NON-NEGATIVE POLYNOMIALS ON GENERALIZED ELLIPTIC CURVES

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Abstract. We study the cone of non-negative polynomials on generalized elliptic curves. We show that the zero set of every extreme ray has dense real points. If a generalized elliptic curve is embedded via a complete linear system, then we show that the convex hull of its real points (taken inside any affine chart containing all real points) is a spectrahedron. On the way, we generalize a result by Geyer–Martens on 2-torsion points in the Picard group of smooth real curves (of arbitrary genus) to possibly singular and reducible ones.

1. Introduction

Given a projective variety X over $\mathbb{R}$, i.e., a reduced and projective scheme of finite type over $\mathbb{R}$, and a line bundle L on X, we consider the cone $\mathrm{P}(X,L)$ of global sections $f \in H^0(X, L\otimes L)$ that are non-negative on $X(\mathbb{R})$. If L is very ample and the corresponding embedding of X to projective space is projectively normal, then $\mathrm{P}(X,L)$ corresponds to the cone of quadrics that are non-negative on X, and this cone has been studied extensively for various different varieties X. For example, in the case of irreducible varieties of minimal degree [4] and, more generally, varieties of Castelnuovo–Mumford regularity 2 [3], it admits the description as all quadrics that can be written as a sum of squares of linear forms, generalizing a classical theorem by Hilbert [8]. In this note, we examine $\mathrm{P}(X,L)$ when X is a *generalized elliptic curve* over $\mathbb{R}$, i.e., a one dimensional, projective, and connected variety over $\mathbb{R}$ with dense real points such that $\omega_X^\circ \cong \mathcal{O}_X$. The prototype of a generalized elliptic curve is a reduced plane cubic. We will see that, even though not every element of $\mathrm{P}(X,L)$ is a sum of squares of elements of $H^0(X,L)$, the cone $\mathrm{P}(X,L)$ still admits a description with several, from the convex algebro-geometric point of view desirable properties. Our results rely on a description of extreme rays:

Theorem 1.1. *Let X be a generalized elliptic curve over $\mathbb{R}$ and let L be a line bundle on X. If $f \in H^0(X, L\otimes L)$ generates an extreme ray of $\mathrm{P}(X,L)$, then the real points are Zariski dense in the zero set of f on X.*

While in the smooth case Theorem 1.1 is a direct consequence of the Riemann–Roch theorem, as observed for very ample L in [1], the singular case is considerably more difficult. A consequence of Theorem 1.1 is that every non-negative global section on a generalized elliptic curve is a sum of squares of global sections of a not necessarily invertible sheaf:

Theorem 1.2. *Let X be a generalized elliptic curve over $\mathbb{R}$ and let L be a line bundle on X. There exists a coherent sheaf $\mathcal{G}$ on X together with a morphism $\varphi\colon \mathcal{G}\otimes\mathcal{G} \to L\otimes L$ such that the following holds:*

(1) The morphism φ is positive semi-definite *in the sense that for every $g \in H^0(X,\mathcal{G})$ the global section $\varphi(g\otimes g)$ of $L\otimes L$ is non-negative on $X(\mathbb{R})$.*

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(2) Every extreme ray of $\mathrm{P}(X, L)$ *is generated by an element of the form* $\varphi(g \otimes g)$ *for some* $g \in H^0(X, \mathcal{G})$.

In particular $\mathrm{P}(X, L)$ *is the cone of sums of squares of global sections of* $\mathcal{G}$*:*

$$\mathrm{P}(X, L) = \{\varphi(g_1 \otimes g_1 + \cdots + g_r \otimes g_r) \mid r \in \mathbb{N} \text{ and } g_1, \ldots, g_r \in H^0(X, \mathcal{G})\}.$$

A direct consequence of Theorem 1.2 is that the dual cone $\mathrm{P}(X, L)^\vee$ can be described as the cone of all linear forms $\ell \colon H^0(X, L \otimes L) \to \mathbb{R}$ such that

$$B_\ell \colon H^0(X, \mathcal{G}) \times H^0(X, \mathcal{G}) \to \mathbb{R},\ (g_1, g_2) \mapsto \ell(\varphi(g_1 \otimes g_2))$$

is positive semi-definite. From this we obtain the following insight on convex hulls of generalized elliptic curves which appears to be new even in the smooth case.

Corollary 1.3. *Let* $X \subseteq \mathbb{P}^n$ *be a generalized elliptic curve over* $\mathbb{R}$ *embedded via a complete linear system and let* $H \subseteq \mathbb{P}^n$ *be a hyperplane with* $H \cap X(\mathbb{R}) = \emptyset$*. Then the convex hull of* $X(\mathbb{R})$ *in* $\mathbb{R}^n = (\mathbb{P}^n \smallsetminus H)(\mathbb{R})$ *is a spectrahedron.*

Spectrahedra are convex semi-algebraic sets defined by a linear matrix inequality. They are fundamental objects in convex algebraic geometry; they are the feasible sets of semi-definite programming and have many desirable properties [2, 11, 14]. Note that, while the convex hull of a curve is always a spectrahedral shadow [13, 15], i.e., the image of a spectrahedron under a linear map, it is rarely a spectrahedron itself. A more careful analysis of the proof of Corollary 1.3 (see Remark 3.8) will show that the description of the convex hull of $X(\mathbb{R})$ in Corollary 1.3 as a spectrahedron is particularly nice in that it is given by block matrices with blocks of size at most $\frac{n+1}{2}$. Convex hulls of elliptic curves were studied before in [10, 12].

In order to deduce Corollary 1.3 from Theorem 1.2, we need a certain divisibility result on the Picard group of a generalized elliptic curve. In Section 2.3 we prove such for arbitrary real curves. From this we also deduce a generalization to singular curves of a result by Geyer–Martens [5] on 2-torsion points in the Picard group of a smooth real curve, which might be of independent interest (Corollary 2.3).

2. Preliminaries

2.1. **Notation.** By a *variety* over a field k we mean a reduced and separated scheme of finite type over k, not necessarily irreducible. If L is a line bundle on a normal irreducible variety X and f a rational section that does not vanish identically on an irreducible component of X, then we denote by

$$\mathrm{div}(f) = \sum_{x \in X} v_x(f) \cdot x$$

its divisor (a Weil divisor); here the sum is taken over all points x of codimension 1 and $v_x(f)$ denotes the valuation of f at x in a trivialization of L around x. A *curve* over k is a projective variety of pure dimension 1 over k. Sometimes we will use the fact that every line bundle L on a curve X over k is isomorphic to $\mathcal{O}_X(D)$ where D is a (Weil) divisor supported on the regular points of X [16, Lemma 0AYM]. Here $\mathcal{O}_X(D)$ denotes the subsheaf of the sheaf of total quotient rings of $\mathcal{O}_X$ associated to D. Further, we say that a line bundle L on a curve X over k has degree 0 if the pull-back to every irreducible component of the normalization of X has degree 0. The subgroup of $\mathrm{Pic}(X)$ of degree 0 line bundles on X is denoted by $\mathrm{Pic}^\circ(X)$.

2.2. **Non-negativity of sections.** Let X be a variety over $\mathbb{R}$ and let L be a line bundle on X. We say that a section $f \in H^0(X, L \otimes L)$ is *non-negative* or *positive* at $x \in X(\mathbb{R})$ if for a trivialization of $L|_U \cong \mathcal{O}_X|_U$ on an open affine neighborhood U of x the section f is mapped by the induced map

$$H^0(X, L \otimes L) \to L(U) \otimes L(U) \to \mathcal{O}_X(U) \otimes \mathcal{O}_X(U) \to \mathcal{O}_X(U)$$

to a regular function on U that is non-negative or positive at x, respectively. Note that the sign at x of this regular function does not depend on the chosen trivialization of L. We denote by $\mathrm{P}(X, L)$ the closed convex cone of all $f \in H^0(X, L \otimes L)$ that are non-negative on $X(\mathbb{R})$.

We recall some straightforward observations that we will use in the proofs. If X is a curve and $f \in H^0(X, L \otimes L)$ is non-negative in a (Euclidean) neighborhood of a regular point $x \in X(\mathbb{R})$, then $v_x(f)$ is even. Moreover, if $g \in H^0(X, L \otimes L)$ is another section which satisfies $v_x(g) \geq v_x(f)$, then, for sufficiently small $\epsilon > 0$, also $f \pm \epsilon g$ is non-negative in a neighborhood of x. If $\pi \colon \tilde{X} \to X$ is the normalization of X, then a section $h \in H^0(X, L \otimes L)$ is non-negative on the set of non-isolated real points if and only if its pull-back $\pi^* h$ is non-negative on $\tilde{X}(\mathbb{R})$. If $L = \mathcal{O}_X(D)$ for a divisor D supported on regular points of X, then $f \in H^0(X, \mathcal{O}_X(2D))$ is non-negative or positive at a real point x of $U = X \setminus \mathrm{Supp}(D)$ if and only if f is non-negative or positive at x as a regular function on U. In particular, if f is non-negative on the real points of U as regular function on U, then $f \in \mathrm{P}(X, L)$ because isolated real points are singular points of X and thus in U.

2.3. **The Picard group of real curves.** In this section we prove some basic facts about the Picard group of not necessarily irreducible or smooth real curves. We start with the following well-known lemma over the complex numbers.

Lemma 2.1. *If X is a curve over $\mathbb{C}$, then $\mathrm{Pic}^\circ(X)$ is divisible.*

Proof. It is shown in the proof of [16, Proposition 0C20] that $\mathrm{Pic}^\circ(X)$ can be obtained from $\mathrm{Pic}^\circ(\tilde{X})$, where $\tilde{X}$ is the normalization of X, by a sequence of extensions by $(\mathbb{C}^\times, \cdot)$ and $(\mathbb{C}, +)$. Now the claim follows because the groups $\mathrm{Pic}^\circ(\tilde{X})$, $(\mathbb{C}^\times, \cdot)$ and $(\mathbb{C}, +)$ are all divisible. $\square$

The following can be interpreted as a divisibility result on the so-called *narrow class group* of a real curve. In the smooth case, it is due to Hanselka [6, Section 4].

Proposition 2.2. *Let X be a curve over $\mathbb{R}$ such that $X(\mathbb{R})$ is Zariski dense in X and let D be a divisor supported on non-real regular points of X. There exists a divisor E, supported on regular points, and a unit f of the total quotient ring of X, which is non-negative on every real point where it is defined, such that*

$$D = 2E + \mathrm{div}(f).$$

Proof. As the support of D does not contain real points, we can write its pullback to $X_{\mathbb{C}}$ as $F + \bar{F}$ for some divisor F where $\bar{F}$ denotes its complex conjugate. Let F_1 be a divisor supported on regular points fixed by the complex conjugation such that $\mathcal{O}_{X_{\mathbb{C}}}(F - F_1) \in \mathrm{Pic}^\circ(X_{\mathbb{C}})$. Since $\mathrm{Pic}^\circ(X_{\mathbb{C}})$ is 2-divisible by Lemma 2.1, there exists a divisor G supported on regular points of $X_{\mathbb{C}}$ and a unit g of the total quotient ring of $X_{\mathbb{C}}$ such that $2G + \mathrm{div}(g) = F - F_1$. This implies that

$$2(G + \bar{G}) + \mathrm{div}(g\bar{g}) = F + \bar{F} - 2F_1$$

and thus $D = 2E + \mathrm{div}(f)$ where E is the divisor and f the rational function whose pullback to $X_{\mathbb{C}}$ is $G + \bar{G} + F_1$ and $g\bar{g}$, respectively. It is clear that f is non-negative at every real point where it is defined. $\square$

Proposition 2.2 can be used to generalize a result by Geyer–Martens [5] on the structure of 2-torsion points on a smooth real curve to singular curves. Namely, for a connected curve X over $\mathbb{R}$ we consider the map

$$\mathrm{sg} \colon \mathrm{Pic}(X)_2 \to \tilde{H}^0(X(\mathbb{R}), \{\pm 1\})$$

which sends a 2-torsion point represented by a divisor D, supported on regular points, to the signs that a rational function, whose divisor is $2D$, takes on the different connected components of $X(\mathbb{R})$. Here $\tilde{H}^0(X(\mathbb{R}),\{\pm 1\})$ denotes the 0th reduced singular cohomology group of $X(\mathbb{R})$ with coefficients in the multiplicative group $\{\pm 1\}$. It is straightforward to check that this is a well-defined group homomorphism. The elements of $\operatorname{Pic}(X)_2^+ = \ker(\mathrm{sg})$ are called *positive* 2*-torsion points* because the associated rational function can be chosen to be non-negative on $X(\mathbb{R})$. The following was shown in the smooth case by Geyer–Martens in [5, Section 5].

Corollary 2.3. *For every connected curve X over $\mathbb{R}$ the map* sg *is surjective.*

Proof. Because X is projective, there exists an open affine subset U of X which contains $X(\mathbb{R})$ and all singular points. Indeed, if X is embedded to $\mathbb{P}^n$, then, after applying a linear change of coordinates if necessary, we can take the principal open subset of all points where $x_0^2+\cdots+x_n^2$ is non-zero. Embedding U to $\mathbb{A}^m$ gives an embedding of $X(\mathbb{R})$ to $\mathbb{R}^m=\mathbb{A}^m(\mathbb{R})$. Let X_1 be a union of connected components of $X(\mathbb{R})$ and $X_2 = X(\mathbb{R}) \smallsetminus X_1$. Since X_1 and X_2 are disjoint compact subsets of $\mathbb{R}^m$, there is a continuous function $\psi\colon \mathbb{R}^m \to \mathbb{R}$ which satisfies $\psi(x) \le -1$ for $x \in X_1$ and $\psi(x) \ge 1$ for $x \in X_2$. By the Stone–Weierstrass theorem we can uniformly approximate ψ on a compact subset of $\mathbb{R}^m$ containing $X(\mathbb{R})$ arbitrarily close by a polynomial. In particular, there exists $g \in \mathbb{R}[x_1,\ldots,x_m]$ with $g(x) < 0$ for $x \in X_1$ and $g(x) > 0$ for $x \in X_2$. After replacing g by a small perturbation if necessary, we can further assume that g does not vanish on any singular point of X. We apply Proposition 2.2 to the principal divisor $D = \operatorname{div}(g)$ and obtain a divisor E, supported on regular points, and a unit f of the total quotient ring of X, which is non-negative on every real point where it is defined, such that $\operatorname{div}(\frac{g}{f}) = 2E$. The 2-torsion point $\mathcal{O}_X(E)$ of $\operatorname{Pic}(X)$ is then mapped by sg to a cohomology class represented by the function which is -1 on X_1 and $+1$ on X_2. Since X_1 was chosen to be an arbitrary union of connected components of $X(\mathbb{R})$ and X_2 its complement, this shows the claim. □

We will see that positive 2-torsion points are closely related to those extreme rays of $\mathrm{P}(X,L)$ with only regular zeros. Corollary 2.3 gives a way to count them.

2.4. **Generalized elliptic curves.** A *generalized elliptic curve* over $\mathbb{R}$ is a connected curve over $\mathbb{R}$ with dense real points such that $\omega_X^\circ \cong \mathcal{O}_X$. Here ω_X° is the dualizing sheaf of X. The prototype of a generalized elliptic curve is a reduced plane cubic curve but there are more examples.

Definition 2.4. For $n \in \mathbb{N}$ the *Néron n-gon* is obtained by taking a copy X_i of $\mathbb{P}^1$ for each $i \in \mathbb{Z}/n\mathbb{Z}$ and gluing the point ∞ of X_i to the point 0 of X_{i+1} such that the intersection points become (ordinary) nodes.

Note that for $n \le 3$ the Néron n-gon is isomorphic to a planar cubic curve. The following characterization is a classical result of Kodaira.

Theorem 2.5 ([9, Theorem 6.2])**.** *Every generalized elliptic curve is either isomorphic to a reduced plane cubic or to the Néron n-gon.*

We list the resulting possible types of generalized elliptic curves over $\mathbb{R}$.

Corollary 2.6. *Let X be a generalized elliptic curve over $\mathbb{R}$. Then X is isomorphic to one of the following: a non-singular elliptic curve, a Néron n-gon for $n \in \mathbb{N}$, a rational curve with one isolated node, a smooth plane conic and a line meeting in two non-real points, a rational curve with one cusp, a smooth plane conic and one of its tangents, or three planar lines that meet in one point.*

Remark 2.7. Let X be a generalized elliptic curve over $\mathbb{R}$ and let $\pi\colon \tilde{X} \to X$ be its normalization. The goal of this remark is to describe the inclusion $\mathcal{O}_X \hookrightarrow \pi_*\mathcal{O}_{\tilde{X}}$ of coherent sheaves on X in detail. At non-singular points, this is an isomorphism.

If $x \in X(\mathbb{C})$ is a point at which m (complex) branches meet transversally, then there are m distinct $\mathbb{C}$-points points $x_1, \dots, x_m$ in $\tilde{X}(\mathbb{C})$ lying over x and the local ring $\mathcal{O}_{X,x}$ consists of all $f \in \mathcal{O}_{\tilde{X},\pi^{-1}(x)}$ such that $f(x_i) = f(x_j)$ for $i, j = 1, \dots, m$. If two branches are tangent to each other at x, then there are two distinct $\mathbb{C}$-points x_1, x_2 in $\tilde{X}(\mathbb{C})$ lying over x and the local ring $\mathcal{O}_{X,x}$ consists of all $f \in \mathcal{O}_{\tilde{X},\pi^{-1}(x)}$ such that $f(x_1) = f(x_2)$ and $f'(x_1) = f'(x_2)$ (the derivative is taken with respect to a suitable local parameter). Finally, if $x \in X(\mathbb{R})$ is a cusp, then there is a unique $\mathbb{R}$-point y in $\tilde{X}(\mathbb{R})$ lying over x and $\mathcal{O}_{X,x}$ consists of all $f \in \mathcal{O}_{\tilde{X},y}$ such that $f'(y) = 0$. This covers all possibilities.

We conclude this section with an estimate of positive 2-torsion points on generalized elliptic curves.

Lemma 2.8. *Let X be a generalized elliptic curve over $\mathbb{R}$. There are at most 2 positive 2-torsion points in* $\mathrm{Pic}(X)$.

Proof. Unless X is smooth, every irreducible component is rational. In this case, we have by [16, Lemma 0CE6] that $|\mathrm{Pic}(X)_2| \leq 2$. If X is smooth with $X(\mathbb{R})$ having r connected components, then

$$r = \dim_{\mathbb{F}_2}(\mathrm{Pic}(X)_2) = \dim_{\mathbb{F}_2}(\mathrm{Pic}(X)_2^+) + r - 1.$$

Here the first equality is a classical fact about real elliptic curves and the second equality holds by Corollary 2.3. This proves that $\dim_{\mathbb{F}_2}(\mathrm{Pic}(X)_2^+) = 1$. □

2.5. **Spectrahedra.** A *spectrahedron* in $\mathbb{R}^n$ is the inverse image of the cone of positive semidefinite matrices under an affine linear map from $\mathbb{R}^n$ to the vector space of real symmetric matrices of size $d \times d$ for some $d \in \mathbb{N}$. Equivalently, a spectrahedron is a set of the form

$$\{x \in \mathbb{R}^n \mid A_0 + x_1 A_1 + \cdots + x_n A_n \text{ is positive semi-definite}\},$$

where $A_0, \dots, A_n$ are real symmetric matrices of the same size. A *spectrahedral shadow* is the image of a spectrahedron under an affine linear map. For reading about the many interesting properties and applications of spectrahedra and their shadows we recommend the books [2, 11, 14].

3. Proofs and examples

Let X be a generalized elliptic curve over $\mathbb{R}$ and let L be a line bundle on X. We let $X = \cup_{i=1}^r X_i$ be the decomposition of X into irreducible components and $\pi\colon \tilde{X} \to X$ the normalization of X. Hence $\tilde{X}$ is a smooth projective real curve with irreducible components $\tilde{X}_1, \dots, \tilde{X}_r$ where each $\tilde{X}_i$ is the normalization of X_i. There exists a short exact sequence

$$0 \to \mathcal{O}_X \to \pi_* \mathcal{O}_{\tilde{X}} \to \mathscr{S} \to 0 \tag{1}$$

of coherent sheaves on X where $\mathscr{S}$ is the cokernel of the map $\mathcal{O}_X \to \pi_* \mathcal{O}_{\tilde{X}}$. The support of $\mathscr{S}$ is the singular locus of X. For a closed point $x \in X$ the dimension (as $\mathbb{R}$-vector space) of the stalk $\mathscr{S}_x$ is denoted by $\delta(x)$. Let L be a line bundle on X and $M = L \otimes L$. Tensoring (1) by M and passing to the long exact sequence of cohomology, we obtain

$$0 \longrightarrow H^0(X, M) \xrightarrow{\ \alpha\ } \oplus_{i=1}^r H^0(\tilde{X}_i, M_i) \xrightarrow{\ \beta\ } H^0(X, \mathscr{S}) \tag{2}$$

where we denote by M_i the pullback of M to $\tilde{X}_i$. For $f \in H^0(X, M)$ and $i = 1, \dots, r$ we let $f_i \in H^0(\tilde{X}_i, M_i)$ be the pullback of f to $\tilde{X}_i$ so that $\alpha(f) = f_1 + \cdots + f_r$.

Remark 3.1. The dimension of the image of $H^0(\tilde{X}_i, M_i)$ under β is bounded from above by the sum $\delta_i = \sum_{x \in X_i} \delta(x)$. Using Remark 2.7 and going through Corollary 2.6 we see that this sum is at most 1 for irreducible X and that in general $\delta_i \leq 2$ for every i. This property can also be deduced directly from the definitions by making use of the fact that $\pi_i^! \omega_X^\circ$, where $\pi_i \colon \tilde{X}_i \to X$ is the inclusion $\tilde{X}_i \hookrightarrow \tilde{X}$ composed with the normalization $\pi \colon \tilde{X} \to X$, is, on the one hand, the canonical sheaf of $\tilde{X}_i$ by [7, Exercise III.7.2], and, on the other hand, the codifferent sheaf $\pi_i^! \mathcal{O}_X$ of π_i.

Lemma 3.2. *Let $f \in H^0(X, M)$ and let $j \in \{1, \ldots, r\}$ such that $f_j \neq 0$. Let $p, q \in \tilde{X}_j$ distinct points with $f_j(p) = f_j(q) = 0$ such that neither $\pi(p)$ nor $\pi(q)$ lies on an irreducible component of X on which f vanishes identically. We consider the vector space V of all $g \in H^0(X, M)$ that satisfy:*

(1) For all $i \neq j$ there exists $\lambda_i \in \mathbb{R}$ such that $g_i = \lambda_i \cdot f_i$.
(2) $\operatorname{div}(g_j) \geq \operatorname{div}(f_j) - p - q$.

The vector space V has dimension at least 2.

Proof. If X is smooth, then the claim follows from the Riemann–Roch theorem. Hence we assume from now on that X is singular. Then each $\tilde{X}_i$ is isomorphic to $\mathbb{P}^1$ and thus the space W of all $h \in H^0(\tilde{X}_j, M_j)$ with $\operatorname{div}(h) \geq \operatorname{div}(f_j) - p - q$ has dimension 3. By Remark 3.1 the dimension of $\beta(H^0(\tilde{X}_j, M_j))$ is at most 2. Since $\dim(W) = 3$, there is $0 \neq h \in W \cap \ker(\beta)$. Thus the claim follows if h is linearly independent from $\alpha(f)$. Hence we assume in the following that h and $\alpha(f)$ are linearly dependent. This can only happen if $f_i = 0$ for all $i \neq j$ and thus $\beta(f_j) = 0$. If $r > 1$, then for $g \in H^0(\tilde{X}_j, M_j)$ the condition $\beta(g) = 0$ is equivalent to a vanishing condition on the points on $\tilde{X}_j$ that are mapped by π to some X_i with $i \neq j$. Since f_j satisfies this condition and since neither p nor q are mapped by π to some X_i with $i \neq j$, the condition $\operatorname{div}(g) \geq \operatorname{div}(f_j) - p - q$ implies that $g \in H^0(\tilde{X}_j, M_j)$ satisfies this vanishing condition as well. Thus W lies in the kernel of β which implies the claim. It remains to treat the case that $r = 1$. Then by Remark 3.1 the dimension of $\beta(H^0(\tilde{X}_j, M_j))$ is 1 and thus the space $W \cap \ker(\beta)$ is at least of 2-dimensional which implies the claim. □

Proof of Theorem 1.1. Let $f \in H^0(X, L \otimes L)$ be non-negative on $X(\mathbb{R})$ and let S be the set of indices $i \in \{1, \ldots, r\}$ for which f vanishes identically on X_i. Assume that there is $x \in X \setminus (\cup_{i \in S} X_i)$ with $f(x) = 0$ such that $x \notin X(\mathbb{R})$. We have to show that f does not generate an extreme ray of $\mathrm{P}(X, L)$.

Let X_j be an irreducible component of X that contains x. Note that this implies $j \notin S$. There exists $p \in \pi^{-1}(x) \cap \tilde{X}_j$ and we let $q = \bar{p}$ be its complex conjugate. Because $x \notin X(\mathbb{R})$, the points p and q are two distinct elements of $\tilde{X}_j$. Now f, p and q satisfy the assumptions of Lemma 3.2. Thus there exists $g \in H^0(X, L \otimes L)$ which is linearly independent of f such that:

(1) For all $i \neq j$ there exists $\lambda_i \in \mathbb{R}$ such that $g_i = \lambda_i \cdot f_i$.
(2) $\operatorname{div}(g_j) \geq \operatorname{div}(f_j) - p - q$.

Let $y \in \tilde{X}$, say $y \in \tilde{X}_i$, such that $\pi(y) \in X(\mathbb{R})$. Conditions (1) and (2) imply, by choice of p and q, that $v_y(g_i) \geq v_y(f_i)$. Therefore, there exists $\epsilon_i > 0$ such that $f_i \pm \epsilon g_i$ is non-negative on $\tilde{X}_i(\mathbb{R})$. This implies that $f \pm \epsilon g$ is non-negative on the set of non-isolated points of $X(\mathbb{R})$ for $\epsilon = \min_{i=1}^r \epsilon_i$.

Finally, we note that if f vanishes at an isolated point z of $X(\mathbb{R})$, then g vanishes this point as well. This follows from $v_y(g_i) \geq v_y(f_i)$ by choosing above y in the preimage of z under π. At an isolated real point where f is positive, we can ensure that $f \pm \epsilon g$ is positive as well by replacing ϵ by a smaller positive number. Since there is at most one isolated point, there exists $\epsilon > 0$ such that $f \pm \epsilon g$ is non-negative on $X(\mathbb{R})$. This shows that f is not an extreme ray of $\mathrm{P}(X, L)$. $\square$

Proof of Theorem 1.2. Let D be a divisor on X supported on regular points such that $L \cong \mathcal{O}_X(D)$, the subsheaf of the sheaf of total quotient rings of $\mathcal{O}_X$ associated to D. Let $s \in H^0(X, L \otimes L)$ generate an extreme ray of $\mathrm{P}(X, L)$. Consider the partial normalization $\tilde{Y} \to X$ at all singular points of X where s vanishes. Further, let $Y \subseteq \tilde{Y}$ be the union of all irreducible components of $\tilde{Y}$ where the pullback of s to $\tilde{Y}$ is not identically zero. Finally, let $\rho\colon Y \to X$ be the composition of the inclusion $Y \hookrightarrow \tilde{Y}$ with $\tilde{Y} \to X$. Then the restriction $D' = D|_Y$ satisfies $\rho^* L \cong \mathcal{O}_Y(D')$. We choose $f \in \mathcal{O}_Y(2D')$, a rational function on Y, which corresponds to $\rho^* s$ under $\rho^* L \cong \mathcal{O}_Y(D')$. We denote by E_1 the restriction of the zero divisor of $\rho^* s$ to the set of all regular points of Y that lie over a singular point of X. Then the principal divisor of f is given by $\operatorname{div}(f) = E_1 + E_2 - 2D'$ for some effective divisor E_2 supported on regular points. By Theorem 1.1 the divisor E_2 is a sum of real non-isolated points of Y. Because f is non-negative on $Y(\mathbb{R})$, it follows that $E_2 = 2F$ for some divisor F because real zeros of non-negative sections must occur with even multiplicity. Consider the bilinear map

$$\mathcal{O}_Y(F) \times \mathcal{O}_Y(F) \longrightarrow \mathcal{O}_Y(E_2) \xrightarrow{\cdot f} \mathcal{O}_Y(2D' - E_1) \subseteq \mathcal{O}_X(2D)$$

where the first arrow is the multiplication map. Letting $A = \mathcal{O}_Y(F)$, the induced morphism $\psi\colon \rho_* A \otimes \rho_* A \to L \otimes L$ of coherent sheaves on X is positive semi-definite and s is the image of a square of a global section of A. We argue that, up to isomorphism and multiplication by a positive scalars (which does not destroy the desired properties), there are only finitely many choices for ψ and A. Indeed, because there are only finitely many singular points, there are only finitely many choices for $\tilde{Y}$. Since there are only finitely many irreducible components, there are only finitely many choices for Y. As the degree of L is finite, there are only finitely many possibilities for the divisor E_1. This implies that there are, up to linear equivalence, only finitely many possibilities for E_2. Since $2F = E_2$ and because the Picard group of Y has only finitely many 2-torsion points, see [16, Lemma 0CE6], there are only finitely many possibilities for the divisor class of F as well. Different resulting maps ψ for the same F can differ only by multiplication with positive scalars. Hence we arrive at finitely many coherent sheaves $A_1, \dots, A_m$ on X and positive semi-definite morphisms $\psi_i\colon A_i \otimes A_i \to L \otimes L$ such that every generator of an extreme ray of $\mathrm{P}(X, L)$ is the image of a square of a global section of one of the A_i. Hence we can choose $\mathcal{G}$ and φ to be the direct sums of the A_i and ψ_i, respectively. The additional statement follows from the fact that every closed convex cone that is pointed is generated by its extreme rays. $\square$

Remark 3.3. The proof of Theorem 1.2 simplifies when the extreme ray s has only regular zeros. In this case, the proof shows that s is the image of a square under a positive semi-definite isomorphism $\psi\colon A \otimes A \to L \otimes L$ where A is a line bundle. Hence we have $A = L \otimes T$ where $T \in \operatorname{Pic}(X)_2$ is an positive 2-torsion point. By Lemma 2.8 there are at most two such points. However, as Example 3.5 shows, line bundles are not enough to describe extreme rays that vanish at singular points.

Example 3.4. Consider the case that X is a smooth elliptic curve over $\mathbb{R}$. A plane affine model of X is given by the equation $y^2 = x \cdot p(x)$ where p is a univariate polynomial of degree 2 with simple zeros which are either positive or non-real. Let P be the point $(0, 0)$ and let O be the point at infinity. Then the unique (see Lemma 2.8) non-trivial positive 2-torsion point is represented by the divisor $T = P - O$. Indeed, we have $\operatorname{div}(x) = 2T$ and x is non-negative on $X(\mathbb{R})$. Hence, if D a divisor on X and $L = \mathcal{O}_X(D)$, then every extreme ray of $\mathrm{P}(X, L)$ is either a square of a global section of $\mathcal{O}_X(D)$ or the image of a square under the map

$$\mathcal{O}_X(D+T) \otimes \mathcal{O}_X(D+T) \longrightarrow \mathcal{O}_X(2D+2T) \xrightarrow{\cdot x} \mathcal{O}_X(2D).$$

Example 3.5. We illustrate the proof of Theorem 1.2 in the case when X is the rational curve with one cusp. A plane affine model of X is given by the equation $y^2 = x^3$. Let O be the point at infinity and $L = \mathcal{O}_X(dO)$ for $d \in \mathbb{N}$. Note that for $d = 1, 2$ the line bundle L is ample but not very ample. The only possibilities for $\rho\colon Y \to X$ are the identity $X \to X$ and the normalization $\mathbb{P}^1 \to X$. In the first case, we have $E_1 = 0$. Since the Picard group of X is the additive group $\mathbb{R} \times \mathbb{Z}$, which is torsion-free, the only possibility, up to a positive scalar, for ψ is the identity on $L \otimes L$. In the case $Y = \mathbb{P}^1$, we denote by P the preimage of the cusp under π. Since E_2 has even degree and E_1 is linearly equivalent to $2dO - E_2$, we either have $E_1 = 2kP$ for $k = 1, \dots, d$. Here, by abuse of notation, we denote the preimage of O under π also by O. Again, as the Picard group of $\mathbb{P}^1$ is torsion-free, we find that F is linearly equivalent to $dO - kP \sim (d-k)O$. In order to describe the resulting map ψ on global sections explicitly, we identify the global sections of $\mathcal{O}_X(mO)$ and $\mathcal{O}_Y(mO)$ with the spaces $\mathbb{R}[t^2, t^3]_{\le m}$ and $\mathbb{R}[t]_{\le m}$ of univariate polynomials in $\mathbb{R}[t^2, t^3]$ and $\mathbb{R}[t]$ that have degree at most m. We get the map

$$\mathbb{R}[t]_{\le d-k} \times \mathbb{R}[t]_{\le d-k} \longrightarrow \mathbb{R}[t]_{\le 2d-2k} \xrightarrow{\cdot t^{2k}} \mathbb{R}[t^2, t^3]_{\le 2d}.$$

The images of squares are thus of the form $(t^k g)^2$ where $g \in \mathbb{R}[t]_{\le d-k}$. For $k \ge 2$ we have $t^k g \in \mathbb{R}[t^2, t^3]$, so that these sections of $L \otimes L$ are actually squares of sections of L. Hence, these are already covered by the case $Y = X$. In conclusion, every extreme ray of $\mathrm{P}(X, L)$ is either the square of a section of $L = \mathcal{O}_X(dO)$ or of $\pi_* \mathcal{O}_{\mathbb{P}^1}((d-1)O)$, and we can choose $\mathcal{G}$ to be the sum of these two sheaves.

Theorem 1.2 gives the following nice description of the dual cone $\mathrm{P}(X, L)^\vee$.

Corollary 3.6. *Let X be a generalized elliptic curve over $\mathbb{R}$ and let L be a line bundle on X. The dual cone $\mathrm{P}(X, L)^\vee$ is the a spectrahedral cone of all linear forms $\ell\colon H^0(X, L \otimes L) \to \mathbb{R}$ such that*

$$B_\ell\colon H^0(X, \mathcal{G}) \times H^0(X, \mathcal{G}) \to \mathbb{R},\ (g_1, g_2) \mapsto \ell(\varphi(g_1 \otimes g_2))$$

is positive semi-definite.

Remark 3.7. The proof of Theorem 1.2 shows that the description of $\mathrm{P}(X, L)^\vee$ in Corollary 3.6 consists of several blocks, one corresponding to each of the sheaves $A_1, \dots, A_m$ on X. A more careful analysis gives bounds on the size of these blocks. By the construction, the size of the block corresponding to A_i is equal to $h^0(X, A_i)$. Using the notation as in the proof of Theorem 1.2, this number is equal to $h^0(Y, \mathcal{O}_Y(F))$. By the Riemann–Roch theorem [16, Lemma 0BS6] we have

$$h^0(Y, \mathcal{O}_Y(F)) = \deg(F) + h^0(Y, \mathcal{O}_Y) - h^1(Y, \mathcal{O}_Y) + h^0(Y, \mathcal{O}_Y(K_Y - F)) \tag{3}$$

where K_Y is a divisor on Y, supported on regular points, such that $\mathcal{O}_Y(K_Y) \cong \omega_Y^\circ$. If $Y = X$, then Equation (3) simplifies to

$$h^0(Y, \mathcal{O}_Y(F)) = \deg(D) + h^0(X, \mathcal{O}_X(-F)).$$

If L is ample, then divisor F has positive degree and we obtain $\deg(D)$ as the size of the block. In the case that Y is different from X, then $0 \le \deg(F) < \deg(D)$ and $\deg(K_Y) < 0$. Hence Equation (3) implies

$$h^0(Y, \mathcal{O}_Y(F)) \le \deg(D) - 1 + h^0(Y, \mathcal{O}_Y).$$

We can further assume that Y is connected because otherwise $\mathcal{O}_Y(F)$ is the direct sum of its restrictions to the connected components of Y and our block decomposes into smaller blocks. In this case $h^0(Y, \mathcal{O}_Y) = 1$ and we find $h^0(Y, \mathcal{O}_Y(F)) \leq \deg(D)$. For later reference we note that if L is ample, the Riemann–Roch theorem implies

$$h^0(X, \mathcal{O}_X(2D)) = 2\deg(D)$$

so that every block is of size at most $\frac{1}{2}h^0(X, L \otimes L)$.

The proof of Corollary 1.3 is a combination of Corollary 3.6 and Proposition 2.2.

Proof of Corollary 1.3. Because $X(\mathbb{R}) \subseteq \mathbb{R}^n = (\mathbb{P}^n \setminus H)(\mathbb{R})$ is compact, there exists an affine linear functional $\mathbb{R}^n \to \mathbb{R}$ which is positive on $X(\mathbb{R})$. Let ℓ be the corresponding global section of $M = \mathcal{O}_X(1)$. By slightly perturbing ℓ if necessary, we can assume that the zero set of ℓ consists only of regular points of X. Let D be its zero divisor. By Proposition 2.2 there exists a divisor E, supported on regular points, and a unit f of the total quotient ring of X, which is non-negative on every real point where it is defined, such that $D = 2E + \operatorname{div}(f)$. We have isomorphisms

$$\mathcal{O}_X(E) \otimes \mathcal{O}_X(E) = \mathcal{O}_X(2E) \xrightarrow{\cdot f} \mathcal{O}_X(D) \xrightarrow{\cdot \ell} M.$$

Letting $L = \mathcal{O}_X(E)$, the induced vector space isomorphism

$$\psi\colon H^0(X, L \otimes L) \to H^0(X, M) = \operatorname{Aff}(\mathbb{R}^n)$$

to the space $\operatorname{Aff}(\mathbb{R}^n)$ of affine linear functionals on $\mathbb{R}^n$ identifies $\mathrm{P}(X, L)$ with the cone K of affine linear functionals that are non-negative on $X(\mathbb{R})$. On the other hand, the affine linear map

$$\operatorname{ev}\colon \mathbb{R}^n \to \operatorname{Aff}(\mathbb{R}^n)^\vee,$$

that maps a point to the corresponding point evaluation, satisfies $\operatorname{ev}^{-1}(K^\vee) = \operatorname{conv}(X(\mathbb{R})) \subseteq \mathbb{R}^n$. Therefore, the set $\operatorname{conv}(X(\mathbb{R})) = \operatorname{ev}^{-1}(\psi^\vee(\mathrm{P}(X, L)^\vee))$ is a spectrahedron by Corollary 3.6. □

Remark 3.8. It follows from Remark 3.7 that, in the situation of Corollary 1.3, the description of $\operatorname{conv}(X(\mathbb{R})) \subseteq \mathbb{R}^n$ consists of blocks, each of size at most $\frac{n+1}{2}$. Note that in general one cannot expect a description that only involves blocks of smaller size, even when allowing a description as a spectrahedral shadow [15].

Example 3.9. Consider the curve X in $\mathbb{P}^3$ defined by

$$x_0^2 - x_1^2 - x_2^2 = 0,$$
$$x_0^2 - 4x_1^2 - x_3^2 = 0.$$

One checks that X is a smooth elliptic curve. The hyperplane defined by $x_0 = 0$ at infinity does not intersect $X(\mathbb{R})$. Thus Corollary 1.3 says that the convex hull of $X(\mathbb{R})$ in the affine chart $x_0 \neq 0$ is a spectrahedron. Indeed, it is the set of all (x_1, x_2, x_3) such that the matrix

$$\begin{pmatrix} 1 - x_1 & x_2 & 0 & 0 \\ x_2 & 1 + x_1 & 0 & 0 \\ 0 & 0 & 1 - 2x_1 & x_3 \\ 0 & 0 & x_3 & 1 + 2x_1 \end{pmatrix}$$

is positive semi-definite. The two blocks correspond to line bundles L_1 and L_2 with $L_i \otimes L_i \cong \mathcal{O}_X(1)$ which differ by the non-trivial positive 2-torsion point of $\operatorname{Pic}(X)$.

Example 3.10. Consider the curve X in $\mathbb{P}^3$ defined by

$$x_0x_1 - x_1^2 - x_2^2 = 0,$$
$$x_0x_2 - x_1x_2 - x_3^2 = 0.$$

Figure 1. The (generalized) elliptic curves from Example 3.9 (left) and Example 3.10 (right).

It has a cusp at $(1:1:0:0)$ and can be parametrized as follows

$$\mathbb{P}^1 \to X,\ (s:t) \mapsto (s^4+t^4 : s^4 : s^2t^2 : st^3).$$

Therefore, it is isomorphic to the curve from Example 3.5 and $\mathcal{O}_X(1)$ is isomorphic to $\mathcal{O}_X(4O)$ where $O = (1:0:0:0)$. We use the notation as in Example 3.5. The hyperplane defined by $x_0 = 0$ at infinity does not intersect $X(\mathbb{R})$. Thus Corollary 1.3 says that the convex hull of $X(\mathbb{R})$ in the affine chart $x_0 \neq 0$ is a spectrahedron. Indeed, it is the set of all (x_1, x_2, x_3) such that the matrix

$$\begin{pmatrix} 1-x_1 & x_2 & 0 & 0 \\ x_2 & x_1 & 0 & 0 \\ 0 & 0 & 1-x_1 & x_3 \\ 0 & 0 & x_3 & x_2 \end{pmatrix}$$

is positive semi-definite. The first block corresponds to the line bundle $\mathcal{O}_X(2O)$ and the second one to the (non-invertible) sheaf $\pi_*\mathcal{O}_{\mathbb{P}^1}(O)$. The determinant of the second block cuts out the cone over X whose apex is the cusp of X.

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