigenvalues $a_1, \ldots, a_\ell$ at $y = 1$. The following lemma shows that such a vector exists.

Lemma 4.1. *There is a column vector $v = \mathrm{col}(v_1, \ldots, v_\ell) \in \mathbb{R}^\ell$, with every $v_j \neq 0$, such that the one-parameter rank-one perturbation $T(y)$ defined in (4.3) satisfies*

$$\det(xI - T(y)) = Q(x) - yR(x), \qquad a_j \leq \lambda_j(T(y)) \leq b_j \quad (0 \leq y \leq 1), \tag{4.4}$$

where eigenvalues are ordered increasingly.

Proof. Applying [HJ13, Theorem 4.3.26] to the reversed ordering $-b_\ell < -a_\ell < \cdots < -b_1 < -a_1$, we obtain $v = \mathrm{col}(v_1, \ldots, v_\ell) \in \mathbb{R}^\ell$ such that $D - vv^{\mathsf{T}}$ has eigenvalues $\{a_1, \ldots, a_\ell\}$. Each v_j is nonzero, since otherwise b_j would remain an eigenvalue, contradicting the prescribed spectrum.

Recall the rank-one determinant identity [HJ13, Equation (0.8.5.11)],

$$\det(A + uv^{\mathsf{T}}) = \det A + v^{\mathsf{T}} \,\mathrm{adj}(A)u. \tag{4.5}$$

Taking $A = xI - D$ and $u = yv \in \mathbb{R}^\ell$, we obtain

$$\det(xI - T(y)) = Q(x) + y\, v^{\mathsf{T}} \,\mathrm{adj}(xI - D)v.$$

At $y = 1$, the left-hand side equals $P(x)$, so

$$v^{\mathsf{T}} \,\mathrm{adj}(xI - D)v = P(x) - Q(x) = -R(x). \tag{4.6}$$

Consequently, $\det(xI - T(y)) = Q(x) - yR(x)$, which proves the first identity. The inequalities follow from $T(1) \preceq T(y) \preceq T(0)$. □

Fix the pencil $T(y)$ as in (4.3). The eigenvector equation $T(y)u(x) = xu(x)$ requires $(xI - D)u(x)$ to be a scalar multiple of v. Away from the b_j, this suggests the direction $(xI - D)^{-1}v$. Clearing the denominator $Q(x)$ leads to the polynomial vector

$$u(x) = \operatorname{adj}(xI - D)v. \tag{4.7}$$

The following lemma shows that $u(x)$ never vanishes on $\mathbb{R}$ and establishes an identity expressing the parameter y as a rational function of x.

Lemma 4.2. *The polynomial vector $u(x)$ defined in (4.7) has no real zero and satisfies*

$$v^{\mathsf{T}}u(x) = -R(x), \qquad (T(y) - xI)u(x) = (yR(x) - Q(x))v. \tag{4.8}$$

The degree of u is $\ell - 1$, and its leading coefficient is v. Consequently, the scalar polynomial

$$w = u^* u \tag{4.9}$$

satisfies

$$w > 0 \text{ on } \mathbb{R}, \qquad \deg w = 2\ell - 2.$$

Proof. The first equality in (4.8) follows directly from (4.6) and (4.7). The adjugate identity $(xI - D)\operatorname{adj}(xI - D) = \det(xI - D)I = Q(x)I$ also gives

$$(D - xI)u(x) = (D - xI)\operatorname{adj}(xI - D)v = -Q(x)v.$$

Therefore,

$$\begin{aligned}(T(y) - xI)u(x) &= (D - yvv^{\mathsf{T}} - xI)u(x)\\ &= (D - xI)u(x) - yv(v^{\mathsf{T}}u(x))\\ &= (yR(x) - Q(x))v.\end{aligned}$$

This proves (4.8).

Since $xI - D$ is diagonal, the entries of $u(x)$ are

$$u_j(x) = v_j \prod_{i \neq j}(x - b_i), \qquad j = 1, \dots, \ell. \tag{4.10}$$

If $x \notin \{b_1, \dots, b_\ell\}$, each entry of $u(x)$ is nonzero. On the other hand, at $x = b_j$, only the jth component can be nonzero, and

$$u(b_j) = v_j \prod_{i \neq j}(b_j - b_i)e_j = Q'(b_j)v_j e_j \neq 0,$$

where e_j is the jth standard basis vector and Q' denotes the derivative of Q. The last inequality follows from the fact that the b_i are distinct and $v_j \neq 0$. Hence $u(x) \neq 0$ for every $x \in \mathbb{R}$.

The formulas (4.10) show that u has degree $\ell - 1$ and leading coefficient v. Since $u(x) \neq 0$ for every $x \in \mathbb{R}$, the polynomial $w(x) = \sum_{j=1}^{\ell} u_j(x)^2$ is strictly positive on $\mathbb{R}$. It has degree $2\ell - 2$ and leading coefficient $v^{\mathsf{T}}v > 0$. □

We now express the parameter y in terms of x. Motivated by (4.8), we set

$$y = t(x) := \frac{Q(x)}{R(x)}, \qquad T(t(x))u(x) = xu(x). \tag{4.11}$$

This gives a single rational substitution for all the interval components.

Lemma 4.3. *The polynomial R defined in (4.2) has degree $\ell - 1$ and exactly one simple zero in each gap (b_j, a_{j+1}), and no other zeros. In particular, $t(x)$ is well-defined on K and satisfies*

$$t(x)(1 - t(x)) = \frac{\Delta(x)}{R(x)^2}, \qquad t(K) \subseteq [0, 1], \qquad t(a_j) = 1, \quad t(b_j) = 0. \tag{4.12}$$

Proof. Recall that $R(x) = Q(x) - P(x)$. Since both P and Q are monic polynomials of degree ℓ, the polynomial R has degree at most $\ell - 1$. The coefficient of $x^{\ell-1}$ in R is $\sum_j a_j - \sum_j b_j < 0$. Thus the degree of R is exactly $\ell - 1$. Since $R(b_j) = -P(b_j)$ and $R(a_{j+1}) = Q(a_{j+1})$ have opposite signs, each gap (b_j, a_{j+1}) has at least one zero of R. The resulting $\ell - 1$ distinct zeros exhaust the degree. The remaining assertions follow from $Q(R - Q) = -PQ = \Delta$ and the endpoint values. □

For $F(x) = \sum_k A_k x^k$, put

$$F(T(y)) = \sum_k A_k \otimes T(y)^k, \qquad \mathbf{u}(x) = I_{\mathcal{H}} \otimes u(x),$$

where $I_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$.

Lemma 4.4. *Let $F \in \mathcal{B}(\mathcal{H})[x]$. Then*

$$F(T(t(x)))\mathbf{u}(x) = \mathbf{u}(x)F(x), \qquad \mathbf{u}(x)^* F(T(t(x)))\mathbf{u}(x) = w(x)F(x) \tag{4.13}$$

hold as rational identities. Moreover, if F is hermitian and $F \succeq 0$ on K, then

$$F(T(y)) \succeq 0 \qquad \text{for all } y \in [0,1].$$

Proof. Fix $0 \le y \le 1$. Since the matrix $T(y)$ is real symmetric, there exists an orthogonal matrix $U_y \in \mathbb{R}^{\ell\times\ell}$ such that $U_y^{\mathsf{T}} T(y) U_y = \operatorname{diag}(\lambda_1, \dots, \lambda_\ell)$. By Lemma 4.1, $\lambda_j \in [a_j, b_j] \subseteq K$, so

$$(I_{\mathcal{H}} \otimes U_y^{\mathsf{T}}) F(T(y)) (I_{\mathcal{H}} \otimes U_y) = \bigoplus_{j=1}^{\ell} F(\lambda_j) \succeq 0.$$

Hence $F(T(y)) \succeq 0$.

Write $F(x) = \sum_{k=0}^{\mathrm{d}} A_k x^k$. For $R(x) \neq 0$, (4.11) gives

$$(T(t(x)))^k u(x) = x^k u(x) \qquad (k \ge 0).$$

Thus,

$$\begin{aligned} F(T(t(x)))\mathbf{u}(x) &= \sum_{k=0}^{\mathrm{d}} A_k \otimes (T(t(x)))^k (I_{\mathcal{H}} \otimes u(x)) \\ &= \sum_{k=0}^{\mathrm{d}} A_k \otimes x^k u(x) = \mathbf{u}(x)F(x), \end{aligned}$$

and

$$\mathbf{u}(x)^* F(T(t(x)))\mathbf{u}(x) = \mathbf{u}(x)^*\mathbf{u}(x)F(x) = (u(x)^{\mathsf{T}} u(x))F(x) = w(x)F(x).$$

These are identities of rational operator functions. □

4.2. **A strictly positive multiplier.** Let $F \in \mathcal{B}(\mathcal{H})[x]$ be hermitian, with $\deg F \le 2$ and $F \succeq 0$ on K. The lifted polynomial $L(y) := F(T(y))$ is quadratic and positive semidefinite on $[0,1]$. By Proposition 2.1, it admits an interval certificate. Substituting $y = t(x) = Q(x)/R(x)$, clearing denominators, and compressing by $\mathbf{u}(x)$ gives a certificate for $R^2 w F$ by (4.13). The difficulty is that, for $\ell \ge 2$, the multiplier $R^2 w$ vanishes precisely at the zeros of R, which lie in the gaps between the interval components of K. To obtain a certificate for F, we would like to remove this multiplier. Lemma 2.5 (see also Remark 2.1) allows us to do so when the multiplier is strictly positive on $\mathbb{R}$. Thus we look for a way to replace $R^2 w$ by a strictly positive polynomial.

To obtain a strictly positive multiplier, we consider modifying the lifted polynomial by a polynomial congruence

$$G(y) = (I_{\mathcal{H}} \otimes C(y)^*) F(T(y)) (I_{\mathcal{H}} \otimes C(y)),$$

which preserves positivity on $[0,1]$. As in (4.13), we want to recover a scalar multiple of $F(x)$ by compressing $G(t(x))$. Recall from (4.11) that $T(t(x))u(x) = xu(x)$. It is this eigenvector relation that allows us to recover $F(x)$ from $F(T(t(x)))$. To retain this property after the congruence, we require

$$C(t(x))u(x) = \beta(x)u(x) \tag{4.14}$$

for some scalar function β. We then have

$$\mathbf{u}(x)^* G(t(x))\mathbf{u}(x) = |\beta(x)|^2 w(x)F(x).$$

Thus, we look for a polynomial matrix $C(y)$ that commutes with $T(y)$, since this ensures that the eigenspaces of $T(y)$ are preserved.

We also want the modified polynomial G to have degree at most two. The reason is that increasing the degree of G may introduce additional powers of R when we clear denominators, leaving zeros in the resulting multiplier. To see this, first consider the original lifted polynomial $L(y) = F(T(y))$. Since L is quadratic, its interval factorization has the form

$$L(y) = H(y)^* H(y) + y(1-y)A^*A, \qquad H(y) = H_0 + yH_1,$$

where A is constant. Substituting $y = t(x) = Q(x)/R(x)$ and using $t(1-t) = \Delta/R^2$, we obtain

$$R^2 wF = R^2 \mathbf{u}^* L\mathbf{u} = \big((RH_0 + QH_1)\mathbf{u}\big)^* \big((RH_0 + QH_1)\mathbf{u}\big) + \Delta(A\mathbf{u})^*(A\mathbf{u}).$$

Thus, R^2 is enough to clear the denominators. Now suppose that $C(y) = C_0 + yC_1$ is affine. Even then, the congruence $G = C^* LC$ may have degree four. Its interval factorization takes the form

$$G(y) = M(y)^* M(y) + y(1-y)N(y)^* N(y),$$

where

$$M(y) = M_0 + yM_1 + y^2 M_2, \qquad N(y) = N_0 + yN_1.$$

This time, clearing the denominators generally requires multiplication by R^4. Indeed, setting

$$\widetilde{M} = R^2 M = R^2 M_0 + RQM_1 + Q^2 M_2, \qquad \widetilde{N} = RN = RN_0 + QN_1,$$

we obtain

$$R^4 \mathbf{u}^* G(t)\mathbf{u} = (\widetilde{M}\mathbf{u})^*(\widetilde{M}\mathbf{u}) + \Delta(\widetilde{N}\mathbf{u})^*(\widetilde{N}\mathbf{u}).$$

Suppose that C satisfies (4.14). The multiplier in the above certificate is then

$$R^4|\beta|^2 w = R^2|R\beta|^2 w.$$

Since C is affine,

$$R(x)C(t(x)) = R(x)C_0 + Q(x)C_1$$

is a polynomial. Moreover,

$$(RC_0 + QC_1)u = (R\beta)u.$$

As u never vanishes on $\mathbb{R}$, the scalar function $R\beta$ has no poles at the real zeros of R. Thus, the multiplier $R^2|R\beta|^2 w$ still vanishes at the zeros of R.

We therefore seek a polynomial matrix C satisfying (4.14) for some scalar function β, such that $\deg G \le 2$ and $R\beta$ is a polynomial with no real zeros. These requirements are not obviously compatible: removing the zeros of the multiplier appears to require division by R, whereas C must remain polynomial and the resulting congruence must remain quadratic. In the following lemma, we characterize all polynomial matrices with the desired properties. As we shall see, the rank-one perturbation structure of $T(y)$ makes this possible and leads to a whole family of admissible matrix polynomials C in Corollary 4.6.

Lemma 4.5. *Let $\ell \ge 2$. For $C \in \mathbb{C}[y]^{\ell\times\ell}$, the following are equivalent:*

(i) *For every $x \in K$, the vector $C(t(x))u(x)$ is a scalar multiple of $u(x)$, and for every quadratic hermitian operator polynomial F, the polynomial*

$$G_C(y) = (I_{\mathcal{H}} \otimes C(y)^*)F(T(y))(I_{\mathcal{H}} \otimes C(y))$$

has degree at most two.

(ii) *C commutes with T, and both C and TC are affine in y.*

(iii) *There is a polynomial $\mathrm{q} \in \mathbb{C}[x]$, with $\deg \mathrm{q} \leq \ell - 1$, such that $C(y) = \mathrm{q}(T(y))R(T(y))^{-1}$.*

Proof. We first show that (1) $\implies$ (2). Taking $F = 1$ and $F = x^2$, we obtain

$$\deg(C^*C) \leq 2, \qquad \deg((TC)^*(TC)) \leq 2.$$

Since $\deg(M^*M) = 2\deg M$ for every nonzero matrix polynomial M, it follows that both C and TC have degree at most one.

Next, fix $y \in (0,1)$. The eigenvalues $\lambda_1(y), \ldots, \lambda_\ell(y)$ of $T(y)$ are distinct and belong to K. Since $t(\lambda_j(y)) = y$, the nonzero vectors $u(\lambda_j(y))$ form an eigenbasis of $T(y)$. By assumption, $C(y)$ preserves each of these eigenspaces, so $C(y)$ commutes with $T(y)$. As this holds for every $y \in (0,1)$ and both C and T are matrix polynomials, we conclude that $CT = TC$ identically.

We next establish (3) $\implies$ (2). Let $\xi_1, \ldots, \xi_{\ell-1}$ be the simple real zeros of R. By (4.4),

$$\det(\xi_j I - T(y)) = Q(\xi_j) - yR(\xi_j) = Q(\xi_j) \neq 0.$$

Thus $T(y) - \xi_j I$ is invertible for every y. Set $A_j = D - \xi_j I$, which is invertible since $Q(\xi_j) \neq 0$. By identity (4.5),

$$\det(T(y) - \xi_j I) = \det(D - yvv^{\mathsf{T}} - \xi_j I\}) = \det(A_j)(1 - yv^{\mathsf{T}}A_j^{-1}v).$$

Since $\det(T(y) - \xi_j I)$ is independent of y, we must have

$$v^{\mathsf{T}}A_j^{-1}v = 0.$$

The Sherman–Morrison formula therefore gives

$$(T(y) - \xi_j I)^{-1} = (A_j - yvv^{\mathsf{T}})^{-1} = A_j^{-1} + yA_j^{-1}vv^{\mathsf{T}}A_j^{-1}.$$

Thus $E_j(y) \coloneqq (T(y) - \xi_j I)^{-1}$ is affine in y. The identity

$$T(y)E_j(y) = I + \xi_j E_j(y)$$

shows that TE_j is affine as well.

For $\deg \mathrm{q} \leq \ell - 1$,

$$\frac{\mathrm{q}(x)}{R(x)} = c + \sum_{j=1}^{\ell-1} \frac{\mathrm{q}(\xi_j)}{R'(\xi_j)(x - \xi_j)}.$$

Substituting $x = T(y)$ yields

$$C_{\mathrm{q}}(y) = cI + \sum_{j=1}^{\ell-1} \frac{\mathrm{q}(\xi_j)}{R'(\xi_j)} E_j(y).$$

Hence C_{q} and TC_{q} are affine, and C_{q} commutes with T. Since $C = C_{\mathrm{q}}$ by assumption, this proves (3) $\implies$ (2).

Now, we prove (2) $\implies$ (3). Write $C(y) = B_0 + yB_1$. Since $CT = TC$, we have $B_0 D = DB_0$. As the diagonal entries of D are distinct, B_0 must be diagonal, say

$$B_0 = \operatorname{diag}(\alpha_1, \ldots, \alpha_\ell).$$

Choose $\mathrm{q} \in \mathbb{C}[x]_{\leq \ell-1}$ interpolating

$$\mathrm{q}(b_j) = R(b_j)\alpha_j, \qquad j = 1, \ldots, \ell.$$

Then $C_{\mathrm{q}}(0) = \mathrm{q}(D)R(D)^{-1} = B_0$. The difference $H = C - C_{\mathrm{q}}$ is therefore of the form $H(y) = yH_1$. Since H commutes with T, comparison of coefficients in $HT = TH$ gives $H_1D = DH_1$, so H_1 is diagonal. Furthermore, TH is affine, whereas

$$T(y)H(y) = yDH_1 - y^2vv^{\mathsf{T}}H_1.$$

Thus $vv^{\mathsf{T}}H_1 = 0$. As every entry of v is nonzero, $H_1 = 0$. Hence $C = C_{\mathrm{q}}$, proving (2) $\Longrightarrow$ (3).

Finally, (2) $\Longrightarrow$ (1). Commutativity preserves the one-dimensional eigenspaces of $T(t(x))$, while the expressions $C^*C, C^*(TC)$ and $(TC)^*(TC)$ have degree at most two. This proves (2) $\Longrightarrow$ (1). □

The preceding lemma identifies all polynomial matrices that preserve the recovery eigenspaces and the quadratic degree. We now use these matrix polynomials to obtain a family of admissible multipliers.

Corollary 4.6. *For every* $\mathrm{q} \in \mathbb{C}[x]$ *with* $\deg \mathrm{q} \le \ell - 1$*, the matrix*

$$C_{\mathrm{q}}(y) := \mathrm{q}(T(y))R(T(y))^{-1} \tag{4.15}$$

is affine. Moreover, for every hermitian $F \in \mathcal{B}(\mathcal{H})[x]$ *with* $\deg F \le 2$*, the polynomial*

$$G_{C_{\mathrm{q}}}(y) = (I_{\mathcal{H}} \otimes C_{\mathrm{q}}(y)^*)F(T(y))(I_{\mathcal{H}} \otimes C_{\mathrm{q}}(y)) \tag{4.16}$$

has degree at most two and satisfies

$$R(x)C_{\mathrm{q}}(t(x))u(x) = \mathrm{q}(x)u(x), \tag{4.17}$$

and

$$R(x)^2\mathbf{u}(x)^*G_{C_{\mathrm{q}}}(t(x))\mathbf{u}(x) = w(x)|\mathrm{q}(x)|^2F(x).$$

These identities extend polynomially across the zeros of R*. In particular, the multiplier* $w|\mathrm{q}|^2$ *is strictly positive on* $\mathbb{R}$ *if and only if* q *has no real zeros.*

Proof. By Lemma 4.5, C_{q} is affine in y, and $G_{C_{\mathrm{q}}}$ has degree at most two. For $C = C_q$, the identity (4.11) gives

$$C(t(x))u(x) = \frac{\mathrm{q}(x)}{R(x)}u(x)$$

whenever $R(x) \neq 0$. Therefore,

$$R(x)^2\mathbf{u}^*(x)G_C(t(x))\mathbf{u}(x) = |\mathrm{q}(x)|^2\mathbf{u}^*(x)F(T(t(x)))\mathbf{u}(x) = w(x)|\mathrm{q}(x)|^2F(x).$$

Both sides extend polynomially across the zeros of R. Since $w > 0$ on $\mathbb{R}$, the multiplier $w|\mathrm{q}|^2$ is strictly positive precisely when q has no real zeros. □

Remark 4.1. An upper bound for the constant c_S in Theorem A depends on the choice of q. It would be interesting to study this dependence and optimize over the family of multipliers $w|\mathrm{q}|^2$. We do not pursue this question here, as any polynomial q of degree at most $\ell - 1$ with no real zeros suffices for our purposes. Henceforth, we take $\mathrm{q} = 1$ and write $C = C_{\mathrm{q}}$ and $G = G_{C_{\mathrm{q}}}$ for this choice.

We apply interval factorization to G, substitute $y = t(x)$, and compress by $\mathbf{u}$.

Lemma 4.7. *Let* $F \in \mathcal{B}(\mathcal{H})[x]$ *be hermitian, with* $\deg F \le 2$ *and* $F \succeq 0$ *on* K*. Let* w *be defined by* (4.9)*. There exist polynomial columns*

$$U = \mathrm{col}(U_1, \dots, U_\ell), \qquad V = \mathrm{col}(V_1, \dots, V_\ell), \tag{4.18}$$

where $U_i, V_i \in \mathcal{B}(\mathcal{H})[x]$ *for* $i = 1, \dots, \ell$*, with* $\deg U \le 2\ell - 1$ *and* $\deg V \le \ell - 1$*, such that*

$$wF = U^*U + \Delta V^*V \in \mathcal{M}_{\mathcal{H}}(\{\Delta\})_{\le 4\ell-2}. \tag{4.19}$$

Proof. By Corollary 4.6 and (ii) of Proposition 2.1, there exist an affine operator polynomial $G_1 \in \mathcal{B}(\mathcal{H}^{\oplus \ell})[y]$ and an operator $H_1 \in \mathcal{B}(\mathcal{H}^{\oplus \ell})$ such that

$$G(y) = G_1(y)^* G_1(y) + y(1-y) H_1^* H_1.$$

Recall that $\mathbf{u}(x) = I_{\mathcal{H}} \otimes u(x)$, where $u(x) = \operatorname{adj}(xI - D)v$ is the polynomial vector from (4.7). Write $G_1(y) = A + yB$, where $A, B \in \mathcal{B}(\mathcal{H}^{\oplus \ell})$, and set

$$U(x) = R(x) G_1(t(x)) \mathbf{u}(x) = \big(R(x)A + Q(x)B\big)\mathbf{u}(x), \qquad V(x) = H_1 \mathbf{u}(x).$$

Thus U and V are polynomial columns of the form (4.18). Since $\deg Q = \ell$ and $\deg R = \deg \mathbf{u} = \ell - 1$, we have $\deg U \le 2\ell - 1$ and $\deg V \le \ell - 1$.

For $x \in \mathbb{R}$ with $R(x) \neq 0$, the identity (4.12) gives

$$U(x)^* U(x) + \Delta(x) V(x)^* V(x) = R(x)^2 \mathbf{u}(x)^* G(t(x)) \mathbf{u}(x).$$

By (4.17),

$$R(x)(I_{\mathcal{H}} \otimes C(t(x)))\mathbf{u}(x) = \mathbf{u}(x).$$

Hence, using (4.16) and (4.13),

$$R(x)^2 \mathbf{u}(x)^* G(t(x)) \mathbf{u}(x) = \mathbf{u}(x)^* F(T(t(x))) \mathbf{u}(x) = w(x) F(x).$$

Since both sides are polynomial expressions, the identity (4.19) holds on all of $\mathbb{R}$.

The degree bounds on U and V yield $\deg(U^*U) \le 4\ell - 2$, $\deg(\Delta V^* V) \le 4\ell - 2$. Moreover, U^*U and V^*V are positive semidefinite on $\mathbb{R}$. By (i) of Proposition 2.1, both belong to $\mathrm{SOS}_{\mathcal{H}}$, with factors of degrees at most $2\ell - 1$ and $\ell - 1$, respectively. Thus, $wF \in \mathcal{M}_{\mathcal{H}}(\{\Delta\})_{\le 4\ell - 2}$. □

4.3. **Quadratic degree bounds.** The polynomial w, defined by (4.9), is strictly positive by Lemma 4.2. By Lemma 2.4, there are $\sigma_0, \sigma_1 \in \mathrm{SOS}_{\mathbb{C}}$ such that

$$\Delta = \sigma_0 + w \Delta \sigma_1. \tag{4.20}$$

We fix such a pair for the remainder of this section. Note that $\sigma_1 \neq 0$, since otherwise $\Delta = \sigma_0$ would be nonnegative on all of $\mathbb{R}$.

Theorem 4.8. *Let K and ℓ be as in (4.1), and let Δ and w be defined by (4.2) and (4.9), respectively. Fix $\sigma_0, \sigma_1 \in \mathrm{SOS}_{\mathbb{C}}$ satisfying (4.20). Then, for every complex Hilbert space $\mathcal{H}$ and every hermitian $F \in \mathcal{B}(\mathcal{H})[x]$ of degree at most 2,*

$$F \succeq 0 \text{ on } K \quad \Longleftrightarrow \quad F \in \mathcal{M}_{\mathcal{H}}(\{\Delta\})_{\le 4\ell - 2 + \deg \sigma_1}.$$

Proof. If $F \succeq 0$ on K, Lemma 4.7 gives $wF \in \mathcal{M}_{\mathcal{H}}(\{\Delta\})_{\le 4\ell - 2}$. By Lemma 2.5,

$$F \in \mathcal{M}_{\mathcal{H}}(\{\Delta\})_{\le \max\{\deg F, 4\ell - 2 + \deg \sigma_1\}} = \mathcal{M}_{\mathcal{H}}(\{\Delta\})_{\le 4\ell - 2 + \deg \sigma_1}.$$

The converse follows by evaluating the certificate on $K = \{\Delta \ge 0\}$. □

Corollary 4.9. *Let K and ℓ be as in (4.1), and let Δ and w be defined by (4.2) and (4.9), respectively. Fix $\sigma_0, \sigma_1 \in \mathrm{SOS}_{\mathbb{C}}$ satisfying (4.20), and let S be a saturated scalar description of K. Then there exists a nonnegative integer η_S, depending only on S, such that for every complex Hilbert space $\mathcal{H}$ and every hermitian $F \in \mathcal{B}(\mathcal{H})[x]$ with $\deg F \le 2$,*

$$F \succeq 0 \text{ on } K \quad \Longrightarrow \quad F \in \mathcal{M}_{\mathcal{H}}(S)_{\le 4\ell - 2 + \deg \sigma_1 + \eta_S}.$$

Proof. Since $\Delta \ge 0$ on K and $M(S)$ is saturated, there exists a representation $\Delta = \sum_{j=0}^{s} g_j \tau_j$, where $g_0 = 1$ and each $\tau_j \in \mathrm{SOS}_{\mathbb{C}}$. Let d_S denote the weighted degree of this certificate and set $\eta_S = \mathrm{d}_S - 2\ell$. Since $\deg \Delta = 2\ell$, we have $\eta_S \ge 0$.

Let $\mathcal{H}$ be a complex Hilbert space and let $F \in \mathcal{B}(\mathcal{H})[x]$ be hermitian, with $\deg F \le 2$ and $F \succeq 0$ on K. By Theorem 4.8, there exist $\Sigma_0, \Sigma_1 \in \mathrm{SOS}_{\mathcal{H}}$ such that $F = \Sigma_0 + \Delta\Sigma_1$, with degree bounds $\deg \Sigma_0, \deg(\Delta\Sigma_1) \le 4\ell - 2 + \deg \sigma_1$. Substituting the scalar certificate for Δ gives

$$F = \Sigma_0 + \sum_{j=0}^{s} g_j \tau_j \Sigma_1.$$

Since each $\tau_j \Sigma_1 \in \mathrm{SOS}_{\mathcal{H}}$, the resulting certificate has weighted degree at most $4\ell - 2 + \deg \sigma_1 + \eta_S$. Thus, $F \in \mathcal{M}_{\mathcal{H}}(S)_{\le 4\ell - 2 + \deg \sigma_1 + \eta_S}$, as claimed. □

4.4. **Explicit multiplier.** Recall that

$$P(x) = \prod_{j=1}^{\ell} (x - a_j), \qquad Q(x) = \prod_{j=1}^{\ell} (x - b_j).$$

By (4.6) and the diagonal form of D,

$$P(x) - Q(x) = \sum_{j=1}^{\ell} v_j^2 \prod_{k \ne j} (x - b_k).$$

Evaluating at $x = b_j$ gives

$$v_j^2 = \frac{P(b_j)}{Q'(b_j)} > 0, \qquad j = 1, \dots, \ell.$$

Thus, using (4.7) the polynomial $w = u^* u$ is given by

$$w(x) = \sum_{j=1}^{\ell} \frac{P(b_j)}{Q'(b_j)} \prod_{k \ne j} (x - b_k)^2.$$

The following lemma will be used in Section 7 to establish uniform degree bounds for operator positivity certificates over families of compact interval unions contained in a fixed compact interval, with component lengths and separating gaps bounded below by positive constants.

Lemma 4.10. *Let K and ℓ be as in (4.1), with $\ell \ge 1$ and pairwise distinct endpoints $a_1, b_1, \dots, a_\ell, b_\ell$. Then the coefficients of w depend continuously on these endpoints.*

Proof. The coefficients of w are rational functions of the endpoints, with denominators

$$Q'(b_j) = \prod_{k \ne j} (b_j - b_k),$$

which are nonzero since the b_j are distinct. □

5. Proof of Theorem A

We extend the positivity certificates of Theorem 4.8 from finite unions of nondegenerate compact intervals to compact semialgebraic sets that also contain isolated points.

Let

$$K_0 = \bigcup_{j=1}^{\ell} [a_j, b_j], \qquad a_1 < b_1 < \cdots < a_\ell < b_\ell, \qquad \ell \ge 1, \tag{5.1}$$

and recall the defining polynomial

$$\Delta(x) = -P(x)Q(x) = -\prod_{j=1}^{\ell} (x - a_j)(x - b_j), \qquad K_0 = \{x \in \mathbb{R} : \Delta(x) \ge 0\}. \tag{5.2}$$

5.1. **Adjoining a single isolated point.** Fix $\xi \in \mathbb{R} \setminus K_0$ and put $\delta = -\Delta(\xi) > 0$. Define

$$G_\xi(x) = \begin{pmatrix} \delta + \Delta(x) & (x-\xi)\Delta(x) \\ (x-\xi)\Delta(x) & (x-\xi)^2\Delta(x) \end{pmatrix}. \tag{5.3}$$

The following lemma describes the positivity set of G_ξ.

Lemma 5.1. *For $\xi \in \mathbb{R} \setminus K_0$, let G_ξ be the 2×2 matrix polynomial defined by (5.3). Then $G_\xi(\xi) = 0$ and*

$$\{x \in \mathbb{R} : \ G_\xi(x) \succeq 0\} \ = \ K_0 \cup \{\xi\}.$$

Proof. The identity

$$G_\xi(x) = \delta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \Delta(x) \begin{pmatrix} 1 \\ x-\xi \end{pmatrix} \begin{pmatrix} 1 & x-\xi \end{pmatrix}$$

gives $G_\xi(x) \succeq 0$ for $x \in K_0$. At $x = \xi$, the equality $\Delta(\xi) = -\delta$ yields $G_\xi(\xi) = 0$. Thus

$$K_0 \cup \{\xi\} \subseteq \{x \in \mathbb{R} : G_\xi(x) \succeq 0\}.$$

Conversely, if $G_\xi(x) \succeq 0$, then

$$\det G_\xi(x) = \delta\Delta(x)(x-\xi)^2 \geq 0.$$

If $x \neq \xi$, then $\delta(x-\xi)^2 > 0$, so $\Delta(x) \geq 0$ and hence $x \in K_0$. □

Positivity at ξ allows a factorization through G_ξ without increasing the degrees of the operator columns.

Lemma 5.2. *Fix $\xi \in \mathbb{R} \setminus K_0$, and let G_ξ be defined by (5.3). Suppose that $F \in \mathcal{B}(\mathcal{H})[x]$ admits a representation*

$$F = A^*A + \Delta B^*B,$$

where $A, B \in \mathcal{B}(\mathcal{H})[x]$ satisfy $\deg A, \deg B \leq \mathtt{d}$ for some integer $\mathtt{d} \geq 0$.

If $F(\xi) \succeq 0$, then there exist operator polynomials $L_\xi, W_\xi, E_\xi \in \mathcal{B}(\mathcal{H})[x]$ such that, defining $V_\xi = \operatorname{col}(W_\xi, E_\xi)$, we have $\deg L_\xi, \deg V_\xi \leq \mathtt{d}$ and

$$F = L_\xi^* L_\xi + V_\xi^*(I_\mathcal{H} \otimes G_\xi)V_\xi. \tag{5.4}$$

Proof. Since $\Delta(\xi) = -\delta$, the assumption $F(\xi) \succeq 0$ gives

$$\delta B(\xi)^* B(\xi) \preceq A(\xi)^* A(\xi).$$

By Douglas' factorization lemma, there exists a contraction $\Gamma : \mathcal{H}_A \to \mathcal{H}_B$ such that

$$\Gamma A(\xi) = \sqrt{\delta}\, B(\xi).$$

Define

$$W_\xi(x) = \frac{\Gamma A(x)}{\sqrt{\delta}}, \qquad E_\xi(x) = \frac{B(x) - W_\xi(x)}{x-\xi},$$
$$L_\xi(x) = (I_{\mathcal{H}_A} - \Gamma^*\Gamma)^{1/2} A(x), \qquad V_\xi(x) = \operatorname{col}(W_\xi(x), E_\xi(x)).$$

Because $W_\xi(\xi) = B(\xi)$, the column E_ξ is a polynomial, with $\deg E_\xi \leq \mathtt{d} - 1$. Consequently, $\deg L_\xi, \deg V_\xi \leq \mathtt{d}$.

The definitions of L_ξ and W_ξ give

$$L_\xi^* L_\xi = A^*(I_{\mathcal{H}_A} - \Gamma^*\Gamma)A = A^*A - \delta W_\xi^* W_\xi.$$

Moreover, $B = W_\xi + (x-\xi)E_\xi$. Hence, by (5.3),

$$\begin{aligned} V_\xi^*(I_{\mathcal{H}_B} \otimes G_\xi)V_\xi &= \delta W_\xi^* W_\xi + \Delta\big(W_\xi + (x-\xi)E_\xi\big)^*\big(W_\xi + (x-\xi)E_\xi\big) \\ &= \delta W_\xi^* W_\xi + \Delta B^* B. \end{aligned}$$

Combining these identities gives

$$L_\xi^* L_\xi + V_\xi^*(I_{\mathcal{H}_B} \otimes G_\xi)V_\xi = A^*A + \Delta B^*B = F,$$

which proves (5.4). □

5.2. **Adjoining finitely many isolated points.** Let $\xi_1, \dots, \xi_r \in \mathbb{R} \setminus K_0$ be distinct, with $r \geq 1$, and put $K = K_0 \cup \{\xi_1, \dots, \xi_r\}$. The following lemma constructs interpolation polynomials for the points $\xi_1, \dots, \xi_r$ whose squared sum is bounded by 1 on K_0.

Lemma 5.3. *There are real polynomials $\psi_1, \dots, \psi_r$, depending only on K, such that*

$$\psi_i(\xi_j) = \delta_{ij}, \qquad \sum_{i=1}^{r} \psi_i(x)^2 \leq 1 \quad (x \in K_0).$$

Proof. Let ℓ_i be the Lagrange interpolation polynomials defined in (3.1). For each i, choose $n_i > 0$ such that $|x - \xi_i| < n_i$ for all $x \in K_0$. For $N \geq 1$, set

$$\psi_{i,N}(x) = \ell_i(x)\left(1 - \frac{(x-\xi_i)^2}{n_i^2}\right)^N.$$

The interpolation conditions hold for every N. Since $\psi_{i,N} \to 0$ uniformly on K_0 as $N \to \infty$, choosing N sufficiently large gives the desired inequality. □

The interpolation polynomials allow us to adjoin all isolated points with a fixed additive increase in the weighted certificate degree.

Proposition 5.4. *Let $\xi_1, \dots, \xi_r$ be distinct points in $\mathbb{R} \setminus K_0$, with $r \geq 1$, and set $K = K_0 \cup \{\xi_1, \dots, \xi_r\}$. Let $S \subset \mathbb{R}[x]$ be finite with $K_S = K$ and $M(S)$ saturated. Then there exists a nonnegative integer η_S, depending only on S, such that, for every Hilbert space $\mathcal{H}$, every nonnegative integer n, and every hermitian $F \in \mathcal{B}(\mathcal{H})[x]$,*

$$F \in \mathcal{M}_{\mathcal{H}}(\{\Delta\})_{\leq n}, \quad F(\xi_i) \succeq 0 \ (i = 1, \dots, r) \quad \Longrightarrow \quad F \in \mathcal{M}_{\mathcal{H}}(S)_{\leq n+\eta_S}.$$

Proof. Let $\psi_1, \dots, \psi_r$ be the interpolation polynomials from Lemma 5.3, and set

$$a = 1 - \sum_{i=1}^{r} \psi_i^2, \qquad \Psi_i = \psi_i^2 G_{\xi_i} \quad (1 \leq i \leq r),$$

where G_{ξ_i} is defined in (5.3). Since $a \geq 0$ on K_0 and $a(\xi_j) = 0$ for every j, both a and $a\Delta$ are nonnegative on K. By Lemma 5.1, each Ψ_i is positive semidefinite on K_0. Moreover, the interpolation conditions imply $\Psi_i(\xi_j) = 0$ for all i, j. Thus $\Psi_i \succeq 0$ on K.

Since $M(S)$ is saturated, both a and $a\Delta$ admit scalar certificates in $M(S)$. By Theorem 2.3, each Ψ_i admits a matrix certificate in $\mathcal{M}_{\mathbb{C}^2}(S)$. Choose a nonnegative integer η_S large enough to bound the weighted degrees of all these certificates and such that $2\deg\psi_i \leq \eta_S$ for every i.

Let $\mathcal{H}$ be a complex Hilbert space and let $F \in \mathcal{B}(\mathcal{H})[x]$ satisfy the assumptions of the proposition. Since $F \in \mathcal{M}_{\mathcal{H}}(\{\Delta\})_{\leq n}$, there exist $A, B \in \mathcal{B}(\mathcal{H})[x]$ such that $F = A^*A + \Delta B^*B$ has weighted degree at most n. Set $\widetilde{n} = \lfloor n/2 \rfloor$. Then $\deg A, \deg B \leq \widetilde{n}$. Applying Lemma 5.2 at each ξ_i gives polynomials $L_i, W_i, E_i \in \mathcal{B}(\mathcal{H})[x]$, with $V_i = \operatorname{col}(W_i, E_i)$ and $\deg L_i, \deg V_i \leq \widetilde{n}$, such that

$$F = L_i^* L_i + V_i^*(I_{\mathcal{H}} \otimes G_{\xi_i})V_i.$$

Using $1 = a + \sum_{i=1}^{r} \psi_i^2$, we obtain

$$F = aF + \sum_{i=1}^{r} \psi_i^2 F$$

$$= aA^*A + (a\Delta)B^*B + \sum_{i=1}^{r}(\psi_i L_i)^*(\psi_i L_i) + \sum_{i=1}^{r} V_i^*(I_{\mathcal{H}} \otimes \Psi_i)V_i. \tag{5.5}$$

Substituting the chosen scalar certificates for a and $a\Delta$, together with the matrix certificates for Ψ_i, into (5.5), we obtain a certificate for F in $\mathcal{M}_{\mathcal{H}}(S)$. Here (i) of Proposition 2.1 allows us to express the resulting positive semidefinite operator polynomial multipliers as hermitian squares without increasing their degrees. Since $\deg A, \deg B, \deg L_i, \deg V_i \le \widetilde{n}$, the resulting certificate has weighted degree at most $\eta_S + 2\widetilde{n} \le n + \eta_S$. Thus $F \in \mathcal{M}_{\mathcal{H}}(S)_{\le n+\eta_S}$. □

5.3. **Completion of Theorem A.**

Proof of Theorem A. We first work over a complex Hilbert space $\mathcal{H}$. If K is finite, Proposition 3.1 gives the required bound with $c_S = \eta_S$. We therefore assume that K contains at least one nondegenerate interval.

Let K_0 be the union of all nondegenerate interval components of K, and let ℓ be their number. Apply Theorem 4.8 to K_0, fix the corresponding polynomial σ_1, and put $\beta = 4\ell - 2 + \deg \sigma_1$. Let $\mathcal{E}$ be an arbitrary complex Hilbert space. By Theorem 4.8, Corollary 4.9, and Proposition 5.4, there exists a nonnegative integer η_S, depending only on S, such that every hermitian $F \in \mathcal{B}(\mathcal{E})[x]$ with $\deg F \le 2$ and $F \succeq 0$ on K satisfies

$$F \in \mathcal{M}_{\mathcal{E}}(S)_{\le N_S}, \qquad N_S = \beta + \eta_S.$$

In particular, N_S is independent of $\mathcal{E}$.

By Theorem 3.4, for every $\mathtt{d} \ge 1$, it follows that

$$F \succeq 0 \text{ on } K, \qquad \deg F \le \mathtt{d} \quad \Longrightarrow \quad F \in \mathcal{M}_{\mathcal{H}}(S)_{\le N_S + 2\lceil \mathtt{d}/2 \rceil - 2} \subseteq \mathcal{M}_{\mathcal{H}}(S)_{\le \mathtt{d} + N_S - 1}.$$

Hence the required degree bound holds with $c_S = N_S - 1$. For $\mathtt{d} = 0$, the positive constant operator F is a single hermitian square. The same bounds for real Hilbert spaces follow from Lemma A.1. The converse follows by evaluating the certificates on K. □

5.4. **Computation of the additive constant.** We now discuss the additive constants obtained in the proof of Theorem A. Suppose K contains $\ell \ge 1$ nondegenerate intervals. Choose $\sigma_0, \sigma_1 \in \mathrm{SOS}_{\mathbb{C}}$ satisfying (4.20). Let η_S be the additive degree increase from Corollary 4.9 or Proposition 5.4, as appropriate. Then the degree bound holds with

$$c_S = 4\ell - 3 + \deg \sigma_1 + \eta_S.$$

If K is finite, Proposition 3.1 gives $c_S = \eta_S$. These constants need not be optimal.

Although Δ and w are explicit, the degrees of the required SOS multipliers are not known in general. If the generators in S have real algebraic coefficients, admissible constants can be computed by increasing the degree bounds and testing semidefinite feasibility. These feasibility conditions are decidable by quantifier elimination over the real algebraic numbers, and saturation guarantees termination. The interpolation polynomials for isolated points can also be constructed effectively. See Example B.1 for two explicit computations.

6. Dependence on the underlying set

The additive constant in Theorem A depends on the compact set and its scalar description. The following examples show that the additive constants can become arbitrarily large, even for quadratic 2×2 matrix polynomials on unions of two compact intervals. In the first example, the right endpoint of one interval tends to infinity. In the second, both intervals remain in $[0, 1]$, but one of them shrinks to a point.

Both examples use the following compactness property of matrix polynomials with uniformly bounded degree.

Lemma 6.1. *Let $I \subset \mathbb{R}$ be a nondegenerate compact interval, and fix integers $m \geq 1$ and $\mathtt{d} \geq 0$. Suppose (F_k) is a sequence in $\mathbb{C}[x]^{m\times m}$ with $\deg F_k \leq \mathtt{d}$ for every k, satisfying*

$$\sup_k \sup_{x\in I} \|F_k(x)\|_2 < \infty, \tag{6.1}$$

where $\|\cdot\|_2$ denotes the spectral norm. Then (F_k) has a subsequence that converges coefficientwise to a polynomial $F \in \mathbb{C}[x]^{m\times m}$ of degree at most $\mathtt{d}$. Moreover, if each F_k is positive semidefinite on $\mathbb{R}$, then so is F.

Proof. The polynomials in $\mathbb{C}[x]^{m\times m}$ of degree at most $\mathtt{d}$ form a finite-dimensional vector space. For $G(x) = \sum_{j=0}^{\mathtt{d}} A_j x^j$, define

$$\|G\|_{\mathrm{coeff}} := \max_{0\leq j\leq \mathtt{d}} \|A_j\|_2, \qquad \|G\|_I := \sup_{x\in I} \|G(x)\|_2.$$

Since I is nondegenerate, a polynomial vanishing on I must vanish identically. Applied entrywise, this shows that $\|\cdot\|_I$ is a norm. The two norms are therefore equivalent by finite dimensionality. Hence (6.1) yields a coefficientwise convergent subsequence, which converges at every $x \in \mathbb{R}$. Since the cone of positive semidefinite matrices is closed, the limit is positive semidefinite on $\mathbb{R}$. □

6.1. **Example I.** Consider the quadratic matrix polynomial

$$F(x) = \begin{pmatrix} x+1 & \sqrt{6} \\ \sqrt{6} & x^2-4x+6 \end{pmatrix} \tag{6.2}$$

and the scalar polynomial $q(x) = (x-1)(x-2)$. The following nonmembership result is a special case of [Zal16, Proposition 3.4, Claim 1].

Lemma 6.2. *Let F be as in (6.2) and $q = (x-1)(x-2)$. Then*

$$F \succeq 0 \text{ on } [0,1]\cup[2,\infty), \qquad F \notin \mathcal{T}_{\mathbb{C}^2}(\{x,q\}).$$

Proof. The diagonal entries of F are positive on $[0,1]\cup[2,\infty)$, and $\det F(x) = x(x-1)(x-2) \geq 0$ on this set. Hence $F \succeq 0$ on $[0,1]\cup[2,\infty)$. The assertion $F \notin \mathcal{T}_{\mathbb{C}^2}(\{x,q\})$ follows from [Zal16, Proposition 3.4, Claim 1]. □

Restricting F to compact truncations gives certificates whose minimum weighted degree must diverge as the truncation expands.

Proposition 6.3. *For $n \geq 3$, let $K_n = [0,1]\cup[2,n]$ and $S_n = \{x, q, 1-x/n\}$. Then, for F as in (6.2),*

$$\min\{k \geq 0 : F \in \mathcal{T}_{\mathbb{C}^2}(S_n)_{\leq k}\} \longrightarrow \infty \qquad (n\to\infty).$$

Proof. For each $n \geq 3$, Theorem 2.2 shows that S_n is a saturated scalar description of K_n. Since $F \succeq 0$ on K_n, Theorem A implies that

$$F \in \mathcal{M}_{\mathbb{C}^2}(S_n) \subseteq \mathcal{T}_{\mathbb{C}^2}(S_n).$$

Suppose, for a contradiction, that there exist an integer $\mathtt{d} \geq 0$ and a sequence $n_j \to \infty$ such that $F \in \mathcal{T}_{\mathbb{C}^2}(S_{n_j})_{\leq \mathtt{d}}$ for every j. Then

$$F = \sum_{a,b,c\in\{0,1\}} x^a q^b (1-x/n_j)^c \Sigma_{j,abc}, \qquad \Sigma_{j,abc} \in \mathrm{SOS}_{\mathbb{C}^2},$$

where each representation has weighted degree at most $\mathtt{d}$. On $I = [1/4, 3/4]$, all eight scalar weights are bounded below by a constant $\delta > 0$ independent of j. Since every summand is positive semidefinite on I, we obtain

$$0 \preceq \Sigma_{j,abc}(x) \preceq \delta^{-1} F(x) \qquad (x \in I).$$

Thus the polynomials $\Sigma_{j,abc}$ are uniformly bounded in spectral norm on I, and $\deg \Sigma_{j,abc} \leq \mathtt{d}$.

Applying Lemma 6.1 successively to the eight sequences, we obtain a common subsequence along which $\Sigma_{j,abc}$ converges coefficientwise to Σ_{abc}. Each limit is positive semidefinite on $\mathbb{R}$ and therefore belongs to $\mathrm{SOS}_{\mathbb{C}^2}$ by (i) of Proposition 2.1. Since $1 - x/n_j$ converges coefficientwise to 1, passing to the limit yields

$$F = \sum_{a,b\in\{0,1\}} x^a q^b (\Sigma_{ab0} + \Sigma_{ab1}).$$

Thus $F \in \mathcal{T}_{\mathbb{C}^2}(\{x, q\})$, contradicting Lemma 6.2. □

6.2. **Example II.** We next construct a family of unions of two disjoint compact intervals for which the gap stays bounded away from zero while one interval component collapses.

For $0 < \varepsilon < 1/2$, define

$$K_\varepsilon = [0, \varepsilon^2] \cup [1/2, 1], \qquad q_\varepsilon(x) = (x - \varepsilon^2)(x - 1/2),$$
$$\theta(x) = x(1 - x), \qquad \rho_\varepsilon = 1 + 2\varepsilon - \varepsilon^3,$$

and let

$$F_\varepsilon(x) = \begin{pmatrix} \rho_\varepsilon(x^2 + \varepsilon\theta(x)) & \rho_\varepsilon\theta(x) \\ \rho_\varepsilon\theta(x) & \varepsilon(1-x)^2 + (1-\varepsilon^2)\theta(x) \end{pmatrix}. \tag{6.3}$$

We first verify positivity of this family on the prescribed sets.

Lemma 6.4. *The quadratic matrix polynomial F_ε is positive semidefinite on K_ε for every $0 < \varepsilon < 1/2$.*

Proof. Since $\rho_\varepsilon > 0$, both diagonal entries are nonnegative on $[0, 1]$. The determinant of F_ε is given by $\det F_\varepsilon = 2\rho_\varepsilon\theta q_\varepsilon$. The right-hand side is nonnegative on K_ε, proving the assertion. □

We include a proof of the following elementary fact.

Lemma 6.5. *Let $H \in \mathbb{C}[x]^{m\times m}$ be hermitian and positive semidefinite on $\mathbb{R}$. If $H(0) = 0$, then $H'(0) = 0$, and x^2 divides H entrywise.*

Proof. For every $\xi \in \mathbb{C}^m$, the scalar polynomial $\xi^* H(x)\xi$ is nonnegative on $\mathbb{R}$ and vanishes at zero. Its derivative therefore vanishes there, so $\xi^* H'(0)\xi = 0$. Polarization gives $H'(0) = 0$. Together with $H(0) = 0$, this implies the asserted divisibility. □

6.3. **Proof of Theorem B.**

Proof of Theorem B. For $0 < \varepsilon < 1/2$, define $K_\varepsilon = [0, \varepsilon^2] \cup [1/2, 1]$ and $q_\varepsilon = (x - \varepsilon^2)(x - 1/2)$, and set $S_{K_\varepsilon} = \{x, 1 - x, q_\varepsilon\}$. By Theorem 2.2, S_{K_ε} is a saturated scalar description of K_ε. For the matrix polynomial F_ε defined in (6.3), Lemma 6.4 gives $F_\varepsilon \succeq 0$ on K_ε. Hence Theorem A yields

$$F_\varepsilon \in \mathcal{M}_{\mathbb{C}^2}(S_{K_\varepsilon}) \subseteq \mathcal{T}_{\mathbb{C}^2}(S_{K_\varepsilon}).$$

Suppose, for a contradiction, that there exist an integer $\mathtt{d} \geq 0$ and a sequence $\varepsilon_j \downarrow 0$ such that $F_{\varepsilon_j} \in \mathcal{T}_{\mathbb{C}^2}(S_{K_{\varepsilon_j}})_{\leq \mathtt{d}}$ for every j. Then

$$F_{\varepsilon_j}(x) = \sum_{a,b,c\in\{0,1\}} x^a (1-x)^b q_{\varepsilon_j}^c \Sigma_{j,abc}, \qquad \Sigma_{j,abc} \in \mathrm{SOS}_{\mathbb{C}^2}, \tag{6.4}$$

where each representation has weighted degree at most $\mathtt{d}$. On $I = [3/4, 7/8]$, all eight scalar weights are bounded below by a constant $\delta > 0$ independent of j. Since every summand is positive semidefinite on I, we obtain

$$0 \preceq \Sigma_{j,abc}(x) \preceq \delta^{-1} F_{\varepsilon_j}(x) \qquad (x \in I).$$

Since the polynomials F_{ε_j} are uniformly bounded in spectral norm on I, the same holds for all $\Sigma_{j,abc}$. Moreover, $\deg \Sigma_{j,abc} \leq \mathtt{d}$.

Applying Lemma 6.1 successively to the eight sequences $(\Sigma_{j,abc})_j$, indexed by $(a,b,c) \in \{0,1\}^3$, we obtain a common subsequence along which $\Sigma_{j,abc}$ converges coefficientwise to Σ_{abc} for every $(a,b,c) \in \{0,1\}^3$. Each limit Σ_{abc} is positive semidefinite on $\mathbb{R}$.

We first extract additional information by evaluating the certificates at zero. We have

$$F_{\varepsilon_j}(0) = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_j \end{pmatrix}, \qquad q_{\varepsilon_j}(0) = \varepsilon_j^2/2 > 0.$$

For $e_1 = (1,0)^{\mathsf{T}}$, equation (6.4) gives

$$0 = \sum_{b,c\in\{0,1\}} q_{\varepsilon_j}(0)^c e_1^{\mathsf{T}} \Sigma_{j,0bc}(0) e_1.$$

Every summand is nonnegative and every scalar weight is positive, so $e_1^{\mathsf{T}}\Sigma_{j,0bc}(0)e_1 = 0$ for all b,c. Positive semidefiniteness then implies $\Sigma_{j,0bc}(0)e_1 = 0$. Passing to the limit yields

$$\Sigma_{0bc}(0)e_1 = 0 \qquad (b,c \in \{0,1\}). \tag{6.5}$$

The coefficientwise limits of F_{ε_j} and q_{ε_j} are

$$F_0(x) = xG_0(x), \qquad G_0(x) = \begin{pmatrix} x & 1-x \\ 1-x & 1-x \end{pmatrix}, \qquad q_0(x) = x(x-1/2).$$

Passing to the limit in (6.4) gives

$$F_0(x) = \sum_{a,b,c\in\{0,1\}} x^a(1-x)^b q_0(x)^c \Sigma_{abc}. \tag{6.6}$$

Evaluating at zero, we obtain $\Sigma_{000}(0) + \Sigma_{010}(0) = 0$. Both matrices are positive semidefinite, so each is zero. Lemma 6.5 therefore shows that x^2 divides both Σ_{000} and Σ_{010} entrywise.

Thus every term in the limiting representation (6.6) is divisible by x. After dividing by x and setting $x = 0$, the terms with $a = c = 0$ vanish by the divisibility established above, while those with $a = c = 1$ retain a factor of x. Hence

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \Sigma_{100}(0) + \Sigma_{110}(0) - \frac{1}{2}\big(\Sigma_{001}(0) + \Sigma_{011}(0)\big).$$

Set $A = \Sigma_{100}(0) + \Sigma_{110}(0) \succeq 0$. Since $\Sigma_{001}(0)e_1 = \Sigma_{011}(0)e_1 = 0$ by (6.5), the preceding identity gives $Ae_1 = (0,1)^{\mathsf{T}}$. But $e_1^{\mathsf{T}}Ae_1 = 0$ and $A \succeq 0$ imply $Ae_1 = 0$, a contradiction. Thus the minimum weighted certificate degree tends to infinity as $\varepsilon \downarrow 0$. □

7. Proof of Theorem C

We prove Theorem C in this section. Fix a compact interval $J \subset \mathbb{R}$ and $\mu, \omega > 0$, as in Theorem C. Since there is nothing to prove if $\mathcal{K}(J,\mu,\omega)$ is empty, we assume that this family is nonempty. Let $|J|$ denote the length of J. Since every $K \in \mathcal{K}(J,\mu,\omega)$ with ℓ components satisfies

$$\ell\mu + (\ell-1)\omega \le |J|,$$

only finitely many values of ℓ occur. For each such ℓ, let $\mathcal{P}_\ell \subset J^{2\ell}$ be the compact set of endpoint tuples $\mathbf{e} = (a_1, b_1, \dots, a_\ell, b_\ell)$ corresponding to sets in $\mathcal{K}(J,\mu,\omega)$ with ℓ components.

For each $\mathbf{e} = (a_1, b_1, \dots, a_\ell, b_\ell) \in \mathcal{P}_\ell$, let $K_{\mathbf{e}}$ denote the corresponding union of intervals and, recalling (4.2), set $\Delta_{\mathbf{e}}(x) = -\prod_{j=1}^{\ell}(x-a_j)(x-b_j)$. Let $w_{\mathbf{e}}$ be obtained from the rank-one pencil associated with $K_{\mathbf{e}}$.

We first show that the degrees of the SOS multipliers in the identity

$$1 = w_{\mathbf{e}}\sigma + \Delta_{\mathbf{e}}\tau, \qquad \sigma, \tau \in \mathrm{SOS}_{\mathbb{C}}, \tag{7.1}$$

can be bounded independently of $\mathbf{e} \in \mathcal{P}_\ell$.

Lemma 7.1. *For every $\mathbf{e} \in \mathcal{P}_\ell$, there exist $\sigma, \tau \in \mathrm{SOS}_{\mathbb{C}}$ satisfying (7.1) and a number $\varepsilon > 0$ such that*

$$\sigma(x), \tau(x) \geq \varepsilon \qquad (x \in \mathbb{R}).$$

Proof. Since $K_{\mathbf{e}}$ is compact, we can choose $\varepsilon > 0$ such that $f = 1 - \varepsilon(w_{\mathbf{e}} + \Delta_{\mathbf{e}})$ is nonnegative on $K_{\mathbf{e}}$. Since $w_{\mathbf{e}} > 0$ on $\mathbb{R}$ and $\Delta_{\mathbf{e}}$ has simple zeros at all the endpoints of $K_{\mathbf{e}}$, the quadratic module $M(\{w_{\mathbf{e}}\Delta_{\mathbf{e}}\})$ is saturated by Theorem 2.2. Hence, as $\Delta_{\mathbf{e}} f \geq 0$ on $K_{\mathbf{e}}$, there exist $\widetilde{\sigma}_0, \widetilde{\sigma}_1 \in \mathrm{SOS}_{\mathbb{C}}$ such that

$$\Delta_{\mathbf{e}} f = \widetilde{\sigma}_0 + w_{\mathbf{e}}\Delta_{\mathbf{e}}\widetilde{\sigma}_1.$$

At every endpoint of $K_{\mathbf{e}}$, the polynomial $\widetilde{\sigma}_0$ vanishes. Since it is nonnegative on $\mathbb{R}$, each such zero has even multiplicity. Therefore, $\widetilde{\sigma}_0 = \Delta_{\mathbf{e}}^2\widetilde{\tau}$ for some $\widetilde{\tau} \in \mathrm{SOS}_{\mathbb{C}}$, since the quotient is nonnegative on $\mathbb{R}$. Dividing by $\Delta_{\mathbf{e}}$ gives

$$f = \Delta_{\mathbf{e}}\widetilde{\tau} + w_{\mathbf{e}}\widetilde{\sigma}_1.$$

Thus, $1 = w_{\mathbf{e}}(\widetilde{\sigma}_1 + \varepsilon) + \Delta_{\mathbf{e}}(\widetilde{\tau} + \varepsilon)$. Taking $\sigma = \widetilde{\sigma}_1 + \varepsilon$ and $\tau = \widetilde{\tau} + \varepsilon$ proves the lemma. □

For $N \in \mathbb{N}$, let

$$V_N = \mathbb{R}[x]_{\leq 2\ell+2N} \oplus \mathbb{R}[x]_{\leq 2\ell-2+2N}, \qquad W_N = \mathbb{R}[x]_{\leq 4\ell-2+2N}.$$

For $\mathbf{e} \in \mathcal{P}_\ell$, define the linear map

$$\mathcal{L}_{\mathbf{e},N} : V_N \longrightarrow W_N; \qquad (f, g) \longmapsto w_{\mathbf{e}} f + \Delta_{\mathbf{e}} g.$$

Lemma 7.2. *For every $\mathbf{e} \in \mathcal{P}_\ell$ and every $N \in \mathbb{N}$, the linear map $\mathcal{L}_{\mathbf{e},N}$ is surjective.*

Proof. Since $w_{\mathbf{e}} > 0$ on $\mathbb{R}$ and all zeros of $\Delta_{\mathbf{e}}$ are real, the polynomials $w_{\mathbf{e}}$ and $\Delta_{\mathbf{e}}$ are coprime. By Bézout's identity, there exist $a, b \in \mathbb{R}[x]$ such that

$$aw_{\mathbf{e}} + b\Delta_{\mathbf{e}} = 1.$$

Let $q \in W_N$. Multiplying this identity by q and dividing qa by $\Delta_{\mathbf{e}}$, we obtain $qa = \widetilde{f}\Delta_{\mathbf{e}} + f$, where $\deg f \leq 2\ell - 1$. Thus,

$$q = w_{\mathbf{e}} f + \Delta_{\mathbf{e}} g, \qquad g = w_{\mathbf{e}}\widetilde{f} + qb. \tag{7.2}$$

Since $\deg w_{\mathbf{e}} = 2\ell - 2$, we have $\deg(w_{\mathbf{e}} f) \leq 4\ell - 3$. Together with $\deg q \leq 4\ell - 2 + 2N$ and $\deg \Delta_{\mathbf{e}} = 2\ell$, this implies $\deg g \leq 2\ell - 2 + 2N$. Thus $(f, g) \in V_N$, proving the surjectivity of $\mathcal{L}_{\mathbf{e},N}$. □

Remark 7.1. The polynomial f in (7.2) satisfies the stronger degree bound $\deg f \leq 2\ell - 1$, whereas the first component of V_N allows degree up to $2\ell + 2N$. This additional degree allowance is not needed for surjectivity. It is used in Lemma 7.3, where the first SOS multiplier is perturbed by $-\delta\Delta_{\mathbf{e}}(1 + x^2)^N$, a polynomial of degree $2\ell + 2N$. Unlike the polynomial f in (7.2), the resulting multiplier must be strictly positive and have the prescribed degree, ensuring that it lies in the interior of the corresponding truncated SOS cone.

Lemma 7.3. *Fix $\mathbf{e} \in \mathcal{P}_\ell$, and let $\sigma, \tau \in \mathrm{SOS}_{\mathbb{C}}$ be as in Lemma 7.1. Suppose $N \in \mathbb{N}$ satisfies*

$$2\ell + 2N > \deg \sigma, \qquad 2\ell - 2 + 2N > \deg \tau.$$

Then there exist $\widehat{\sigma}, \widehat{\tau} \in \mathrm{SOS}_{\mathbb{C}}$, both strictly positive on $\mathbb{R}$, such that

$$1 = w_{\mathbf{e}}\widehat{\sigma} + \Delta_{\mathbf{e}}\widehat{\tau},$$

with $\deg \widehat{\sigma} = 2\ell + 2N$ and $\deg \widehat{\tau} = 2\ell - 2 + 2N$.

Proof. Set

$$(\kappa_0, \kappa_1) = \big(-\Delta_{\mathbf{e}}(1 + x^2)^N,\ w_{\mathbf{e}}(1 + x^2)^N\big).$$

For $\delta > 0$, define

$$\sigma_\delta = \sigma + \delta\kappa_0, \qquad \tau_\delta = \tau + \delta\kappa_1.$$

Since $(\kappa_0, \kappa_1) \in \ker \mathcal{L}_{\mathbf{e},N}$, we have

$$1 = w_{\mathbf{e}}\sigma_\delta + \Delta_{\mathbf{e}}\tau_\delta.$$

Moreover, by the choice of N, $\deg \sigma_\delta = 2\ell + 2N$ and $\deg \tau_\delta = 2\ell - 2 + 2N$. We show that both σ_δ and τ_δ are strictly positive on $\mathbb{R}$ for sufficiently small $\delta > 0$.

By Lemma 7.1, there exists $\varepsilon > 0$ such that $\sigma, \tau \geq \varepsilon$ on $\mathbb{R}$. Since $w_{\mathbf{e}} > 0$ on $\mathbb{R}$,

$$\tau_\delta = \tau + \delta w_{\mathbf{e}}(1 + x^2)^N \geq \varepsilon.$$

On the other hand,

$$\sigma_\delta = \sigma - \delta \Delta_{\mathbf{e}}(1 + x^2)^N.$$

If $x \notin K_{\mathbf{e}}$, then $\Delta_{\mathbf{e}}(x) < 0$, so $\sigma_\delta(x) > 0$. Since $K_{\mathbf{e}}$ is compact, the number

$$\omega_N = \max_{x \in K_{\mathbf{e}}} \Delta_{\mathbf{e}}(x)(1 + x^2)^N$$

is finite and strictly positive. Hence

$$\sigma_\delta(x) \geq \varepsilon - \delta\omega_N \qquad (x \in K_{\mathbf{e}}).$$

Choosing $0 < \delta < \varepsilon/\omega_N$ ensures that $\sigma_\delta > 0$ on $\mathbb{R}$. Since every nonnegative real polynomial in one variable belongs to $\mathrm{SOS}_{\mathbb{C}}$, we conclude that $\sigma_\delta, \tau_\delta \in \mathrm{SOS}_{\mathbb{C}}$. Taking $\widehat{\sigma} = \sigma_\delta$ and $\widehat{\tau} = \tau_\delta$ completes the proof. □

Recall that a strictly positive real polynomial of exact even degree $2\mathtt{d}$ is an interior point of $\mathrm{SOS}_{\mathbb{C}} \cap \mathbb{R}[x]_{\leq 2\mathtt{d}}$ in $\mathbb{R}[x]_{\leq 2\mathtt{d}}$. Indeed, its quotient by $(1 + x^2)^{\mathtt{d}}$ is bounded below by a positive constant on $\mathbb{R}$, so sufficiently small coefficient perturbations preserve positivity.

For $N \in \mathbb{N}$, define

$$C_N = \{(\sigma, \tau) \in V_N : \sigma, \tau \in \mathrm{SOS}_{\mathbb{C}}\}.$$

By the preceding observation, the pair $(\widehat{\sigma}, \widehat{\tau})$ constructed in Lemma 7.3 is an interior point of C_N in V_N.

Lemma 7.4. *Fix $\mathbf{e}_0 \in \mathcal{P}_\ell$ and $N \in \mathbb{N}$. Suppose there exists (σ_0, τ_0) in the interior of C_N such that $\mathcal{L}_{\mathbf{e}_0,N}(\sigma_0, \tau_0) = 1$. Then there is a neighborhood $\mathcal{U}$ of $\mathbf{e}_0$ in $\mathcal{P}_\ell$ such that, for every $\mathbf{e} \in \mathcal{U}$, there exist $(\sigma_{\mathbf{e}}, \tau_{\mathbf{e}}) \in C_N$ satisfying $\mathcal{L}_{\mathbf{e},N}(\sigma_{\mathbf{e}}, \tau_{\mathbf{e}}) = 1$.*

Proof. By Lemma 7.2, the map $\mathcal{L}_{\mathbf{e}_0,N}$ is surjective. Hence it admits a right inverse

$$\mathcal{R}_0 : W_N \longrightarrow V_N.$$

By Lemma 4.10, the coefficients of $w_{\mathbf{e}}$ depend continuously on $\mathbf{e}$. The same holds for $\Delta_{\mathbf{e}}$, since its coefficients depend polynomially on the endpoints by (4.2). Since V_N and W_N are fixed finite-dimensional spaces, $\mathcal{L}_{\mathbf{e},N}$ also depends continuously on $\mathbf{e}$. Now

$$\mathcal{L}_{\mathbf{e}_0,N}\mathcal{R}_0 = \mathrm{id}_{W_N}.$$

By continuity, $\mathcal{L}_{\mathbf{e},N}\mathcal{R}_0$ remains invertible for $\mathbf{e}$ sufficiently close to $\mathbf{e}_0$. For such $\mathbf{e}$, define

$$\mathcal{R}_{\mathbf{e}} = \mathcal{R}_0\big(\mathcal{L}_{\mathbf{e},N}\mathcal{R}_0\big)^{-1}.$$

Then $\mathcal{R}_{\mathbf{e}}$ is a right inverse of $\mathcal{L}_{\mathbf{e},N}$, and it depends continuously on $\mathbf{e}$. Set

$$(\sigma_{\mathbf{e}}, \tau_{\mathbf{e}}) = (\sigma_0, \tau_0) + \mathcal{R}_{\mathbf{e}}\big(1 - \mathcal{L}_{\mathbf{e},N}(\sigma_0, \tau_0)\big).$$

By construction, $\mathcal{L}_{\mathbf{e},N}(\sigma_{\mathbf{e}}, \tau_{\mathbf{e}}) = 1$. Furthermore, since $\mathcal{L}_{\mathbf{e}_0,N}(\sigma_0, \tau_0) = 1$ and the maps $\mathcal{R}_{\mathbf{e}}$ are continuous,

$$(\sigma_{\mathbf{e}}, \tau_{\mathbf{e}}) - (\sigma_0, \tau_0) = \mathcal{R}_{\mathbf{e}}\big(1 - \mathcal{L}_{\mathbf{e},N}(\sigma_0, \tau_0)\big) \longrightarrow 0$$

as $\mathbf{e} \to \mathbf{e}_0$. Since (σ_0, τ_0) is an interior point of C_N, the perturbed pair also belongs to C_N for $\mathbf{e}$ sufficiently close to $\mathbf{e}_0$. Shrinking the neighborhood if necessary proves the lemma. □

Lemma 7.5. *For each fixed $\ell \geq 1$ with $\mathcal{P}_\ell \neq \emptyset$, there exists $N_{1,\ell} \in \mathbb{N}$ such that, for every $\mathbf{e} \in \mathcal{P}_\ell$, the identity (7.1) admits SOS multipliers satisfying*

$$\deg \sigma \leq 2\ell + 2N_{1,\ell}, \qquad \deg \tau \leq 2\ell - 2 + 2N_{1,\ell}.$$

Proof. Fix $\mathbf{e}_0 \in \mathcal{P}_\ell$. By Lemmas 7.1 and 7.3, we can choose $N_{\mathbf{e}_0} \in \mathbb{N}$ and multipliers $\sigma_0, \tau_0 \in \mathrm{SOS}_{\mathbb{C}}$ satisfying

$$1 = w_{\mathbf{e}_0}\sigma_0 + \Delta_{\mathbf{e}_0}\tau_0,$$

such that (σ_0, τ_0) is an interior point of $C_{N_{\mathbf{e}_0}}$.

By Lemma 7.4, there exists a neighborhood $U_{\mathbf{e}_0}$ of $\mathbf{e}_0$ on which (7.1) admits SOS multipliers with degrees at most $2\ell + 2N_{\mathbf{e}_0}$ and $2\ell - 2 + 2N_{\mathbf{e}_0}$, respectively. These neighborhoods form an open cover of $\mathcal{P}_\ell$. By compactness, finitely many of them cover $\mathcal{P}_\ell$. Taking $N_{1,\ell}$ to be the largest of the corresponding values of $N_{\mathbf{e}_0}$ gives the required uniform degree bounds. □

Proof of Theorem C. Fix ℓ such that $\mathcal{P}_\ell \neq \emptyset$, and let $N_{1,\ell}$ be as in Lemma 7.5. Then, for every $\mathbf{e} \in \mathcal{P}_\ell$, there exist $\sigma, \tau \in \mathrm{SOS}_{\mathbb{C}}$ such that

$$1 = w_{\mathbf{e}}\sigma + \Delta_{\mathbf{e}}\tau, \qquad \deg \sigma \leq 2\ell + 2N_{1,\ell}.$$

Multiplying by $\Delta_{\mathbf{e}}$ gives

$$\Delta_{\mathbf{e}} = \Delta_{\mathbf{e}}^2\tau + w_{\mathbf{e}}\Delta_{\mathbf{e}}\sigma.$$

Set $\beta_\ell = 6\ell - 2 + 2N_{1,\ell}$. By Theorem 4.8, for every complex Hilbert space $\mathcal{H}$ and every hermitian $F \in \mathcal{B}(\mathcal{H})[x]$ with $\deg F \leq 2$, we have

$$F \succeq 0 \text{ on } K_{\mathbf{e}} \quad \Longrightarrow \quad F \in \mathcal{M}_{\mathcal{H}}(\{\Delta_{\mathbf{e}}\})_{\leq \beta_\ell}. \tag{7.3}$$

Since only finitely many values of ℓ occur, we set

$$N = \max_{\ell:\, \mathcal{P}_\ell \neq \emptyset} \beta_\ell.$$

By (7.3), every positive semidefinite hermitian quadratic operator polynomial on $K_{\mathbf{e}}$ admits a certificate in $\mathcal{M}_{\mathcal{H}}(\{\Delta_{\mathbf{e}}\})$ of weighted degree at most N, independently of $\mathbf{e}$ and the Hilbert space $\mathcal{H}$.

Theorem 3.4, applied separately to each $K_{\mathbf{e}}$, therefore gives for every $\mathtt{d} \geq 1$,

$$F \succeq 0 \text{ on } K_{\mathbf{e}} \quad \Longrightarrow \quad F \in \mathcal{M}_{\mathcal{H}}(\{\Delta_{\mathbf{e}}\})_{\leq N+2\lceil \mathtt{d}/2\rceil - 2} \subseteq \mathcal{M}_{\mathcal{H}}(\{\Delta_{\mathbf{e}}\})_{\leq \mathtt{d}+N-1}.$$

Thus the required degree bound holds with $C = N - 1$, which depends only on J, μ, ω. For $\mathtt{d} = 0$, the conclusion follows by taking the square root of the positive constant operator.

Finally, since $\Delta_K = \prod_{g \in S_K} g$, every certificate in $\mathcal{M}_{\mathcal{H}}(\{\Delta_K\})$ is also a certificate in $\mathcal{T}_{\mathcal{H}}(S_K)$ with the same weighted degree. The real Hilbert space case follows from Lemma A.1, and the reverse implications follow by evaluating the certificates on K. □

Declaration of AI use

During the preparation of this manuscript, the authors used large language models (LLMs) to identify typographical errors. LLM assistance also contributed to the development of Example II in Subsection 6.2, using Example I (Section 6) as a starting point to construct a family of examples contained in a fixed bounded interval that establishes the nonuniformity result. In addition, LLMs assisted in working out the examples in Appendix B by applying methods developed in this paper. The authors have independently verified all mathematical arguments in Example II and Appendix B and take full responsibility for the content and conclusions of the manuscript.

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Appendix A. The real Hilbert space case

The degree bounds in Theorems A and C are established for complex Hilbert spaces. In this appendix, we show that the same bounds hold for real Hilbert spaces. Recall that the operator quadratic modules and preorderings over real Hilbert spaces are defined in Subsection 1.2, where each SOS multiplier is a sum of at most two hermitian squares.

The complexification of $\mathcal{H}$ is the complex Hilbert space $\mathcal{H}_{\mathbb{C}} = \mathcal{H} \oplus i\mathcal{H}$. Every bounded real-linear operator $A \in \mathcal{B}(\mathcal{H})$ extends to a bounded complex-linear operator $A_{\mathbb{C}} \in \mathcal{B}(\mathcal{H}_{\mathbb{C}})$ defined by

$$A_{\mathbb{C}}(u + iv) = Au + iAv, \qquad u, v \in \mathcal{H}.$$

For an operator polynomial $F(x) = \sum_k A_k x^k \in \mathcal{B}(\mathcal{H})[x]$, we define its coefficientwise complexification by

$$F_{\mathbb{C}}(x) \;=\; \sum_k (A_k)_{\mathbb{C}}\, x^k \in \mathcal{B}(\mathcal{H}_{\mathbb{C}})[x].$$

We also recall the decomposition of a complex operator into its real and imaginary parts. For every $A \in \mathcal{B}(\mathcal{H}_{\mathbb{C}})$ and $v \in \mathcal{H}$, write uniquely

$$Av = Bv + iDv, \qquad Bv, Dv \in \mathcal{H}.$$

This defines bounded real-linear operators $B, D \in \mathcal{B}(\mathcal{H})$ satisfying $A = B_{\mathbb{C}} + iD_{\mathbb{C}}$. We define the conjugate of A by $\overline{A} = B_{\mathbb{C}} - iD_{\mathbb{C}}$.

Every polynomial G with coefficients in $\mathcal{B}(\mathcal{H}_{\mathbb{C}})$ admits a unique decomposition

$$G = Y_{\mathbb{C}} + iZ_{\mathbb{C}},$$

where $Y, Z \in \mathcal{B}(\mathcal{H})[x]$ satisfy $\deg Y, \deg Z \le \deg G$, and $Y_{\mathbb{C}}, Z_{\mathbb{C}}$ denote their coefficientwise complexifications. The conjugate polynomial is then given by $\overline{G} = Y_{\mathbb{C}} - iZ_{\mathbb{C}}$.

Lemma A.1. *Let $\mathcal{H}$ be a real Hilbert space, and let $F \in \mathcal{B}(\mathcal{H})[x]$ be hermitian. Then, for every finite $S \subset \mathbb{R}[x]$ and every nonnegative integer n,*

$$F \in \mathcal{M}_{\mathcal{H}}(S)_{\le n} \quad \Longleftrightarrow \quad F_{\mathbb{C}} \in \mathcal{M}_{\mathcal{H}_{\mathbb{C}}}(S)_{\le n}.$$

Proof. Complexification preserves adjoints, products, and polynomial degrees. Thus, complexifying each square factor in a real operator certificate proves the forward implication.

Conversely, suppose that

$$F_{\mathbb{C}} = F_0^* F_0 + \sum_{j=1}^{s} g_j F_j^* F_j, \qquad F_j \in \mathcal{B}(\mathcal{H}_{\mathbb{C}})[x], \quad j = 0, 1, \dots, s,$$

where $S = \{g_1, \dots, g_s\}$, and the certificate has weighted degree at most n.

Let $Y_j, Z_j \in \mathcal{B}(\mathcal{H})[x]$ satisfy $F_j = (Y_j)_{\mathbb{C}} + i(Z_j)_{\mathbb{C}}$. Since $F_{\mathbb{C}}$ and the scalar generators g_j are invariant under conjugation, averaging the certificate with its conjugate gives

$$\begin{aligned} F_{\mathbb{C}} &= \frac{1}{2}\big(F_0^* F_0 + \overline{F}_0^{\,*}\overline{F}_0\big) + \frac{1}{2}\sum_{j=1}^{s} g_j\big(F_j^* F_j + \overline{F}_j^{\,*}\overline{F}_j\big) \\ &= (Y_0^* Y_0 + Z_0^* Z_0)_{\mathbb{C}} + \left(\sum_{j=1}^{s} g_j (Y_j^* Y_j + Z_j^* Z_j)\right)_{\mathbb{C}}. \end{aligned}$$

Restricting to $\mathcal{H}$ we get

$$F = Y_0^* Y_0 + Z_0^* Z_0 + \sum_{j=1}^{s} g_j\big(Y_j^* Y_j + Z_j^* Z_j\big),$$

which is a real operator certificate. Since $\deg Y_j, \deg Z_j \le \deg F_j$, the resulting certificate has weighted degree at most n. □

Appendix B. Explicit Examples

The following example illustrates the computation of the additive constants obtained in the proof of Theorem A. We first consider a union of two disjoint compact intervals and then examine how adjoining an isolated point affects the resulting degree bound. The Mathematica code for these computations is available in the accompanying repository [JZ26].

Example B.1. Let

$$K_0 = [0,1] \cup [2,3], \qquad K_1 = K_0 \cup \{4\},$$

with natural descriptions

$$\begin{aligned} S_0 &= \{x,\ (x-1)(x-2),\ 3-x\}, \\ S_1 &= \{x,\ (x-1)(x-2),\ (x-3)(x-4),\ 4-x\}. \end{aligned}$$

By Theorem 2.2, both $M(S_0)$ and $M(S_1)$ are saturated. The union of the nondegenerate interval components is K_0 in both cases. Thus $\ell = 2$, and the polynomials from (4.2) are

$$P = x(x-2), \qquad Q = (x-1)(x-3), \qquad R = 3-2x, \qquad \Delta = -x(x-1)(x-2)(x-3).$$

Recall from (4.3) that the rank-one pencil is $T(y) = D - yvv^{\mathsf{T}}$, where

$$D = \operatorname{diag}(b_1, b_2) = \operatorname{diag}(1, 3),$$

and the entries of v may be chosen positive, with $v_j^2 = \frac{P(b_j)}{Q'(b_j)}$, $j = 1, 2$. In the present example, $v_1^2 = \frac{1}{2}$, $v_2^2 = \frac{3}{2}$, so we can choose

$$v = \begin{pmatrix} 1/\sqrt{2} \\ \sqrt{3/2} \end{pmatrix}.$$

Claim 1. The polynomial w from (4.9) is

$$w(x) = 2\left((x - \tfrac{3}{2})^2 + \tfrac{3}{4}\right).$$

In particular, $w > 0$ on $\mathbb{R}$.

Proof of Claim 1. Since $v_1^2 = \frac{1}{2}$ and $v_2^2 = \frac{3}{2}$, we have

$$w(x) = \frac{1}{2}(x-3)^2 + \frac{3}{2}(x-1)^2 = 2\left((x - \tfrac{3}{2})^2 + \tfrac{3}{4}\right),$$

which proves the claim. □

Claim 2. There exist $\sigma_0, \sigma_1 \in \mathrm{SOS}_{\mathbb{C}}$ satisfying

$$\Delta = \sigma_0 + w\Delta\sigma_1,$$

with $\deg \sigma_1 = 4$. Moreover, no such identity is possible with $\deg \sigma_1 < 4$.

Proof of Claim 2. Set

$$\sigma_1(x) = \frac{1}{4}\left((x-\tfrac{3}{2})^2 - \frac{19}{12}\right)^2 + \frac{1}{18},$$
$$\tau(x) = \frac{1}{2}(x-\tfrac{3}{2})^2 + \frac{1}{24},$$

and $\sigma_0 = \Delta^2\tau$. Direct expansion gives

$$1 = w\sigma_1 + \Delta\tau,$$

and hence

$$\Delta = \sigma_0 + w\Delta\sigma_1.$$

Clearly, $\sigma_0, \sigma_1 \in \mathrm{SOS}_{\mathbb{C}}$ and $\deg \sigma_1 = 4$.

We now show that, the degree 4 of σ_1 is the smallest possible among all SOS identities of the form $\Delta = \sigma_0 + w\Delta\sigma_1$. Suppose that

$$\Delta = \widetilde{\sigma}_0 + w\Delta\widetilde{\sigma}_1, \qquad \widetilde{\sigma}_0, \widetilde{\sigma}_1 \in \mathrm{SOS}_{\mathbb{C}}\,.$$

Since Δ changes sign at $0, 1, 2, 3$, we must have

$$w(x)\widetilde{\sigma}_1(x) = 1 \qquad (x = 0, 1, 2, 3).$$

The unique polynomial of degree at most three satisfying these conditions is

$$p(x) = \frac{1 + 3x - x^2}{6}.$$

However, $p(x) < 0$ for sufficiently large $|x|$, so $p \notin \mathrm{SOS}_{\mathbb{C}}$. Consequently, no such identity is possible with $\deg \widetilde{\sigma}_1 < 4$. □

By Claim 2 and Theorem 4.8, the quadratic certificate bound for $\mathcal{M}_{\mathcal{H}}(\{\Delta\})$ is

$$4\ell - 2 + \deg \sigma_1 = 6 + 4 = 10.$$

We next transfer this bound to the natural description S_0.

Claim 3. The polynomial Δ admits a certificate in $M(S_0)$ of weighted degree 7, and no certificate of smaller weighted degree exists.

Proof of Claim 3. A computation verifies that

$$\Delta = \frac{1}{6}x\big[(x-1)(x-2)(x-3)\big]^2 + \frac{1}{2}(x-1)(x-2)\big[x(x-3)\big]^2 + \frac{1}{6}(3-x)\big[x(x-1)(x-2)\big]^2. \quad \text{(B.1)}$$

Each multiplier is a scalar sum of squares, and the largest weighted degree is 7. Thus $\Delta \in M(S_0)_{\leq 7}$.

To prove minimality, suppose that Δ admits a certificate of weighted degree at most 6. Write

$$\Delta = \tau_0 + x\tau_1 + (x-1)(x-2)\tau_2 + (3-x)\tau_3,$$

where $\tau_j \in \mathrm{SOS}_{\mathbb{C}}$.

At each zero of Δ, every summand is nonnegative. Consequently, whenever a generator is strictly positive at such a zero, its SOS multiplier must vanish there. Since a nonnegative real polynomial has only zeros of even multiplicity, every nonzero multiplier must satisfy $\deg \tau_0 \geq 8$, $\deg \tau_1 \geq 6$, $\deg \tau_2 \geq 4$, and $\deg \tau_3 \geq 6$. Indeed, these bounds correspond to the required double zeros at the endpoints where the respective generators are strictly positive.

A certificate of weighted degree at most 6 therefore forces $\tau_0 = \tau_1 = \tau_3 = 0$. The remaining multiplier must have the form $\tau_2 = c\,x^2(x-3)^2$, $c \geq 0$. Hence

$$\Delta = c(x-1)(x-2)x^2(x-3)^2.$$

If $c > 0$, the right-hand side has degree 6, whereas $\deg \Delta = 4$. If $c = 0$, the identity is also impossible. Thus no certificate of weighted degree at most 6 exists. Together with (B.1), this proves that the minimum weighted degree is 7. □

By Claim 3, the minimum weighted degree of a certificate for Δ in $M(S_0)$ is 7. Hence, in Corollary 4.9, we may take $\eta_{S_0} = 7 - 2\ell = 3$. Combining this with Claim 2 gives the quadratic bound $N_{S_0} = 10 + 3 = 13$. Therefore, by Theorem 3.4, every hermitian $F \in \mathcal{B}(\mathcal{H})[x]$ with $\deg F \leq \mathtt{d}$, $\mathtt{d} \geq 1$, and $F \succeq 0$ on K_0 satisfies

$$F \in \mathcal{M}_{\mathcal{H}}(S_0)_{\leq 11+2\lceil \mathtt{d}/2 \rceil} \subseteq \mathcal{M}_{\mathcal{H}}(S_0)_{\leq \mathtt{d}+12}.$$

Thus, we obtain the additive constant $C_{S_0} = 12$.

Adjoining the isolated point. We now pass from K_0 to $K_1 = K_0 \cup \{4\}$. In the notation of Subsection 5, the isolated point is $\xi = 4$, and

$$\delta = -\Delta(4) = 24.$$

Hence the associated matrix polynomial from (5.3) is

$$G_4(x) = \begin{pmatrix} 24 + \Delta(x) & (x-4)\Delta(x) \\ (x-4)\Delta(x) & (x-4)^2\Delta(x) \end{pmatrix}.$$

For this single isolated point, we may choose the interpolation polynomial $\psi_1 = 1$ in Proposition 5.4. Then $a = 1 - \psi_1^2 = 0$, so it suffices to find a matrix certificate for G_4 in $\mathcal{M}_{\mathbb{C}^2}(S_1)$.

Claim 4. The matrix polynomial G_4 admits a certificate in $\mathcal{M}_{\mathbb{C}^2}(S_1)$ of weighted degree 9.

Proof of Claim 4. Define polynomials

$$P_1 = (x-1)(x-2)(x-3)(x-4), \qquad P_2 = x(x-3)(x-4),$$
$$P_3 = x(x-1)(x-2), \qquad P_4 = x(x-1)(x-2)(x-3),$$

and set

$$\tau = x^2 - 3x + \frac{27}{8}, \qquad \rho_1 = \frac{1}{4}(4-x)^2\tau, \qquad \rho_4 = x^2 + \frac{3}{2} + \frac{1}{4}x^2\tau,$$

Define the polynomial columns

$$u_3 = \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{6}P_3 \end{pmatrix}, \qquad u_4 = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{6}P_4 \end{pmatrix}.$$

With these choices, direct expansion gives

$$\begin{aligned} G_4 = x\begin{pmatrix} \rho_1 & 0 \\ 0 & \frac{1}{6}P_1^2 \end{pmatrix} + (x-1)(x-2)\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}P_2^2 \end{pmatrix} \\ + 6(x-3)(x-4)u_3u_3^{\mathsf{T}} + (4-x)\left(6u_4u_4^{\mathsf{T}} + \begin{pmatrix} \rho_4 & 0 \\ 0 & 0 \end{pmatrix}\right). \end{aligned} \tag{B.2}$$

To verify that this is an SOS certificate, observe that $\tau = (x - \frac{3}{2})^2 + \frac{9}{8} > 0$ on $\mathbb{R}$. Therefore, $\rho_1, \rho_4 \in \mathrm{SOS}_{\mathbb{C}}$. The two rank-one matrices $u_3u_3^{\mathsf{T}}$ and $u_4u_4^{\mathsf{T}}$ are also hermitian squares. Thus every matrix multiplier in (B.2) belongs to $\mathrm{SOS}_{\mathbb{C}^2}$. Finally, inspection of the degrees in (B.2) shows that the largest weighted degree is 9. Hence $G_4 \in \mathcal{M}_{\mathbb{C}^2}(S_1)_{\leq 9}$, as claimed. □

The choice $\psi_1 = 1$ therefore gives an additive increase of at most 9 in Proposition 5.4. We now show that allowing a different interpolation polynomial cannot reduce this degree within the construction.

Claim 5. For every real polynomial ψ satisfying $\psi(4) = 1$, the matrix polynomial $\psi^2 G_4$ admits no certificate in $\mathcal{M}_{\mathbb{C}^2}(S_1)$ of weighted degree at most 8. Consequently, the minimum weighted degree achievable for this fixed matrix construction is 9.

Proof of Claim 5. Suppose, for a contradiction, that $\psi(4) = 1$ and $\psi^2 G_4$ has a certificate of weighted degree at most 8. Taking the $(2,2)$ entry gives a scalar certificate in $M(S_1)$ for

$$f(x) = \psi(x)^2 (x-4)^2 \Delta(x).$$

Since $\deg f = 6 + 2\deg\psi$, the assumed weighted degree bound forces $\deg\psi \le 1$. Write the resulting scalar certificate as

$$f = \tau_0 + x\tau_1 + (x-1)(x-2)\tau_2 + (x-3)(x-4)\tau_3 + (4-x)\tau_4,$$

where every $\tau_j \in \mathrm{SOS}_{\mathbb{C}}$. The polynomial f vanishes at $0, 1, 2, 3, 4$. As in Claim 3, at any such point where a generator is strictly positive, the corresponding SOS multiplier must have a double zero. The resulting minimum degrees for nonzero multipliers are $\deg\tau_0 \ge 10$, $\deg\tau_1 \ge 8$, $\deg\tau_2 \ge 6$, $\deg\tau_3 \ge 6$, and $\deg\tau_4 \ge 8$. Since the certificate has weighted degree at most 8, we must have $\tau_0 = \tau_1 = \tau_4 = 0$. Moreover, the remaining two multipliers must be of the form

$$\begin{aligned} \tau_2 &= a\,x^2(x-3)^2(x-4)^2, \\ \tau_3 &= b\,x^2(x-1)^2(x-2)^2, \end{aligned} \qquad a, b \ge 0.$$

Substituting these expressions into the scalar certificate for f, and using $\Delta = -x(x-1)(x-2)(x-3)$, we obtain

$$\begin{aligned} -\psi^2 x(x-1)(x-2)(x-3)(x-4)^2 = a\,x^2(x-1)(x-2)(x-3)^2(x-4)^2 \\ + b\,x^2(x-1)^2(x-2)^2(x-3)(x-4). \end{aligned}$$

Cancelling the common polynomial factor gives

$$-\psi^2(x-4) = a\,x(x-3)(x-4) + b\,x(x-1)(x-2). \tag{B.3}$$

We distinguish the two possible degrees of ψ.

If ψ is constant, then $\psi(4) = 1$ implies $\psi = 1$. Evaluating (B.3) at $x = 0$ gives $4 = 0$, a contradiction.

If $\deg\psi = 1$, the leading coefficient of the left-hand side of (B.3) is strictly negative, whereas the leading coefficient of the right-hand side is $a + b \ge 0$. This is again impossible.

Thus no interpolation polynomial ψ with $\psi(4) = 1$ can give a certificate of weighted degree at most 8. Together with Claim 4, this establishes the asserted minimum degree 9. □

We can therefore apply Proposition 5.4 with $\eta_{S_1} = 9$. Combining this with the quadratic bound 10 from Claim 2 gives $N_{S_1} = 10 + 9 = 19$. Therefore, for every hermitian $F \in \mathcal{B}(\mathcal{H})[x]$ with $\deg F \le \mathtt{d}$, $\mathtt{d} \ge 1$, and $F \succeq 0$ on K_1, Theorem 3.4 gives

$$F \in \mathcal{M}_{\mathcal{H}}(S_1)_{\le 17+2\lceil \mathtt{d}/2 \rceil} \subseteq \mathcal{M}_{\mathcal{H}}(S_1)_{\le \mathtt{d}+18}.$$

Thus, we obtain the additive constant $C_{S_1} = 18$.

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