POSITIVE POLYNOMIALS AND THE TRUNCATED MOMENT PROBLEM ON PLANE CUBICS

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ABSTRACT. The truncated moment problem supported on a given closed set K in $\mathbb{R}^2$ (K–TMP) asks to characterize conditions for a given linear functional on bivariate polynomials of bounded degree to have an integral representation with respect to a Borel measure μ with $\operatorname{supp}\mu \subseteq K$. The solutions to the K–TMP are known for K, which is a line, a quadratic curve, and for some cases of cubic curves. In this paper, we solve the C–TMP for every cubic curve C. Our first result states that the extreme rays of the cone of polynomials of bounded degree, nonnegative on C, have only real zeroes. This result allows us to establish certificates for the positivity of polynomials on C with degree bounds. To obtain concrete forms of these certificates, a case-by-case analysis is required. Up to affine linear change of variables, we divide cubics into 29 cases, 13 irreducible and 16 reducible ones. Using the certificates, we concretely solve the nonsingular C–TMP in terms of positive semidefiniteness of two or three localizing moment matrices. In most irreducible cases, we also provide constructive solutions to the nonsingular and singular C–TMPs, which can be used to compute a representing measure concretely. Upper bounds for the Carathéodory number are also obtained, which in some cases are sharp or differ by at most 1 from the sharp bound.

1. INTRODUCTION

Let $k \in \mathbb{N}_0$ be a non-negative integer. We denote by $\mathbb{R}[x,y]_{\leq k}$ the vector space of polynomials of (total) degree at most k. Let

$$L : \mathbb{R}[x,y]_{\leq 2k} \to \mathbb{R}$$

be a linear functional and $K \subseteq \mathbb{R}^2$ a closed set. The **truncated moment problem on** K **(K–TMP)** asks to characterize conditions for the existence of the Borel measure μ, supported on K, such that

$$L(f) = \int_K f d\mu \quad \text{for every } f \in \mathbb{R}[x,y]_{\leq 2k}.$$

A measure μ is called a K–**representing measure (K–rm)** for L. If L has some K–rm, then it is called a K–**moment functional (K–mf)**.

The moment problem (MP) is a classical question in analysis that has been studied since the end of the 19th century, appearing first in the memoir of the famous Dutch mathematician Stieltjes in 1894. The fact being especially fascinating about the MP is its interplay with many different areas of mathematics and a broad range of applications, such

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as operator theory, probability and statistics, numerical analysis and, more recently, real algebraic geometry, polynomial optimization, control theory, partial differential equations, machine learning, data analysis and others [13, 14, 30, 39, 42, 43].

The TMP is a variant of the MP that occurs even more frequently in applications, since only finitely many data is given. Apart from the applications, the TMP is also more general than the full version, which is a remarkable result of Stochel [61]. A lot of work on univariate TMPs was done in the early second half of the 20th century by Akhiezer, Krein and Nudelman [1, 2, 41], while in the last three decades renown interest in TMPs started with a series of papers by Curto and Fialkow [15–19, 21, 22], leaning on the interplay of TMPs with *real algebraic geometry (RAG).* RAG studies certificates, called *Positivstellensätze*, for positivity of polynomials on positivity sets of other polynomials [12, 46, 51, 52, 57]. One of the roots of RAG is Hilbert's 17th problem from 1900, which asked whether every positive polynomial is a sum of squares of rational functions, and was answered in the affirmative by Artin (1926). The connection between the MP and RAG is Haviland's theorem from 1935 [38], which states that the MP with a distribution μ supported on a closed set K in $\mathbb{R}^d$ has a solution if and only if the corresponding functional, defined on the vector space of all polynomials, maps polynomials, positive on K, to $[0, \infty)$. This interplay received new attention with Schmüdgen's solution of the multidimensional MP in 1991 [58], which combines the Positivstellensatz with ideas from functional analysis. However, in order to apply this interplay to the *truncated* case, one needs Positivstellensätze for positive polynomials with degree bounds. Such certificates are difficult to obtain and this is the reason why concrete solutions are only known in very special cases, such as when K is a quadratic plane curve. Besides some easily derived properties of the *moment matrices*, i.e., matrices with a special arrangement of moments $\beta_{ij} = L(x^i y^j)$, concrete solutions contain other conditions that are difficult to obtain in general. The TMPs on plane curves of degree more than two and also on the whole plane $\mathbb{R}^2$ are widely open.

A **concrete solution** to the TMP is a set of necessary and sufficient conditions for the existence of a K–rm μ, that can be tested in numerical examples. Let $\mathcal{Z}(P)$ stand for the vanishing set of a polynomial $P \in \mathbb{R}[x, y]_{\leq k}$. The bivariate K–TMP is concretely solved in the following cases:

(A) $K = \mathcal{Z}(P)$ for a polynomial P with $1 \leq \deg p \leq 2$.

For $\deg P = 1$ the solution is [21, Proposition 3.11] and uses the far-reaching **flat extension theorem (FET)** [16, Theorem 7.10] (see also [20, Theorem 2.19] and [44] for an alternative proof). Alternatively, it can be also obtained by reducing the problem to the univariate setting (see [69, Remark 3.3.(4)])

Assume that $\deg P = 2$. By applying an affine linear transformation it suffices to consider one of the canonical cases: $x^2 + y^2 = 1$, $y = x^2$, $xy = 1$, $xy = 0$, $y^2 = y$. The case $x^2 + y^2 = 1$ is equivalent to the univariate trigonometric moment problem, solved in [17]. The other four cases were settled in [17–19, 33] by applying the FET. For an alternative approach by reducing the problem to the univariate setting see [47, Theorem 4.4] (for $x^2 + y^2 = 1$), [6, Section 6] (for $xy = 0$), [68] (for $y^2 = y$), [67] (for $xy = 1$) and [69] (for $y = x^2$).

(B) $K = \mathbb{R}^2$, $k = 2$ and the moment matrix is positive definite.

This case was first solved nonconstructively using convex geometry techniques in [35] and later on constructively in [27] by a novel rank reduction technique.

(C) K is one of $\mathcal{Z}(y-x^3)$ [32, 36, 66], $\mathcal{Z}(y^2-x^3)$ [66], $\mathcal{Z}(y(y-a)(y-b))$, $a, b \in \mathbb{R} \setminus \{0\}$, $a \neq b$ [62, 68], $\mathcal{Z}(xy^2-1)$ [67], $\mathcal{Z}(y(ay+x^2+y^2))$, $a \in \mathbb{R} \setminus \{0\}$ [65], $\mathcal{Z}(y(x-y^2))$ [65], $\mathcal{Z}(xy-y^4-q(x))$, $q \in \mathbb{R}[x]_{\leq 3}$ [64] or $\mathcal{Z}(y-x^4)$ [36]. The main technique in [32] is the FET, in [64–68] the reduction to the univariate TMP is applied, while in [36] the core variety approach (see (b) below) and the results on positive semidefinite matrix completions are used.

(D) The moment matrix has a special feature called *recursive determinateness* [22] or *extremality* [23].

(E) $k = 3$ on a special cubic curve K [25].

The solutions to the K–TMP, which are not concrete in the sense of the definition above, but are partly algorithmic, are known in the following cases:

(a) $K = \mathcal{Z}(P) \subset \mathbb{R}^2$ with $P(x,y) = y - q(x)$ or $P(x,y) = yq(x) - 1$, where $q \in \mathbb{R}[x]$.
[32, Section 6] gives a solution in terms of the bound on the degree m, quadratic in k and $\deg p$, for which the existence of a positive semidefinite (psd) extension of the moment matrix is equivalent to the existence of a K–rm. In [69] the bound on m is improved to $\deg P$ for $P(x,y) = y - q(x)$ and $P(x,y) = yx^\ell$ by working with the corresponding univariate TMPs.

(b) $K = \mathbb{R}^2$, $k = 3$ and the moment matrix is positive definite.
In [34] this case is approached via a new notion, called the *core variety* $\mathcal{V}$. It is shown that for the sextic nonsingular case a rm exists if and only if $\mathcal{V}$ is nonempty, the result extended to the general case in [9]. In contrast to the nonsingular quartic case (see (B) above), positive definiteness of the moment matrix does not guarantee a rm exists. Moreover, if $\mathcal{V}$ is nonempty, then either $\mathcal{V} = \mathbb{R}^2$ or $|\mathcal{V}| = 10$, with the latter case having a unique rm. However, the solution based on the core variety does not belong to the class of concrete solutions, since $\mathcal{V}$ might be very difficult to compute in general.

(c) The moment matrix satisfies special cubic relations [25].

(d) Special cases of the sextic two–dimensional TMP [26, 62, 63], i.e., $2k = 6$.

(e) For an arbitrary closed set $K \subseteq \mathbb{R}^d$, a solution in terms of positive extensions of the linear functional is the *truncated Riesz–Haviland theorem* [21].

(f) $K = \mathcal{Z}(xy - y^m - q(x))$, $m \in \mathbb{N}$, $m > 4$, $q \in \mathbb{R}[x]_{\leq m-1}$ [64].

Some other special cases of the TMP have also been studied in [8, 9, 29, 34, 40], while [49] considers subspaces of the polynomial algebra and [24] the TMP for commutative $\mathbb{R}$–algebras. For an excellent monograph with a recent development in the area we refer a reader to [59].

A **constructive solution** to the K–TMP is a solution, where not only the existence of a K–rm is characterized, but a concrete K–rm is explicitly constructed.

The **psd=sos** question for a given plane curve C asks whether every positive polynomial on C is a sum of squares of polynomials. The complete answer to this question was given by Scheiderer for irreducible curves [55,56] and Plaumann for reducible curves [50]. The psd=sos question has an affirmative answer if and only if C is one of the following:

(i) A smooth affine rational curve [55, Proposition 2.17].

(ii) A non-rational, irreducible, *virtually compact* (see [56, Definition 4.8]) curve that is either smooth or has only singularities, which are ordinary multiple points with independent tangents [56, Theorem 4.18].

(iii) A reducible curve such that: a) all singularities are ordinary multiple points with independent tangents, b) there are no non-real intersection points, c) all irreducible components of C', which is the union of all irreducible components of C that are not virtually compact, are smooth and rational, and d) the configuration of the irreducible components of C' contains no loops [50].

Let us mention here that the psd=sos question for homogenuous polynomials on projective varieties is solved in [10], while the question of degree bounds in positivity certificates is studied in [11].

The **Carathéodory number** $\mathrm{Car}_{2k}(K)$ of the **moment cone**

$$\mathcal{M}_{2k}(K) := \{L : \mathbb{R}[x,y]_{\leq 2k} \to \mathbb{R} \colon L \text{ is a } K\text{–mf}\},$$

is the smallest natural number such that every $L \in \mathcal{M}_{2k}(K)$ can be represented as a conic combination of at most $\mathrm{Car}_{2k}(K)$ point evaluations. By Richter's result [53] (see also [59, Theorem 1.24]) $\mathrm{Car}_{2k}(K)$ is finite. Estimates of $\mathrm{Car}_{2k}(C)$ for arbitrary affine curves C were recently obtained in [54], while for compact curves sharper asymptotic estimates appear in [28, Theorem 4.8]. Recently, Baldi, Blekherman and Sinn showed that $\mathrm{Car}_{2k}(C)$ is the smallest possible (i.e., $3k$) for any connected plane cubic C with smooth projective closure and only one point at infinity, while for every disconnected plane cubic with a smooth projectivization it is 1 more than the smallest possible (i.e., $3k+1$). This gives an abstract solution to the C–TMP in terms of the existence of positive rank–preserving extensions of the moment matrix, i.e., in the connected case the solution to the C–TMP is equivalent to the existence of a flat extension of the moment matrix [4, Theorem 6.2.2], while in the second case the moment matrix must have a positive semidefinite extension which in turn admits a flat extension of rank at most 1 greater than the rank of the original matrix [4, Theorem 6.2.4]. See also [5, Example 5.15] for a concrete numerical example demonstrating that the existence of a flat extension is not sufficient in the disconnected case.

The motivation for this paper is the following problem:

Problem. *Let C be a plane cubic curve.*

(1) Obtain algebraic certificates for positivity of polynomials on plane cubics with degree bounds.

(2) Solve the C–TMP concretely.

(3) Solve the C–TMP constructively.

Let $\Phi(x,y) = (ax+by+c, dx+ey+f)$, $a,b,c,d,e,f \in \mathbb{R}$, $ae-bd \neq 0$, be an invertible affine linear transformation (alt). Since the solution to the C–TMP implies the solution to the $\Phi(C)$–TMP, it suffices to solve Problem for chosen representatives of equivalence classes of plane cubic curves with respect to the relation

$$C_1 \sim C_2 \quad \Leftrightarrow \quad C_2 = \Phi(C_1) \quad \text{where } \Phi \text{ is some invertible alt.}$$

Up to $\sim$ every irreducible cubic has one of the forms (e.g., [48])

$$(I)\ y = p(x), \quad (II)\ xy = p(x), \quad (III)\ y^2 = p(x), \quad (IV)\ xy^2 + ay = p(x), \tag{1.1}$$

where $p(x) = bx^3 + cx^2 + dx + e$. Form (I) can be further transformed into $y = x^3$, being one of the solved cases (see (C) above).

Up to $\sim$ every reducible cubic is of the form $yc(x, y)$ for some $c \in \mathbb{R}[x, y]$ of degree 2. Namely, it is a union of the x–axis and a conic, where the conic is either a circle, a parabola, a hyperbola or two lines. Depending on the position of the line and a conic there are 16 cases to consider:

- A line and a circle intersect at a double real point or intersect at two non-real points or intersect at two real points.
- A line and a parabola intersect at a double real point or intersect at a real point and a point at infinity or intersect at two non-real points or intersect at two real points.
- A line and a hyperbola intersect at a point at infinity with multiplicity 2 or intersect at a real point and a point at infinity or intersect at a double real point or intersect at two non-real points or intersect at two real points.
- Three parallel lines.
- Three lines that intersect at a real point.
- Two parallel lines and one line intersecting both in different points.
- Three lines that intersect at 3 real points.

In this paper we settle (1) and (2) of Problem completely, while (3) of Problem for most irreducible cases.

1.1. **Reader's guide.** Let C be a plane cubic.

In Section 2 we first introduce the notation and definitions. Then we state our first two main results, which are positivity certificates for polynomials on C with degree bounds, i.e., Theorems 2.3 and 2.4 for cubics without (resp. with) non-real intersections points. In Tables 1–4 the abstract elements appearing in both theorems are specified for all different choices of C up to affine linear equivalence. As a consequence of positivity certificates, we obtain solution to the nonsingular C–TMP, e.g., Corollaries 2.6 and 2.7.

Section 3 is devoted to the properties of the extreme rays of the cone of positive polynomials on C. The main result, Theorem 3.1, states that each extreme ray consists only of real points. From this it follows as a corollary that for irreducible C with a smooth projective closure there are exactly two types of extreme rays, i.e., Corollary 3.5.

Sections 4–8 are devoted to proving explicit versions of Theorems 2.3 and 2.4 and solving the C–TMP for all possible C up to affine linear equivalence. Namely:

- In Section 4 we study C defined by an irreducible cubic polynomial $P(x, y)$ in the Weierstraß form. There are five such cases to be considered depending on the smoothness and the number of connected components. We solve both singular and nonsingular C–TMPs, constructively for all singular cases and concretely in all nonsingular cases. In the case of a nodal curve $y^2 = x(x-1)^2$ and a cubic with an isolated point $y^2 = x^2(x-1)$ we also solve the nonsingular case constructively.

- In Section 5 we study C defined by an irreducible cubic polynomial $P(x, y)$ of the form $xy^2 + ay - bx^3 - cx^2 - dx - e$, $a, b, c, d, e \in \mathbb{R}$. We distinguish six such cases according to the signs of a and b. In each case, explicit descriptions of the pair $(f, V^{(k)})$ from Theorem 2.3 are given, while a constructive solution to the C–TMP for the curve $xy^2 + ay - dx - e = 0$ is established.
- In Section 6 we study C defined by an irreducible cubic polynomial $P(x, y)$ of the form $xy - c(x)$, with $\deg c = 3$. An explicit description of the pair $(f, V^{(k)})$ from Theorem 2.3 is given and also a constructive solution to the C–TMP is derived.
- In Section 7 the pair $(f, V^{(k)})$ from Theorem 2.3 is explicitly described for the curve $y = x^3$.
- Finally, in Section 8 we study reducible cubic curves. In Proposition 8.1 we divide them into 16 equivalence classes up to affine linear equivalence. Then explicit descriptions of the pair $(f, V^{(k)})$ from Theorem 2.3 are established for each of these classes.

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2. Notation and main results

In this paper closed sets $K \subseteq \mathbb{R}^2$ in K–TMP will be cubic curves and we will denote them with C. Namely, $C \subseteq \mathbb{R}^2$ stands for an algebraic subset whose vanishing ideal

$$I = \{f \in \mathbb{R}[x, y] \colon f(x) = 0 \text{ for all } x \in C\} \tag{2.1}$$

is generated by a polynomial $P \in \mathbb{R}[x, y]$ of degree three. Note also that $C = \mathcal{Z}(P)$. We denote by $\mathbb{R}[C]$ the ring of polynomial functions $C \to \mathbb{R}$. We have $\mathbb{R}[C] = \mathbb{R}[x, y]/I$ because I is the kernel of the restriction map

$$\mathbb{R}[x, y] \to \mathbb{R}[C],\ f \mapsto f|_C.$$

We write $Q(\mathbb{R}[C])$ for the total ring of fractions of $\mathbb{R}[C]$. If C is irreducible, then $Q(\mathbb{R}[C])$ is the usual quotient field $\mathbb{R}(C)$ of $\mathbb{R}[C]$. Let $m \in \mathbb{N}_0$ be a non-negative integer and let $\mathbb{R}[C]_{\leq m}$ be the image of $\mathbb{R}[x, y]_{\leq m}$ under the restriction map $\mathbb{R}[x, y] \to \mathbb{R}[C]$ and

$$I_{\leq m} := \mathbb{R}[x, y]_{\leq m} \cap I.$$

Let $\widetilde{C} \subseteq C$ and $k \in \mathbb{N}_0$. We write

$$\mathrm{Pos}_{2k}(\widetilde{C}) = \{f \in \mathbb{R}[C]_{\leq 2k} \mid f(x) \geq 0 \text{ for all } x \in \widetilde{C}\}.$$

Let $L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ be a linear functional and

$$\overline{L} : \mathbb{R}[x, y]_{\leq k} \times \mathbb{R}[x, y]_{\leq k} \to \mathbb{R}, \quad \overline{L}(f, g) := L(fg)$$

the corresponding bilinear form. A necessary condition for L to be a C–mf is that it vanishes on all polynomials $p \in I_{\leq 2k}$. The induced functional

$$L_C : \mathbb{R}[C]_{\leq 2k} \to \mathbb{R}, \quad L_C(f + I) := L(f).$$

is then well–defined.

Assume from now on that L_C is well–defined. We call L_C:

(1) **positive** if $L_C(f) \geq 0$ for every $0 \neq f \in \mathrm{Pos}_{\leq 2k}(C)$.

(2) **strictly positive** if $L_C(f) > 0$ for every $0 \neq f \in \mathrm{Pos}_{\leq 2k}(C)$.

(3) **square positive** if $L_C(f^2) \geq 0$ for every $0 \neq f \in \mathbb{R}[C]_{\leq k}$.

(4) **strictly square positive** $L_C(f^2) > 0$ for every $0 \neq f \in \mathbb{R}[C]_{\leq k}$.

Remark 2.1. (1) Note that the square positivity of L_C is generally a weaker condition than the positivity of L_C, but the positivity of L_C is a necessary condition for L_C to be a C–mf.

(2) The verification of the positivity of L_C is generally a hard problem, since Positivstellensätze (see §1), which are "simple enough", are hard to establish. Simple enough means that they contain only finitely many summands whose degree is bounded as a function of the degree of the given polynomial, a property called *stability*.

(3) In contrast to the verification of positivity, checking square positivity is easy. Let $\overline{L}_C$ be a bilinear form

$$\overline{L}_C : \mathbb{R}[C]_{\leq k} \times \mathbb{R}[C]_{\leq k} \to \mathbb{R}, \quad \overline{L}_C(f,g) := L_C(fg)$$

induced by L_C. Choose a basis $\mathcal{B}_k$ for $\mathbb{R}[C]_{\leq k}$ and let M be a matrix representing $\overline{L}_C$ in this basis. The square positivity (resp. strict square positivity) of L_C is equivalent to M being psd (resp. pd).

Let $B : V \times V \to \mathbb{R}$ be a bilinear form on a vector space V. We write $\ker B$ to denote the **kernel** of B, i.e.,

$$\ker B := \{v \in V \colon B(u,v) = 0 \text{ for every } u \in V\}.$$

Note that if B is a semi-inner product, then by the Cauchy-Schwartz inequality we have $\ker B = \{v \in V \colon B(v,v) = 0\}$. The **rank** of a bilinear form B, denoted by $\operatorname{rank} B \in \mathbb{Z}_+ \cup \{\infty\}$, is equal to the rank of a matrix, representing B in some basis. We denote by $B|_U : U \times U \to \mathbb{R}$ the restriction of B to a vector subspace $U \subseteq V$.

Let $\widetilde{V}$ be a finite dimensional vector space in $Q(\mathbb{R}[C])$ and denote by $\widetilde{U}$ the vector space

$$\widetilde{U} := \mathrm{Span}\{gh \colon g,h \in \widetilde{V}\} \subseteq Q(\mathbb{R}[C]).$$

Let $f \in \mathbb{R}[C]$ and assume that

$$\widetilde{U}_f := \left\{fgh \colon g,h \in \widetilde{V}\right\} \subseteq \mathbb{R}[C]_{\leq 2k}.$$

Then the functional

$$L_{C,\widetilde{V},f} : \widetilde{U} \to \mathbb{R}, \quad L_{C,\widetilde{V},f}(g) := L_C(fg)$$

if well–defined. Let

$$\overline{L}_{C,\widetilde{V},f} : \widetilde{V} \times \widetilde{V} \to \mathbb{R}, \quad \overline{L}_{C,\widetilde{V},f}(g,h) := L_{C,\widetilde{V},f}(gh)$$

be the corresponding bilinear form. We call L_C:

(1) $(\widetilde{V}, f)$–**locally square positive** if $L_{C,\widetilde{V},f}(g^2) \geq 0$ for every $0 \neq g \in \widetilde{V}$ or equivalently, if $\overline{L}_{C,\widetilde{V},f}$ is a semi-inner product.

(2) $(\widetilde{V}, f)$–**locally strictly square positive** if $L_{C,\widetilde{V},f}(g^2) > 0$ for every $0 \neq g \in \widetilde{V}$ or equivalently, if $\overline{L}_{C,\widetilde{V},f}$ is an inner product.

(3) **singular** if $\ker \overline{L}_C \neq \{0\}$.
(4) $(\widetilde{V}, f)$–**locally singular** if $\ker \overline{L}_{C,\widetilde{V},f} \neq \{0\}$.

Remark 2.2. As in Remark 2.1, checking the $(\widetilde{V}, f)$–local square positivity of L_C is easy as one only needs to consider positive semidefiniteness of a matrix, representing $\overline{L}_{C,\widetilde{V},f}$ in some basis for $\widetilde{V}$.

The first two main results of the paper are certificates of belonging to $\mathrm{Pos}_{2k}(\widetilde{C})$ where $\widetilde{C}$ is the set of non-isolated points of C (see Theorems 2.3 and 2.4). They imply solutions to the corresponding nonsingular C–TMPs (see Corollaries 2.6 and 2.7). For C without isolated points, $C = \widetilde{C}$ and we will show that deciding positivity of the functional L_C translates to checking its square positivity and $(\widetilde{V}, f)$–local square positivity for a certain choices of the pair $(\widetilde{V}, f)$. The choice of $(\widetilde{V}, f)$ depends on the polynomial, defining the curve.

Theorem 2.3. *Let C be a cubic curve for which every non-real point lies on a unique irreducible component and $\widetilde{C}$ the set of non-isolated points of C. There exists $f \in Q(\mathbb{R}[C])$ such that for every $k \in \mathbb{N}$ there is a vector subspace $V^{(k)} \subseteq Q(\mathbb{R}[C])$ of dimension $3k$ so that the following are equivalent:*

(1) $p \in \mathrm{Pos}_{2k}(\widetilde{C})$.
(2) There exist finitely many $g_i \in \mathbb{R}[C]_{\leq k}$ and $h_j \in V^{(k)}$ satisfying

$$p = \sum\nolimits_i g_i^2 + f \sum\nolimits_j h_j^2.$$

In the paper we consider each possible C up to invertible affine linear change of variables and establish appropriate choices of f and $V^{(k)}$. See Tables 1–4 below.

Theorem 2.4. *Let C be a cubic curve defined by a polynomial $P = P_1 P_2 \in \mathbb{R}[x, y]$ with* $\deg P = 3$, $\deg P_1 = 1$ *and* $\deg P_2 = 2$, *such that the zero sets of P_1 and P_2 intersect in a pair of non-real points. Then the following are equivalent:*

(1) $p \in \mathrm{Pos}_{2k}(C)$.
(2) Then there exist finitely many $f_i \in \mathbb{R}[C]_{\leq k}$, $h_j \in \mathbb{R}[C]_{\leq k-1}$ and $g_\ell \in \mathbb{R}[C]_{\leq k-1}$ such that

$$p = \sum\nolimits_i f_i^2 + \chi_1 P_1 \sum\nolimits_j h_j^2 + \chi_2 P_2 \sum\nolimits_\ell g_\ell^2,$$

where

$$\chi_1 = \begin{cases} 1, & \text{if } P_1 \text{ is nonnegative on } \mathcal{Z}(P_2), \\ -1, & \text{if } P_1 \text{ is nonpositive on } \mathcal{Z}(P_2), \\ 0, & \text{if } P_1 \text{ changes sign on } \mathcal{Z}(P_2), \end{cases}$$

$$\chi_2 = \begin{cases} 1, & \text{if } P_2 \text{ is nonnegative on } \mathcal{Z}(P_1), \\ -1, & \text{if } P_2 \text{ is nonpositive on } \mathcal{Z}(P_1). \end{cases}$$

Remark 2.5. Note that $\mathcal{Z}(P_2)$ in Theorem 2.4 is always an irreducible conic. Applying an affine linear change of variables, it is either a circle, a parabola or a hyperbola. In the case of a circle or a parabola, $\chi_1 \in \{-1, 1\}$, while in the case of a hyperbola $\chi_1 = 0$, as P_1 then changes sign on $\mathcal{Z}(P_2)$.

A functional $L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ is C–**nonsingular**, if $\ker \overline{L} = I_{\leq k}$ and:

(1) If C is without non-real intersection points and f, $V^{(k)}$ are as in Theorem 2.3 above, then $\ker \overline{L}_{C,V^{(k)},f} = \{0\}$.
(2) If C has a pair of non-real intersection points and P_1, P_2 are as in Theorem 2.4 above, then $\ker \overline{L}_{C,\mathbb{R}[C]_{\leq k-1},P_1} = \{0\}$ and $\ker \overline{L}_{C,\mathbb{R}[C]_{\leq k-1},P_2} = \{0\}$.

A **nonsingular** C–TMP refers to the C–TMP for C–nonsingular functionals.

A functional $L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ is C–**singular**, if L_C is well-defined, but L is not C–nonsingular. A **singular** C–TMP refers to the C–TMP for C–singular functionals.

Theorems 2.3 and 2.4 imply the following solutions to the nonsingular C–TMP.

Corollary 2.6. *Let C, $\widetilde{C}$, f and $V^{(k)}$ be as in Theorem 2.3. Let $L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ be a C–nonsingular linear functional. Then the following are equivalent:*

(1) L is a $\widetilde{C}$–moment functional.
(2) L_C is strictly square positive and $(V^{(k)}, f)$–locally strictly square positive.

Proof. The equivalence (1) ⇔ (2) follows using Theorem 2.3 above and [29, Proposition 2 and Corollary 6] (or [59, Theorem 1.30]). □

Corollary 2.7. *Let C, P_1, P_2, χ_1 and χ_2 be as in Theorem 2.4. Let $L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ be a C–nonsingular linear functional. Then the following are equivalent:*

(1) L is a C–moment functional.
(2) L_C is strictly square positive, $(\mathbb{R}[C]_{\leq k-1}, \chi_1 P_1)$–locally strictly square positive and $(\mathbb{R}[C]_{\leq k-1}, \chi_2 P_2)$–locally strictly square positive.

Proof. The equivalence (1) ⇔ (2) follows using Theorem 2.4 above and [29, Proposition 2 and Corollary 6] (or [59, Theorem 1.30]). □

In some cases of Table 1 we need the definition of Riemann–Roch spaces, which we now recall. Let Y be a smooth projective and irreducible curve defined over $\mathbb{R}$. Then by $\mathrm{Div}(Y)$ we denote the divisor group of Y, i.e., the free abelian group generated by the (real and complex) points of Y. This means that $\mathrm{Div}(Y)$ consists of all formal sums $D := \sum_{Q\in Y} n_Q Q$ with $n_Q \in \mathbb{Z}$ and $n_Q = 0$ for all but finitely many $Q \in Y$. We call a divisor D effective if $n_Q \geq 0$ for every $Q \in Y$. For $D_1, D_2 \in \mathrm{Div}(Y)$, we write $D_1 \geq D_2$ if and only if $D_1 - D_2$ is effective. Each rational function $0 \neq f \in \mathbb{R}(Y)$ on Y determines a divisor $\mathrm{div}(f) = \sum_{Q\in Y} \mathrm{ord}_Q(f)Q$ of $\mathrm{Div}(Y)$, where $\mathrm{ord}_Q(f)$ is the order of f at Q, called the principal divisor. One assigns a real vector space of functions to $D \in \mathrm{Div}(Y)$, defined by

$$\mathcal{L}(D) := \{f \in \mathbb{R}(Y) \setminus \{0\} : \ \mathrm{div}(f) \geq -D\} \cup \{0\},$$

called the Riemann–Roch space of D.

TABLE 1. Irreducible polynomial P defining a smooth C, f a polynomial from Theorem 2.3, $\mathcal{B}_k$ and $\mathcal{B}_{V^{(k)}}$ bases of $\mathbb{R}[C]_{\leq k}$ and $V^{(k)}$, respectively.

P	f	$\mathcal{B}_k$	$\mathcal{B}_{V^{(k)}}$
Smooth cubic in the Weierstraß form: $y^2 - x(x^2+c)$ or $y^2 - x(x-a)(x-b)$, $a, b \in \mathbb{R},\ 0 < a < b$, $c \in [0, \infty)$	x	$\{1, x, y, x^2, xy, y^2, \ldots,$ $x^2y^{i-2}, xy^{i-1}, y^i, \ldots,$ $x^2y^{k-2}, xy^{k-1}, y^k\}$	$\mathcal{B}_k \setminus \{y^k\} \cup \{\frac{y}{x}\}$
Smooth cubic in the non-Weierstraß form type 1: $xy^2 + ay - x^2 - dx - e$ $a, d, e \in \mathbb{R}$,	$x - \alpha_1$[a]	$\{1, x, y, x^2, xy, y^2,$ $x^3, x^2y, y^3, \ldots,$ $x^i, x^{i-1}y, y^i, \ldots,$ $x^k, x^{k-1}y, y^k\}$	$\mathcal{B}_k \setminus \{y^k\} \cup$ $\{\frac{2xy+a}{x-\alpha_1}\}$
Smooth cubic in the non-Weierstraß form type 2: $xy^2 - x^3 - cx^2 - dx - e$ $c, d, e \in \mathbb{R}$,	$\frac{e}{\lvert e \rvert}\left(\frac{1}{x} - \alpha_2\right)$[b]	$\{1, x, y, x^2, xy, y^2,$ $x^3, x^2y, y^3, \ldots,$ $x^i, x^{i-1}y, y^i, \ldots,$ $x^k, x^{k-1}y, y^k\}$	$\mathcal{B}_k \setminus \{y^k\} \cup$ $\{\frac{xy}{1-\alpha_2 x}\}$
Smooth cubic in the non-Weierstraß form type 3: $xy^2 + x^3 - cx^2 - dx - e$ $c, d, e \in \mathbb{R}$,	$\frac{e}{\lvert e \rvert}\left(\frac{1}{x} - \alpha_3\right)$[c]	$\{1, x, y, x^2, xy, y^2,$ $x^3, x^2y, y^3, \ldots,$ $x^i, x^{i-1}y, y^i, \ldots,$ $x^k, x^{k-1}y, y^k\}$	$\mathcal{B}_k \setminus \{y^k\} \cup$ $\{\frac{xy}{1-\alpha_3 x}\}$
Smooth cubic in the non-Weierstraß form type 2: $xy^2 + ay - x^3 - cx^2 - dx - e$ $a, c, d, e \in \mathbb{R}$,	$y^2 - x^2 + cx - \alpha_4$[d]	$\{1, x, y, x^2, xy, y^2,$ $x^3, x^2y, y^3, \ldots,$ $x^i, x^{i-1}y, y^i, \ldots,$ $x^k, x^{k-1}y, y^k\}$	$\mathcal{B}_k \setminus \{y^k\} \cup$ $\{r\}$[e]
Smooth cubic in the non-Weierstraß form type 3: $xy^2 + ay + x^3 - cx^2 - dx - e$ $a, c, d, e \in \mathbb{R}$,	$y^2 + x^2 + cx - \alpha_5$[f]	$\{1, x, y, x^2, xy, y^2,$ $x^3, x^2y, y^3, \ldots,$ $x^i, x^{i-1}y, y^i, \ldots,$ $x^k, x^{k-1}y, y^k\}$	$\mathcal{B}_k \setminus \{y^k\} \cup$ $\{s\}$[g]

---

[a] α_1 is the smallest zero of $q(t) = t^3 + dt + et + \frac{a^2}{4}$.

[b] α_2 is the smallest (resp. largest) zero of $q(t) = et^3 + dt^2 + ct + 1$ if $e > 0$ (resp. $e < 0$).

[c] α_3 is the smallest (resp. largest) zero of $q(t) = et^3 + dt^2 + ct - 1$ if $e > 0$ (resp. $e < 0$).

[d] α_4 is the smallest zero of $q(t) = t^3 - 2dt^2 + (d^2 - a^2 + ce)t + \frac{a^2c^2}{4} - cde + e^2$.

[e] $r \in \mathcal{L}(D)$ with $D := k[-1:1:0] + (k-1)[0:1:0] + k[1:1:0] + Q$, where Q is a zero of f on C.

[f] α_5 is the smallest zero of $q(t) = t^3 - 2dt^2 + (d^2 + a^2 + ce)t + \frac{a^2c^2}{4} - cde - e^2$.

[g] $s \in \mathcal{L}(D)$ with $D := k[-1:1:0] + (k-1)[0:1:0] + k[1:1:0] + Q$, where Q is a zero of f on C.

Table 2. Irreducible polynomial P defining a rational C, f a polynomial from Theorem 2.3, $\mathcal{B}_k$ and $\mathcal{B}_{V^{(k)}}$ bases of $\mathbb{R}[C]_{\leq k}$ and $V^{(k)}$, respectively.

P	f	$\mathcal{B}_k$	$\mathcal{B}_{V^{(k)}}$
Neile's parabola: $y^2 - x^3$	1	$\{1, x, y, x^2, xy, y^2, \ldots,$ $x^2y^{i-2}, xy^{i-1}, y^i, \ldots,$ $x^2y^{k-2}, xy^{k-1}, y^k\}$	$\mathcal{B}_k \setminus \{1\} \cup \{\frac{y}{x}\}$
Nodal cubic: $y^2 - x(x-1)^2$	1	$\Phi_1^{-1}(\{1, t^2-1, t^3-t,$ $t^4-t^2, \ldots, t^{k-1}-t^{k-3},$ $t^k - t^{k-2}\})$[h]	$\mathcal{B}_k \setminus \{1\} \cup \{\frac{y}{x-1}\}$
Cubic with an isolated point: $y^2 - x^2(x-1)$	1	$\Phi_2^{-1}(\{1, t^2+1, t^3+t,$ $t^4+t^2, \ldots, t^{k-1}+t^{k-3},$ $t^k + t^{k-2}\})$[i]	$\mathcal{B}_k \setminus \{1\} \cup \{\frac{y}{x}\}$
Rational cubic type 1: $xy^2 + ax - by - c$ $a, b, c \in \mathbb{R}$, $c \neq 0$ or $ab \neq 0$	1	$\{x^k, x^{k-1}, x^{k-1}y,$ $x^{k-2}, x^{k-2}y, \ldots,$ $x, xy, 1, y, \ldots, y^k\}$	$\mathcal{B}_k \setminus \{x^k\} \cup \{x^k y\}$
Rational cubic type 2: $yx - c(x)$, c of degree $3, c(0) \neq 0$	1	$\{1, x, y, x^2, xy, y^2, \ldots,$ $x^2y^{i-2}, xy^{i-1}, y^i, \ldots$ $x^2y^{k-2}, xy^{k-1}, y^k\}$	$\mathcal{B}_k \setminus \{y^k\} \cup \{y^k - 2g\}$[j]
Rational cubic type 3: $y = x^3$	1	$\{1, x, y, x^2, xy, y^2, \ldots,$ $x^2y^{i-2}, xy^{i-1}, y^i, \ldots,$ $x^2y^{k-2}, xy^{k-1}, y^k\}$	$\mathcal{B}_k \setminus \{y^k\} \cup \{x^2y^{k-1}\}$

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[h] $\Phi_1 : \mathbb{R}[C] \to \mathcal{R}_1 := \{p \in \mathbb{R}[t] \colon p(1) = p(-1)\}$, $\Phi_1(f(x,y)) = f(t^2, t^3 - t)$.

[i] $\Phi_2 : \mathbb{R}[C] \to \mathcal{R}_2 := \{p \in \mathbb{R}[t] \colon p(\mathfrak{i}) = p(-\mathfrak{i})\}$, $\Phi_2(f(x,y)) = f(t^2+1, t^3+t)$, $\mathfrak{i}$ is an imaginary unit.

[j] g is a representative of x^{2k} in $\mathbb{R}[C]_{\leq k}$.

Table 3. Reducible polynomial P with two irreducible factors defining C without non-real intersection points, $\mathcal{B}_k$ and $\mathcal{B}_{V^{(k)}}$ bases of $\mathbb{R}[C]_{\leq k}$ and $V^{(k)}$. A polynomial f from Theorem 2.3 is equal to 1 in all cases.

P	$\mathcal{B}_k$	$\mathcal{B}_{V^{(k)}}$
A line and a circle with a double real intersection point: $y(ay+x^2+y^2)$, $a \in \mathbb{R}\setminus\{0\}$	$\{1, x, y, x^2, xy, y^2, \ldots,$ $x^j, x^{j-1}y, x^{j-2}y^2, \ldots,$ $x^k, x^{k-1}y, x^{k-2}y^2\}$	$\mathcal{B}_k \setminus \{1\} \cup$ $\{\frac{ay+x^2+y^2}{x}\}$
A line and a circle with two simple real intersection points: $y(1+ay-x^2-y^2)$, $a \in \mathbb{R}$	$\{1, x+1, x^2-1, x(x^2-1), \ldots,$ $x^{k-2}(x^2-1), y, yx, \ldots, yx^{k-1},$ $y^2, y^2x, \ldots, y^2x^{k-2}\}$	$\mathcal{B}_k \setminus \{1\} \cup$ $\{\frac{-1-2ay+x^2+2y^2}{1+x}\}$
A line and a parabola with a double real intersection point: $y(y-x^2)$	$\{1, x, x^2, \ldots, x^k, y, y^2, \ldots, y^k,$ $yx, y^2x, \ldots, y^{k-1}x\}$	$\mathcal{B}_k \setminus \{1\} \cup \{\frac{y}{x}\}$
A line and a parabola with one simple real intersection point: $y(x-y^2)$	$\{1, x, \ldots, x^k,$ $y, y^2, yx, y^2x, \ldots, yx^j,$ $y^2x^j, \ldots, y^2x^{k-2}, yx^{k-1}\}$	$\mathcal{B}_k \setminus \{x^k\} \cup$ $\{x^{k-1}(x-2y^2)\}$
A line and a parabola with two simple real intersection points: $y(1+y-x^2)$	$\{1, x+1, x^2-1, x(x^2-1), \ldots,$ $x^{k-2}(x^2-1), y, yx, y^2, y^2x, \ldots,$ $y^{k-1}, y^{k-1}x, y^k\}$	$\mathcal{B}_k \setminus \{1\} \cup$ $\{\frac{-1-2y+x^2}{1+x}\}$
A line and a hyperbola with a double intersection point at ∞: $y(1-xy)$	$\{1, x, y, x^2, xy, y^2, \ldots,$ $x^j, x^{j-1}y, y^j, \ldots,$ $x^k, x^{k-1}y, y^k\}$	$\mathcal{B}_k \setminus \{x^k\} \cup \{yx^k\}$
A line and a hyperbola with a simple real intersection point: $y(x+y+axy)$, $a \in \mathbb{R}\setminus\{0\}$	$\{1, x, y, x^2, xy, y^2, \ldots,$ $x^j, x^{j-1}y, y^j, \ldots,$ $x^k, x^{k-1}y, y^k\}$	$\mathcal{B}_k \setminus \{x^k\} \cup$ $\{x^{k-1}(x+2y(1+ax))\}$
A line and a hyperbola with a double real intersection point: $y(ay+x^2-y^2)$, $a \in \mathbb{R}\setminus\{0\}$	$\{1, x, y, x^2, xy, y^2, \ldots,$ $x^j, x^{j-1}y, x^{j-2}y^2, \ldots,$ $x^k, x^{k-1}y, x^{k-2}y^2\}$	$\mathcal{B}_k \setminus \{1\} \cup$ $\{\frac{ay+x^2-y^2}{x}\}$
A line and a hyperbola with two simple real intersection points: $y(1+ay-x^2+y^2)$, $a \in \mathbb{R}$	$\{1, x+1, x^2-1, x(x^2-1), \ldots,$ $x^{k-2}(x^2-1), y, yx, \ldots, yx^{k-1},$ $y^2, y^2x, \ldots, y^2x^{k-2}\}$	$\mathcal{B}_k \setminus \{1\} \cup$ $\{\frac{-1-2ay+x^2-2y^2}{1+x}\}$

Table 4. Reducible polynomial P with three irreducible factors defining C, $\mathcal{B}_k$ and $\mathcal{B}_{V^{(k)}}$ bases of $\mathbb{R}[C]_{\leq k}$ and $V^{(k)}$. A polynomial f from Theorem 2.3 is equal to 1 in all cases.

P	$\mathcal{B}_k$	$\mathcal{B}_{V^{(k)}}$
3 parallel lines: $y(a+y)(b+y)$, $a,b \in \mathbb{R}\setminus\{0\}$, $a \neq b$	$\{1, x, y, x^2, xy, y^2, \ldots,$ $x^j, x^{j-1}y, x^{j-2}y^2, \ldots,$ $x^k, x^{k-1}y, x^{k-2}y^2\}$	$\mathcal{B}_k \setminus \{x^k\} \cup$ $\{y(y+a)x^{k-1}\}$
3 lines with one real intersection point: $y(x-y)(x+y)$	$\{1, x, y, x^2, xy, y^2, \ldots,$ $x^j, x^{j-1}y, y^j, \ldots,$ $x^k, x^{k-1}y, y^k\}$	$\mathcal{B}_k \setminus \{1\} \cup$ $\{\frac{x^2-y^2}{x}\}$
3 lines with two real intersection points: $yx(y+1)$	$\{1, x, y, x^2, xy, y^2, \ldots,$ $x^j, x^{j-1}y, y^j, \ldots,$ $x^k, x^{k-1}y, y^k\}$	$\mathcal{B}_k \setminus \{x^k\} \cup$ $\{x^k(1+2y)\}$
3 lines with three real intersection points: $y(1-x+y)(1+x+y)$	$\{1, x, y, x^2, xy, y^2, \ldots,$ $x^j, x^{j-1}y, x^{j-2}y^2, \ldots,$ $x^k, x^{k-1}y, x^{k-2}y^2\}$	$\mathcal{B}_k \setminus \{1\} \cup$ $\{\frac{-x+x^3+y+xy-y^2}{x}\}$

3. EXTREME RAYS OF $\mathrm{Pos}_{2k}(C)$

Assume the notation from §2. In particular, $C = \mathcal{Z}(P)$ for some $P \in \mathbb{R}[x, y]$ of degree 3 and $\widetilde{C}$ is the set of non-isolated points of C. We denote by X the Zariski closure of C in $\mathbb{C}^2$ and by $\bar{X} \subseteq \mathbb{P}^2$ the projective closure of X. Let

$$P = \prod_{i=1}^{r} P_i$$

be the decomposition of P into irreducible factors and denote by C_i, X_i and $\bar{X}_i$ the zero set of P_i in $\mathbb{R}^2$, $\mathbb{C}^2$ and its projective closure, respectively. Let $d \in \mathbb{N}_0$ be a non-negative integer. The sets $\mathrm{Pos}_{2d}(C)$ and $\mathrm{Pos}_{2d}(\widetilde{C})$ are closed convex cones and the goal of this section is to understand their extreme rays. We will first prove the following theorem.

Theorem 3.1. *Assume that the restriction of $Q \in \mathbb{R}[x, y]_{\leq 2d}$, $d \geq 1$, to C or $\widetilde{C}$ generates an extreme ray of $\mathrm{Pos}_{2d}(C)$ or $\mathrm{Pos}_{2d}(\widetilde{C})$, respectively, and let S be the set of indices $i \in \{1, \dots, r\}$ for which Q is divisible by P_i. Then, for every $j \in \{1, \dots, r\} \setminus S$, the set*

$$\{x \in \mathbb{P}^2 \mid Q^h(x) = P_j^h(x) = 0 \text{ and } P_i^h(x) \neq 0 \text{ for all } i \in S\}$$

consists only of real points or of real non-isolated points, respectively. Here we denote $Q^h(x, y, z) = z^{2d} \cdot Q(\frac{x}{z}, \frac{y}{z})$ and for $i = 1, \dots, r$ we let $P_i^h(x, y, z) = z^{\deg(P_i)} \cdot P_i(\frac{x}{z}, \frac{y}{z})$.

Remark 3.2. By [4, Example 3.3.3], Theorem 3.1 fails for curves of higher degree, i.e., there are extreme rays of $\mathrm{Pos}_{2d}(C)$ not having only real zeroes. However, for d large enough there exist extreme rays with only real zeros [28].

For the proof of Theorem 3.1 we need some preparation. We let $\pi \colon \tilde{X} \to \bar{X}$ be the normalization of the projective curve $\bar{X}$. Hence $\tilde{X}$ is a smooth projective real curve with irreducible components $\tilde{X}_1, \dots, \tilde{X}_r$. For every $i = 1, \dots, r$ we denote by H_i the divisor on $\tilde{X}_i$ defined by the line at infinity in $\mathbb{P}^2$. We obtain a natural embedding

$$\iota \colon \mathbb{R}[C]_{\leq m} \hookrightarrow \mathcal{L}(mH_1) \times \cdots \times \mathcal{L}(mH_r)$$

as follows: We consider the unique extension $\bar{f}$ of $f \in \mathbb{R}[C]_{\leq m}$ to a rational function on $\bar{X}$ and define $\iota(f) = (f_1, \dots, f_r)$ where f_i is the restriction of $\bar{f} \circ \pi$ to $\tilde{X}_i$. Here $\mathcal{L}(mH_i)$ denotes the Riemann–Roch space associated to mH_i (see §2).

Lemma 3.3. *Let $f \in \mathbb{R}[C]_{\leq m}$ where $m \geq 2$ and $\iota(f) = (f_1, \dots, f_r)$. Further, let $j \in \{1, \dots, r\}$ such that $f_j \neq 0$ and $p, q \in \tilde{X}_j$ distinct points with $f_j(p) = f_j(q) = 0$ and neither $\pi(p)$ nor $\pi(q)$ lie on a component of $\bar{X}$ on which f vanishes identically. Finally, we consider the vector space V of all $g \in \mathbb{R}[C]_{\leq m}$ that satisfy the following:*

1. *For all $i \neq j$ there exists $\lambda_i \in \mathbb{R}$ such that $g_i = \lambda_i \cdot f_i$.*
2. $\mathrm{div}(g_j) \geq \mathrm{div}(f_j) - p - q$.

Here we denote $\iota(g) = (g_1, \dots, g_r)$. The vector space V has dimension at least two.

Proof. We first consider the case when $\bar{X}$ is smooth. Then $\bar{X}$ is an irreducible smooth curve of genus one and the claim follows immediately from the Riemann–Roch theorem. If $\bar{X}$ is not smooth, then each $\tilde{X}_i$ is isomorphic to $\mathbb{P}^1$. For $i = 1, \dots, r$ let d_i be the degree of P_i. Note that $d_i = \deg(H_i)$. Further let S be the set of indices $i \in \{1, \dots, r\}$ for which f_i vanishes identically on $\tilde{X}_i$ and $d = \sum_{i \in S} d_i$.

Next we define V_1 to be the linear subspace of $\mathbb{R}[C]_{\leq m}$ that consists of all g such that $g_i = 0$ for all $i \in S$. We clearly have $V \subseteq V_1$ and a straight-forward calculation shows that $\dim(V_1) = \frac{1}{2}(3-d)(2m-d)$. Indeed, V_1 is the image of the linear map

$$\mathbb{R}[x,y]_{\leq m-d} \to \mathbb{R}[C]_{\leq m},\ f \mapsto (f \cdot \prod_{i \in S} P_i)|_C$$

whose kernel consists of all $f \in \mathbb{R}[x,y]_{\leq m-d}$ that are divisible by $\prod_{i \notin S} P_i$.

Now for $i \notin S$ we let D_i be the pullback of the intersection divisor of $\bar{X}_i$ with $\cup_{k \in S} \bar{X}_k$ on $\tilde{X}_i$. Note that $\deg(D_i) = d \cdot d_i$. By construction, for every $g \in V_1$ we have $g_i \in \mathcal{L}(mH_i - D_i)$ for all $i \notin S$. We let $V_2 = \prod_{i \in S}\{0\} \times \prod_{i \notin S} \mathcal{L}(mH_i - D_i)$. Because each $\tilde{X}_i$ is isomorphic to $\mathbb{P}^1$, we find that

$$\dim(V_2) = \sum_{i \notin S}(md_i + 1 - d \cdot d_i).$$

Finally, we consider the linear space $V_3 \subseteq \prod_{i=1}^r \mathcal{L}(mH_i)$ of all $(h_1, \ldots, h_r)$ such that for $i \neq j$ there exists $\lambda_i \in \mathbb{R}$ with $h_i = \lambda_i \cdot f_i$ and $\operatorname{div}(h_j) \geq \operatorname{div}(f_j) - p - q$. We have that

$$\dim(V_3) = \sum_{i \in S} 0 + \sum_{i \notin S \cup \{j\}} 1 + 3 = 2 + r - |S|$$

because $\tilde{X}_j$ is isomorphic to $\mathbb{P}^1$. As neither $\pi(p)$ nor $\pi(q)$ lie on a component of $\bar{X}$ on which f vanishes identically, we moreover have that $V_3 \subseteq V_2$.

Since V is the preimage of V_3 under $\iota\colon V_1 \to V_2$ we have:

$$\begin{aligned}
\dim(V) &\geq \dim(V_1) - (\dim(V_2) - \dim(V_3)) \\
&= \frac{1}{2}(3-d)(2m-d) - \sum_{i \notin S}(md_i + 1 - d \cdot d_i) + (2 + r - |S|) \\
&= \frac{1}{2}(3-d)(2m-d) - \sum_{i \notin S}(md_i - d \cdot d_i) + 2 \\
&= \frac{1}{2}(3-d)(2m-d) - (m-d) \cdot \sum_{i \notin S} d_i + 2 \\
&= \frac{1}{2}(3-d)(2m-d) - (m-d) \cdot (3-d) + 2 \\
&= (3-d) \cdot (\frac{1}{2}(2m-d) - (m-d)) + 2 \\
&= \frac{1}{2}d(3-d) + 2 \\
&\geq 2.
\end{aligned}$$

This proves the claim. □

Proof of Theorem 3.1. Let $Q \in \mathbb{R}[x,y]_{\leq 2d}$ be non-negative on C (resp. on $\widetilde{C}$) and let S be the set of indices $i \in \{1, \ldots, r\}$ for which Q is divisible by P_i. Assume that there exists $j \in \{1, \ldots, r\} \smallsetminus S$ and a non-real (resp. isolated real or non-real) $x \in \mathbb{P}^2$ such that $Q^h(x) = P_j^h(x) = 0$ and $P_i^h(x) \neq 0$ for all $i \in S$. We have to show that the restriction $f \in \mathbb{R}[C]_{\leq 2d}$ of Q to C does not generate an extreme ray of $\operatorname{Pos}_{2d}(C)$.

There exists $p \in \pi^{-1}(x) \cap \tilde{X}_j$ and we let be $q = \bar{p}$ its complex conjugate. Because x is either not real or isolated real, these are two distinct points on $\tilde{X}_j$. Now f, p and q satisfy the assumptions of Lemma 3.3. Thus by Lemma 3.3 there exists $g \in \mathbb{R}[C]_{\leq 2d}$ which is linearly independent of f such that $g \circ \pi$ has at every real zero and real pole of $f \circ \pi$ on $\tilde{X}$, that is mapped by π to a real point (resp. to a non-isolated real point), a zero of at least the same multiplicity and a pole of at most the same multiplicity, respectively. In particular, in both cases, this applies to all real zeros and poles. Therefore, there exists $\epsilon > 0$ such that $(f \pm \epsilon g) \circ \pi$ is non-negative on the real part of $\tilde{X}$. This shows that $f \pm \epsilon g$ is non-negative on $\widetilde{C}$ and thus f is not an extreme ray of $\mathrm{Pos}_{2d}(\widetilde{C})$.

To treat the other case, we note that if f vanishes at an isolated real point, then g vanishes by construction at this point as well. At isolated real points where f is positive, we can ensure that $f \pm \epsilon g$ is positive as well by replacing ϵ by a smaller positive number. Since there are at most finitely many isolated points, there exists $\epsilon > 0$ such that $f \pm \epsilon g$ is non-negative on C. This shows that f is not an extreme ray of $\mathrm{Pos}_{2d}(C)$. □

Corollary 3.4. *Assume that* $f \in \mathrm{Pos}_{2d}(\widetilde{C})$, $d \geq 1$, *generates an extreme ray and denote* $\iota(f) = (f_1, \ldots, f_r)$. *Let* S *be the set of indices* $i \in \{1, \ldots, r\}$ *for which* f_i *vanishes identically. For every* $j \in \{1, \ldots, r\} \smallsetminus S$ *and every* $p \in \tilde{X}_j$, *such that either* p *is real or* $\pi(p) \notin \cup_{i \in S} \bar{X}_i$, *the rational function* f_j *has a pole or a zero of even order at* p.

Proof. If f_j has a zero at p, then $\pi(p)$ is non-isolated real by Theorem 3.1 and hence p is real. Therefore, the multiplicity has to be even because f is non-negative. Assume that f_j has a pole of odd order at p. Let $Q \in \mathbb{R}[x, y]_{\leq 2d}$ be such that its restriction to C is equal to f. Then Q^h (defined as in Theorem 3.1) vanishes at p because Q^h is homogeneous of even degree. Thus, as above, Theorem 3.1 shows that p is real. But then, because f_j is non-negative, the order of the pole at p must be even. □

Corollary 3.5. *Let* $r = 1$ *and* $\bar{X}$ *be smooth. Let* g *be a non-negative rational function on* $\bar{X}$ *such that* $\mathrm{div}(g) = 2D$ *for some divisor* D *whose equivalence class is non-trivial. If* $f \in \mathrm{Pos}_{2d}(C)$, $d \geq 1$, *generates an extreme ray, then either* f *or* $g \cdot f$ *is the square of a rational function.*

Proof. If $\bar{X}$ is smooth, then $C = \widetilde{C}$. By Corollary 3.4 we have $\mathrm{div}(f) = 2E$ for some divisor E on $\bar{X}$. It follows for example from [37, §5] that either E has trivial equivalence class or is linearly equivalent to D. In the first case f is a square and in the second case $g \cdot f$ is a square. □

4. THE $\mathcal{Z}(P)$–TMP FOR IRREDUCIBLE $P(x, y) = y^2 - xq(x)$, $q \in \mathbb{R}[x]_{\leq 2}$, $\deg q = 2$

Assume the notation as in §2, §3. Throughout the section P will be as in the title of the section.

Proposition 4.1 ([7, §III.8])**.** *Up to invertible affine linear change of variables* P *has one of the following forms:*

(i) Disconnected Weierstraß form: $P_1(x, y) = y^2 - x(x - a)(x - b)$, $a, b \in \mathbb{R}_{>0}, a < b$.
(ii) Connected Weierstraß form: $P_2(x, y) = y^2 - x(x^2 + c^2)$, $c \in \mathbb{R}$, $c \neq 0$.
(iii) Neile's parabola: $P_3(x, y) = y^2 - x^3$.
(iv) Nodal curve: $P_4(x, y) = y^2 - x(x - 1)^2$.
(v) Cubic with an isolated point: $P_5(x, y) = y^2 - x^2(x - 1)$.

Let P_i be as in Proposition 4.1. The main results of this section are the following:

1. Explicit descriptions of the pair $(f, V^{(k)})$ in Theorem 2.3 for each $C = \mathcal{Z}(P_i)$.
2. Constructive solutions to each singular $\mathcal{Z}(P_i)$–TMP.
3. Constructive solutions to the nonsingular $\mathcal{Z}(P_4)$–TMP and $\mathcal{Z}(P_5)$–TMP.

Remark 4.2. A constructive solution to the $\mathcal{Z}(P_3)$–TMP can be found in [66].

4.1. Smooth Weierstraß form $\mathcal{Z}(P)$–TMP.

Let $C = \mathcal{Z}(P_1)$ or $C = \mathcal{Z}(P_2)$ for P_1, P_2 as in Proposition 4.1. Throughout the whole subsection assume that $V^{(k)}$ is a vector space with a basis $\mathcal{B}_{V^{(k)}}$ as stated in Table 1. Theorem 2.3 has the following concrete form.

Theorem 4.3. *The following statements are equivalent:*

(1) $p \in \mathrm{Pos}_{2k}(C)$.

(2) There exist finitely many $g_i \in \mathbb{R}[C]_{\leq k}$ *and* $h_j \in V^{(k)}$ *such that* $p = \sum_i g_i^2 + x \sum_j h_j^2$.

Proof. The nontrivial implication is (1) ⇒ (2). It suffices to prove the statement for every extreme ray p of $\mathrm{Pos}_{2k}(C)$. The rational function $g = \frac{1}{x}$ clearly satisfies the assumption of . It follows that p is of the form h^2 or $xh^2 = x\big(\frac{h_1}{x}\big)^2$ for some rational functions h, h_1. Let $Q = (0, 0)$ and O the point on C at infinity. Then $\mathrm{div}(x) = 2 \cdot (Q - O)$. Because $p \in \mathcal{L}(6k \cdot O)$, we have that $h \in \mathcal{L}(3k \cdot O)$ in the first case and $h \in \mathcal{L}(3k \cdot O + (Q - O))$ in the second case. We clearly have that $\mathbb{R}[C]_{\leq k} = \mathcal{L}(3k \cdot O)$ and that $\mathcal{L}(3k \cdot O + (Q - O))$ has the desired basis. □

Let $\mathcal{B} := \mathcal{B}_k \cup \mathcal{B}_{V^{(k)}}$, where $\mathcal{B}_k$, $\mathcal{B}_{V^{(k)}}$ are as in Table 1. Let us define a C–degree function $\deg_C$ on $\mathcal{B}$ by $\deg_C(x^i y^j) := 2i + 3j$. We say an expression $p := \sum_{x^i y^j \in \mathcal{B}} a_{ij} x^i y^j$ has C–degree d, if $a_{i_0 j_0} \neq 0$ for $\deg_C(x^{i_0} y^{j_0}) = d$ and $\deg_C(x^i y^j) < d$ for all other $x^i y^j$ with $a_{ij} \neq 0$. We denote the C–degree of p by $\deg_C p$.

Assume that L_C is singular and let $0 \neq p_{\mathrm{gen}}$ be a polynomial with the smallest C–degree among all nonzero polynomials from $\ker \overline{L}_C$. We call p_{gen} the **generating polynomial** of L_C.

Lemma 4.4. *Let* $L_C : \mathbb{R}[C]_{\leq 2k} \to \mathbb{R}$ *be a singular and square positive linear functional. If* $L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ *is a C–moment functional, then* L_C *has a unique square positive extension* $L_C^{(2k+2)} : \mathbb{R}[C]_{\leq 2k+2} \to \mathbb{R}$, *determined by*

$$L_C^{(2k+2)}(x^i y^j p_{\mathrm{gen}}) = 0 \tag{4.1}$$

for $i \in \{0, 1, 2\}$, $j \in \mathbb{Z}_0$ *such that*

$$6k + 1 - \deg_C(p_{\mathrm{gen}}) \leq 2i + 3j \leq 6k + 6 - \deg_C(p_{\mathrm{gen}}).$$

Proof. Since L has a C–rm, it clearly has an extension $L^{(2k+2)} : \mathbb{R}[x, y]_{\leq 2k+2} \to \mathbb{R}$, which is a C–mf. In particular, $L_C^{(2k+2)} : \mathbb{R}[C]_{\leq 2k+2} \to \mathbb{R}$ is square positive. By the Cauchy-Schwartz inequality, we have that

$$\big|L_C^{(2k+2)}(x^i y^j p_{\mathrm{gen}})\big|^2 \leq L_C^{(2k+2)}(x^{2i} y^{2j}) \underbrace{L_C^{(2k+2)}(p_{\mathrm{gen}}^2)}_{0} = 0,$$

for each i, j. This proves the lemma. □

Assume that L_C is $(V^{(k)}, x)$–locally singular and let $p_{\ell\mathrm{gen}}$ be an element with the smallest C–degree among all nonzero elements from $\ker \overline{L}_{C,V^{(k)},x}$. We call $p_{\ell\mathrm{gen}}$ a $(V^{(k)}, x)$–**locally generating element** of L_C.

Lemma 4.5. *Let $L_C : \mathbb{R}[C]_{\leq 2k} \to \mathbb{R}$ be a $(V^{(k)}, x)$–locally singular and $(V^{(k)}, x)$–locally square positive linear functional. If $L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ is a moment functional, then L_C has a unique square positive extension $L_C^{(2k+2)} : \mathbb{R}[C]_{\leq 2k+2} \to \mathbb{R}$, determined by*

$$L^{(2k+2)}_{C,V^{(k)},x}\Big(\frac{y}{x}p_{\ell\mathrm{gen}}\Big) = L^{(2k+2)}_{C,V^{(k)},x}\Big(x^i y^j p_{\ell\mathrm{gen}}\Big) = 0 \tag{4.2}$$

for $i \in \{0, 1, 2\}$, $j \in \mathbb{Z}_0$ such that

$$6k - 1 - \deg_C(p_{\ell\mathrm{gen}}) \leq 2i + 3j \leq 6k + 4 - \deg_C(p_{\ell\mathrm{gen}}).$$

Proof. The proof is analogous to the proof of Lemma 4.4. □

The following theorem solves the singular C–TMP.

Theorem 4.6 (Singular C–TMP)**.** *Let $L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ be a C–singular linear functional. Then the following are equivalent:*

(1) L is a C–moment functional.

(2) L_C is square positive and $(V^{(k)}, x)$–locally square positive, and the extension $L_C^{(2k+2)}$: $\mathbb{R}[C]_{\leq 2k+2} \to \mathbb{R}$, obtained by either (4.1) *if L_C is singular or* (4.2) *if L_C is $(V^{(k)}, x)$–locally singular, is square positive and $(V^{(k+1)}, x)$–locally square positive.*

Proof. By [21, Theorem 1.2], L is a C–mf iff it admits a C–positive extension $L^{(2k+2)}$: $\mathbb{R}[x, y]_{\leq 2k+2} \to \mathbb{R}$, i.e., $L^{(2k+2)}(p) \geq 0$ for every $p \in \mathbb{R}[x, y]_{\leq 2k+2}$ with $p|_C \geq 0$. By Lemmas 4.4 and 4.5, a candidate for $L^{(2k+2)}$ is uniquely determined. By Theorem 4.3 above, the C–positivity of $L^{(2k+2)}$ is equivalent to the square positivity and the $(V^{(k+1)}, x)$–local square positivity of $L_C^{(2k+2)}$. □

Example 4.7. Let $k = 3$ and $\beta_{ij} = L(x^i y^j)$ for $0 \leq i, j \leq 6$ such that $i + j \leq 6$. Then the square positivity and the $(V^{(3)}, x)$–local square positivity of L_C are equivalent to the positive semidefiniteness of the following matrices:

$$\begin{array}{cc}
 & \begin{array}{ccccccccc} \mathit{1} & X & Y & X^2 & XY & Y^2 & X^2Y & XY^2 & Y^3 \end{array} \\
\begin{array}{c} \mathit{1} \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \\ X^2Y \\ XY^2 \\ Y^3 \end{array} &
\left(\begin{array}{ccccccccc}
\beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} \\
\beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} \\
\beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} \\
\beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{23} \\
\beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{14} \\
\beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} & \beta_{23} & \beta_{14} & \beta_{05} \\
\beta_{21} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{23} & \beta_{42} & \beta_{33} & \beta_{24} \\
\beta_{12} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{14} & \beta_{33} & \beta_{24} & \beta_{15} \\
\beta_{03} & \beta_{13} & \beta_{04} & \beta_{23} & \beta_{14} & \beta_{05} & \beta_{24} & \beta_{15} & \beta_{06}
\end{array}\right)
\end{array},$$

and

$$\begin{array}{c}
 \\ \mathit{1} \\ Y/X \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \\ X^2Y \\ XY^2
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc} X & Y & X^2 & XY & X^3 & X^2Y & XY^2 & X^3Y & X^2Y^2 \end{array} \\
\left(\begin{array}{ccccccccc}
\beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} \\
\beta_{01} & L(q(x)) & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} \\
\beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} \\
\beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} \\
\beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{50} & \beta_{41} & \beta_{32} & \beta_{51} & \beta_{42} \\
\beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{23} & \beta_{42} & \beta_{33} \\
\beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{14} & \beta_{33} & \beta_{24} \\
\beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{51} & \beta_{42} & \beta_{33} & \beta_{52} & \beta_{43} \\
\beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{42} & \beta_{33} & \beta_{24} & \beta_{43} & \beta_{34}
\end{array}\right)
\end{array}.$$

Note that $\beta_{52}, \beta_{43}, \beta_{34}$ are equal to

$$\beta_{52} = \begin{cases} \beta_{24} + (a+b)\beta_{42} - ab\beta_{32}, & \text{if } q(x) = (x-a)(x-b), \\ \beta_{24} - c^2\beta_{32}, & \text{if } q(x) = x^2 + c^2, \end{cases}$$

$$\beta_{43} = \begin{cases} \beta_{15} + (a+b)\beta_{33} - ab\beta_{23}, & \text{if } q(x) = (x-a)(x-b), \\ \beta_{15} - c^2\beta_{23}, & \text{if } q(x) = x^2 + c^2, \end{cases}$$

$$\beta_{34} = \begin{cases} \beta_{06} + (a+b)\beta_{24} - ab\beta_{13}, & \text{if } q(x) = (x-a)(x-b), \\ \beta_{06} - c^2\beta_{13}, & \text{if } q(x) = x^2 + c^2. \end{cases}$$

4.2. **TMP on Neile's parabola.** Let P_3 be as in Proposition 4.1 above. Recall that

$$(x(t), y(t)) = (t^2, t^3), \quad t \in \mathbb{R}$$

is a parametrization of $\mathcal{Z}(P_3)$. Let

$$\begin{aligned} \mathrm{Neile} &:= \{s \in \mathbb{R}[t] \colon s'(0) = 0\}, \quad \mathrm{Neile}_{\leq i} = \{s \in \mathrm{Neile} \colon \deg s \leq i\}, \\ \mathrm{Pos}(\mathrm{Neile}_{\leq i}) &:= \{f \in \mathrm{Neile}_{\leq i} \colon f(t) \geq 0 \text{ for every } t \in \mathbb{R}\}, \\ \widetilde{\mathrm{Neile}}_{\leq i} &:= \{s \in \mathbb{R}[t]_{\leq i} \colon s(0) = 0\}. \end{aligned}$$

Theorem 4.8. *The following statements are equivalent:*

(1) $p \in \mathrm{Pos}(\mathrm{Neile}_{\leq 6k})$.

(2) There exist finitely many $f_i \in \mathrm{Neile}_{\leq 3k}, g_j \in \widetilde{\mathrm{Neile}}_{\leq 3k}$ *such that* $p = \sum_i f_i^2 + \sum_j g_j^2$.

Moreover, for $C = \mathcal{Z}(P_3)$ *the appropriate choices of* f *and* $V^{(k)}$ *in Theorem 2.3 are as stated in Table 2.*

Proof. The nontrivial implication is (1) ⇒ (2). Let $\Phi : \mathbb{R}[\mathcal{Z}(P_3)] \to \mathrm{Neile}$ be a map defined by $\Phi(p(x,y)) := p(t^2, t^3)$. Clearly Φ is a well–defined ring homomorphism, because $p(t^2, t^3) = 0$ for every $p \in I$ and $(\Phi(p))'(0) = 0$ for every $p \in \mathbb{R}[\mathcal{Z}(P_3)]$. From $\Phi(\mathbb{R}[\mathcal{Z}(P_3)]_{\leq i}) \subseteq \mathrm{Neile}_{\leq 3i}$ and $\dim \Phi(\mathbb{R}[\mathcal{Z}(P_3)]_{\leq i}) = \dim \mathrm{Neile}_{\leq 3i} = 3i$, it follows that Φ is a ring isomorphism. Using Corollary 3.4, every extreme ray p of the cone $\mathrm{Pos}(\mathrm{Neile}_{\leq 6k})$ is of the form u^2 for some $u \in \mathbb{R}[t]_{\leq 3k}$ satisfying $0 = (u^2)'(0) = 2u(0)u'(0)$. If $u'(0) = 0$, then $u \in \mathrm{Neile}_{\leq 3k}$. Else $u(0) = 0$ and $u \in \widetilde{\mathrm{Neile}}_{\leq 3k}$.

It remains to prove the moreover part. Let $\mathcal{B}_{\widetilde{\mathrm{Neile}}_{\leq i}} := \{t, t^2, \ldots, t^i\}$ be the basis for $\widetilde{\mathrm{Neile}}_{\leq i}$. Extending the ring isomorphism Φ to the isomorphism between quotient fields

$\mathbb{R}(\mathcal{Z}(P_3))$ and Quot(Neile) of $\mathbb{R}[\mathcal{Z}(P_3)]$ and Neile, respectively, note that $\Phi^{-1}(\mathcal{B}_{\widetilde{\mathrm{Neile}}_{\leq 3k}})$ is equal to $\mathcal{B}_{V^{(k)}}$ from Table 2. $\square$

Example 4.9. Let $k = 3$ and $\beta_{ij} = L(x^i y^j)$ for $i, j \geq 0$, $i + j \leq 6$. Then the square positivity and the $(V^{(3)}, 1)$–local square positivity of $L_{\mathcal{Z}(P_3)}$ are equivalent to the partial positive semidefiniteness of the following Hankel matrix (i.e., all fully determined principal submatrices are psd):

$$\begin{array}{c|cccccc} & \mathit{1} & T & T^2 & T^3 & \cdots & T^9 \\ \hline \mathit{1} & \gamma_0 & ? & \gamma_2 & \gamma_3 & \cdots & \gamma_9 \\ T & ? & \gamma_2 & \gamma_3 & \gamma_4 & & \gamma_{10} \\ T^2 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & & \gamma_{11} \\ T^3 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & & \gamma_{12} \\ \vdots & \vdots & & & & \ddots & \vdots \\ T^9 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & \cdots & \gamma_{18} \end{array},$$

where $\gamma_i := \beta_{2i \bmod 3, \lfloor \frac{i}{3} \rfloor}$ for each i. Note that the missing entries are at the positions $(1, 2)$ and $(2, 1)$, since the value $L_{\mathcal{Z}(P_3)}(\frac{y}{x})$ is unknown. The matrix representation of $\overline{L}_{\mathcal{Z}(P_3)}$ (resp., $\overline{L}_{\mathcal{Z}(P_3), V^{(k)}, 1}$) is the restriction of this matrix to a submatrix an all rows and columns but the one indexed with T (resp., but the one indexed with *1*).

Remark 4.10. A constructive approach to solve the $\mathcal{Z}(P_3)$–TMP (nonsingular and singular) by solving the equivalent $\mathbb{R}$–TMP from Example 4.9 is presented in [66, Section 4].

4.3. **TMP on the nodal curve.**

Let P_4 be as in Proposition 4.1 above. Recall that

$$(x(t), y(t)) = (t^2, t^3 - t), \quad t \in \mathbb{R},$$

is a parametrization of $\mathcal{Z}(P_4)$. Let

$$\begin{aligned} \mathrm{Nodal} &:= \{s \in \mathbb{R}[t] \colon s(1) = s(-1)\}, \quad \mathrm{Nodal}_{\leq i} := \{s \in \mathrm{Nodal} \colon \deg s \leq i\}, \\ \mathrm{Pos}(\mathrm{Nodal}_{\leq i}) &:= \{f \in \mathrm{Nodal}_{\leq i} \colon f(t) \geq 0 \text{ for every } t \in \mathbb{R}\}, \\ \widetilde{\mathrm{Nodal}}_{\leq i} &:= \{s \in \mathbb{R}[t]_{\leq i} \colon s(1) = -s(-1)\}. \end{aligned}$$

Theorem 4.11. *The following statements are equivalent:*

(1) $p \in \mathrm{Pos}(\mathrm{Nodal}_{\leq 6k})$.

(2) Then there exist finitely many $f_i \in \mathrm{Nodal}_{\leq 3k}$ *and* $g_j \in \widetilde{\mathrm{Nodal}}_{\leq 3k}$ *such that* $p = \sum_i f_i^2 + \sum_j g_j^2$.

Moreover, for $C = \mathcal{Z}(P_4)$ *the appropriate choices of* f *and* $V^{(k)}$ *in Theorem 2.3 are as stated in Table 2.*

Proof. The nontrivial implication is (1) ⇒ (2). Let $\Phi : \mathbb{R}[\mathcal{Z}(P_4)] \to \mathrm{Nodal}$ be a map defined by $\Phi(p(x, y)) = p(t^2, t^3 - t)$. Analogously as in the proof of Theorem 4.8 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_4)]_{\leq i}$ is in one-to-one correspondence with the set $\mathrm{Nodal}_{\leq 3i}$ under Φ. Using Corollary 3.4, every extreme ray p of the cone $\mathrm{Pos}(\mathrm{Nodal}_{\leq 6k})$ is of the form u^2 for some $u \in \mathbb{R}[t]_{\leq 3t}$ satisfying $(u^2)(1) =$

$(u^2)(-1)$. It follows that $u(1) = u(-1)$ or $u(1) = -u(-1)$ and so $u \in \mathrm{Nodal}_{\leq 3k}$ or $u \in \widetilde{\mathrm{Nodal}}_{\leq 3k}$.

It remains to prove the moreover part. Let $\mathcal{B}_{\widetilde{\mathrm{Nodal}}_{\leq i}} := \cup_{j=2}^{i}\{t^j - t^{j-2}\} \cup \{1\}$ be the basis for $\widetilde{\mathrm{Nodal}}_{\leq i}$. Extending the ring isomorphism Φ to the isomorphism between quotient fields $\mathbb{R}(\mathcal{Z}(P_4))$ and $\mathrm{Quot}(\mathrm{Nodal})$ of $\mathbb{R}[\mathcal{Z}(P_4)]$ and Nodal, respectively, note that $\Phi^{-1}(\mathcal{B}_{\widetilde{\mathrm{Nodal}}_{\leq 3k}})$ is equal to $\mathcal{B}_{V^{(k)}}$ from Table 2. ☐

Below we present a constructive solution to the nonsingular $\mathcal{Z}(P_4)$–TMP via the solution to the corresponding $\mathbb{R}$–TMP.

Constructive proof of Corollary 2.6 for $C = \mathcal{Z}(P_4)$. Using the correspondence as in the proof of Theorem 4.11, the $\mathcal{Z}(P_4)$–TMP for L is equivalent to the $\mathbb{R}$–TMP for $L_{\mathrm{Nodal}_{\leq 6k}} : \mathrm{Nodal}_{\leq 6k} \to \mathbb{R}$, $L_{\mathrm{Nodal}_{\leq 6k}}(p) := L_{\mathcal{Z}(P_4)}(\Phi^{-1}(p))$. If $L_{\mathrm{Nodal}_{\leq 6k}}$ is a $\mathbb{R}$–mf, then it extends to the $\mathbb{R}$–mf $\widehat{L} \equiv L_{\mathbb{R}[t]_{\leq 6k}} : \mathbb{R}[t]_{\leq 6k} \to \mathbb{R}$. In the ordered basis $\{\mathit{1}, T, T^2-1, T^3-T, \ldots, T^{3k} - T^{3k-2}\}$ of rows and columns, the strict square positivity and the $(V^{(k)}, 1)$–local strict square positivity of $L_{\mathcal{Z}(P_4)}$ are equivalent to the partial positive definiteness of the matrix

$$\begin{array}{cccccc}
1 & T & T^2-1 & T^3-T & \cdots & T^{3k}-T^{3k-2} \\
\end{array}$$
$$\begin{pmatrix}
\widehat{L}(1) & ? & \widehat{L}(t^2-1) & \widehat{L}(t^3-t) & \cdots & \widehat{L}(t^{3k}-t^{3k-2}) \\
? & \widehat{L}(t^2) & \widehat{L}(t^3-t) & \widehat{L}(t^4-t^2) & & \widehat{L}(t^{3k+1}-t^{3k-1}) \\
\widehat{L}(t^2-1) & \widehat{L}(t^3-t) & \widehat{L}((t^2-1)^2) & \widehat{L}(t(t^2-1)^2) & & \widehat{L}(t^{3k-2}(t^2-1)^2) \\
\widehat{L}(t^3-t) & \widehat{L}(t(t^3-t)) & \widehat{L}(t(t^2-1)^2) & \widehat{L}((t^3-t)^2) & \cdots & \widehat{L}(t^{3k-1}(t^2-1)^2) \\
\vdots & & & & \ddots & \vdots \\
\widehat{L}(t^{3k}-t^{3k-2}) & \cdots & & & \cdots & \widehat{L}((t^{3k}-t^{3k-2})^2)
\end{pmatrix}.$$

The missing entries are at the positions $(1, 2)$ and $(2, 1)$, since the value $\widehat{L}(t)$ is unknown. Then there exists an interval $(a, b) \subset \mathbb{R}$, such that for every $\widehat{L}(t) \in (a, b)$, the completion is positive definite (see e.g., [65, Lemma 2.4]) and $\widehat{L} : \mathbb{R}[t]_{\leq 6k} \to \mathbb{R}$ admits a $(3k+1)$–atomic $\mathbb{R}$–rm by [15, Theorem 3.9]. ☐

Remark 4.12. It is not clear if in the case that $L_{\mathrm{Nodal}_{\leq 6k}}$ is a $\mathbb{R}$–mf, there exists a $(3k)$–atomic $\mathbb{R}$–rm. This is equivalent to the fact that for one of the choices $\widehat{L}(t) = a$ or $\widehat{L}(t) = b$ in the proof above, $\widehat{L}$ is a $\mathbb{R}$–mf. Since the rank of the completed moment matrix is $3k$ for $\widehat{L}(t) = a$ or $\widehat{L}(t) = b$, a $(3k)$–atomic $\mathbb{R}$–rm would exist by [15, Theorem 3.9].

The following theorem solves the singular $\mathcal{Z}(P_4)$–TMP.

Theorem 4.13 (Singular $\mathcal{Z}(P_4)$–TMP)**.** *Let $L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ be a $\mathcal{Z}(P_4)$–singular linear functional. Let $V^{(k)}$ be as in Table 2. Then the following are equivalent:*

(1) L is a $\mathcal{Z}(P_4)$–moment functional.

(2) $L_{\mathcal{Z}(P_4)}$ is square positive, $(V^{(k)}, 1)$–locally square positive and one of the following holds:

(a) For $U := \mathrm{Span}(\Phi^{-1}(\mathcal{B}_{\mathrm{Nodal}_{\leq 3k-1}}))$ it holds that

$$\operatorname{rank} \overline{L}_{\mathcal{Z}(P_4)} = \operatorname{rank}\big((\overline{L}_{\mathcal{Z}(P_4)})|_U\big).$$

(b) For $W := \mathrm{Span}(\Phi^{-1}(\mathcal{B}_{\widetilde{\mathrm{Nodal}}_{\leq 3k-1}}))$ it holds that

$$\operatorname{rank} \overline{L}_{\mathcal{Z}(P_4), V^{(k)}, 1} = \operatorname{rank}\big((\overline{L}_{\mathcal{Z}(P_4), V^{(k)}, 1})|_W\big).$$

Proof. Following the constructive proof of Corollary 2.6 for $\mathcal{Z}(P_4)$ via the solution to the $\mathbb{R}$–TMP above and replacing positive definiteness with positive semidefiniteness, the only addition is that a psd extension $L_{\mathbb{R}[t]_{\leq 6t}} : \mathbb{R}[t]_{\leq 6t} \to \mathbb{R}$ is not necessarily a $\mathbb{R}$–mf. Since L is $\mathcal{Z}(P_4)$–singular and using [15, Theorem 3.9], (1) is equivalent to:

$$\begin{aligned}&\text{There is a square positive extension } L_{\mathbb{R}[t]_{\leq 6t}} \text{ of } L_{\mathrm{Nodal}_{\leq 6k}} \text{ such that}\\ &\operatorname{rank} \overline{L}_{\mathbb{R}[t]_{\leq 6k}} = \operatorname{rank} \overline{L}_{\mathbb{R}[t]_{\leq 6k-2}} \text{ holds.}\end{aligned} \tag{4.3}$$

Let us denote by M the matrix, which represents $\overline{L}_{\mathbb{R}[t]_{\leq 6k}}$ in the basis $\mathcal{B}_{\mathrm{Nodal}_{\leq 3k}} \cup \{t\}$ (see the proof of Corollary 2.6 above). The rank condition in (4.3) means that there are some $\alpha_i \in \mathbb{R}$, not all equal to 0, such that

$$T^{3k} - T^{3k-2} = \alpha_0 \mathit{1} + \alpha_1 T + \sum_{i=2}^{3k-1} \alpha_i (T^i - T^{i-2}) \tag{4.4}$$

Let us prove the implication (4.3) $\Rightarrow$ (2). The square positivity and the $(V^{(k)}, 1)$–local square positivity of $L_{\mathcal{Z}(P_4)}$ are clear. If $\alpha_1 = 0$ in (4.4), then (2a) holds, while $\alpha_0 = 0$ implies (2b). It remains to study the case: $\alpha_0 \neq 0$ and $\alpha_1 \neq 0$. We separate two cases according to the $\mathcal{Z}(P_4)$–singularity of L:

Case 1: $L_{\mathcal{Z}(P_4)}$ is singular. There are some $\beta_i \in \mathbb{R}$ such that in the matrix representation of $\overline{L}_{\mathrm{Nodal}_{\leq 6k}}$ with respect to the basis $\mathcal{B}_{\mathrm{Nodal}_{\leq 3k}}$, the relation

$$T^{3j} - T^{3j-2} = \beta_0 \mathit{1} + \sum_{i=2}^{3j-1} \beta_i (T^i - T^{i-2}) \tag{4.5}$$

holds. If $j = k$, then (2a) holds. Otherwise $j < k$. By the extension principle [31, Proposition 2.4], (4.5) also holds in the matrix M. We separate two subcases according to the value of β_0.

Case 1.1: $\beta_0 = 0$. Since $L_{\mathbb{R}[t]_{\leq 6t}}$ is a $\mathbb{R}$–mf, multiplying the relation (4.5) with $T^{3(k-j)}$ gives a column relation of M:

$$T^{3k} - T^{3k-2} = \sum_{i=2}^{3j-1} \beta_i \big(T^{i+3(k-j)} - T^{i-2+(3k-j)}\big). \tag{4.6}$$

Since the matrix, representing $\overline{L}_{C, V^{(k)}, 1}$ in the basis $\mathcal{B}_{\widetilde{\mathrm{Nodal}}_{\leq 3k}}$, is a submatrix of M, it follows that (2b) holds.

Case 1.2: $\beta_0 \neq 0$. In this case we can express *1* out of (4.5) and plug it into (4.4), whence we end up with a relation of the form (4.4) with $\alpha_0 = 0$ and hence (2b) holds.

Case 2: $L_{\mathcal{Z}(P_4)}$ is not singular, but $L_{\mathcal{Z}(P_4)}$ is $(V^{(k)}, 1)$–locally singular. The proof is analogous to the proof of Case 1, only that one starts with the relation

$$T^{3j} - T^{3j-2} = \beta_1 T + \sum_{i=2}^{j} \beta_i (T^i - T^{i-2}) \tag{4.7}$$

in the matrix, representing $\overline{L}_{\mathcal{Z}(P_4),V^{(k)},1}$ in the basis $\mathcal{B}_{\widetilde{\text{Nodal}}_{\leq 3k}}$ Then three cases need to be considered: (i) $j = k$; (ii) $j < k$, $\beta_1 = 0$; and (iii) $j < k, \beta_1 \neq 0$.

It remains to prove the implication (2) $\Rightarrow$ (4.3). The existence of a square positive extension is clear from the positivity assumptions on $L_{\mathcal{Z}(P_4)}$ (see the first paragraph of the proof). The rank condition in (4.3) follows from either (4.5) used for $j = k$ under the assumption (2a), or (4.7) used for $j = k$ under the assumption (2b). $\square$

4.4. **TMP on the cubic curve with an isolated point.** Let P_5 be as in Proposition 4.1 above. Note that

$$(x(t), y(t)) = (t^2 + 1, t^3 + t), \quad t \in \mathbb{R},$$

is a parametrization of $\mathcal{Z}(P_5) \setminus \{(0,0)\}$. Let $\mathfrak{i}$ stand for the imaginary unit and

$$\begin{aligned} \text{Isol} &:= \{s \in \mathbb{R}[t] \colon s(\mathfrak{i}) = s(-\mathfrak{i})\}, \quad \text{Isol}_{\leq i} := \{s \in \text{Isol} \colon \deg s \leq i\}, \\ \text{Pos}(\text{Isol}_{\leq i}) &:= \{f \in \text{Isol}_{\leq i} \colon f(t) \geq 0 \text{ for every } t \in \mathbb{R}\}, \\ \widetilde{\text{Isol}}_{\leq i} &:= \{s \in \mathbb{R}[t]_{\leq i} \colon s(\mathfrak{i}) = -s(-\mathfrak{i})\}. \end{aligned}$$

Theorem 4.14. *The following statements are equivalent:*

(1) $p \in \text{Pos}(\text{Isol}_{\leq 6k})$.

(2) There exist finitely many $f_i \in \text{Isol}_{\leq 3k}$ *and* $g_j \in \widetilde{\text{Isol}}_{\leq 3k}$ *such that* $p = \sum_i f_i^2 + \sum_j g_j^2$.

Moreover, for $C = \mathcal{Z}(P_5)$ *the appropriate choices of* f *and* $V^{(k)}$ *in Theorem 2.3 are as stated in Table 2.*

Proof. The nontrivial implication is (1) $\Rightarrow$ (2). Let $\Phi : \mathbb{R}[\mathcal{Z}(P_5)] \to \text{Isol}$ be a map defined by $\Phi(p(x, y)) = p(t^2 + 1, t^3 + t)$. Analogously as in the proof of Theorem 4.8 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_5)]_{\leq i}$ is in one-to-one correspondence with the set $\text{Isol}_{\leq 3i}$ under Φ. Using Corollary 3.4, every extreme ray p of the cone $\text{Pos}(\text{Isol}_{\leq 6k})$ is of the form u^2 for some $u \in \mathbb{R}[t]_{\leq 3t}$ satisfying $(u^2)(\mathfrak{i}) = (u^2)(-\mathfrak{i})$. It follows that $u(\mathfrak{i}) = u(-\mathfrak{i})$ or $u(\mathfrak{i}) = -u(-\mathfrak{i})$ and so $u \in \text{Isol}_{\leq 3k}$ or $u \in \widetilde{\text{Isol}}_{\leq 3k}$.

It remains to prove the moreover part. Let $\mathcal{B}_{\widetilde{\text{Isol}}_{\leq i}} := \cup_{j=2}^{i}\{t^j + t^{j-2}\} \cup \{1\}$. be the base for $\widetilde{\text{Isol}}_{\leq i}$. Extending the ring isomorphism Φ to the isomorphism between quotient fields $\mathbb{R}(\mathcal{Z}(P_5))$ and Quot(Isol) of $\mathbb{R}[\mathcal{Z}(P_5)]$ and Isol, respectively, note that $\Phi^{-1}(\mathcal{B}_{\widetilde{\text{Isol}}_{\leq 3k}})$ is equal to $\mathcal{B}_{V^{(k)}}$ from Table 2. $\square$

Remark 4.15. Note that in our definition of nonsingularity, $\mathcal{Z}(P_5)$–nonsingular L admits a $\mathcal{Z}(P_5)$–rm if and only if L admits a $(\mathcal{Z}(P_5) \setminus \{0\})$–rm. Namely, there always exists a $\mathcal{Z}(P_5)$–rm without $(0, 0)$ in the support.

Below we present a constructive solution to the nonsingular $\mathcal{Z}(P_5)$–TMP via the solution to the corresponding $\mathbb{R}$–TMP.

Constructive proof of Corollary 2.6 for $C = \mathcal{Z}(P_5)$. The proof is verbatim the same to the constructive proof of Corollary 2.6 for $C = \mathcal{Z}(P_4)$ above, only that the corresponding univariate moment matrix is

$$\begin{array}{cccccc}
1 & T & T^2+1 & T^3+T & \cdots & T^{3k}+T^{3k-2} \\
\hline
\widehat{L}(1) & ? & \widehat{L}(t^2+1) & \widehat{L}(t^3+t) & & \widehat{L}(t^{3k}+t^{3k-2}) \\
? & \widehat{L}(t^2) & \widehat{L}(t^3+t) & \widehat{L}(t^4+t^2) & & \widehat{L}(t^{3k+1}+t^{3k-1}) \\
\widehat{L}(t^2+1) & \widehat{L}(t^3+t) & \widehat{L}((t^2+1)^2) & \widehat{L}(t(t^2+1)^2) & & \widehat{L}(t^{3k-2}(t^2+1)^2) \\
\widehat{L}(t^3+t) & \widehat{L}(t(t^3+t)) & \widehat{L}(t(t^2+1)^2) & \widehat{L}((t^3+t)^2) & \cdots & \widehat{L}(t^{3k-1}(t^2+1)^2) \\
\vdots & & & & \ddots & \vdots \\
\widehat{L}(t^{3k}+t^{3k-2}) & \cdots & & & \cdots & \widehat{L}((t^{3k}+t^{3k-2})^2)
\end{array}.$$

All the other arguments remain the same. $\square$

Remark 4.16. Similarly as in Remark 4.12 it is not clear whether a $(3k)$–atomic $\mathcal{Z}(P_5)$–rm for L exists.

Let us denote by $O := (0,0)$ the isolated point of $\mathcal{Z}(P_5)$. The following theorem solves the singular $\mathcal{Z}(P_5)$–TMP where O is not in the support of the measure.

Theorem 4.17 (Singular $\mathcal{Z}(P_5)$–TMP avoiding O)**.** *Let $L : \mathbb{R}[x,y]_{\leq 2k} \to \mathbb{R}$ be a $\mathcal{Z}(P_5)$–singular linear functional. Let $V^{(k)}$ be as in Table 2. Then the following are equivalent:*

(1) L is a $(\mathcal{Z}(P_5) \setminus O)$–moment functional.

(2) $L_{\mathcal{Z}(P_5)}$ is square positive and $(V^{(k)}, 1)$–locally square positive and one of the following holds:

(a) For $U := \mathrm{Span}(\Phi^{-1}(\mathcal{B}_{\mathrm{Isol}_{\leq 3k-1}}))$ it holds that

$$\mathrm{rank}\, \overline{L}_{\mathcal{Z}(P_5)} = \mathrm{rank}\, \big((\overline{L}_{\mathcal{Z}(P_5)})|_U\big).$$

(b) For $W := \mathrm{Span}(\Phi^{-1}(\mathcal{B}_{\widetilde{\mathrm{Isol}}_{\leq 3k-1}}))$ it holds that

$$\mathrm{rank}\, \overline{L}_{\mathcal{Z}(P_5),V^{(k)},1} = \mathrm{rank}\, \big((\overline{L}_{\mathcal{Z}(P_5),V^{(k)},1})|_W\big).$$

Proof. The proof is verbatim the same to the proof of Theorem 4.13 above after replacing $\mathrm{Nodal}_{\leq i}$, $\widetilde{\mathrm{Nodal}}_{\leq i}$, $T^i - T^{i-2}$ with $\mathrm{Isol}_{\leq i}$, $\widetilde{\mathrm{Isol}}_{\leq i}$ and $T^i + T^{i-2}$, respectively. $\square$

It remains to characterize the cases when L is a $\mathcal{Z}(P_5)$–mf but not a $(\mathcal{Z}(P_5) \setminus O)$–mf. Namely, O necessarily lies in the support of any rm. Let

$$L_{\mathrm{ev}_O} : \mathbb{R}[x,y]_{\leq 2k} \to \mathbb{R}, \quad L_{\mathrm{ev}_O}(p) := p(0,0) \ \text{ for } p \in \mathbb{R}[x,y]_{\leq 2k}$$

be a functional, which evaluates a polynomial in the point O. The functional L_{ev_O} is clearly a mf having a Dirac measure δ_O as its rm. If L is a $\mathcal{Z}(P_5)$–mf, which is not a $(\mathcal{Z}(P_5) \setminus O)$–mf, then there exists $\lambda > 0$ such that

$$L^{(\lambda)} := L - \lambda L_{\mathrm{ev}_O} \tag{4.8}$$

is a $(\mathcal{Z}(P_5) \setminus O)$–mf. Let us denote by

$$\widehat{L} := L_{\mathbb{R}[t]_{\leq 6t}} : \mathbb{R}[t]_{\leq 6t} \to \mathbb{R} \qquad \text{and} \qquad \widehat{L}^{(\lambda)} := L^{(\lambda)}_{\mathbb{R}[t]_{\leq 6t}} : \mathbb{R}[t]_{\leq 6t} \to \mathbb{R} \tag{4.9}$$

the univariate linear functionals, corresponding to $L_{\mathcal{Z}(P_5)}$ and $L_{\mathcal{Z}(P_5)} - \lambda L_{\mathrm{ev}_O}$, respectively. In the ordered basis $\{\mathit{1}, T, T^2+1, T^3+T, \ldots, T^{3k}+T^{3k-2}\}$ of rows and columns, the matrix

M representing the bilinear form $\overline{L}^{(\lambda)}_{\mathbb{R}[t]_{\leq 6t}}$ is the following:

$$\begin{array}{c}
\begin{array}{cccccc}
\mathit{1} & T & T^2+1 & T^3+T & \cdots & T^{3k}+T^{3k-2}
\end{array} \\
\left(\begin{array}{cccccc}
\widehat{L}(1)-\lambda & ? & \widehat{L}(t^2+1) & \widehat{L}(t^3+t) & \cdots & \widehat{L}(t^{3k}+t^{3k-2}) \\
? & \widehat{L}(t^2)+\lambda & \widehat{L}(t^3+t) & \widehat{L}(t^4+t^2) & & \widehat{L}(t^{3k+1}+t^{3k-1}) \\
\widehat{L}(t^2+1) & \widehat{L}(t^3+t) & \widehat{L}((t^2+1)^2) & \widehat{L}(t(t^2+1)^2) & & \widehat{L}(t^{3k-2}(t^2+1)^2) \\
\widehat{L}(t^3+t) & \widehat{L}(t(t^3+t)) & \widehat{L}(t(t^2+1)^2) & \widehat{L}((t^3+t)^2) & \cdots & \widehat{L}(t^{3k-1}(t^2+1)^2) \\
\vdots & & & & \ddots & \vdots \\
\widehat{L}(t^{3k}+t^{3k-2}) & \cdots & & & \cdots & \widehat{L}((t^{3k}+t^{3k-2})^2)
\end{array}\right)
\end{array}.$$

We write

$$\widehat{L}_1^{(\lambda)} := \begin{pmatrix} \widehat{L}(1)-\lambda & a \\ a & A \end{pmatrix} \quad \text{and} \quad \widehat{L}_2^{(\lambda)} := \begin{pmatrix} \widehat{L}(t^2)+\lambda & b \\ b & B \end{pmatrix} \tag{4.10}$$

for the restrictions of M to all rows and columns except T in case of $\widehat{L}_1^{(\lambda)}$, and except *1* in case of $\widehat{L}_2^{(\lambda)}$. Namely, the rows and columns of $\widehat{L}_1^{(\lambda)}$ are all elements of $\mathcal{B}_{\mathrm{Isol}_{\leq 3k}}$, while the rows and columns of $\widehat{L}_2^{(\lambda)}$ are all elements of $\mathcal{B}_{\widetilde{\mathrm{Isol}}_{\leq 3k}}$.

Let

$$\mathcal{M} = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)},$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $D \in \mathbb{R}^{m\times m}$. The **generalized Schur complement** $\mathcal{M}/D$ [70] of D in $\mathcal{M}$ is defined by

$$\mathcal{M}/D = A - BD^{\dagger}B^T,$$

where $D^{\dagger}$ stands for the Moore–Penrose inverse of D.

For a matrix $X \in \mathbb{R}^{n\times m}$ we call the linear span of its columns a **column space** and denote it by $\mathcal{C}(X)$.

A complete solution to the nonsingular $\mathcal{Z}(P_5)$–TMP where also atom O is allowed, is the following.

Theorem 4.18 (Nonsingular $\mathcal{Z}(P_5)$–TMP)**.** *Let $L : \mathbb{R}[x,y]_{\leq 2k} \to \mathbb{R}$ be a $\mathcal{Z}(P_5)$–nonsingular linear functional. Assume the notation from* (4.8), (4.9) *and* (4.10)*. Then the following are equivalent:*

1. *L is a $\mathcal{Z}(P_5)$–moment functional.*
2. *$L_{\mathcal{Z}(P_5)}$ is strictly square positive, B is positive semidefinite, $b \in \mathcal{C}(B)$ and one of the following holds:*
   - *(a) $\widehat{L}_2^{(0)}/B > 0$.*
   - *(b) $\widehat{L}_1^{(0)}/A > -(\widehat{L}_2^{(0)}/B) \geq 0$.*
   - *(c) $L^{(\lambda_0)}$ is a $(\mathcal{Z}(P_5)\setminus O)$–moment functional for $\lambda_0 = \widehat{L}_1^{(0)}/A$.*

Proof. First we prove the implication (1) $\Rightarrow$ (2). As explained above there exists $\widetilde{\lambda} \geq 0$ such that $L^{(\widetilde{\lambda})}$ is a $(\mathcal{Z}(P_5)\setminus O)$–mf. In particular, $\widehat{L}_1^{(\widetilde{\lambda})}$ and $\widehat{L}_2^{(\widetilde{\lambda})}$ must satisfy conditions coming from Corollary 2.6 for $C = \mathcal{Z}(P_5)$ or Theorem 4.17. In particular, $\widehat{L}_1^{(\widetilde{\lambda})}$ and $\widehat{L}_2^{(\widetilde{\lambda})}$ are psd and hence by [3], B is psd, $b \in \mathcal{C}(B)$ and

$$0 \leq \widehat{L}_1^{(\widetilde{\lambda})}/A = \widehat{L}_1^{(0)}/A - \tilde{\lambda} \quad \text{and} \quad 0 \leq \widehat{L}_2^{(\widetilde{\lambda})}/B = \widehat{L}_2^{(0)}/B + \tilde{\lambda}.$$

Hence,

$$-(\widehat{L}_2^{(0)}/B) \leq \widetilde{\lambda} \leq \widehat{L}_1^{(0)}/A. \tag{4.11}$$

Note that $\widehat{L}_1^{(\widetilde{\lambda})} \preceq \widehat{L}_1^{(0)}$ and hence $\overline{L}_{\mathcal{Z}(P_5)}$ is psd. Since by assumption $\ker \overline{L}_{\mathcal{Z}(P_5)} = \{0\}$, it follows that $\overline{L}_{\mathcal{Z}(P_5)}$ is positive definite. If none of (2a), (2b) above holds, then there must be equalities in (4.11) and $L^{(\widetilde{\lambda})}$ is a $(\mathcal{Z}(P_5) \setminus O)$–mf. This is precisely (2c).

It remains to prove the implication (2) ⇒ (1). If (2a) holds, the L is a $(\mathcal{Z}(P_5) \setminus O)$–mf by Corollary 2.6. If (2b) holds, then for $\widetilde{\lambda} := \frac{1}{2}(\widehat{L}_1^{(0)}/A + \widehat{L}_2^{(0)}/B)$ both $\widehat{L}_1^{(\widetilde{\lambda})}/A$, $\widehat{L}_2^{(\widetilde{\lambda})}/B$ are positive definite and hence they satisfy Corollary 2.6 for $C = \mathcal{Z}(P_5)$, whence L is a $(\mathcal{Z}(P_5) \setminus O)$–mf. Assume now that (2c) holds. Then L is clearly a $\mathcal{Z}(P_5)$–mf. □

The following theorem solves the singular $\mathcal{Z}(P_5)$–TMP, where also atom O is allowed.

Theorem 4.19 (Singular $\mathcal{Z}(P_5)$–TMP)**.** *Let $L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ be a $\mathcal{Z}(P_5)$–singular linear functional. Let $V^{(k)}$ be as in Table 2. Assume the notation in* (4.8)*,* (4.9) *and* (4.10)*. Then the following are equivalent:*

1. *L is a $\mathcal{Z}(P_5)$–moment functional.*
2. *One of the following holds:*
   - *(a) $\overline{L}_{\mathcal{Z}(P_5)}$ is singular and for $\lambda_0 := \widehat{L}_1^{(0)}/A$ we have that $\lambda_0 \geq 0$ and $L^{(\lambda_0)}$ is a $(\mathcal{Z}(P_5) \setminus O)$–moment functional.*
   - *(b) $\overline{L}_{\mathcal{Z}(P_5)}$ is positive definite.*

Proof. First we prove the implication (1) ⇒ (2). As explained in the paragraph after the proof of Theorem 4.17 above, there exists $\widetilde{\lambda} \geq 0$ such that $L^{(\widetilde{\lambda})}$ is a $(\mathcal{Z}(P_5) \setminus O)$–mf. Since L is $\mathcal{Z}(P_5)$–singular, it follows that at least one of $\widehat{L}_1^{(0)}$, $\widehat{L}_2^{(0)}$ has a nontrivial kernel. Note that $\widehat{L}_1^{(0)}$ has a nontrivial kernel if and only if $\overline{L}_{\mathcal{Z}(P_5)}$ is singular. If $\overline{L}_{\mathcal{Z}(P_5)}$ is nonsingular, then it is positive definite and we are in the case (2b). Otherwise $\widehat{L}_1^{(0)}$ is singular. If $\lambda_0 := \widehat{L}_1^{(0)}/A = 0$, then $\widehat{L}_1^{(\lambda)}$ has a negative eigenvalue for every $\lambda > 0$. Hence, the only option for $\tilde{\lambda}$ is 0 and (2a) holds. If $\widehat{L}_1^{(0)}/A > 0$, then there is a nontrivial relation in $\widehat{L}_1^{(\widetilde{\lambda})}$ of the form

$$T^{3j} + T^{3j-2} = \sum_{i=2}^{3j-1} \beta_i (T^i + T^{i-2}) \tag{4.12}$$

Multiplying (4.12) with $T^{3(k-j)}$,

$$T^{3k} + T^{3k-2} = \sum_{i=2}^{3j-1} \beta_i (T^{i+3(k-j)} + T^{i-2+(3k-j)}) \tag{4.13}$$

is also a relation in $\widehat{L}_1^{(\widetilde{\lambda})}$. But this is then a relation in $\widehat{L}_1^{(\lambda)}$ for every λ. Since $\tilde{\lambda} \leq \lambda_0$, it follows that $L^{(\lambda_0)}$ is also a $(\mathcal{Z}(P_5) \setminus O)$–mf by Theorem 4.17. Hence, (2a) holds.

It remains to prove the implication (2) ⇒ (1). If (2a) holds, the L is clearly a $\mathcal{Z}(P_5)$–mf. If (2b) holds, then $\widehat{L}_1^{(0)}$ is invertible and $\widehat{L}_2^{(0)}$ is singular. Since $\widehat{L}_1^{(0)}$ is positive definite, B

is also positive definite. Hence, $\widehat{L}_2^{(0)}/B = 0$. It follows that for every $\lambda > 0$, $\widehat{L}_2^{(\lambda)}$ is positive definite. For $\lambda_0 > 0$ small enough both $\widehat{L}_1^{(\lambda_0)}$ and $\widehat{L}_2^{(\lambda_0)}$ are positive definite and $L^{(\lambda_0)}$ is $\mathcal{Z}(P_5)$–nonsingular. By Corollary 2.6 used for $C = \mathcal{Z}(P_5)$, $L^{(\lambda_0)}$ is a $(\mathcal{Z}(P_5) \setminus O)$–mf, whence L is $\mathcal{Z}(P_5)$–mf and (1) holds. □

5. THE $\mathcal{Z}(P)$–TMP FOR IRREDUCIBLE $P(x,y) = xy^2 + ay - bx^3 - cx^2 - dx - e$

Assume the notation as in §2, §3. Let P be as in the title of the section.

Proposition 5.1. *Up to invertible affine linear change of variables P has one of the following forms:*

(i) Rational cubic type 1: $P_6(x,y) = xy^2 + ay - dx - e$, $a, d, e \in \mathbb{R}$.

(ii) Non-Weierstraß type 1: $P_7(x,y) = xy^2 + ay - x^2 - dx - e$, $a, d, e \in \mathbb{R}$.

(iii) Non-Weierstraß type 2: $P_8(x,y) = xy^2 - x^3 - cx^2 - dx - e$, $c, d \in \mathbb{R}$, $e \in \mathbb{R}^*$.

(iv) Non-Weierstraß type 3: $P_9(x,y) = xy^2 + x^3 - cx^2 - dx - e$, $c, d \in \mathbb{R}$, $e \in \mathbb{R}^*$.

(v) Non-Weierstraß type 4: $P_{10}(x,y) = xy^2 + ay - x^3 - cx^2 - dx - e$, $c, d \in \mathbb{R}$, $a, e \in \mathbb{R}^*$.

(vi) Non-Weierstraß type 5: $P_{11}(x,y) = xy^2 + ay + x^3 - cx^2 - dx - e$, $c, d \in \mathbb{R}$, $a, e \in \mathbb{R}^*$.

Proof. Let P be as in the title of the section. If $b = c = 0$, then P is as in (i). If $b = 0$ and $c > 0$, then after applying the alt $(x,y) \mapsto (x, \frac{y}{\sqrt{c}})$, P gets the form (ii). If $b = 0$ and $c < 0$, then applying the alt $(x,y) \mapsto (-x, y)$, we can assume that $b = 0$ and $c > 0$, which transforms to (ii) as just described. If $b > 0$, then after applying the alt $(x,y) \mapsto (x, \frac{y}{\sqrt{b}})$, P gets the form (iii) or (v). Finally, if $b < 0$, then after applying the alt $(x,y) \mapsto (x, \frac{y}{\sqrt{-b}})$, P gets the form (iv) or (vi). □

The main results of this section are the following:

1. Explicit descriptions of the pair $(f, V^{(k)})$ in Theorem 2.3 for each $C = \mathcal{Z}(P_i)$ from Proposition 5.1.
2. A constructive solution to the $\mathcal{Z}(P_6)$–TMP.

5.1. The $\mathcal{Z}(P_6)$–TMP.

Let P_6 be as in Proposition 5.1 above. Note that a rational parametrization of $\mathcal{Z}(P_6)$ is given by

$$(x(t), y(t)) = \Big(\frac{-at+e}{t^2-d}, t\Big), \quad t \in \begin{cases} \mathbb{R}, & \text{if } d < 0, \\ \mathbb{R} \setminus \{\sqrt{d}, -\sqrt{d}\}, & \text{if } d \geq 0. \end{cases}$$

Remark 5.2. Note that every point of $\mathcal{Z}(P_6)$ is indeed parametrized with the parametrization above. If $d \geq 0$ and $y_0 \in \{\sqrt{d}, -\sqrt{d}\}$, then $P_6(x, y_0) = ay_0 - e = 0$ would imply that $y - y_0$ divides P_6, which is a contradiction with the irreducibility of P_6.

Let us write $h_1(t) := -at + e$, $h_2(t) := t^2 - d$ and $h_3(t) := \frac{h_1(t)}{h_2(t)}$. Let $\mathcal{S}_{\leq 3i}$ and $\mathcal{T}_{\leq 3i}$ be vector spaces in $\mathbb{R}(t)$ with the bases

$$\begin{aligned} \mathcal{B}_{\mathcal{S}_{\leq 3i}} &:= \left\{h_3^i, h_3^{i-1}, h_3^{i-1}t, h_3^{i-2}, h_3^{i-2}t, \ldots, h_3, h_3t, 1, t, \ldots, t^i\right\} \quad \text{and} \\ \mathcal{B}_{\mathcal{T}_{\leq 3i}} &:= \left\{h_3^i t, h_3^{i-1}, h_3^{i-1}t, h_3^{i-2}, h_3^{i-2}t, \ldots, h_3, h_3t, 1, t, \ldots, t^i\right\}. \end{aligned} \tag{5.1}$$

Let

$$\mathrm{Pos}(\mathcal{S}_{\leq 3i}) := \{f \in \mathcal{S}_{\leq 3i} \colon f(t) \geq 0 \text{ for every } t \in \mathbb{R}\} \quad \text{and} \quad \mathcal{S} := \bigcup_{i=0}^{\infty} \mathcal{S}_{\leq 3i}.$$

Theorem 5.3. *Let $p \in \mathrm{Pos}(\mathcal{S}_{\leq 6k})$. Then there exist finitely many $f_i \in \mathcal{S}_{\leq 3k}$ and $g_j \in \mathcal{T}_{\leq 3k}$ such that $p = \sum_i f_i^2 + \sum_j g_j^2$.*

Moreover, for $C = \mathcal{Z}(P_6)$ the appropriate choices of f and $V^{(k)}$ in Theorem 2.3 are as stated in Table 2.

Proof. Let $\Phi : \mathbb{R}[\mathcal{Z}(P_6)] \to \mathcal{S}$ be a map defined by $\Phi(p(x, y)) = p(h_3(t), t)$. Analogously as in the proof of Theorem 4.8 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_6)]_{\leq i}$ is in one-to-one correspondence with the set $\mathcal{S}_{\leq 3i}$ under Φ. We separate three cases according to the sign of a.

Case 1: $d < 0$. In this case note that

$$\mathcal{S}_{\leq 3i} = \left\{ \frac{q(t)}{h_2^i} \colon q \in \mathbb{R}[t]_{\leq 3i},\ (e + \mathrm{i}a\sqrt{-d})^i q(\mathrm{i}\sqrt{-d}) = (e - \mathrm{i}a\sqrt{-d})^i q(-\mathrm{i}\sqrt{-d}) \right\},$$
$$\mathcal{T}_{\leq 3i} := \left\{ \frac{q(t)}{h_2^i} \colon q \in \mathbb{R}[t]_{\leq 3i},\ (e + \mathrm{i}a\sqrt{-d})^i q(\mathrm{i}\sqrt{-d}) = -(e - \mathrm{i}a\sqrt{-d})^i q(-\mathrm{i}\sqrt{-d}) \right\},$$

where both equalities follow by a short computation. From here on the proof is analogous to the proof of Theorem 4.11, only that the condition $(u^2)(1) = (u^2)(-1)$ is replaced by the condition $(e + \mathrm{i}a\sqrt{-d})^{2k}(u^2)(\mathrm{i}\sqrt{-d}) = (e - \mathrm{i}a\sqrt{-d})^{2k}(u^2)(-\mathrm{i}\sqrt{-d})$, whence $(e + \mathrm{i}a\sqrt{-d})^k u(\mathrm{i}\sqrt{-d}) = \pm(e - \mathrm{i}a\sqrt{-d})^k u(-\mathrm{i}\sqrt{-d})$. So $u \in \mathcal{S}_{\leq 3k}$ or $u \in \mathcal{T}_{\leq 3k}$.

Case 2: $d = 0$. In this case first note that $e \neq 0$, otherwise P_6 was reducible. Then it turns out by a short computation that

$$\mathcal{S}_{\leq 3i} = \left\{ \frac{q(t)}{h_2^i} \colon q \in \mathbb{R}[t]_{\leq 3i},\ q'(0) = -\frac{ia}{e} q(0) \right\},$$
$$\mathcal{T}_{\leq 3i} := \left\{ \frac{q(t)}{h_2^i} \colon q \in \mathbb{R}[t]_{\leq 3i},\ q(0) = 0 \right\}.$$

From here on the proof is again analogous to the proof of Theorem 4.11, only that the condition $(u^2)(1) = (u^2)(-1)$, is replaced by the condition $(u^2)'(0) = 2u'(0)u(0) = -\frac{2ka}{e}(u^2)(0)$, and hence $u'(0) = -\frac{ka}{e}u(0)$ or $u(0) = 0$. So $u \in \mathcal{S}_{\leq 3k}$ or $u \in \mathcal{T}_{\leq 3k}$.

Case 3: $d > 0$. In this case note that

$$\mathcal{S}_{\leq 3i} = \left\{ \frac{q(t)}{h_2^i} \colon q \in \mathbb{R}[t]_{\leq 3i},\ (e + a\sqrt{d})^i q(\sqrt{d}) = (e - a\sqrt{d})^i q(-\sqrt{d}) \right\},$$
$$\mathcal{T}_{\leq 3i} := \left\{ \frac{q(t)}{h_2^i} \colon q \in \mathbb{R}[t]_{\leq 3i},\ (e + a\sqrt{d})^i q(\sqrt{d}) = -(e - a\sqrt{d})^i q(-\sqrt{d}) \right\}.$$

From here on the proof is analogous to the proof of Theorem 4.11, only that the condition $(u^2)(1) = (u^2)(-1)$, is replaced by the condition

$$(e + a\sqrt{d})^{2k}(u^2)(\sqrt{d}) = (e - a\sqrt{d})^{2k}(u^2)(-\sqrt{d}),$$

whence $(c - b\sqrt{-a})^k u(\sqrt{-a}) = \pm(c + b\sqrt{-a})^k u(-\sqrt{-a})$. So $u \in \mathcal{S}_{\leq 3k}$ or $u \in \mathcal{T}_{\leq 3k}$.

The moreover part follows by noting that $\Phi^{-1}(\mathcal{B}_{\mathcal{T}_{\leq 3k}})$ is equal to $\mathcal{B}_{V^{(k)}}$ from Table 1. $\square$

For $x \in \mathbb{C}$, we write Q_x for the divisor $[x : 1]$ on $\mathbb{P}^1$. We also write Q_∞ for the divisor $[1 : 0]$.

Below we present a constructive solution to the nonsingular $\mathcal{Z}(P_6)$–TMP via the corresponding univariate moment problem.

Constructive proof of Corollary 2.6 for $C = \mathcal{Z}(P_6)$. Using the correspondence as in the proof of Theorem 5.3, the $\mathcal{Z}(P_6)$–TMP for L is equivalent to the rational $\mathbb{R}$–TMP for $L_{\mathcal{S}_{\leq 6k}} : \mathcal{S}_{\leq 6k} \to \mathbb{R}$. Let

$$D = \begin{cases} Q_{-\mathrm{i}\sqrt{-d}} + Q_{\mathrm{i}\sqrt{-d}} + Q_\infty, & \text{if } d < 0, \\ 2Q_0 + Q_\infty, & \text{if } d = 0, \\ Q_{-\sqrt{d}} + Q_{\sqrt{d}} + Q_\infty, & \text{if } d > 0. \end{cases}$$

If $L_{\mathcal{S}_{\leq 6k}}$ is a mf, then it extends to the $\mathbb{R}$–mf $\widehat{L} : \mathcal{L}(2kD) \to \mathbb{R}$. In the ordered basis

$$\{(h_3(T))^k, (h_3(T))^k T, (h_3(T))^{k-1}, \ldots, \mathit{1}, T, \ldots, T^k\}$$

of rows and columns, the strict square positivity and the $(V^{(k)}, 1)$–local strict square positivity of $L_{\mathcal{Z}(P_6)}$ are equivalent to the partial positive definiteness of the matrix

$$\begin{array}{c}
\begin{array}{cccccccc}
(h_3(T))^k & (h_3(T))^k T & (h_3(T))^{k-1} & \cdots & \mathit{1} & T & \cdots & T^k
\end{array} \\
\begin{pmatrix}
\widehat{L}(h_3^{2k}) & ? & \widehat{L}(h_3^{2k-1}) & \cdots & \widehat{L}(h_3^k) & \cdots & \cdots & \widehat{L}(h_3^k t^k) \\
? & \widehat{L}(h_3^{2k} t^2) & \widehat{L}(h_3^{2k-1} t) & \cdots & \widehat{L}(h_3^k t) & & & \widehat{L}(h_3^k t^{k+1}) \\
\widehat{L}(h_3^{2k-1} t) & \widehat{L}(h_3^{2k-1} t^2) & \widehat{L}(h_3^{2k-2} t) & \cdots & \widehat{L}(h_3^{k-1} t) & \cdots & \cdots & \widehat{L}(h_3^{k-1} t^{k+1}) \\
\vdots & & & & & \ddots & & \vdots \\
\widehat{L}(h_3^k) & & & & & & \ddots & \widehat{L}(t^k) \\
\vdots & & & & & & & \vdots \\
\widehat{L}(h_3^k t^k) & \cdots & & \cdots & \widehat{L}(t^k) & \cdots & \cdots & \widehat{L}(t^{2k})
\end{pmatrix}
\end{array}.$$

The missing entries are at the positions $(1, 2)$ and $(2, 1)$, since the value $\widehat{L}(h_3^{2k} t)$ is unknown. Then there exists an interval $(a, b) \subset \mathbb{R}$, such that for every $\widehat{L}(h_3^{2k} t) \in (a, b)$, the completion is positive definite (see e.g., [65, Lemma 2.4]) and for every such completion the functional $\widehat{L}$ has a $(3k + 1)$–atomic $\mathbb{R}$–rm by [47, Theorem 3.1]. □

Remark 5.4. Similarly as in Remark 4.12 it is not clear whether a $(3k)$–atomic $\mathcal{Z}(P_6)$–rm for L exists.

Note that $|\mathcal{B}_{\mathcal{S}_{\leq 3i}}| = |\mathcal{B}_{\mathcal{T}_{\leq 3i}}| = 3i$ and

$$\mathcal{B}_i = \Phi^{-1}(\mathcal{B}_{\mathcal{S}_{\leq 3i}}) = \{x^i, x^{i-1}, x^{i-1}y, x^{i-2}, x^{i-2}y, \ldots, x, xy, 1, y, \ldots, y^i\},$$

$$\widetilde{\mathcal{B}}_i = \Phi^{-1}(\mathcal{B}_{\mathcal{T}_{\leq 3i}}) = \{x^i y, x^{i-1}, x^{i-1}y, x^{i-2}, x^{i-2}y, \ldots, x, xy, 1, y, \ldots, y^i\}.$$

The following theorem solves the singular $\mathcal{Z}(P_6)$–TMP with $d < 0$.

Theorem 5.5 (Singular $\mathcal{Z}(P_6)$–TMP with $d < 0$)**.** *Let $d < 0$ and $L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ be a $\mathcal{Z}(P_6)$–singular linear functional. Let $V^{(k)}$ is as in Table 2. Then the following are equivalent:*

(1) L *is a* $\mathcal{Z}(P_6)$*–moment functional.*

(2) $L_{\mathcal{Z}(P_6)}$ *is square positive and* $(V^{(k)},1)$*–locally square positive, and one of the following holds:*

(a) For $U := \mathrm{Span}(\mathcal{B}_k \setminus \{y^k\})$ *it holds that*

$$\mathrm{rank}\, \overline{L}_{\mathcal{Z}(P_6)} = \mathrm{rank}\, \big((\overline{L}_{\mathcal{Z}(P_6)})|_U\big).$$

(b) For $W := \mathrm{Span}(\widetilde{\mathcal{B}}_k \setminus \{y^k\})$ *it holds that*

$$\mathrm{rank}\, \overline{L}_{\mathcal{Z}(P_6),V^{(k)},1} = \mathrm{rank}\, \big((\overline{L}_{\mathcal{Z}(P_6),V^{(k)},1})|_W\big).$$

Proof. Following the constructive proof of Corollary 2.6 for $C = \mathcal{Z}(P_6)$ via the solution to the rational $\mathbb{R}$–TMP above and replacing positive definiteness with positive semidefiniteness, the only addition is that psd extension $\widehat{L}$ is not necessarily a $\mathbb{R}$–mf. Since L is $\mathcal{Z}(P_6)$–singular and using [47, Theorem 3.2], (1) is equivalent to:

$$\begin{aligned}&\text{There is a square positive extension } L_{\mathcal{L}(2kD)} \text{ of } L_{\mathcal{S}_{\leq 6k}}\\ &\text{such that } \mathrm{rank}\, \overline{L}_{\mathcal{L}(2kD)} = \mathrm{rank}\, \overline{L}_{\mathcal{L}(2kD-2Q_\infty)} \text{ holds.}\end{aligned} \tag{5.2}$$

Note that the rank condition in (5.2) means that in the matrix M, representing the bilinear form $\overline{L}_{\mathcal{L}(2kD)}$ in the basis $\mathcal{B} = \mathcal{B}_{\mathcal{S}_{\leq 3k}} \cup \mathcal{B}_{\mathcal{T}_{\leq 3k}}$, there is a relation

$$T^k = \alpha_0 h_3(T)^k + \alpha_1 h_3(T)^k T + \alpha_2 h_3(T)^{k-1} + \ldots + \alpha_{3k-1} T^{k-1} \tag{5.3}$$

with $\alpha_i \in \mathbb{R}$ not all zero. Let (5.3) be a relation with the largest index i_0 such that $\alpha_0 = \alpha_1 = \ldots = \alpha_{i_0-1} = 0$.

Let us now prove the implication (5.2) $\Rightarrow$ (2). The square positivity and the $(V^{(k)},1)$–local square positivity of $L_{\mathcal{Z}(P_6)}$ are clear. If $\alpha_0 = 0$ in (5.3), then (2b) holds. If $\alpha_1 = 0$ in (5.3), then (2a) holds. It remains to study the case: $\alpha_0 \neq 0$ and $\alpha_1 \neq 0$. We separate two cases according to the $\mathcal{Z}(P_6)$–singularity of L:

Case 1: $L_{\mathcal{Z}(P_6)}$ is singular. There are some $\beta_i \in \mathbb{R}$, not all equal to 0, such that in the matrix N, representing $\overline{L}_{\mathcal{Z}(P_6)}$ with respect to the ordered basis $\mathcal{B}_{\mathcal{S}_{\leq 3k}}$ (see (5.1)), the relation

$$0 = \beta_1 h_3(T)^k + \beta_2 h_3(T)^{k-1} + \ldots + \beta_{3k-1} T^{k-1} + \beta_{3k} T^k \tag{5.4}$$

holds. If $\beta_{3k} \neq 0$, then (2a) holds. Assume that $\beta_{3k} = 0$. By the extension principle [31, Proposition 2.4], (5.4) is also a relation in M. If $\beta_1 \neq 0$, then we can express the column $h_3(T)^k$ out of (5.4) and plug it into (5.3). Then we end up with a relation of the form (5.3) with $\alpha_0 = 0$ and $\alpha_1 \neq 0$, whence (2b) holds. Assume now that $\beta_1 = 0$. Let Col_1, Col_2, $\ldots$, Col_{3k} be the columns of N enumerated from the left to the right side.

Claim. For $2 \leq i \leq 3k-1$ we have that

$$T \cdot \mathrm{Col}_i = \left\{ \begin{array}{rl} \mathrm{Col}_{i+1}, & \text{if } \mathrm{Col}_i \in \{(h_3(T))^j, T^j\}, \\ -a\,\mathrm{Col}_{i+2} + e\,\mathrm{Col}_{i+1} + d\,\mathrm{Col}_{i-1}, & \text{if } \mathrm{Col}_i = h_3(T)^j T. \end{array} \right. .$$

Proof of Claim. The case $\mathrm{Col}_i \in \{(h_3(T))^j, T^j\}$ is clear. Assume now that $\mathrm{Col}_i = h_3(T)^j T$ for some j. Then

$$T \cdot \mathrm{Col}_i = (h_3(T))^j T^2 = (h_3(T))^j (T^2 - d) + (h_3(T))^j d$$

$$= -ah_3(T)^{j-1}T + eh_3(T)^{j-1} + d(h_3(T))^j$$
$$= -a\operatorname{Col}_{i+2} + e\operatorname{Col}_{i+1} + d\operatorname{Col}_{i-1}.$$

This proves the Claim. ■

Let Col_{j_0} be the first column in N, counted from the left side, which is linearly dependent from columns to the left of this column, i.e.,

$$\operatorname{Col}_{j_0} = \beta_2\operatorname{Col}_2 + \ldots + \beta_{j_0-1}\operatorname{Col}_{j_0-1} \tag{5.5}$$

for some $\beta_j \in \mathbb{R}$ not all equal to 0. Recall from above that we assumed $\beta_1 = 0$ in every such relation. We separate three cases:

Case 1.1: Col_{j_0} is on the right side of the column corresponding to the basis element *1*. Multiplying (5.5) by T and using Claim we get a relation of the form (5.5) with j_0 replaced by $j_0 + 1$. Note also that $\operatorname{Col}_2 = h_3(T)^{k-1}$ so $T\operatorname{Col}_2 = \operatorname{Col}_3$ and Col_1 does not appear in (5.5). We continue until $j_0 = 3k$. But this then gives (2a).

Case 1.2: Col_{j_0} is equal to the column corresponding to the basis element *1*. Multiplying (5.5) by T and using Claim we get a relation of the form

$$T = \widetilde{\beta}_2\operatorname{Col}_2 + \ldots + \beta_{j_0-2}h_3(T)T + \beta_{j_0-1}(-aT + e\mathit{1} + dh_3(T)).$$

If $-\beta_{j_0-1}a \neq 1$, then we are in Case 1.1 and continue as above. If $-\beta_{j_0-1}a = 1$, then we get a relation of the form

$$-\beta_{j_0-1}e\mathit{1} = \widetilde{\beta}_2\operatorname{Col}_2 + \ldots + \beta_{j_0-2}h_3(T)T + \beta_{j_0-1}dh_3(T). \tag{5.6}$$

We know that $\beta_{j_0-1} \neq 0$ (by $-\beta_{j_0-1}a = 1$). If also $e \neq 0$, then we can express *1* from (5.6) and plug it into (5.5). Then we get a relation of the form (5.5) with smaller j_0, which is a contradiction. Here note that the relation we would get cannot be a trivial one. Indeed, let $p(X,Y) = 0$ be a representation of the column relation (5.5) of $\overline{L}_{\mathcal{Z}(P_6)}$. Then $Yp(X,Y) = 0$ is a representation of the column relation (5.6) of $\overline{L}_{\mathcal{Z}(P_6)}$. If $yp(x,y) - p(x,y)$ is in the ideal I generated by $P_6(x,y)$, then one of $y-1$ or p must be divisible by P_6, which is not true due to the irreducibility of P_6. It remains to study the case $e = 0$ in (5.6). However, this case cannot appear since then (5.6) would be a relation of the form (5.5) with smaller j_0. As above note that the relation (5.6) cannot be a trivial one in this case, since it was obtained (after using the identification with $\mathbb{R}[\mathcal{Z}(P_6)]$) from some nonzero polynomial in $\mathbb{R}[\mathcal{Z}(P_6)]$ after multiplying with y. Since P_6 is irreducible, this polynomial cannot be divisible by P_6.

Case 1.3: Col_{j_0} is on the left side of the column corresponding to the basis element *1*. We multiply the relation (5.5) by T until we come into one of the Cases 1.1, 1.2 above. By the same reasoning as in Case 1.2 above, after multiplying with T the leading term must have larger index than j_0.

Case 2: $L_{\mathcal{Z}(P_6)}$ is not singular, but $L_{\mathcal{Z}(P_6)}$ is $(V^{(k)},1)$–locally singular. This case is analogous to Case 1 only that the relation (5.4) is replaced by

$$0 = \beta_1 h_3(T)^{k-1}T + \beta_2 h_3(T)^{k-1} + \ldots + \beta_{3k-1}T^{k-1} + \beta_{3k}T^k. \tag{5.7}$$

Namely, the term $\beta_1 h_3(T)^k$ is replaced by the term $\beta_1 h_3(T)^{k-1}T$.

It remains to prove the implication (2) ⇒ (5.2). The existence of a square positive extension is clear from the positivity assumptions on $L_{\mathcal{Z}(P_6)}$. The rank condition in (5.2) follows from either (5.4), where $\beta_{3k} \neq 0$, under the assumption (2a), or from (5.7), where $\beta_{3k} \neq 0$, under the assumption (2b). □

The following is the solution to the singular $\mathcal{Z}(P_6)$–TMP with $d = 0$.

Theorem 5.6 (Singular case $\mathcal{Z}(P_6)$–TMP with $d = 0$)**.** *Let* $d = 0$ *and* $L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ *be a* $\mathcal{Z}(P_6)$*–singular linear functional. Let* $V^{(k)}$ *is as in Table 2 and*

$$U := \mathrm{Span}(\mathcal{B}_k \setminus \{x^k\}), \quad W := \mathrm{Span}(\mathcal{B}_k \setminus \{y^k\}) \quad \textit{and} \quad Z := \mathrm{Span}(\widetilde{\mathcal{B}}_k \setminus \{y^k\}).$$

Then the following are equivalent:

(1) L *is a* $\mathcal{Z}(P_6)$*–moment functional.*

(2) $L_{\mathcal{Z}(P_6)}$ *is square positive and* $(V^{(k)}, 1)$*–locally square positive, and*

$$\mathrm{rank}\, \overline{L}_{\mathcal{Z}(P_6)} = \mathrm{rank}\,\big((\overline{L}_{\mathcal{Z}(P_6)})|_U\big) \tag{5.8}$$

and one of the following holds:

  (a) $\mathrm{rank}\, \overline{L}_{\mathcal{Z}(P_6)} = \mathrm{rank}\,\big((\overline{L}_{\mathcal{Z}(P_6)})|_W\big)$.

  (b) $\mathrm{rank}\, \overline{L}_{\mathcal{Z}(P_6),V^{(k)},1} = \mathrm{rank}\,\big((\overline{L}_{\mathcal{Z}(P_6),V^{(k)},1})|_Z\big)$.

Proof. Following the constructive proof of Corollary 2.6 for $\mathcal{Z}(P_6)$ via the solution to the rational $\mathbb{R}$–TMP above and replacing positive definiteness with positive semidefiniteness, the only addition is that the psd extension $\widehat{L}$ is not necessarily a $(\mathbb{R} \setminus \{0\})$–mf. Since L is $\mathcal{Z}(P_6)$–singular and using [67, Theorem 3.1], (1) is equivalent to:

$$\begin{aligned}&\text{There is a square positive extension } L_{\mathcal{L}(2kD)} \text{ of } L_{\mathcal{S}_{\leq 6k}} \text{ with}\\ &\mathrm{rank}\, \overline{L}_{\mathcal{L}(2kD)} = \mathrm{rank}\, \overline{L}_{\mathcal{L}(2kD-2Q_0)} = \mathrm{rank}\, \overline{L}_{\mathcal{L}(2kD-2Q_\infty)}.\end{aligned} \tag{5.9}$$

Let Col_1, Col_2, …, Col_{3k} be the columns of the matrix N, representing the bilinear form $\overline{L}_C$ with respect to the basis $\mathcal{B}_{\mathcal{S}_{\leq 3k}}$, ordered as in (5.1).

Claim 1. For $2 \leq i \leq 3k + 1$ we have that

$$T^{-1} \cdot \mathrm{Col}_i = \left\{ \begin{array}{rl} \mathrm{Col}_{i-1}, & \text{if } \mathrm{Col}_i \in \{(h_3(T))^j T, T^j : j \in \mathbb{N}\}, \\ \frac{1}{e}\,\mathrm{Col}_{i-1} + \frac{a}{e}\,\mathrm{Col}_i, & \text{if } \mathrm{Col}_i = \{h_3(T)^j : j \in \mathbb{N} \cup \{0\}\}. \end{array} \right. .$$

Proof of Claim 1. The case $\mathrm{Col}_i \in \{(h_3(T))^j T, T^j : j \in \mathbb{N}\}$ is clear. Assume now that $\mathrm{Col}_i = h_3(T)^j$ for some $j \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned} T^{-1} \cdot \mathrm{Col}_i = (h_3(T))^j T^{-1} &= (h_3(T))^j \frac{(-aT + e)T + aT^2}{eT^2} \\ &= \frac{1}{e} h_3(T)^{j+1} T + \frac{a}{e} (h_3(T))^j \\ &= \frac{1}{e}\,\mathrm{Col}_{i-1} + \frac{a}{e}\,\mathrm{Col}_i \,. \end{aligned}$$

This proves Claim 1. ■

Claim 2. Assume that L is a $\mathcal{Z}(P_6)$–singular and $\mathcal{Z}(P_6)$–mf. Then (5.8) holds.

Proof of Claim 2. Assume first that $L_{\mathcal{Z}(P_6)}$ is singular. There are some $\beta_i \in \mathbb{R}$, not all equal to 0, such that in the matrix representation of $\overline{L}_{\mathcal{Z}(P_6)}$ with respect to the basis $\mathcal{B}_{\mathcal{S}_{\leq 3k}}$, the following relation

$$0 = \beta_1 h_3(T)^k + \beta_2 h_3(T)^{k-1} + \ldots + \beta_{3k-1} T^{k-1} + \beta_{3k} T^k \tag{5.10}$$

holds. If $\beta_1 \neq 0$, this implies Claim 2. Otherwise assume that $\beta_1 = 0$. By the extension principle [31, Proposition 2.4], (5.10) is also a relation in the matrix M representing $\widehat{L}$. Multiplying with T^{-1} successively and using Claim 1 we conclude that there is a relation of the form (5.10) with $\beta_1 \neq 0$ in M, and hence also in $\overline{L}_{\mathcal{Z}(P_6)}$. Note that first we get a relation containing $h_3(T)^k T$ instead of $h_3(T)^k$. After another step of the procedure we get $h_3(T)^k$ and express $h_3(T)^k T$ using the previous step of the procedure. This gives (5.8).

It remains to prove Claim 2 in the case $L_{\mathcal{Z}(P_6)}$ is $(V^{(k)}, 1)$–locally singular. This case is analogous to the case in the first paragraph above only that the relation (5.10) is replaced by the relation

$$0 = \beta_1 h_3(T)^{k-1} T + \beta_2 h_3(T)^{k-1} + \ldots + \beta_{3k-1} T^{k-1} + \beta_{3k} T^k.$$

Namely, the term $\beta_1 h_3(T)^k$ is replaced by the term $\beta_1 h_3(T)^{k-1} T$. ■

Let us now prove the implication (5.9) ⇒ (2). The square positivity and the $(V^{(k)}, 1)$–local square positivity of $L_{\mathcal{Z}(P_6)}$ are clear. The equality (5.8) follows from Claim 2. The statements (2a) or (2b) follow as in the proof of Theorem 5.5 above.

It remains to prove the implication (2) ⇒ (5.9). The existence of a square positive extension $\widehat{L}$ is clear from the positivity assumptions on L_C as explained in the first paragraph of the proof. The rank condition in (5.9) follows from rank conditions in (2). ☐

Let $d > 0$. A functional $\widehat{L} := L_{\mathcal{L}(2kD)}$ induces a functional

$$L_{\mathbb{R}[t]_{\leq 6k}} : \mathbb{R}[t]_{\leq 6k} \to \mathbb{R} \quad \text{by} \quad L_{\mathbb{R}[t]_{\leq 6k}}(f) := \widehat{L}\Big(\frac{f}{(t^2-d)^{2k}}\Big). \tag{5.11}$$

Let p be a nonzero polynomial of the lowest degree in $\ker \overline{L}_{\mathbb{R}[t]_{\leq 6k}}$. The polynomial p is called a **generating polynomial** of $L_{\mathbb{R}[t]_{\leq 6k}}$.

The following is the solution to the singular $\mathcal{Z}(P_6)$–TMP with $d > 0$.

Theorem 5.7 (Singular $\mathcal{Z}(P_6)$–TMP with $d > 0$)**.** *Let $d > 0$ and $L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ be a $\mathcal{Z}(P_6)$–singular linear functional. Let $V^{(k)}$ is as in Table 2. Let $L_{\mathbb{R}[t]_{\leq 6k}}$ be the induced functional as in* (5.11)*. Then the following are equivalent:*

(1) L is a $\mathcal{Z}(P_6)$–moment functional.

(2) $L_{\mathcal{Z}(P_6)}$ is square positive and $(V^{(k)}, 1)$–locally square positive, $\sqrt{d}$ and $-\sqrt{d}$ are not zeroes of the generating polynomial of $L_{\mathbb{R}[t]_{\leq 6k}}$ and one of the following holds:

(a) *For* $U := \operatorname{Span}(\mathcal{B}_k \setminus \{y^k\})$ *it holds that*

$$\operatorname{rank} \overline{L}_{\mathcal{Z}(P_6)} = \operatorname{rank}\big((\overline{L}_{\mathcal{Z}(P_6)})|_U\big).$$

(b) *For* $W := \operatorname{Span}(\widetilde{\mathcal{B}}_k \setminus \{y^k\})$ *it holds that*

$$\operatorname{rank} \overline{L}_{\mathcal{Z}(P_6),V^{(k)},1} = \operatorname{rank}\big((\overline{L}_{\mathcal{Z}(P_6),V^{(k)},1})|_W\big).$$

Proof. This follows by [47, Theorem 3.2], where note that by the $\mathcal{Z}(P_6)$–singularity of L, the condition $L_{\mathbb{R}[t]_{\leq 6k}}(t^{6k-\deg p}p) = 0$ is equivalent to one of (2a) or (2b). □

5.2. **Positivstellensatz on $C = \mathcal{Z}(P_i)$, $i = 7, \ldots, 11$.** Let P_i be as in Proposition 5.1. We denote by H_i the divisor on $\bar{X}$ at infinity, i.e., writing $Q_1 = [1 : 0 : 0]$, $Q_2 = [0 : 1 : 0]$, $Q_3 = [-1 : 1 : 0]$, $Q_4 = [1 : 1 : 0]$, $Q_5 = [\mathfrak{i} : 1 : 0]$ and $Q_6 = [-\mathfrak{i} : 1 : 0]$, we have

$$H_i = \begin{cases} 2Q_1 + Q_2, & \text{if } i = 7, \\ Q_2 + Q_3 + Q_4, & \text{if } i \in \{8, 10\}, \\ Q_2 + Q_5 + Q_6, & \text{if } i \in \{9, 11\}. \end{cases} \tag{5.12}$$

Theorem 5.8. *For* $i = 7, \ldots, 11$*, let* $C = \mathcal{Z}(P_i)$ *and* H_i *be as in* (5.12)*. Assume that the projective closure* $\bar{X}$ *of* C *is smooth. Let* $f_i \in \mathbb{R}[\mathcal{Z}(P_i)]$ *be equal to*

(i) $f_7 = x - \alpha_7$*, where* α_7 *is the smallest zero of*

$$q_7(t) := t^3 + dt^2 + et + \frac{a^2}{4}.$$

(ii)

$$\begin{aligned} f_8 = \frac{e}{|e|}\Big(\frac{1}{x} - \alpha_8\Big) &= \frac{1}{|e|}(xy^2 - x^3 - cx^2 - dx)\frac{1 - \alpha_8 x}{x} \\ &= \frac{1}{|e|}(y^2 - x^2 - cx - d)(1 - \alpha_8 x), \end{aligned}$$

where α_8 *is the smallest (resp. largest) zero of*

$$q_8(t) := et^3 + dt^2 + ct + 1$$

if $e > 0$ *(resp.* $e < 0$*).*

(iii)

$$\begin{aligned} f_9 = \frac{e}{|e|}\Big(\frac{1}{x} - \alpha_9\Big) &= \frac{1}{|e|}(xy^2 + x^3 - cx^2 - dx)\frac{1 - \alpha_9 x}{x} \\ &= \frac{1}{|e|}(y^2 + x^2 - cx - d)(1 - \alpha_9 x), \end{aligned}$$

where α_9 *is the smallest (resp. largest) zero of*

$$q_9(t) := et^3 + dt^2 + ct - 1$$

if $e > 0$ *(resp.* $e < 0$*).*

(iv) $f_{10} = y^2 - x^2 - cx - \alpha_{10}$*, where* α_{10} *is the smallest zero of*

$$q_{10}(t) := t^3 - 2dt^2 + (d^2 - a^2 + ce)t + \Big(\frac{a^2c^2}{4} - cde + e^2\Big).$$

(v) $f_{11} = y^2 + x^2 - cx - \alpha_{11}$, *where* α_{11} *is the smallest zero of*

$$q_{11}(t) = t^3 - 2dt^2 + (d^2 + a^2 + ce)t + \Big(\frac{a^2c^2}{4} - cde - e^2\Big).$$

Let

$$R_i = \begin{cases} Q_1, & \text{if } i = 7,\\ Q_2, & \text{if } i \in \{8, 9, 10, 11\},\end{cases}$$

be the unique pole of f_i *and* S_i *its unique zero on* $\bar{X}$*. The following statements are equivalent:*

(1) $p \in \mathrm{Pos}_{2k}(\mathcal{Z}(P_i))$.

(2) There exist finitely many $g_j \in \mathbb{R}[C]_{\leq k}$ *and* $h_l \in \mathcal{L}(kH_i - R_i + S_i)$ *satisfying*

$$p = \sum\nolimits_j g_j^2 + f_i \sum\nolimits_l h_l^2.$$

Moreover, choosing the basis

$$\mathcal{B}_k := \{1, x, y, x^2, xy, y^2, x^3, x^2y, y^3, \ldots, x^i, x^{i-1}y, y^i, \ldots, x^k, x^{k-1}y, y^k\}$$

for $\mathbb{R}[C]_{\leq k}$*, the elements* r_i*, such that* $\mathcal{B} := \mathcal{B}_k \setminus \{y^k\} \cup \{r_i\}$ *is a basis for* $\mathcal{L}(kH_i - R_i + S_i)$*, are equal to:*

$$r_7 = \frac{xy + \frac{a}{2}}{x - \alpha_7}, \qquad r_8 = \frac{xy}{1 - \alpha_8 x}, \qquad r_9 = \frac{xy}{1 - \alpha_9 x},$$

$$r_{10} = \gamma_1 + \gamma_2(y^2 - x^2 - cx) + \gamma_3\Big(y^2 - x^2 - cx - \frac{e^2}{a^2}\Big)\frac{x-1}{x}$$

$$r_{11} = \widetilde{\gamma}_1 + \widetilde{\gamma}_2(y^2 + x^2 - cx) + \widetilde{\gamma}_3\Big(y^2 + x^2 - cx - \frac{e^2}{a^2}\Big)\frac{x-1}{x}$$

for γ_i*,* $\widetilde{\gamma}_i$ *such that* $r_i \in \mathcal{L}(H_i - Q_2 + S_i)$ *with* $S_i = \mathcal{Z}(f_i) \cap \mathcal{Z}(P_i)$*,* $i = 10, 11$*.*

Proof. In each case we bring the equation to Weierstraß form so that the point R_i gets mapped to the point at infinity as in [60, Proposition III.3.1]. Then the proof follows analogously to the proof of Theorem 4.3.

Namely, following [60, Proposition III.3.1], let $u, v \in \mathbb{R}(\mathcal{Z}(P_i))$ such that $\{1, u\}$, $\{1, u, v\}$ are bases for $\mathcal{L}(2R_i)$, $\mathcal{L}(3R_i)$, respectively. Then the map $\Phi : \mathcal{Z}(P_i) \to \mathbb{P}^2$, $(x, y) \mapsto [u(x, y) : v(x, y) : 1]$ is an isomorphism onto a curve in the Weierstraß form $W(u, v) = a_1 + a_2u + a_3v + a_4u^2 + a_5uv + a_6v^2 + a_7u^3$ of some $a_i \in \mathbb{R}$ with $a_6a_7 \neq 0$. Moreover, Φ maps R_i to Q_2.

For $C = \mathcal{Z}(P_7)$ a proper choices of u and v are x and xy. Indeed, if $a = 0$, then $\mathrm{div}(x) = 2Q_2 - 2Q_1$ and $\mathrm{div}(y) = -Q_1 - Q_2 + [x_1 : 0 : 1] + [x_2, 0, 1]$, where x_1 and x_2 are zeroes of $x^2 + dx + e$. In the case $a \neq 0$, we have that $\mathrm{div}(x) = [0 : \frac{e}{a} : 1] + Q_2 - 2Q_1$, while $\mathrm{div}(y)$ is the same as in the $a = 0$ case. Hence, $x \in \mathcal{L}(2Q_1)$ and $xy \in \mathcal{L}(3Q_1)$. The Weierstraß form of C is $(xy + \frac{a}{2})^2 = x^3 + dx^2 + ex + \frac{a^2}{4}$, whence α_7 is as stated in the theorem.

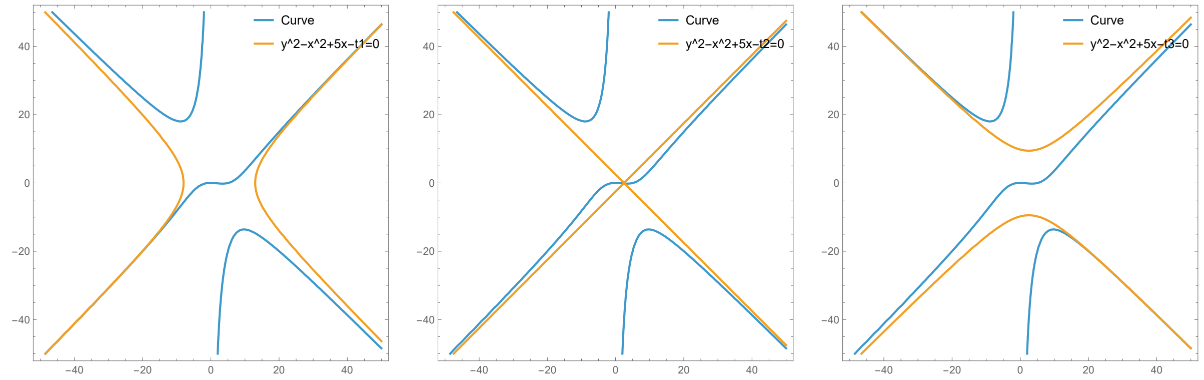
For $C = \mathcal{Z}(P_8)$ a proper choices of u and v are $\frac{1}{x}$ and $\frac{y}{x}$. Indeed, $\mathrm{div}(x) = 2Q_2 - Q_3 - Q_4$ and $\mathrm{div}(y) = -Q_2 - Q_3 - Q_4 + [x_1 : 0 : 0] + [x_2 : 0 : 1] + [x_3 : 0 : 1]$, where x_1, x_2, x_3 are zeroes of $x^3 + cx^2 + dx + e$. Hence, $\frac{1}{x} \in \mathcal{L}(2Q_2)$ and $\frac{y}{x} \in \mathcal{L}(3Q_2)$. The Weierstraß form of C is $y^2 = ex^3 + dx^2 + cx + 1$, whence α_8 is as stated in the theorem.

For $C = \mathcal{Z}(P_9)$ the proof is analogous to the proof for $C = \mathcal{Z}(P_8)$.

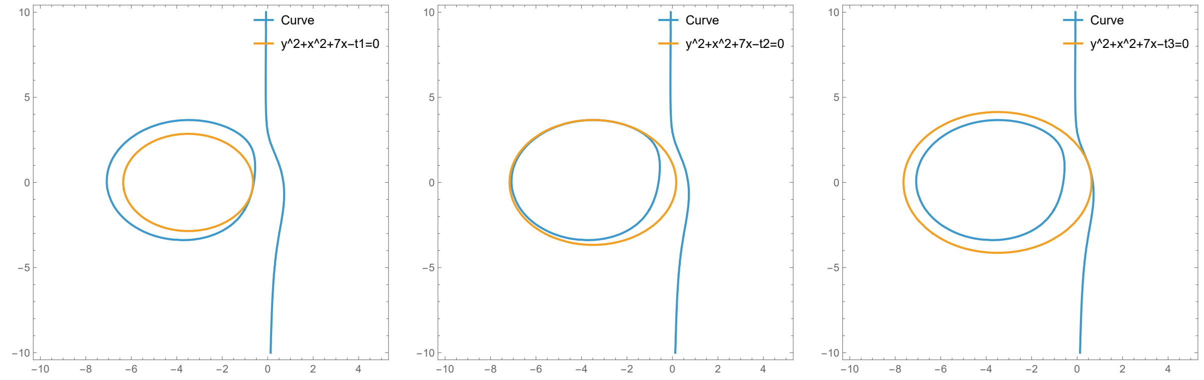
Let $C = \mathcal{Z}(P_{10})$. Note that the only pole of $y^2 - x^2 - cx$ is Q_2 and $\{1, y^2 - x^2 - cx\}$ is a basis for $\mathcal{L}(2Q_2)$. Further on, $\operatorname{div}(x) = [0 : \frac{e}{a} : 1] + Q_2 - Q_3 - Q_4$ and for $x_0 \neq 0$, $\operatorname{div}(x - x_0) = [x_0 : y_1 : 1] + [x_0 : y_2 : 1] - Q_3 - Q_4$, where y_1 and y_2 are zeroes of $x_0y^2 - ay - x_0^3 - x_0^2c - x_0d - e$. Hence, $\{1, u, v\} := \{1, y^2 - x^2 - cx - \gamma, (y^2 - x^2 - cx - \frac{e^2}{a^2})\frac{x-x_0}{x}\}$ is a basis for $\mathcal{L}(3Q_2)$ for any $\gamma \in \mathbb{R}$. Let $t \in \mathbb{R}$ such that the line $u - t$ has a double zero with the curve $W(u, v) = 0$. Namely, $y^2 - x^2 - cx - t = 0$ and $\frac{ay-dx-e}{x} + t = 0$ have a double intersection point. Expressing y out of the second equation, plugging into the first and solving the quadratic equation in x we get that the discriminant multiplied with $\frac{a^2}{4}$ is precisely $q_{10}(t)$ in the statement of the theorem.

For $C = \mathcal{Z}(P_{11})$ the proof is analogous to the proof for $C = \mathcal{Z}(P_{10})$. $\square$

Example 5.9. Let $P_{10} = xy^2 + 100y - x^3 + 5x^2 + x - 3$. Then $q_{10} = t^3 + 2t^2 - 10014t + 62494$ with zeroes $t_1 \approx -104.033$, $t_2 \approx 6.273$, $t_3 \approx 95.76$. By Theorem 5.8 above, $f_{10} = y^2 - x^2 + 5x + 104.033$. Indeed, the right choice for α_{10} is t_1 as presented on the following figures:



Example 5.10. Let $P_{11} = xy^2 + y + x^3 + 7x^2 - x - 3$. Then $q_{11} = t^3 - 2t^2 - 19t + \frac{97}{4}$ with zeroes $t_1 \approx -4.091$, $t_2 \approx 1.22$, $t_3 \approx 4.88$. By Theorem 5.8 above, $f_{11} = y^2 + x^2 + 7x + 4.091$. Indeed, the right choice for α_{11} is t_1 as presented on the following figures:



Remark 5.11. Every irreducible plane cubic curve with a singularity as a projective curve has a rational parametrization. (See e.g., [45, Section 15.2] for explicit formulas.) In case the projective closure of $\mathcal{Z}(P_i)$, $i = 7, \ldots, 11$, is not smooth, we can rationally

parametrize the curve and then appy the extreme ray machinery to describe a pair $(f, V^{(k)})$ in Theorem 2.3 for this curve.

6. THE $\mathcal{Z}(P_{12})$–TMP FOR IRREDUCIBLE $P_{12}(x,y) = xy - c(x)$, $c \in \mathbb{R}[x]_{\leq 3}$, $\deg c = 3$

Assume the notation as in Sections 2, 3. Let P_{12} be as in the title of the section.

The main results of this section are the following:

(1) Explicit description of the pair $(f, V^{(k)})$ in Theorem 2.3 for $C = \mathcal{Z}(P_{12})$.
(2) A constructive solution to the $\mathcal{Z}(P_{12})$–TMP.

After applying an affine linear transformation we can assume that $c_3 = 1$ in $c(x) = \sum_{i=0}^{3} c_i x^i$. The rational parametrization of $\mathcal{Z}(P_{12})$ is given by

$$(x(t), y(t)) = \Big(t, \frac{c(t)}{t}\Big), \quad t \in \mathbb{R} \setminus \{0\}.$$

Let

$$\mathcal{Q}_{\leq 3i} := \left\{ \frac{\sum_{j=0}^{3i} p_j t^j}{t^i} : p_0 = p_{3i} c_0^i,\ p_j \in \mathbb{R} \right\}, \quad \mathcal{Q} := \bigcup_{i=0}^{\infty} \mathcal{Q}_{\leq 3i},$$
$$\operatorname{Pos}(\mathcal{Q}_{\leq 3i}) := \{f \in \mathcal{Q}_{\leq 3i} : f(t) \geq 0 \text{ for every } t \in \mathbb{R}\},$$
$$\mathcal{R}_{\leq 3i} := \left\{ \frac{\sum_{j=0}^{3i} p_j t^j}{t^i} : p_0 = -p_{3i} c_0^i,\ p_j \in \mathbb{R} \right\}.$$

Theorem 6.1. *Let $p \in \operatorname{Pos}(\mathcal{Q}_{\leq 6k})$. Then there exist finitely many $f_i \in \mathcal{Q}_{\leq 3k}$ and $g_j \in \mathcal{R}_{\leq 3k}$ such that $p = \sum_i f_i^2 + \sum_j g_j^2$.*

Moreover, for $C = \mathcal{Z}(P_{12})$ the appropriate choices of f and $V^{(k)}$ in Theorem 2.3 are as stated in Table 2.

Proof. Let $\Phi : \mathbb{R}[\mathcal{Z}(P_{12})] \to \mathcal{Q}$ be a map defined by $\Phi(p(x,y)) = p\big(t, \frac{c(t)}{t}\big)$. Analogously as in the proof of Theorem 4.8 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_{12})]_{\leq i}$ is in one-to-one correspondence with the set $\mathcal{Q}_{\leq 3i}$ under Φ. Using Corollary 3.4, every extreme ray p of the cone $\operatorname{Pos}(\mathcal{Q}_{\leq 6k})$ is of the form $\frac{u^2}{t^{2k}}$ for some $u = \sum_{j=0}^{3k} u_j t^j \in \mathbb{R}[t]_{\leq 3k}$ such that $u_0^2 = u_{3k}^2 c_0^{2k}$. It follows that we either have $u_0 = u_{3k} c_0^k$ and $\frac{u}{t^k} \in \mathcal{Q}_{\leq 3k}$ or $u_0 = -u_{3k} c_0^k$ and $\frac{u}{t^k} \in \mathcal{R}_{\leq 3k}$.

It remains to prove the moreover part. Let $d_i(t) := c^i - 2t^{3i}$ and

$$\mathcal{B}_{\mathcal{R}_{\leq 3i}} := \left\{ \frac{d_i(t)}{t^i}, \frac{c(t)^{i-1}}{t^{i-1}}, \ldots, \frac{c(t)}{t}, 1, t, \ldots, t^{2i-1} \right\}$$

be the basis for $\mathcal{R}_{\leq 3i}$. Extending the ring isomorphism Φ to the isomorphism between quotient fields $\mathbb{R}(\mathcal{Z}(P_{12}))$ and $\operatorname{Quot}(\mathcal{Q})$ of $\mathbb{R}[\mathcal{Z}(P_{12})]$ and $\mathcal{Q}$, respectively, note that $\Phi^{-1}(\mathcal{B}_{\mathcal{R}_{\leq 3k}})$ is equal to $\mathcal{B}_{V^{(k)}}$ from Table 2. □

For $x \in \mathbb{C}$, we write Q_x for the divisor $[x : 1]$ on $\mathbb{P}^1$. We also write Q_∞ for the divisor $[1 : 0]$.

Below we present a constructive solution to the nonsingular $\mathcal{Z}(P_{12})$–TMP via the solution to the corresponding $(\mathbb{R} \setminus \{0\})$–TMP.

Constructive proof of Corollary 2.6 for $C = \mathcal{Z}(P_{12})$. Using the correspondence as in the proof of Theorem 6.1, the $\mathcal{Z}(P_{12})$–TMP for L is equivalent to the $(\mathbb{R} \setminus \{0\})$–TMP for $L_{\mathcal{Q}_{\leq 6k}} : \mathcal{Q}_{\leq 6k} \to \mathbb{R}$, $L_{\mathcal{Q}_{\leq 6k}}(p) := L_{\mathcal{Z}(P_{12})}(\Phi^{-1}(p))$. If $L_{\mathcal{Q}_{\leq 6k}}$ is a $(\mathbb{R} \setminus \{0\})$–mf, then it extends to the $(\mathbb{R} \setminus \{0\})$–mf

$$\widehat{L} : \mathcal{L}(2kQ_0 + 4kQ_\infty) \to \mathbb{R}.$$

In the ordered basis

$$\Big\{\frac{c(T)^k}{T^k}, \frac{d_k(T)}{T^k}, \frac{c(T)^{k-1}}{T^{k-1}}, \frac{c(T)^{k-2}}{T^{k-2}}, \ldots, \frac{c(T)}{T}, \mathit{1}, T, \ldots, T^{2k-1}\Big\}$$

of rows and columns, the strict square positivity and zhe $(V^{(k)}, 1)$–local strict square positivity of $L_{\mathcal{Z}(P_{12})}$ are equivalent to the partial positive definiteness of the matrix

$$\left(\begin{array}{ccccccccc}
\frac{c(T)^k}{T^k} & \frac{d_k(T)}{T^k} & \frac{c(T)^{k-1}}{T^{k-1}} & \cdots & \frac{c(T)}{T} & 1 & T & \cdots & T^{2k-1} \\
\widehat{L}(\frac{c^{2k}}{t^{2k}}) & ? & \widehat{L}(\frac{c^{2k-1}}{t^{2k-1}}) & \cdots & \widehat{L}(\frac{c^{k+1}}{t^{k+1}}) & \widehat{L}(\frac{c^k}{t^k}) & \widehat{L}(\frac{c^k}{t^{k-1}}) & \cdots & \widehat{L}(t^{k-1}c^k) \\
? & \widehat{L}(\frac{d_k^2}{t^{2k}}) & \widehat{L}(\frac{c^{k-1}d_k}{t^{2k-1}}) & \cdots & & \widehat{L}(\frac{d_k}{t^k}) & & & \widehat{L}(t^{k-1}d_k) \\
\widehat{L}(\frac{c^{2k-1}}{t^{2k-1}}) & \widehat{L}(\frac{c^{k-1}d_k}{t^{2k-1}}) & \widehat{L}(\frac{c^{2k-2}}{t^{2k-2}}) & \cdots & & \widehat{L}(\frac{c^k}{t^k}) & \cdots & \cdots & \widehat{L}(t^{k-1}c^{k-1}) \\
\vdots & & & & & & \ddots & & \vdots \\
\widehat{L}(\frac{c^k}{t^k}) & & & & & & & & \widehat{L}(t^{2k-1}) \\
\vdots & & & & & & & \ddots & \vdots \\
\widehat{L}(t^{k-1}c^k) & \cdots & & & \cdots & \widehat{L}(t^{2k-1}) & \cdots & \cdots & \widehat{L}(t^{2k-2})
\end{array}\right).$$

The missing entries are at the positions $(1, 2)$ and $(2, 1)$, since the value $\widehat{L}(\frac{c^k d_k}{t^{2k}})$ is unknown. Then there exists an interval $(a, b) \subset \mathbb{R}$, such that for every $\widehat{L}(\frac{c^k d_k}{t^{2k}}) \in (a, b)$, the completion is positive definite (see e.g., [65, Lemma 2.4]) and for every such completion the functional $\widehat{L}$ has a $(3k + 1)$–atomic $(\mathbb{R} \setminus \{0\})$–rm by [67, Theorem 3.1]. ☐

Remark 6.2. By [64, Theorem 3.1], it follows that for one of the two positive semidefinite completions of the partial matrix above, the $(\mathbb{R}\setminus\{0\})$–rm is $(3k)$–atomic. The proof is by analysing the completion of the matrix of $L_{\mathcal{Z}(P_{12})}$ in the usual ordered basis $\{T^i \colon -k \leq i \leq 2k\}$ of rows and columns. Then the variable is the moment of t^{4k} and it also occurs linearly in the left-upper corner of the matrix. This makes the analysis of the existence of a $(\mathbb{R} \setminus \{0\})$–rm in the psd cases tractable.

Let

$$\mathcal{B}_{\mathcal{Q}_{\leq 3i}} := \Big\{\frac{c(t)^i}{t^i}, \frac{c(t)^{i-1}}{t^{i-1}}, \ldots, \frac{c(t)}{t}, 1, t, \ldots, t^{2i-1}\Big\}$$

be a basis for $\mathcal{Q}_{\leq 3i}$.

The following theorem solves the singular $\mathcal{Z}(P_{12})$–TMP.

Theorem 6.3 (Singular $\mathcal{Z}(P_{12})$–TMP)**.** *Let $L : \mathbb{R}[x, y]_{\leq 2k} \to \mathbb{R}$ be a $\mathcal{Z}(P_{12})$–singular linear functional. Let $V^{(k)}$ is as in Table 2 above. Then the following are equivalent:*

(1) L is a $\mathcal{Z}(P_{12})$–moment functional.

(2) $L_{\mathcal{Z}(P_{12})}$ is square positive and $(V^{(k)}, 1)$–locally square positive, and one of the following holds:

(a) *For* $U := \operatorname{Span}\left(\Phi^{-1}\big(\mathcal{B}_{\mathcal{Q}_{\leq 3k}} \setminus \{\frac{c^k}{t^k}\}\big)\right)$ *it holds that*

$$\operatorname{rank} \overline{L}_{\mathcal{Z}(P_{12})} = \operatorname{rank}\big((\overline{L}_{\mathcal{Z}(P_{12})})|_U\big).$$

(b) *For* $W := \operatorname{Span}\big(\Phi^{-1}\big(\mathcal{B}_{\mathcal{R}_{\leq 3k}} \setminus \{\frac{d_k}{t^k}\}\big)\big)$ *it holds that*

$$\operatorname{rank} \overline{L}_{\mathcal{Z}(P_{12}),V^{(k)},1} = \operatorname{rank}\Big((\overline{L}_{\mathcal{Z}(P_{12}),V^{(k)},1})|_W\Big).$$

Proof. Following the constructive proof of Corollary 2.6 for $C = \mathcal{Z}(P_{12})$ via the solution to the $(\mathbb{R}\backslash\{0\})$–TMP above and replacing positive definiteness with positive semidefiniteness, the only addition is that a psd extension $\widehat{L}$ is not necessarily a $(\mathbb{R} \setminus \{0\})$–mf. Let $D := Q_0 + 2Q_\infty$. Since L is $\mathcal{Z}(P_{12})$–singular and using [67, Theorem 3.1], (1) is equivalent to:

$$\begin{aligned}&\text{There is a square positive extension } L_{\mathcal{L}(2kD)} : \mathcal{L}(2kD) \to \mathbb{R} \text{ of } L_{\mathcal{Q}_{\leq 6k}} \text{ with}\\ &\operatorname{rank} \overline{L}_{\mathcal{L}(2kD)} = \operatorname{rank} \overline{L}_{\mathcal{L}(2kD-2Q_0)} = \operatorname{rank} \overline{L}_{\mathcal{L}(2kD-2Q_\infty)}.\end{aligned} \tag{6.1}$$

Let

$$\mathcal{B} := \Big\{\frac{c(t)^k}{t^k}, \frac{d_k(t)}{t^k}, \frac{c(t)^{k-1}}{t^{k-1}}, \ldots, \frac{c(t)}{t}, 1, t, \ldots, t^{2k-1}\Big\} \tag{6.2}$$

be a basis of $\mathcal{L}(kD)$. Let $\widetilde{D} := kD - Q_0 - Q_\infty$. We have that

$$\mathcal{L}(\widetilde{D}) = \operatorname{Span}\Big\{\mathcal{B} \setminus \Big\{\frac{c(t)^k}{t^k}, \frac{d_k(t)}{t^k}\Big\}\Big\}.$$

[67, Theorem 3.1] also implies the following:

Fact. If $(L_{\mathcal{L}(2kD)})|_{\mathcal{L}(2\widetilde{D})} : \mathcal{L}(2\widetilde{D}) \to \mathbb{R}$ is singular and admits some representing measure, then its extension to $\mathcal{L}(2kD)$, generated by any measure, does not increase the rank of the corresponding bilinear form.

Note that the rank conditions in (6.1) mean that in the matrix M, representing the bilinear form of the extension $L_{\mathcal{L}(2kD)}$ in the basis $\mathcal{B}$, there is a relation

$$\gamma\frac{c(T)^k}{T^k} + \delta\frac{d_k(T)}{T^k} = \alpha_0 \mathit{1} + \sum_{j=1}^{2k-1} \alpha_j T^j + \sum_{\ell=1}^{k-1} \beta_k \frac{c(T)^\ell}{T^\ell} \tag{6.3}$$

with γ, δ not both equal to 0.

Let us now prove the implication (6.1) ⇒ (2). The square positivity and the $(V^{(k)}, 1)$–local square positivity of $L_{\mathcal{Z}(P_{12})}$ are clear. If $\delta = 0$ in (6.3), then (2a) holds. If $\gamma = 0$, then (2b) holds. It remains to study the case: $\gamma \neq 0$ and $\delta \neq 0$. We separate two cases according to the $\mathcal{Z}(P_{12})$–singularity of L:

Case 1: $L_{\mathcal{Z}(P_{12})}$ is singular. There are some $\widetilde{\alpha}_j, \widetilde{\beta}_k, \widetilde{\gamma} \in \mathbb{R}$, not all equal to 0, such that in the matrix representation of $\overline{L}_{\mathcal{Z}(P_{12})}$ with respect to the basis $\mathcal{B}_{\mathcal{Q}_{\leq 3k}}$, the relation

$$\widetilde{\gamma}\frac{c(T)^k}{T^k} = \sum_{j=0}^{2k-1} \widetilde{\alpha}_j T^j + \sum_{\ell=1}^{k-1} \widetilde{\beta}_k \frac{c(T)^\ell}{T^\ell} \tag{6.4}$$

holds. If $\widetilde{\gamma} \neq 0$, then (2a) holds. Otherwise, $\widetilde{\gamma} = 0$. By the extension principle [31, Proposition 2.4], $M|_{\mathcal{L}(\widetilde{D})}$ is singular. By Fact above, (2a) and (2b) hold.

Case 2: $L_{\mathcal{Z}(P_{12})}$ is not singular, but $L_{\mathcal{Z}(P_{12})}$ is $(V^{(k)}, 1)$–locally singular. The proof is analogous to Case 1, only that one starts with the relation

$$\widetilde{\delta}\frac{c(T)^k}{T^k} = \sum_{j=0}^{2k-1} \widetilde{\alpha}_j T^j + \sum_{\ell=1}^{k-1} \widetilde{\beta}_k \frac{c(T)^\ell}{T^\ell} \tag{6.5}$$

in the matrix, representing $\overline{L}_{\mathcal{Z}(P_{12}), V^{(k)}, 1}$, in the basis $\mathcal{B}_{\mathcal{R}_{\leq 3k}}$.

It remains to prove the implication (2) $\Rightarrow$ (6.1). The existence of a square positive extension is clear from the positivity assumptions on L_C. The rank condition in (6.1) follows from either (6.4), where $\widetilde{\gamma} \neq 0$ under the assumption (2a), or (6.5), where $\widetilde{\delta} \neq 0$ under the assumption (2b). □

7. $\mathcal{Z}(P_{13})$–TMP FOR $P_{13}(x,y) = y - x^3$

Note that $(x(t), y(t)) = (t, t^3)$, $t \in \mathbb{R}$ is a parametrization of $\mathcal{Z}(P)$. Let

$$\mathcal{P}_{\leq 3i} := \left\{ \sum_{j=0}^{3i} p_j t^j : p_{3i-1} = 0,\ p_j \in \mathbb{R} \right\}, \quad \mathcal{P} := \bigcup_{i=0}^{\infty} \mathcal{P}_{\leq 3i},$$
$$\mathrm{Pos}(\mathcal{P}_{\leq 3i}) := \{ f \in \mathcal{P}_{\leq 3i} : f(t) \geq 0 \text{ for every } t \in \mathbb{R} \}.$$

Theorem 7.1. *The following statements are equivalent:*

(1) $p \in \mathrm{Pos}(\mathcal{P}_{\leq 6k})$.

(2) There exist finitely many $f_i \in \mathcal{P}_{\leq 3k}, g_j \in \mathbb{R}[t]_{\leq 3k-1}$ *such that* $p = \sum_i f_i^2 + \sum_j g_j^2$.

Moreover, for $C = \mathcal{Z}(P_{13})$ *the appropriate choices of* f *and* $V^{(k)}$ *in Theorem 2.3 are as stated in Table 2.*

Proof. The nontrivial implication is (1) $\Rightarrow$ (2). Let $\Phi : \mathbb{R}[\mathcal{Z}(P_{13})] \to \mathcal{P}$ be a map defined by $\Phi(p(x,y)) = p\big(t, t^3\big)$. Analogously as in the proof of Theorem 4.8 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_{13})]_{\leq i}$ is in one-to-one correspondence with the set $\mathcal{P}_{\leq 3i}$ under Φ. Using Corollary 3.4, every extreme ray p of the cone $\mathrm{Pos}(\mathcal{P}_{\leq 6k})$ is of the form u^2 for some $u = \sum_{j=0}^{3k} u_j t^j \in \mathbb{R}[t]_{\leq 3k}$ such that $0 = (u^2)_{6k-1} = 2u_{3k-1}u_{3k}$. If $u_{3k-1} = 0$, then $u \in \mathcal{Q}_{\leq 3k}$. Else $u_{3k} = 0$ and $u \in \mathbb{R}[x]_{\leq 3k-1}$.

It remains to prove the moreover part. Notice that $\Phi^{-1}(\mathbb{R}[x]_{\leq 3k-1}) = V^{(k)}$ is equal to $\mathcal{B}_{V^{(k)}}$ from Table 2. □

Example 7.2. Let $k = 3$ and $\beta_{ij} = L(x^i y^j)$ for $i, j \geq 0$, $i + j \leq 6$. Then the square positivity and the $(V^{(3)}, 1)$–local square positivity of $L_{\mathcal{Z}(P_{13})}$ are equivalent to the partial positive semidefiniteness of the following Hankel matrix:

$$\begin{array}{cc} & \begin{array}{cccccc} \mathit{1} & T & \cdots & T^7 & T^8 & T^9 \end{array} \\ \begin{array}{c} \mathit{1} \\ T \\ \vdots \\ T^7 \\ T^8 \\ T^9 \end{array} & \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_7 & \gamma_8 & \gamma_9 \\ \gamma_1 & \gamma_2 & \cdots & \gamma_8 & \gamma_9 & \gamma_{10} \\ \vdots & & \ddots & \vdots & & \vdots \\ \gamma_7 & \gamma_8 & \cdots & \gamma_{14} & & \gamma_{16} \\ \vdots & & & & \ddots & ? \\ \gamma_9 & \gamma_{10} & \cdots & \gamma_{16} & ? & \gamma_{18} \end{pmatrix} \end{array},$$

where $\gamma_i := \beta_{i \bmod 3, \lfloor \frac{i}{3} \rfloor}$ for each i. Note that the missing entries are at the positions $(9,10)$ and $(10,9)$, since the value $L_{\mathcal{Z}(P_{13})}(y^5x^2)$ is unknown. The matrix representation of $\overline{L}_{\mathcal{Z}(P_{13})}$ (resp., $\overline{L}_{\mathcal{Z}(P_{13}),V^{(k)},1}$) is the restriction of this matrix to a submatrix an all rows and columns but the one indexed with T^8 (resp., but the one indexed with T^9).

Remark 7.3. The first solution to the $\mathcal{Z}(P)$–TMP is [32] and is based on the flat extension theorem. Another approach by solving the equivalent $\mathbb{R}$–TMP from Example 7.2 is presented in [66, Section 3]. The third approach using the result on the core variety is [36].

8. $\mathcal{Z}(P)$–TMP FOR REDUCIBLE CUBIC POLYNOMIAL $P(x,y) \in \mathbb{R}[x,y]$, $\deg P = 3$

Assume the notation as in §2, §3. Let P be as in the title of the section.

Proposition 8.1. *Up to invertible affine linear change of variables every reducible cubic polynomial P such that $\mathcal{Z}(P) \not\subseteq \mathcal{Z}(Q)$ for any $Q \in \mathbb{R}[x,y]_{\leq 2}$, has one of the following forms:*

- *(i) Circular type 1: $P_{14}(x,y) = y(ay + x^2 + y^2)$, $a \in \mathbb{R} \setminus \{0\}$.*
- *(ii) Circular type 2: $P_{15}(x,y) = y(1 + ay + x^2 + y^2)$, $|a| > 2$.*
- *(iii) Circular type 3: $P_{16}(x,y) = y(1 + ay - x^2 - y^2)$, $a \in \mathbb{R}$.*
- *(iv) Parabolic type 1: $P_{17}(x,y) = y(x^2 - y)$.*
- *(v) Parabolic type 2: $P_{18}(x,y) = y(x - y^2)$.*
- *(vi) Parabolic type 3: $P_{19}(x,y) = y(1 + y + x^2)$.*
- *(vii) Parabolic type 4: $P_{20}(x,y) = y(1 + y - x^2)$.*
- *(viii) Hyperbolic type 1: $P_{21}(x,y) = y(1 - xy)$.*
- *(ix) Hyperbolic type 2: $P_{22}(x,y) = y(x + y + axy)$, $a \in \mathbb{R} \setminus \{0\}$.*
- *(x) Hyperbolic type 3: $P_{23}(x,y) = y(ay + x^2 - y^2)$, $a \in \mathbb{R} \setminus \{0\}$.*
- *(xi) Hyperbolic type 4: $P_{24}(x,y) = y(1 + ay + x^2 - y^2)$, $|a| \neq 2$.*
- *(xii) Hyperbolic type 5: $P_{25}(x,y) = y(1 + ay - x^2 + y^2)$.*
- *(xiii) Parallel lines type: $P_{26}(x,y) = y(a + y)(b + y)$, $a, b \in \mathbb{R} \setminus \{0\}$, $a \neq b$.*
- *(xiv) Intersecting lines type 1: $P_{27}(x,y) = y(x - y)(x + y)$,*
- *(xv) Intersecting lines type 2: $P_{28}(x,y) = yx(y + 1)$,*
- *(xvi) Intersecting lines type 3: $P_{29}(x,y) = y(1 + x - y)(1 - x - y)$.*

Proof. If we combine types (ii), (iii), (vi), (vii), (xi), (xii) and (xvi) into a common type $y(1+ay+bx^2+cy^2)$, $b \neq 0$, called *mixed type*, then this is [65, Proposition 3.1]. (Note that in the statement of the latter type (iv) is missing, but in the proof it is Case 2.1.1.1. Also the types (x) and (xiv) are combined under Hyperbolic type 3. Here we separate the case $a \neq 0$, which is the type (x), from $a = 0$, which is the type (xiv).

We will show that the above mixed type actually decomposes into the types mentioned in the first sentence of the proof. Applying an alt $(x, y) \mapsto (\sqrt{b}x, y)$ if $b > 0$ and $(x, y) \mapsto (\sqrt{-b}x, y)$ if $b < 0$, we can first split the mixed type into two types, i.e., $M_1 : y(1+ay+x^2+cy^2)$ and $M_2 : y(1+ay-x^2+cy^2)$. Further on, according to the sign of c we separate three cases: $c = 0$, $c > 0$ and $c < 0$.

Case 1: $c = 0$. We have $M_{1.1} : y(1+ay+x^2)$ or $M_{2.1} : y(1+ay-x^2)$. Further, if $a = 0$, then we have $M_{1.1.1} : y(1+x^2)$ or $M_{2.1.1} : y(1-x^2)$. The case $M_{1.1.1}$ does not fulfil the assumption $\mathcal{Z}(P) \not\subseteq \mathcal{Z}(Q)$, $Q \in \mathbb{R}[x, y]_{\leq 2}$, while in the case $M_{2.1.1}$, after applying an alt $(x, y) \mapsto (x - y, y)$, we get the type (xvi) in the statement of the proposition. If $a \neq 0$, then we can apply an alt $(x, y) \mapsto (x, ay)$ and get one of $M_{1.1.2} : y(1+y+x^2)$ and $M_{2.1.2} : y(1+y-x^2)$, which are the types (vi) and (vii).

Case 2: $c \neq 0$. We can apply an alt $(x, y) \mapsto (x, \sqrt{c}y)$ if $c > 0$ and $(x, y) \mapsto (x, \sqrt{-c}y)$ if $c < 0$, and get one of the types $M_{1.2.1} : y(1+ay+x^2+y^2)$, $M_{1.2.2} : y(1+ay+x^2-y^2)$, $M_{2.2.1} : y(1+ay-x^2+y^2)$ or $M_{2.2.2} : y(1+ay-x^2-y^2)$. Type $M_{1.2.1}$ with $|a| > 2$ give the type (ii) in the statement of the proposition. The type $M_{1.2.1}$ with $|a| \leq 2$ does not fulfil the assumption $\mathcal{Z}(P) \not\subseteq \mathcal{Z}(Q)$, $Q \in \mathbb{R}[x, y]_{\leq 2}$, since $\mathcal{Z}(y(1+ay+x^2+y^2)) = \mathcal{Z}(y((y+\frac{a}{2})^2+x^2+1-\frac{a^2}{4}))$ is a union of a line and at most one point in $\mathbb{R}^2$. The type $M_{1.2.2}$ gives the type (xi) in the statement of the proposition, type $M_{2.2.1}$ gives type (xii) if $|a| \neq -2$ and type (xv) if $|a| = 2$ (after possibly applying an alt $(x, y) \mapsto (x, -y)$), while type $M_{2.2.2}$ gives type (iii). ☐

The main results of this section are explicit descriptions of the pair $(f, V^{(k)})$ in Theorem 2.3 for each $C = \mathcal{Z}(P_i)$ from Proposition 8.1 above.

Throughout the whole section, for $x \in \mathbb{C}$ we write Q_x for the divisor $[x : 1]$ on $\mathbb{P}^1$. We also write Q_∞ for the divisor $[1 : 0]$.

8.1. **Circular type 1.**

Let P_{14} be as in Proposition 8.1 above. A circle $ay + x^2 + y^2 = 0$, centered in $(0, -\frac{a}{2})$ and having radius $-\frac{a}{2}$, has a rational parametrization

$$(x(t), y(t)) = \Big(-\frac{a}{2}\frac{t^2-1}{t^2+1}, -\frac{a}{2}\frac{(t+1)^2}{t^2+1}\Big), \quad t \in \mathbb{R}.$$

Let $D := Q_{\mathrm{i}} + Q_{-\mathrm{i}}$ and

$$\begin{aligned}
\mathrm{Circ}_1 &:= \left\{(f, g) \in \mathbb{R}[s] \times \mathbb{R}\Big[\frac{1}{t^2+1}, \frac{t}{t^2+1}\Big] : f(0) = g(-1), f'(0) = \frac{2g'(-1)}{a}\right\},\\
(\mathrm{Circ}_1)_{\leq i} &:= \left\{(f, g) \in \mathbb{R}[s]_{\leq i} \times \mathcal{L}(iD) \colon f(0) = g(-1),\ f'(0) = \frac{2g'(-1)}{a}\right\},\\
\mathrm{Pos}((\mathrm{Circ}_1)_{\leq i}) &:= \{(f, g) \in (\mathrm{Circ}_1)_{\leq i} \colon f(s) \geq 0,\ g(t) \geq 0 \text{ for every } (s, t) \in \mathbb{R}^2\},
\end{aligned}$$

$$(\widetilde{\mathrm{Circ}_1})_{\leq i} := \{(f,g) \in \mathbb{R}[s]_{\leq i} \times \mathcal{L}(iD) \colon f(0) = g(-1) = 0\}.$$

Theorem 8.2. *Let $(p_1, p_2) \in \mathrm{Pos}((\mathrm{Circ}_1)_{\leq 2k})$. Then there exist finitely many $(f_{1;i}, f_{2,i}) \in (\mathrm{Circ}_1)_{\leq k}$ and $(g_{1;j}, g_{2;j}) \in (\widetilde{\mathrm{Circ}_1})_{\leq k}$ such that*

$$(p_1, p_2) = \sum\nolimits_i (f_{1;i}^2, f_{2;i}^2) + \sum\nolimits_j (g_{1;j}^2, g_{2;j}^2).$$

Moreover, for $C = \mathcal{Z}(P_{14})$ the appropriate choices of f and $V^{(k)}$ in Theorem 2.3 are as stated in Table 3.

Proof. Let $\Phi : \mathbb{R}[\mathcal{Z}(P_{14})] \to \mathrm{Circ}_1$ be a map defined by

$$\Phi(p(x,y)) := \Big(\underbrace{p(s,0)}_{\Phi_1(p)}, \underbrace{p\Big(-\frac{a}{2}\frac{t^2-1}{t^2+1}, -\frac{a}{2}\frac{(t+1)^2}{t^2+1}\Big)}_{\Phi_2(p)}\Big).$$

Clearly Φ is a well–defined ring homomorphism, because $\Phi(p) = 0$ for every $p \in I$ and

$$\begin{aligned}(\Phi_1(p))(0) &= (\Phi_2(p))(-1) = p(0,0),\\ (\Phi_1(p))'(0) &= (p(s,0))'(0) = \text{the coefficient of } p \text{ at } x,\\ (\Phi_2(p))'(-1) &= \Big(p\Big(-\frac{a}{2}\frac{t^2-1}{t^2+1}, -\frac{a}{2}\frac{(t+1)^2}{t^2+1}\Big)\Big)'(-1) = \frac{a}{2} \cdot (\Phi_1(p))'(0)\end{aligned}$$

for every $p \in \mathbb{R}[C]$. The inclusion $\Phi(\mathbb{R}[C]_{\leq i}) \subseteq (\mathrm{Circ}_1)_{\leq i}$ is clear. Since

$$\dim \Phi(\mathbb{R}[C]_{\leq i}) = \dim(\mathrm{Circ}_1)_{\leq i} = 3i,$$

we have equality for every i and Φ is also one-to-one.

Let $p = (p_1, p_2)$ be an extreme ray the cone $\mathrm{Pos}((\mathrm{Circ}_1)_{\leq 2k})$. Using Corollary 3.4, $(p_1, p_2) = (u_1^2, u_2^2)$ for some $u_1 \in \mathbb{R}[s]_{\leq k}$, $u_2 \in \mathcal{L}(kD)$, such that $u_1^2(0) = u_2^2(-1)$ and $(u_1^2)'(0) = \frac{2(u_2^2)'(-1)}{a}$. So $u_1(0) = \pm u_2(-1)$ and $2u_1(0)u_1'(0) = \frac{2}{a}u_2(-1)u_2'(-1)$. Upon multiplying u_2 with -1 if necessary we may assume $u_1(0) = u_2(-1)$. If $u_1(0) = u_2(-1) \neq 0$, then $u_1'(0) = \frac{2u_2'(-1)}{a}$, in which case $(u_1, u_2) \in (\mathrm{Circ}_1)_{\leq k}$. Otherwise $u_1(0) = u_2(-1) = 0$ in which case $(u_1, u_2) \in (\widetilde{\mathrm{Circ}_1})_{\leq k}$.

It remains to prove the moreover part. Let

$$\mathcal{B}_{\leq i} = \{1, x, y, x^2, xy, y^2, \ldots, x^j, x^{j-1}y, x^{j-2}y^2, \ldots x^i, x^{i-1}y, x^{i-2}y^2\}$$

be a basis for $\mathbb{R}[C]_{\leq i}$. Let $h(x,y) = \frac{ay+x^2+y^2}{x}$ and extend Φ to $h(x,y)$ by the same rule. Note that: $\Phi(h) = (s, 0) \in (\widetilde{\mathrm{Circ}_1})_{\leq i}$. Replace 1 by h to obtain $\widetilde{\mathcal{B}}_{\leq i} = \mathcal{B}_{\leq i} \setminus \{1\} \cup \{h\}$. Note that $\Phi(\widetilde{\mathcal{B}}_{\leq i})$ is a basis for $(\widetilde{\mathrm{Circ}_1})_{\leq i}$. □

8.2. **Circular type 2.**

Let P_{15} be as in Proposition 8.1 above. Upon applying and alt $(x,y) \mapsto (x,-y)$ we may assume that $a < 0$. A circle $1 + ay + x^2 + y^2 = 0$, centered in $(0, -\frac{a}{2})$ and having radius $r := \sqrt{-1 + \frac{a^2}{4}}$, has a rational parametrization

$$(x(t), y(t)) = \Big(r\frac{2t}{t^2+1}, r\frac{t^2-1}{t^2+1} - \frac{a}{2}\Big), \quad t \in \mathbb{R}.$$

A short computation shows that

$$(x(t_0), y(t_0)) = (\mathfrak{i}, 0) \quad \text{for } t_0 = -\frac{\mathfrak{i}}{2}(a + \sqrt{-4 + a^2}).$$

Let $D := Q_{\mathfrak{i}} + Q_{-\mathfrak{i}}$ and

$$\begin{aligned}\mathrm{Circ}_2 &:= \Big\{(f,g) \in \mathbb{R}[s] \times \mathbb{R}\Big[\frac{1}{t^2+1}, \frac{t}{t^2+1}\Big] : f(\mathfrak{i}) = g(t_0)\Big\},\\ (\mathrm{Circ}_2)_{\leq i} &:= \{(f,g) \in \mathbb{R}[s]_{\leq i} \times \mathcal{L}(iD) \colon f(\mathfrak{i}) = g(t_0)\},\\ \mathrm{Pos}((\mathrm{Circ}_2)_{\leq i}) &:= \{(f,g) \in (\mathrm{Circ}_2)_{\leq i} \colon f(s) \geq 0,\ g(t) \geq 0 \text{ for every } (s,t) \in \mathbb{R}^2\}.\end{aligned}$$

Theorem 8.3. *Let $(p_1,p_2) \in \mathrm{Pos}((\mathrm{Circ}_2)_{\leq 2k})$. Then there exist finitely many $(f_{1;i}, f_{2,i}) \in (\mathrm{Circ}_2)_{\leq k}$, $g_{1;j} \in \mathbb{R}[s]_{\leq k-1}$, $h_{2;\ell} \in \mathcal{L}((k-1)D)$ such that*

$$(p_1,p_2) = \sum\nolimits_i (f_{1;i}^2, f_{2;i}^2) + \sum\nolimits_j ((1+s^2)g_{1;j}^2, 0) + \sum\nolimits_\ell (0, y(t)h_{2;\ell}^2).$$

Moreover, for $C = \mathcal{Z}(P_{15})$ the appropriate choices of P_1 and P_2 in Theorem 2.4 are y and $1 + ay + x^2 + y^2$.

Proof. Let $\Phi : \mathbb{R}[\mathcal{Z}(P_{15})] \to \mathrm{Circ}_2$ be a map defined by

$$\Phi(p(x,y)) = \Big(p(s,0), p\Big(r\frac{2t}{t^2+1},\ r\frac{t^2-1}{t^2+1} - \frac{a}{2}\Big)\Big)$$

Analogously as in the proof of Theorem 8.2 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_{15})]_{\leq i}$ is in one-to-one correspondence with the set $(\mathrm{Circ}_2)_{\leq 3i}$ under Φ.

Let $p = (p_1,p_2)$ be an extreme ray of the cone $\mathrm{Pos}((\mathrm{Circ}_2)_{\leq 2k})$. Using Corollary 3.4, one of the following cases occurs:

Case 1: *$p_1 \neq 0$, $p_2 \neq 0$ and each component p_1, p_2 has all zeroes and poles of even order.*

This implies that $(p_1,p_2) = (u_1^2, u_2^2)$ for some $u_1 \in \mathbb{R}[s]_{\leq k}$, $u_2 \in \mathcal{L}(kD)$ such that $u_1^2(\mathfrak{i}) = u_2^2(t_0)$. Further on, $u_1(\mathfrak{i}) = \pm u_2(t_0)$. Upon multiplying with -1 we may assume that $u_1(\mathfrak{i}) = u_2(t_0)$, whence $(u_1,u_2) \in (\mathrm{Circ}_2)_{\leq k}$.

Case 2: *$p_1 = 0$ or $p_2 = 0$.*

Case 2.1: $p_1 = 0$. In this case p_2 vanishes in t_0 and $-t_0$, since these two points correspond to $(\mathfrak{i},0)$ and $(-\mathfrak{i},0)$ in the ambient curve. Since $y(t)$ vanishes in $t_0, -t_0$ and has a quadratic numerator, if follows that $p_2(t) = y(t)^m r_2(t)$ for some m and r_2 does not vanish in $\pm t_0$. Since $y(t)$ is positive on the circle, it follows that $r_2(t) \geq 0$ for every $t \in \mathbb{R}$. Moreover, r_2 has only real zeroes and poles of even order. So it is of the form $r_2 = u_2^2$ for some $u_2 \in \mathcal{L}\big((k-m)D\big)$. Hence, $p_2 = (y(t)^{m/2}u_2)^2$ if m is even and $p_2 = y(t)(y(t)^{\lfloor m/2 \rfloor}u_2)^2$ if m is odd, whence $(0,p_2)$ is of the desired form.

Case 2.2: $p_2 = 0$. In this case p_1 vanishes in $\mathfrak{i}$, $-\mathfrak{i}$. It follows that $p_1(s) = (1+s^2)^m q_1(s)$, $q_2(s) \geq 0$ for every $s \in \mathbb{R}$ and q has only real zeroes. Hence, there is $u_1 \in \mathbb{R}[t]$ such that $p_1(s) = ((1+s^2)^{m/2}u_1)^2$ if m is even and $p_1(s) = (1+s^2)((1+s^2)^{\lfloor m/2 \rfloor}u_1)^2$ if m is odd, whence $(p_1,0)$ is of the desired form.

The moreover part is clear. □

8.3. **Circular type 3.** Let P_{16} be as in Proposition 8.1 above. A circle $1+ay-x^2-y^2=0$, centered in $(0,\frac{a}{2})$ and having radius $r:=\sqrt{1+\frac{a^2}{4}}$, has a rational parametrization

$$(x(t),y(t)=\Big(r\frac{2t}{t^2+1},r\frac{t^2-1}{t^2+1}+\frac{a}{2}\Big),\quad t\in\mathbb{R}.$$

A short computation shows that for

$$t_-=\frac{1}{2}\big(a-\sqrt{4+a^2}\big),\quad t_+=\frac{1}{2}\big(-a+\sqrt{4+a^2}\big)$$

we have that

$$(x(t_-),y(t_-))=(-1,0),\quad (x(t_+),y(t_+))=(1,0).$$

Let $D:=Q_{\mathrm{i}}+Q_{-\mathrm{i}}$ and

$$\begin{aligned}
\mathrm{Circ}_3&:=\Big\{(f,g)\in\mathbb{R}[s]\times\mathbb{R}\Big[\frac{1}{t^2+1},\frac{t}{t^2+1}\Big]\colon f(-1)=g(t_-),\ f(1)=g(t_+)\Big\},\\
(\mathrm{Circ}_3)_{\leq i}&:=\{(f,g)\in\mathbb{R}[s]_{\leq i}\times\mathcal{L}(iD)\colon f(-1)=g(t_-),\ f(1)=g(t_+)\},\\
\mathrm{Pos}((\mathrm{Circ}_3)_{\leq i})&:=\{(f,g)\in(\mathrm{Circ}_3)_{\leq i}\colon f(s)\geq 0,\ g(t)\geq 0 \text{ for every } (s,t)\in\mathbb{R}^2\},\\
(\widetilde{\mathrm{Circ}}_3)_{\leq i}&:=\{(f,g)\in\mathbb{R}[s]_{\leq i}\times\mathcal{L}(iD)\colon f(-1)=-g(t_-),\ f(1)=g(t_+)\}.
\end{aligned}$$

Theorem 8.4. *Let $(p_1,p_2)\in\mathrm{Pos}((\mathrm{Circ}_3)_{\leq 2k})$. Then there exist finitely many $(f_{1;i},f_{2,i})\in(\mathrm{Circ}_3)_{\leq k}$ and $(g_{1;j},g_{2;j})\in(\widetilde{\mathrm{Circ}}_3)_{\leq k}$ such that*

$$(p_1,p_2)=\sum\nolimits_i(f_{1;i}^2,f_{2;i}^2)+\sum\nolimits_j(g_{1;j}^2,g_{2;j}^2).$$

Moreover, for $C=\mathcal{Z}(P_{16})$ the appropriate choices of f and $V^{(k)}$ in Theorem 2.3 are as stated in Table 3.

Proof. Let $\Phi:\mathbb{R}[\mathcal{Z}(P_{16})]\to\mathrm{Circ}_2$ be a map defined by

$$\Phi(p(x,y))=\Big(p(s,0),p\Big(r\frac{2t}{t^2+1},\ r\frac{t^2-1}{t^2+1}+\frac{a}{2}\Big)\Big)$$

Analogously as in the proof of Theorem 8.2 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_{16})]_{\leq i}$ is in one-to-one correspondence with the set $(\mathrm{Circ}_3)_{\leq 3i}$ under Φ.

Let $p=(p_1,p_2)$ be an extreme ray of the cone $\mathrm{Pos}((\mathrm{Circ}_3)_{\leq 2k})$. Using Corollary 3.4, $(p_1,p_2)=(u_1^2,u_2^2)$ for some $u_1\in\mathbb{R}[s]_{\leq k}$, $u_2\in\mathcal{L}(iD)$ with $u_1^2(-1)=u_2^2(t_-)$ and $u_1^2(1)=u_2^2(t_+)$. Hence, $u_1(-1)=\pm u_2(t_-)$ and $u_1(1)=\pm u_2(t_+)$. Upon multiplying with -1 we may assume that $u_1(1)=u_2(t_+)$. If $u_1(-1)=u_2(t_-)$, then $(u_1,u_2)\in(\mathrm{Circ}_3)_{\leq k}$. Otherwise $u_1(-1)=-u_2(t_-)$ and $(u_1,u_2)\in(\widetilde{\mathrm{Circ}}_3)_{\leq k}$.

It remains to prove the moreover part. Let

$$\mathcal{B}_{\leq i}=\{1,x+1,x^2-1,x(x^2-1),\ldots,x^{k-2}(x^2-1),y,yx,\ldots,yx^{k-1},y^2,y^2x,\ldots,y^2x^{k-2}\}$$

be a basis for $\mathbb{R}[\mathcal{Z}(P_{16})]_{\leq i}$. Let $h(x,y):=1-x-2\frac{1+ay-x^2-y^2}{1+x}$ and extend Φ to $h(x,y)$ by the same rule. Note that: $\Phi(h)=(-(1-x),1-x)\in(\widetilde{\mathrm{Circ}}_3)_{\leq i}$. Replace 1 by h to obtain $\widetilde{\mathcal{B}}_{\leq i}=\mathcal{B}_{\leq i}\setminus\{1\}\cup\{h\}$. Note that $\Phi(\widetilde{\mathcal{B}}_{\leq i})$ is a basis for $(\widetilde{\mathrm{Circ}}_3)_{\leq i}$. □

8.4. **Parabolic type 1.** Let P_{17} be as in Proposition 8.1 above. Let

$$\begin{aligned}
\mathrm{Par}_1 &:= \{(f,g) \in \mathbb{R}[s] \times \mathbb{R}[t] \colon f(0) = g(0),\ f'(0) = g'(0)\}, \\
(\mathrm{Par}_1)_{\leq i} &:= \{(f,g) \in \mathbb{R}[s]_{\leq i} \times \mathbb{R}[t]_{\leq 2i} \colon f(0) = g(0),\ f'(0) = g'(0)\}, \\
\mathrm{Pos}((\mathrm{Par}_1)_{\leq i}) &:= \{(f,g) \in (\mathrm{Par}_1)_{\leq i} \colon f(s) \geq 0,\ g(t) \geq 0 \text{ for every } (s,t) \in \mathbb{R}^2\}, \\
(\widetilde{\mathrm{Par}_1})_{\leq i} &:= \{(f,g) \in \mathbb{R}[s]_{\leq i} \times \mathbb{R}[t]_{\leq 2i} \colon f(0) = g(0) = 0\}.
\end{aligned}$$

Theorem 8.5. *Let $(p_1,p_2) \in \mathrm{Pos}((\mathrm{Par}_1)_{\leq 2k})$. Then there exist finitely many $(f_{1;i}, f_{2,i}) \in (\mathrm{Par}_1)_{\leq k}$ and $(g_{1;j}, g_{2;j}) \in (\widetilde{\mathrm{Par}_1})_{\leq k}$ such that*

$$(p_1,p_2) = \sum\nolimits_i (f_{1;i}^2, f_{2;i}^2) + \sum\nolimits_j (g_{1;j}^2, g_{2;j}^2).$$

Moreover, for $C = \mathcal{Z}(P_{17})$ the appropriate choices of f and $V^{(k)}$ in Theorem 2.3 are as stated in Table 3.

Proof. Let $\Phi : \mathbb{R}[\mathcal{Z}(P_{17})] \to \mathrm{Par}_1$ be a map defined by $\Phi(p(x,y)) = (p(s,0), p(t,t^2))$. Analogously as in the proof of Theorem 8.2 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_{17})]_{\leq i}$ is in one-to-one correspondence with the set $(\mathrm{Par}_1)_{\leq 3i}$ under Φ.

Let $p = (p_1,p_2)$ be an extreme ray the cone $\mathrm{Pos}((\mathrm{Par}_1)_{\leq 2k})$. Using Corollary 3.4, $(p_1,p_2) = (u_1^2, u_2^2)$ for some $u_1 \in \mathbb{R}[t]_{\leq k}$, $u_2 \in \mathbb{R}[s]_{\leq 2k}$ such that $u_1^2(0) = u_2^2(0)$ and $2u_1(0)u_1'(0) = (u_1^2)'(0) = (u_2^2)'(0) = 2u_2(0)u_2'(0)$. So $u_1(0) = \pm u_2(0)$. Multiplying u_2 with -1 if necesarry we may assume that $u_1(0) = u_2(0)$. If $u_1(0) = u_2(0) = 0$, then $(u_1,u_2) \in (\widetilde{\mathrm{Par}_1})_{\leq k}$. Else $u_1'(0) = u_2'(0)$, and $(u_1,u_2) \in (\mathrm{Par}_1)_{\leq k}$.

It remains to prove the moreover part. Let

$$\mathcal{B}_{\leq i} = \{1, x, x^2, \ldots, x^k, y, y^2, \ldots, y^k, yx, y^2, y^2x, \ldots, y^{k-1}x\}$$

be a basis for $\mathbb{R}[\mathcal{Z}(P_{17})]_{\leq i}$. Let $h(x,y) := \frac{y}{x}$ and extend Φ to $h(x,y)$ by the same rule. Note that: $\Phi(h) = (0,t) \in (\widetilde{\mathrm{Par}_2})_{\leq i}$. Replace 1 by h to obtain $\widetilde{\mathcal{B}}_{\leq i} = \mathcal{B}_{\leq i} \setminus \{1\} \cup \{h\}$. Note that $\Phi(\widetilde{\mathcal{B}}_{\leq i})$ is a basis for $(\widetilde{\mathrm{Par}_2})_{\leq i}$. □

8.5. **Parabolic type 2.** Let P_{18} be as in Proposition 8.1 above. Let

$$\begin{aligned}
\mathrm{Par}_2 &:= \{(f,g) \in \mathbb{R}[s] \times \mathbb{R}[t] \colon f(0) = g(0)\}, \\
(\mathrm{Par}_2)_{\leq i} &:= \{(f,g) \in \mathbb{R}[s]_{\leq i} \times \mathbb{R}[t]_{\leq 2i} \colon f(0) = g(0),\ f_i = g_{2i}\}, \\
\mathrm{Pos}((\mathrm{Par}_2)_{\leq i}) &:= \{(f,g) \in \mathrm{Par}_{\leq i} \colon f(s) \geq 0,\ g(t) \geq 0 \text{ for every } (s,t) \in \mathbb{R}^2\}, \\
(\widetilde{\mathrm{Par}_2})_{\leq i} &:= \{(f,g) \in \mathbb{R}[s]_{\leq i} \times \mathbb{R}[t]_{\leq 2i} \colon f(0) = g(0),\ f_i = -g_{2i}\}.
\end{aligned}$$

Theorem 8.6. *Let $(p_1,p_2) \in \mathrm{Pos}((\mathrm{Par}_2)_{\leq 2k})$. Then there exist finitely many $(f_{1;i}, f_{2,i}) \in (\mathrm{Par}_2)_{\leq k}$ and $(g_{1;j}, g_{2;j}) \in (\widetilde{\mathrm{Par}_2})_{\leq k}$ such that*

$$(p_1,p_2) = \sum\nolimits_i (f_{1;i}^2, f_{2;i}^2) + \sum\nolimits_j (g_{1;j}^2, g_{2;j}^2).$$

Moreover, for $C = \mathcal{Z}(P_{18})$ the appropriate choices of f and $V^{(k)}$ in Theorem 2.3 are as stated in Table 3.

Proof. Let $\Phi : \mathbb{R}[\mathcal{Z}(P_{18})] \to \mathrm{Par}_2$ be a map defined by $\Phi(p(x,y)) = (p(s,0), p(t^2,t))$. Analogously as in the proof of Theorem 8.2 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_{18})]_{\leq i}$ is in one-to-one correspondence with the set $(\mathrm{Par}_2)_{\leq 3i}$ under Φ.

Let $p = (p_1,p_2)$ be an extreme ray the cone $\mathrm{Pos}((\mathrm{Par}_2)_{\leq 2k})$. Using Corollary 3.4, $(p_1,p_2) = (u_1^2,u_2^2)$ for some $u_1 \in \mathbb{R}[t]_{\leq k}$, $u_2 \in \mathbb{R}[s]_{\leq 2k}$ such that $u_1^2(0) = u_2^2(0)$ and $(u_1)_{2k}^2 = (u_2)_{4k}^2$. So $u_1(0) = \pm u_2(0)$ and $(u_1)_k = \pm(u_2)_{2k}$. Multiplying u_2 with -1 if necesarry we may assume that $u_1(0) = u_2(0)$. Then $(u_1,u_2) \in (\mathrm{Par}_2)_{\leq k}$ if $(u_1)_k = (u_2)_{2k}$ and $(u_1,u_2) \in (\widetilde{\mathrm{Par}_2})_{\leq k}$ if $(u_1)_k = -(u_2)_{2k}$.

It remains to prove the moreover part. Let

$$\mathcal{B}_{(\mathrm{Par}_2)_{\leq i}} := \{(1,1),(s,t),(s^2,t^2),\ldots,(s^i,t^{2i}),(0,t),(0,t^2),\ldots,(0,t^{2i-1})\}.$$

be a basis for $\mathbb{R}[\mathcal{Z}(P_{18})]_{\leq i}$. Replacing (s^i,t^{2i}) by $(s^i,-t^{2i})$ we get a basis $\mathcal{B}_{(\widetilde{\mathrm{Par}_2})_{\leq i}}$ for $(\widetilde{\mathrm{Par}_2})_{\leq i}$. Note that $\widetilde{\mathcal{B}}_i = \Phi^{-1}(\mathcal{B}_{(\widetilde{\mathrm{Par}_2})_{\leq i}}) = \mathcal{B}_i \setminus \{x^i\} \cup \{x^i - 2y^2x^{i-1}\}$, which concludes the proof of the theorem. ☐

8.6. **Parabolic type 3.** Let P_{19} be as in Proposition 8.1 above. A parametrization of the parabola $1 + y + x^2 = 0$ is

$$(x(t),y(t)) = (t,-t^2-1), \quad t \in \mathbb{R}.$$

Notice that $(x(\mathfrak{i}),y(\mathfrak{i})) = (\mathfrak{i},0)$. Let

$$\begin{aligned}
\mathrm{Par}_3 &:= \{(f,g) \in \mathbb{R}[s] \times \mathbb{R}[t] \colon f(\mathfrak{i}) = g(\mathfrak{i})\},\\
(\mathrm{Par}_3)_{\leq i} &:= \{(f,g) \in \mathbb{R}[s]_{\leq i} \times \mathbb{R}[t]_{\leq 2i} \colon f(\mathfrak{i}) = g(\mathfrak{i})\},\\
\mathrm{Pos}((\mathrm{Par}_3)_{\leq i}) &:= \{(f,g) \in (\mathrm{Par}_3)_{\leq i} \colon f(s) \geq 0,\ g(t) \geq 0 \text{ for every } (s,t) \in \mathbb{R}^2\}.
\end{aligned}$$

Theorem 8.7. *Let* $(p_1,p_2) \in \mathrm{Pos}((\mathrm{Par}_3)_{\leq 2k})$. *Then there exist finitely many* $(f_{1;i}, f_{2,i}) \in (\mathrm{Par}_3)_{\leq k}$, $g_{1;j} \in \mathbb{R}[s]_{\leq k-1}$ *and* $h_{2;\ell} \in \mathbb{R}[t]_{\leq 2k-1}$ *such that*

$$(p_1,p_2) = \sum\nolimits_i (f_{1;i}^2, f_{2;i}^2) + \sum\nolimits_j ((1+s^2)g_{1;j}^2, 0) + \sum\nolimits_\ell (0,(1+t^2)h_{2;\ell}^2).$$

Moreover, for $C = \mathcal{Z}(P_{19})$ *the appropriate choices of* P_1 *and* P_2 *in Theorem 2.4 are* y *and* $1 + y + x^2$.

Proof. Let $\Phi : \mathbb{R}[\mathcal{Z}(P_{19})] \to \mathrm{Par}_3$ be a map defined by

$$\Phi(p(x,y)) = (p(s,0), p(t,-t^2-1))$$

Analogously as in the proof of Theorem 8.2 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_{19})]_{\leq i}$ is in one-to-one correspondence with the set $(\mathrm{Par}_3)_{\leq 3i}$ under Φ.

Let $p = (p_1,p_2)$ be an extreme ray of the cone $\mathrm{Pos}((\mathrm{Par}_3)_{\leq 2k})$. Using Corollary 3.4, one of the following cases occurs:

Case 1: $p_1 \neq 0$, $p_2 \neq 0$ *and each component* p_1, p_2 *has all zeroes and poles of even order.*

This implies that $(p_1,p_2) = (u_1^2,u_2^2)$ for some $u_1 \in \mathbb{R}[s]_{\leq k}$, $u_2 \in \mathbb{R}[t]_{\leq 2k}$. such that $u_1^2(\mathfrak{i}) = u_2^2(\mathfrak{i})$. So $u_1(\mathfrak{i}) = \pm u_2(\mathfrak{i})$. Upon multiplying with -1 we may assume that

$u_1(\mathfrak{i}) = u_2(\mathfrak{i})$. Hence, $(u_1, u_2) \in (\mathrm{Par}_3)_{\leq k}$.

Case 2: $p_1 = 0$ *or* $p_2 = 0$.

Case 2.1: $p_1 = 0$. In this case p_2 vanishes in $\mathfrak{i}$ and $-\mathfrak{i}$. It follows that $p_2(t) = (1+t^2)^m q_2(t)$, $m \in \mathbb{N}$, $q_2(t) \geq 0$ for every $t \in \mathbb{R}$ and q has only real zeroes. Hence, there is $u_2 \in \mathbb{R}[t]$ such that $p_2(t) = ((1+t^2)^{m/2} u_2)^2$ if m is even and $p_2(t) = (1+t^2)((1+t^2)^{\lfloor m/2 \rfloor} u_2)^2$ if m is odd, whence $(0, p_2)$ is of the desired form.

Case 2.2: $p_2 = 0$. In this case p_1 vanishes in $\mathfrak{i}$, $-\mathfrak{i}$. If follows that $p_1(s) = (1+s^2)^m q_1(s)$, $m \in \mathbb{N}$, $q_1(s) \geq 0$ for every $s \in \mathbb{R}$ and q_1 has only real zeroes. Hence, there is $u_1 \in \mathbb{R}[s]$ such that $p_1(s) = ((1+s^2)^{m/2} u_1)^2$ if m is even and $p_1(s) = (1+s^2)((1+s^2)^{\lfloor m/2 \rfloor} u_1)^2$ if m is odd, whence $(p_1, 0)$ is of the desired form.

The moreover part is clear. ☐

8.7. **Parabolic type 4.** Let P_{20} be as in Proposition 8.1 above. A parametrization of the parabola $1 + y - x^2 = 0$ is

$$(x(t), y(t)) = (t, t^2 - 1), \quad t \in \mathbb{R}.$$

Let

$$\begin{aligned}
\mathrm{Par}_4 &:= \{(f,g) \in \mathbb{R}[s] \times \mathbb{R}[t] \colon f(-1) = g(-1),\ f(1) = g(1)\},\\
(\mathrm{Par}_4)_{\leq i} &:= \{(f,g) \in \mathbb{R}[s]_{\leq i} \times \mathbb{R}[t]_{\leq 2i} \colon f(-1) = g(-1),\ f(1) = g(1)\},\\
\mathrm{Pos}((\mathrm{Par}_4)_{\leq i}) &:= \{(f,g) \in (\mathrm{Par}_4)_{\leq i} \colon f(s) \geq 0,\ g(t) \geq 0\ \ \forall (s,t) \in \mathbb{R}^2\},\\
(\widetilde{\mathrm{Par}}_4)_{\leq i} &:= \{(f,g) \in \mathbb{R}[s]_{\leq i} \times \mathbb{R}[s]_{\leq 2i} \colon f(1) = g(1),\ f(-1) = -g(-1)\}.
\end{aligned}$$

Theorem 8.8. *Let* $(p_1, p_2) \in \mathrm{Pos}((\mathrm{Par}_4)_{\leq 2k})$. *Then there exist finitely many* $(f_{1;i}, f_{2,i}) \in (\mathrm{Par}_4)_{\leq k}$ *and* $(g_{1;j}, g_{2;j}) \in (\widetilde{\mathrm{Par}}_4)_{\leq k}$ *such that*

$$(p_1, p_2) = \sum\nolimits_i (f_{1;i}^2, f_{2;i}^2) + \sum\nolimits_j (g_{1;j}^2, g_{2;j}^2).$$

Moreover, for $C = \mathcal{Z}(P_{20})$ *the appropriate choices of* f *and* $V^{(k)}$ *in Theorem 2.3 are as stated in Table 3.*

Proof. Let $\Phi : \mathbb{R}[\mathcal{Z}(P_{20})] \to \mathrm{Par}_4$ be a map defined by

$$\Phi(p(x,y)) = \Phi(p(x,y)) = \big(p(s,0), p(t, t^2 - 1)\big)$$

Analogously as in the proof of Theorem 8.2 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_{20})]_{\leq i}$ is in one-to-one correspondence with the set $(\mathrm{Par}_4)_{\leq 3i}$ under Φ.

Let $p = (p_1, p_2)$ be an extreme ray the cone $\mathrm{Pos}((\mathrm{Par}_4)_{\leq 2k})$. Using Corollary 3.4, $(p_1, p_2) = (u_1^2, u_2^2)$ for some $u_1 \in \mathbb{R}[s]_{\leq k}$, $u_2 \in \mathbb{R}[t]_{\leq 2k}$ such that $u_1^2(\pm) = u_2^2(\pm)$. So $u_1(-1) = \pm u_2(-1)$ and $u_1(1) = \pm u_2(1)$. Upon multiplying with -1 we may assume that $u_1(1) = u_2(1)$. If $u_1(-1) = u_2(-1)$, then $(u_1, u_2) \in (\mathrm{Par}_4)_{\leq k}$. Otherwise $u_1(-1) = -u_2(-1)$ and $(u_1, u_2) \in (\widetilde{\mathrm{Par}}_4)_{\leq k}$.

It remains to prove the moreover part. Let

$$\mathcal{B}_{\leq i} = \{1, x+1, x^2-1, x(x^2-1), \ldots, x^{k-2}(x^2-1), y, yx, y^2, y^2x, \ldots, y^{k-1}, y^{k-1}x, y^k\}$$

be a basis for $\mathbb{R}[\mathcal{Z}(P_{20})]_{\leq i}$. Let $h(x,y) := 1 - x - 2\frac{1+y-x^2}{1+x}$ and extend Φ to $h(x,y)$ by the same rule. Note that: $\Phi(h) = (-(1-x), 1-x) \in \widetilde{(\mathrm{Par}_4)}_{\leq i}$. Replace 1 by h to obtain $\widetilde{\mathcal{B}}_{\leq i} = \mathcal{B}_{\leq i} \setminus \{1\} \cup \{h\}$. Note that $\Phi(\widetilde{\mathcal{B}}_{\leq i})$ is a basis for $\widetilde{(\mathrm{Par}_4)}_{\leq i}$. □

8.8. **Hyperbolic type 1.** Let P_{21} be as in Proposition 8.1 above. A rational parametrization of the hyperbola $1 - xy = 0$ is given by $(x(t), y(t)) = \big(t, \frac{1}{t}\big)$, $t \in \mathbb{R} \setminus \{0\}$. Let $D = Q_0 + Q_\infty$ and

$$\begin{aligned}
\mathrm{Hyp}_1 &:= \mathbb{R}[s] \times \mathbb{R}\big[t, t^{-1}\big],\\
(\mathrm{Hyp}_1)_{\leq i} &:= \Big\{(f,g) \in \mathbb{R}[s]_{\leq i} \times \mathcal{L}(iD)\colon\ f_{i-1} = g_{i-1},\ f_i = g_i,\\
&\qquad \text{where } f(s) = \sum\nolimits_{j=0}^{i} f_j s^j,\ f_j \in \mathbb{R} \text{ and } g(t) = \sum\nolimits_{j=-i}^{i} g_j t^j,\ g_j \in \mathbb{R}\Big\},\\
\mathrm{Pos}((\mathrm{Hyp}_1)_{\leq i}) &:= \{(f,g) \in (\mathrm{Hyp}_1)_{\leq i}\colon\ f(s) \geq 0,\ g(t) \geq 0 \text{ for every } (s,t) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})\},\\
\widetilde{(\mathrm{Hyp}_1)}_{\leq i} &= \mathbb{R}[s]_{\leq i-1} \times \mathcal{L}(iD - Q_\infty).
\end{aligned}$$

Theorem 8.9. *Let $(p_1, p_2) \in \mathrm{Pos}((\mathrm{Hyp}_1)_{\leq 2k})$. Then there exist finitely many $(f_{1;i}, f_{2,i}) \in (\mathrm{Hyp}_1)_{\leq k}$ and $(g_{1;j}, g_{2;j}) \in \widetilde{(\mathrm{Hyp}_1)}_{\leq k}$ such that*

$$(p_1, p_2) = \sum\nolimits_i (f_{1;i}^2, f_{2;i}^2) + \sum\nolimits_j (g_{1;j}^2, g_{2;j}^2).$$

Moreover, for $C = \mathcal{Z}(P_{21})$ the appropriate choices of f and $V^{(k)}$ in Theorem 2.3 are as stated in Table 3.

Proof. Let $\Phi : \mathbb{R}[\mathcal{Z}(P_{21})] \to \mathrm{Hyp}_1$ be a map with $\Phi(p(x,y)) = (p(s,0), p(t,t^{-1}))$. Analogously as in the proof of Theorem 8.2 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_{21})]_{\leq i}$ is in one-to-one correspondence with the set $(\mathrm{Hyp}_1)_{\leq 3i}$ under Φ.

Let $p = (p_1, p_2)$ be an extreme ray the cone $\mathrm{Pos}((\mathrm{Hyp}_1)_{\leq 2k})$. Using Corollary 3.4, $(p_1, p_2) = (u_1^2, u_2^2)$ for some $u_1 \in \mathbb{R}[s]_{\leq k}$, $u_2 = \frac{\widetilde{u}_2}{t^i}$, $\widetilde{u}_2 \in \mathbb{R}[t]_{\leq i}$, $i \leq k$, such that $(u_1^2)_{2k-1} = (\widetilde{u}_2^2)_{2k-1}$, $(u_1^2)_{2k} = (\widetilde{u}_2^2)_{2k}$. So $(u_1)_k = \pm(\widetilde{u}_2)_k$ and

$$2(u_1)_k (u_1)_{k-1} = 2(\widetilde{u}_2)_k (\widetilde{u}_2)_{k-1}.$$

Upon multiplying u_2 with -1 if necessary we may assume $(u_1)_k = (\widetilde{u}_2)_k$. If $(u_1)_k = (\widetilde{u}_2)_k \neq 0$, then $(u_1)_{k-1} = (\widetilde{u}_2)_{k-1}$, in which case $(u_1, u_2) \in (\mathrm{Hyp}_1)_{\leq k}$. Otherwise $(u_1)_k = (\widetilde{u}_2)_k = 0$ in which case $(u_1, u_2) \in \widetilde{(\mathrm{Hyp}_1)}_{\leq k}$.

It remains to prove the moreover part. Let

$$\mathcal{B}_{\leq i} = \{1, x, y, x^2, xy, y^2, \ldots, x^j, x^{j-1}y, y^j, \ldots x^i, x^{i-1}y, y^i\}$$

be a basis for $\mathbb{R}[\mathcal{Z}(P_{21})]_{\leq i}$. Replace x^i by yx^i to obtain $\widetilde{\mathcal{B}}_{\leq i} = \mathcal{B}_{\leq i} \setminus \{x^i\} \cup \{yx^i\}$. Note that $\Phi(\widetilde{\mathcal{B}}_{\leq i})$ is a basis for $\widetilde{(\mathrm{Hyp}_1)}_{\leq i}$. □

8.9. **Hyperbolic type 2.** Let P_{22} be as in Proposition 8.1 above. Let $D = Q_{-\frac{1}{a}} + Q_\infty$ and

$$\begin{aligned}
\mathrm{Hyp}_2 &:= \{(f,g) \in \mathbb{R}[s] \times \mathbb{R}\Big[t, \frac{1}{1+at}\Big] : f(0) = g(0)\},\\
(\mathrm{Hyp}_2)_{\leq i} &:= \Big\{(f,g) \in \mathbb{R}[s]_{\leq i} \times \mathcal{L}(iD) \colon f(0) = g(0),\ f_i = \frac{g_{2i}}{a^i},\\
&\qquad\qquad \text{where } g(t) = \frac{\sum_{j=0}^{2i} g_j t^j}{(1+at)^i}\Big\},\\
\mathrm{Pos}((\mathrm{Hyp}_2)_{\leq i}) &:= \{(f,g) \in (\mathrm{Hyp}_2)_{\leq i} \colon f(s) \geq 0,\ g(t) \geq 0 \text{ for every } (s,t) \in \mathbb{R}^2\},\\
(\widetilde{\mathrm{Hyp}_2})_{\leq i} &:= \Big\{(f,g) \in \mathbb{R}[s]_{\leq i} \times \mathcal{L}(iD) \colon f(0) = g(0),\ f_i = -\frac{g_{2i}}{a^i},\\
&\qquad\qquad \text{where } g(t) = \frac{\sum_{j=0}^{2i} g_j t^j}{(1+at)^i}\Big\}.
\end{aligned}$$

Theorem 8.10. *Let $(p_1, p_2) \in \mathrm{Pos}((\mathrm{Hyp}_2)_{\leq 2k})$. Then there exist finitely many $(f_{1;i}, f_{2,i}) \in (\mathrm{Hyp}_2)_{\leq k}$ and $(g_{1;j}, g_{2;j}) \in (\widetilde{\mathrm{Hyp}_2})_{\leq k}$ such that*

$$(p_1, p_2) = \sum\nolimits_i (f_{1;i}^2, f_{2;i}^2) + \sum\nolimits_j (g_{1;j}^2, g_{2;j}^2).$$

Moreover, for $C = \mathcal{Z}(P_{22})$ the appropriate choices of f and $V^{(k)}$ in Theorem 2.3 are as stated in Table 3.

Proof. Let $\Phi : \mathbb{R}[\mathcal{Z}(P_{22})] \to \mathrm{Hyp}_1$ be a map with $\Phi(p(x,y)) = \big(p(s,0), p\big(t, -\frac{t}{1+at}\big)\big)$. Analogously as in the proof of Theorem 8.2 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_{22})]_{\leq i}$ is in one-to-one correspondence with the set $(\mathrm{Hyp}_2)_{\leq 3i}$ under Φ.

Let $p = (p_1, p_2)$ be an extreme ray the cone $\mathrm{Pos}((\mathrm{Hyp}_2)_{\leq 2k})$. Using Corollary 3.4, $(p_1, p_2) = (u_1^2, u_2^2)$ for some $u_1 \in \mathbb{R}[s]_{\leq k}$, $u_2 \in \mathcal{L}(kD)$ such that $u_1^2(0) = u_2^2(0)$ and $(u_1^2)_{2k} = \frac{(u_2^2)_{4k}}{a^{2k}}$. So $u_1(0) = \pm u_2(0)$ and $(u_1)_k = \pm\frac{(u_2)_{2k}}{a^k}$. Multiplying u_2 by -1 if necessary, we may assume that $u_1(0) = u_2(0)$. Then $u_1 \in (\mathrm{Hyp}_2)_{\leq k}$ if $(u_1)_k = \frac{(u_2)_{2k}}{a^k}$ and $u_1 \in (\widetilde{\mathrm{Hyp}_2})_{\leq k}$ if $(u_1)_k = -\frac{(u_2)_{2k}}{a^k}$.

It remains to prove the moreover part. Let

$$\mathcal{B}_{\leq i} = \{1, x, y, x^2, xy, y^2, \ldots, x^j, x^{j-1}y, y^j, \ldots x^i, x^{i-1}y, y^i\}$$

be a basis for $\mathbb{R}[C]_{\leq i}$. Let $h(x,y) = x^i + 2y(1+ax)x^{i-1}$ and extend Φ to $h(x,y)$ by the same rule. Note that $\Phi(h) = (s^i, -t^i) \in (\widetilde{\mathrm{Hyp}_2})_{\leq i}$. Replace x^i by h to obtain $\widetilde{\mathcal{B}}_{\leq i} = \mathcal{B}_{\leq i} \setminus \{x^i\} \cup \{h\}$. Note that $\Phi(\widetilde{\mathcal{B}}_{\leq i})$ is a basis for $(\widetilde{\mathrm{Hyp}_2})_{\leq i}$. ☐

8.10. **Hyperbolic type 3.** Let P_{23} be as in Proposition 8.1 above. The rational parametrization of the hyperbola $ay + x^2 - y^2 = 0$ is given by

$$(x(t), y(t) = \Big(a\frac{t}{t^2-1}, a\frac{t^2}{t^2-1}\Big) \quad t \in \mathbb{R} \setminus \{-1, 1\}.$$

Let $D := Q_1 + Q_{-1}$ and

$$\mathrm{Hyp}_3 := \Big\{(f,g) \in \mathbb{R}[s] \times \mathbb{R}\Big[\frac{1}{t^2-1}, \frac{t}{t^2-1}\Big] : f(0) = g(0),\ f'(0) = -\frac{g'(0)}{a}\Big\},$$

$$\begin{aligned}(\mathrm{Hyp}_3)_{\leq i} &:= \Big\{(f,g) \in \mathbb{R}[s]_{\leq i} \times \mathcal{L}(iD)\colon\, f(0) = g(0),\ f'(0) = -\frac{g'(0)}{a}\Big\},\\ \mathrm{Pos}((\mathrm{Hyp}_3)_{\leq i}) &:= \{(f,g) \in (\mathrm{Hyp}_3)_{\leq i}\colon\, f(s) \geq 0,\ g(t) \geq 0\\ &\qquad\qquad \text{for every } (s,t) \in \mathbb{R} \times (\mathbb{R} \setminus \{-1,1\})\},\\ (\widetilde{\mathrm{Hyp}}_3)_{\leq i} &:= \{(f,g) \in \mathbb{R}[s]_{\leq i} \times \mathcal{L}(iD)\colon\, f(0) = g(0) = 0\}.\end{aligned}$$

Theorem 8.11. *Let $(p_1,p_2) \in \mathrm{Pos}((\mathrm{Hyp}_3)_{\leq 2k})$. Then there exist finitely many $(f_{1;i}, f_{2,i}) \in (\mathrm{Hyp}_3)_{\leq k}$ and $(g_{1;j}, g_{2;j}) \in (\widetilde{\mathrm{Hyp}}_3)_{\leq k}$ such that*

$$(p_1,p_2) = \sum\nolimits_i (f_{1;i}^2, f_{2;i}^2) + \sum\nolimits_j (g_{1;j}^2, g_{2;j}^2).$$

Moreover, for $C = \mathcal{Z}(P_{23})$ the appropriate choices of f and $V^{(k)}$ in Theorem 2.3 are as stated in Table 3.

Proof. Let $\Phi : \mathbb{R}[\mathcal{Z}(P_{23})] \to \mathrm{Hyp}_1$ be a map with

$$\Phi(p(x,y)) = \Big(p(s,0), p\Big(a\frac{t}{t^2-1}, a\frac{t^2}{t^2-1}\Big)\Big).$$

Analogously as in the proof of Theorem 8.2 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_{23})]_{\leq i}$ is in one-to-one correspondence with the set $(\mathrm{Hyp}_3)_{\leq 3i}$ under Φ.

Let $p = (p_1,p_2)$ be an extreme ray the cone $\mathrm{Pos}((\mathrm{Hyp}_3)_{\leq 2k})$. Using Corollary 3.4, $(p_1,p_2) = (u_1^2, u_2^2)$ for some $u_1 \in \mathbb{R}[s]_{\leq k}$, $u_2 \in \mathcal{L}(kD)$ such that $u_1^2(0) = u_2^2(0)$ and $(u_1^2)'(0) = \frac{(u_2^2)'(0)}{-a}$. So $u_1(0) = \pm u_2(0)$ and $2u_1(0)u_1'(0) = -\frac{2}{a}u_2(0)u_2'(0)$. Upon multiplying u_2 with -1 if necessary we may assume $u_1(0) = u_2(0)$. If $u_1(0) = u_2(0) \neq 0$, then $u_1'(0) = -\frac{u_2'(0)}{a}$, in which case $(u_1,u_2) \in (\mathrm{Hyp}_3)_{\leq k}$. Otherwise $u_1(0) = u_2(0) = 0$ in which case $(u_1,u_2) \in (\widetilde{\mathrm{Hyp}}_3)_{\leq k}$.

It remains to prove the moreover part. Let

$$\mathcal{B}_{\leq i} = \{1, x, y, x^2, xy, y^2, \ldots, x^j, x^{j-1}y, x^{j-2}y^2, \ldots x^i, x^{i-1}y, x^{i-2}y^2\}$$

be a basis for $\mathbb{R}[\mathcal{Z}(P_{23})]_{\leq i}$. Let $h(x,y) = \frac{ay+x^2-y^2}{x}$ and extend Φ to $h(x,y)$ by the same rule note that: $\Phi(h) = (s,0) \in (\widetilde{\mathrm{Hyp}}_3)_{\leq i}$. Replace 1 by h to obtain $\widetilde{\mathcal{B}}_{\leq i} = \mathcal{B}_{\leq i} \setminus \{1\} \cup \{h\}$. Note that $\Phi(\widetilde{\mathcal{B}}_{\leq i})$ is a basis for $(\widetilde{\mathrm{Hyp}}_3)_{\leq i}$. □

8.11. **Hyperbolic type 4.**

Let P_{24} be as in Proposition 8.1 above. Using an affine linear transformation $(x,y) \mapsto (x,-y)$ we may assume that $a < 0$. A rational parametrization of the hyperbola $1 + ay + x^2 - y^2 = 0$, is

$$(x(t), y(t)) = \Big(r\frac{2t}{t^2-1}, r\frac{t^2+1}{t^2-1} + \frac{a}{2}\Big),\quad t \in \mathbb{R} \setminus \{-1,1\},$$

where $r = \sqrt{1 + \frac{a^2}{4}}$. A short computation shows that for

$$t_0 = -\frac{\mathfrak{i}}{2}(-a + \sqrt{4+a^2})$$

we have that

$$(x(t_0), y(t_0)) = (\mathfrak{i}, 0).$$

Let $D := Q_1 + Q_{-1}$ and

$$\begin{aligned}\mathrm{Hyp}_4 &:= \{(f,g) \in \mathbb{R}[s] \times \mathbb{R}\Big[\frac{1}{t^2-1}, \frac{t}{t^2-1}\Big] \colon f(\mathfrak{i}) = g(t_0)\},\\ (\mathrm{Hyp}_4)_{\leq i} &:= \{(f,g) \in \mathbb{R}[s]_{\leq i} \times \mathcal{L}(iD) \colon f(\mathfrak{i}) = g(t_0)\},\\ \mathrm{Pos}((\mathrm{Hyp}_4)_{\leq i}) &:= \{(f,g \in (\mathrm{Hyp}_4)_{\leq i} \colon f(s) \geq 0,\ g(t) \geq 0 \text{ for every } (s,t) \in \mathbb{R}^2\}\end{aligned}$$

Theorem 8.12. *Let $(p_1,p_2) \in \mathrm{Pos}((\mathrm{Hyp}_4)_{\leq 2k})$. Then there exist finitely many $(f_{1;i}, f_{2,i}) \in (\mathrm{Hyp}_4)_{\leq k}$ and $g_{1;j} \in \mathbb{R}[s]_{k-1}$ such that*

$$(p_1,p_2) = \sum\nolimits_i (f_{1;i}^2, f_{2;i}^2) + \sum\nolimits_j ((1+s^2)g_{1;j}^2, 0).$$

Moreover, for $C = \mathcal{Z}(P_{24})$ the appropriate choices of χ_1, χ_2 and P_2 in Theorem 2.4 are 0, 1 and $1 + ay + x^2 - xy^2$, respectively.

Proof. Let $\Phi : \mathbb{R}[\mathcal{Z}(P_{24})] \to \mathrm{Hyp}_4$ be a map defined by

$$\Phi(p(x,y)) = \left(p(s,0), p\Big(r\frac{2t}{t^2-1}, r\frac{t^2+1}{t^2-1} + \frac{a}{2}\Big)\right).$$

Analogously as in the proof of Theorem 8.2 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_{24})]_{\leq i}$ is in one-to-one correspondence with the set $(\mathrm{Hyp}_4)_{\leq 3i}$ under Φ. Let $p = (p_1,p_2)$ be an extreme ray of the cone $\mathrm{Pos}((\mathrm{Hyp}_4)_{\leq 2k})$. Using Corollary 3.4, one of the following cases occurs:

Case 1: *$p_1 \neq 0$, $p_2 \neq 0$ and each component p_1, p_2 has all zeroes and poles of even order.*

This implies that $(p_1,p_2) = (u_1^2, u_2^2)$ for some $u_1 \in \mathbb{R}[s]_{\leq k}$, $u_2 \in \mathcal{L}(kD)$ such that $u_1^2(\mathfrak{i}) = u_2^2(t_0)$. So $u_1(\mathfrak{i}) = \pm u_2(t_0)$. Upon multiplying with -1 we may assume that $u_1(\mathfrak{i}) = u_2(t_0)$. Hence, $(u_1,u_2) \in (\mathrm{Hyp}_4)_{\leq k}$.

Type 2: *$p_1 = 0$ or $p_2 = 0$.*

Case 2.1: $p_1 = 0$**.** In this case p_2 vanishes in t_0 and $-t_0$, since these two points correspond to $(\mathfrak{i},0)$ and $(-\mathfrak{i},0)$ in the ambient curve. Since $y(t)$ vanishes in $t_0, -t_0$ and has a quadratic numerator, it follows that $p_2(t) = y(t)^m h_2(t)$, $m \in \mathbb{N}$, $h(t) \in \mathbb{R}[t]$ and h does not vanish in t_0, $-t_0$. Since $y(t)$ changes sign on the hyperbola, it follows that $h_2(t)$ must change sign on the hyperbola as well. Moreover, h_2 has only real zeroes and poles of even order. As in the reasoning for Case 1 above, it is a square of an element from $\mathcal{L}((k-m)D)$. It follows that the only option is $h_2 = 0$.

Case 2.2: $p_2 = 0$**.** In this case p_1 vanishes in $\mathfrak{i}$, $-\mathfrak{i}$. If follows that $p_1(s) = (1+s^2)g_1(s)$, $g_1(s) \geq 0$ for every $s \in \mathbb{R}$ and g_1 has only real zeroes. Hence $p_1(s) = (1+s^2)u_1^2(s)$ and $(p_1, 0)$ satisfies the statement in the first sentence of the proof.

The moreover part is clear. □

8.12. **Hyperbolic type 5.** Let P_{25} be as in Proposition 8.1 above. A parametrization of the hyperbola $1 + ay - x^2 + y^2 = 0$ is

$$(x(t), y(t)) = \Big(r\frac{2t}{t^2-1}, r\frac{t^2+1}{t^2-1} - \frac{a}{2}\Big), \quad t \in \mathbb{R} \setminus \{-1, 1\},$$

where $r = \sqrt{-1 + \frac{a^2}{4}}$. A short computation shows that for

$$t_- = \frac{1}{2}(-a - \sqrt{-4 + a^2}), \quad t_+ = \frac{1}{2}(a + \sqrt{-4 + a^2})$$

we have that

$$(x(t_-), y(t_-)) = (-1, 0), \quad (x(t_+), y(t_+)) = (1, 0).$$

Let $D := Q_1 + Q_{-1}$ and

$$\begin{aligned}
\mathrm{Hyp}_5 &:= \{(f, g) \in \mathbb{R}[s] \times \mathbb{R}\Big[\frac{1}{t^2-1}, \frac{t}{t^2-1}\Big] \colon f(-1) = g(t_-),\ f(1) = g(t_+)\},\\
(\mathrm{Hyp}_5)_{\leq i} &:= \{(f, g) \in \mathbb{R}[s]_{\leq i} \times \mathcal{L}(iD) \colon f(-1) = g(t_-),\ f(1) = g(t_+)\},\\
\mathrm{Pos}((\mathrm{Hyp}_5)_{\leq i}) &:= \{(f, g) \in (\mathrm{Hyp}_3)_{\leq i} \colon f(s) \geq 0,\ g(t) \geq 0\\
&\qquad\qquad\qquad\qquad \text{for every } (s, t) \in \mathbb{R} \times (\mathbb{R} \setminus \{-1, 1\})\},\\
(\widetilde{\mathrm{Hyp}}_3)_{\leq i} &:= \{(f, g) \in \mathbb{R}[s]_{\leq i} \times \mathcal{L}(iD) \colon f(-1) = -g(t_-),\ f(1) = g(t_+)\}.
\end{aligned}$$

Theorem 8.13. *Let* $(p_1, p_2) \in \mathrm{Pos}((\mathrm{Hyp}_5)_{\leq 2k})$. *Then there exist finitely many* $(f_{1;i}, f_{2,i}) \in (\mathrm{Hyp}_5)_{\leq k}$ *and* $(g_{1;j}, g_{2;j}) \in (\widetilde{\mathrm{Hyp}}_5)_{\leq k}$ *such that*

$$(p_1, p_2) = \sum\nolimits_i (f_{1;i}^2, f_{2;i}^2) + \sum\nolimits_j (g_{1;j}^2, g_{2;j}^2).$$

Moreover, for $C = \mathcal{Z}(P_{25})$ *the appropriate choices of* f *and* $V^{(k)}$ *in Theorem 2.3 are as stated in Table 3.*

Proof. Let $\Phi : \mathbb{R}[\mathcal{Z}(P_{25})] \to \mathrm{Hyp}_1$ be a map with

$$\Phi(p(x, y)) = \Big(p(s, 0), p\Big(r\frac{2t}{t^2-1}, r\frac{t^2+1}{t^2-1} - \frac{a}{2}\Big)\Big).$$

Analogously as in the proof of Theorem 8.2 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_{25})]_{\leq i}$ is in one-to-one correspondence with the set $(\mathrm{Hyp}_5)_{\leq 3i}$ under Φ. Let $p = (p_1, p_2)$ be an extreme ray the cone $\mathrm{Pos}((\mathrm{Hyp}_5)_{\leq 2k})$. Using Corollary 3.4, $(p_1, p_2) = (u_1^2, u_2^2)$ for some $u_1 \in \mathbb{R}[s]_{\leq k}$, $u_2 \in \mathcal{L}(kD)$ such that $u_1^2(-1) = u_2^2(t_-)$ and $u_1^2(1) = u_2^2(t_+)$. So $u_1(-1) = \pm u_2(t_-)$ and $u_1(1) = \pm u_2(t_+)$. Upon multiplying with -1 we may assume that $u_1(1) = u_2(t_+)$. If $u_1(-1) = u_2(t_-)$, then $(u_1, u_2) \in (\mathrm{Hyp}_5)_{\leq k}$. Otherwise $u_1(-1) = -u_2(t_-)$ and $(u_1, u_2) \in (\widetilde{\mathrm{Hyp}}_5)_{\leq k}$.

It remains to prove the moreover part. Let

$$\mathcal{B}_{\leq i} = \{1, x+1, x^2-1, x(x^2-1), \ldots, x^{k-2}(x^2-1), y, yx, \ldots, yx^{k-1}, y^2, y^2x, \ldots, y^2x^{k-2}\}$$

be a basis for $\mathbb{R}[\mathcal{Z}(P_{25})]_{\leq i}$. Let $h(x, y) := 1 - x - 2\frac{1+ay-x^2+y^2}{1+x}$ and extend Φ to h by the same rule. Note that: $\Phi(h) = (-(1-x), 1-x) \in ((\widetilde{\mathrm{Hyp}})_5)_{\leq i}$. Replace 1 by h to obtain $\widetilde{\mathcal{B}}_{\leq i} = \mathcal{B}_{\leq i} \setminus \{1\} \cup \{h\}$. Note that $\Phi(\widetilde{\mathcal{B}}_{\leq i})$ is a basis for $(\widetilde{\mathrm{Hyp}}_5)_{\leq i}$. $\square$

8.13. **Parallel lines type.** Let P_{26} be as in Proposition 8.1 above.

Let

$$\begin{aligned}
\mathrm{PLines} &:= \mathbb{R}[s] \times \mathbb{R}[t] \times \mathbb{R}[u],\\
\mathrm{PLines}_{\leq i} &:= \{(f,g,h) \in \mathbb{R}[s]_{\leq i} \times \mathbb{R}[t]_{\leq i} \times \mathbb{R}[u]_{\leq i} \colon f_i = g_i = h_i,\\
&\qquad b(g_{i-1} - f_{i-1}) = a(h_{i-1} - f_{i-1}),\ \text{where}\\
&\qquad f(s) = \sum\nolimits_{j=0}^{i} f_j s^j, g(t) = \sum\nolimits_{j=0}^{i} g_j t^j, h(u) = \sum\nolimits_{j=0}^{i} h_j u^j\},\\
\mathrm{Pos}(\mathrm{PLines}_{\leq i}) &:= \{(f(s), g(t), h(u)) \in \mathrm{PLines}_{\leq i} \colon f(s) \geq 0,\ g(t) \geq 0,\ h(u) \geq 0\\
&\qquad\qquad \text{for every } (s,t,u) \in \mathbb{R}^3\},\\
\widetilde{\mathrm{PLines}}_{\leq i} &:= \{(f,g,h) \in \mathbb{R}[s]_{\leq i} \times \mathbb{R}[t]_{\leq i} \times \mathbb{R}[u]_{\leq i} \colon f_i = g_i = h_i = 0,\\
&\qquad \text{where } f(s) = \sum\nolimits_{j=0}^{i} f_j s^j, g(t) = \sum\nolimits_{j=0}^{i} g_j t^j, h(u) = \sum\nolimits_{j=0}^{i} h_j u^j\}.
\end{aligned}$$

Theorem 8.14. *Let* $(p_1, p_2, p_3) \in \mathrm{Pos}(\mathrm{PLines}_{\leq 2k})$. *Then there exist finitely many*

$$(f_{1;i}, f_{2;i}, f_{3;i}) \in \mathrm{PLines}_{\leq k} \quad \text{and} \quad (g_{1;j}, g_{2;j}, g_{3;j}) \in \widetilde{\mathrm{PLines}}_{\leq k}$$

such that

$$(p_1, p_2, p_3) = \sum\nolimits_i (f_{1;i}^2, f_{2;i}^2, f_{3;i}^2) + \sum\nolimits_j (g_{1;j}^2, g_{2;j}^2, g_{3;j}^2).$$

Moreover, for $C = \mathcal{Z}(P_{26})$ *the appropriate choices of* f *and* $V^{(k)}$ *in Theorem 2.3 are as stated in Table 4.*

Proof. Let $\Phi : \mathbb{R}[\mathcal{Z}(P_{26})] \to \mathrm{PLines}$ be a map with

$$\Phi(p(x,y)) = (p(s,0), p(t,-a), p(u,-b)).$$

Let us write $\Phi = (\Phi_1, \Phi_2, \Phi_3)$. Clearly Φ is a well–defined ring homomorphism, because $\Phi(p) = 0$ for every $p \in I$ and

$$\begin{aligned}
\Phi_1\Big(\sum_{\substack{i_1,i_2=0,\ldots,i,\\ i_1+i_2\leq i}} a_{i_1,i_2} x^{i_1} y^{i_2}\Big) &= \sum_{i_1=0}^{i} a_{i_1,0} s^{i_1},\\
\Phi_2\Big(\sum_{\substack{i_1,i_2=0,\ldots,i,\\ i_1+i_2\leq i}} a_{i_1,i_2} x^{i_1} y^{i_2}\Big) &= \sum_{\substack{i_1,i_2=0,\ldots,i,\\ i_1+i_2\leq i}} a_{i_1,i_2} t^{i_1} (-a)^{i_2},\\
\Phi_3\Big(\sum_{\substack{i_1,i_2=0,\ldots,i,\\ i_1+i_2\leq i}} a_{i_1,i_2} x^{i_1} y^{i_2}\Big) &= \sum_{\substack{i_1,i_2=0,\ldots,i,\\ i_1+i_2\leq i}} a_{i_1,i_2} t^{i_1} (-b)^{i_2}
\end{aligned}$$

for every $p \in \mathbb{R}[C]$. Thus

$$\begin{aligned}
(\Phi_1(p))_i = (\Phi_2(p))_i &= (\Phi_3(p))_i = a_{i,0},\\
(\Phi_1(p))_{i-1} &= a_{i_1-1,0},\\
(\Phi_2(p))_{i-1} &= a_{i_1-1,0} - a a_{i_1-1,1},\\
(\Phi_3(p))_{i-1} &= a_{i_1-1,0} - b a_{i_1-1,1}.
\end{aligned}$$

The inclusion $\Phi(\mathbb{R}[C]_{\leq i}) \subseteq \mathrm{PLines}_{\leq i}$ is clear. Since $\dim \Phi(\mathbb{R}[C]_{\leq i}) = \dim \mathrm{PLines}_{\leq i} = 3i$, we have equality for every i and Φ is also one-to-one.

Let $p = (p_1, p_2, p_3)$ be an extreme ray the cone $\text{Pos}(\text{PLines}_{\leq 2k})$. Using Corollary 3.4, $(p_1, p_2, p_3) = (u_1^2, u_2^2, u_3^2)$ for some $u_1 \in \mathbb{R}[s]_{\leq k}$, $u_2 \in \mathbb{R}[t]_{\leq k}$ $u_3 \in \mathbb{R}[u]_{\leq k}$ such that $(u_1^2)_{2k} = (u_2^2)_{2k} = (u_3^2)_{2k}$ and

$$b((u_2^2)_{2k-1} - (u_1^2)_{2k-1}) = a((u_3^2)_{2k-1} - (u_1^2)_{2k-1}).$$

Hence, $(u_1)_k = \pm(u_2)_k$, $(u_1)_k = \pm(u_3)_k$, and

$$2b((u_2)_k(u_2)_{k-1} - (u_1)_k(u_1)_{k-1}) = 2a((u_3)_k(u_3)_{k-1} - (u_1)_k(u_1)_{k-1}).$$

Upon multiplying by -1 if necessary, we may assume that $(u_1)_k = (u_2)_k = (u_3)_k$. If $(u_1)_k = (u_2)_k = (u_3)_k \neq 0$, then

$$b((u_2)_{k-1} - (u_1)_{k-1}) = a((u_3)_{k-1} - (u_1)_{k-1})$$

and $(u_1, u_2, u_3) \in \text{PLines}_{\leq k}$. Else $(u_1)_k = (u_2)_k = (u_3)_k = 0$ and $(u_1, u_2, u_3) \in \widetilde{\text{PLines}}_{\leq k}$.

It remains to prove the moreover part. Let

$$\mathcal{B}_{\leq i} = \{1, x, y, x^2, xy, y^2, \ldots, x^j, x^{j-1}y, x^{j-2}y^2, \ldots x^i, x^{i-1}y, x^{i-2}y^2\}$$

be a basis for $\mathbb{R}[C]_{\leq i}$. Note that:

$$\Phi(y(y+a)x^{i-1}) = (0, 0, -b(-b+a)u^{i-1}) \in \widetilde{\text{PLines}}_{\leq i}.$$

Replace x^i by $y(y+a)x^{i-1}$ to obtain $\widetilde{\mathcal{B}}_{\leq i} = \mathcal{B}_{\leq i} \setminus \{x^i\} \cup \{y(y+a)x^{i-1}\}$. Note that $\Phi(\widetilde{\mathcal{B}}_{\leq i})$ is a basis for $\widetilde{\text{PLines}}_{\leq i}$. □

8.14. **Intersecting lines type 1.** Let P_{27} be as in Proposition 8.1 above. Let

$$\begin{aligned}
\text{ILines}_1 &:= \{(f, g, h) \in \mathbb{R}[s] \times \mathbb{R}[t] \times \mathbb{R}[u] \colon f(0) = g(0) = h(0)\},\\
(\text{ILines}_1)_{\leq i} &:= \{(f, g, h) \in \mathbb{R}[s]_{\leq i} \times \mathbb{R}[t]_{\leq i} \times \mathbb{R}[u]_{\leq i} \colon f(0) = g(0) = h(0),\\
&\qquad g'(0) - f'(0) = f'(0) - h'(0)\},\\
\text{Pos}((\text{ILines}_1)_{\leq i}) &:= \{(f, g, h) \in (\text{ILines}_1)_{\leq i} \colon f(s) \geq 0,\ g(t) \geq 0,\ h(u) \geq 0\\
&\qquad \text{for every } (s, t, u) \in \mathbb{R}^3\},\\
(\widetilde{\text{ILines}}_1)_{\leq i} &:= \{(f, g, h) \in \mathbb{R}[s]_{\leq i} \times \mathbb{R}[t]_{\leq i} \times \mathbb{R}[u]_{\leq i} \colon f(0) = h(0) = g(0) = 0\}.
\end{aligned}$$

Theorem 8.15. *Let $(p_1, p_2, p_3) \in \text{Pos}((\text{ILines}_1)_{\leq 2k})$. Then there exist finitely many $(f_{1;i}, f_{2;i}, f_{3;i}) \in (\text{ILines}_1)_{\leq k}$ and $(g_{1;j}, g_{2;j}, g_{3;j}) \in (\widetilde{\text{ILines}}_1)_{\leq k}$ such that*

$$(p_1, p_2, p_3) = \sum\nolimits_i (f_{1;i}^2, f_{2;i}^2, f_{3;i}^2) + \sum\nolimits_j (g_{1;j}^2, g_{2;j}^2, g_{3;j}^2).$$

Moreover, for $C = \mathcal{Z}(P_{27})$ the appropriate choices of f and $V^{(k)}$ in Theorem 2.3 are as stated in Table 4.

Proof. Let $\Phi : \mathbb{R}[\mathcal{Z}(P_{27})] \to \text{ILines}_1$ be a map with

$$\Phi(p(x, y)) = (p(s, 0), p(t, t), p(u, -u)).$$

Analogously as in the proof of Theorem 8.14 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_{27})]_{\leq i}$ is in one-to-one correspondence with the set $(\text{ILines}_1)_{\leq 3i}$

under Φ. Let $p = (p_1, p_2, p_3)$ be an extreme ray the cone $\mathrm{Pos}((\mathrm{ILines}_1)_{\le 2k})$. Using Corollary 3.4, $(p_1, p_2, p_3) = (u_1^2, u_2^2, u_3^2)$ for some $u_1 \in \mathbb{R}[s]_{\le k}$, $u_2 \in \mathbb{R}[t]_{\le k}$ $u_3 \in \mathbb{R}[u]_{\le k}$ such that $(u_1^2)(0) = (u_2^2)(0) = (u_3^2)(0)$ and

$$(u_2^2)'(0) - (u_1^2)'(0) = (u_1^2)'(0) - (u_3^2)'(0).$$

The second equality is equivalent to

$$2u_2(0)u_2'(0) - 2u_1(0)u_1'(0) = 2u_1(0)u_1'(0) - 2u_3(0)u_3'(0).$$

From the first equality we conclude $u_1(0) = \pm u_2(0)$ and $u_1(0) = \pm u_3(0)$. Upon multiplying u_2, u_3 by -1 if necessary, we may assume that $u_1(0) = u_2(0) = u_3(0)$. If $u_1(0) = u_2(0) = u_3(0) \neq 0$, then we must also have $u_2'(0) - u_1'(0) = u_1'(0) - u_3'(0)$, in which case $(u_1, u_2, u_3) \in (\mathrm{ILines}_1)_{\le k}$. Else $u_1(0) = u_2(0) = u_3(0) = 0$ and $(u_1, u_2, u_3) \in (\widetilde{\mathrm{ILines}}_1)_{\le k}$.

It remains to prove the moreover part. Let

$$\mathcal{B}_{\le i} = \{1, x, y, x^2, xy, y^2, \ldots, x^j, x^{j-1}y, y^j, \ldots x^i, x^{i-1}y, y^i\}$$

be a basis for $\mathbb{R}[\mathcal{Z}(P_{27})]_{\le i}$. Let $h(x, y) = \frac{x^2 - y^2}{x}$ and extend Φ to $h(x, y)$ by the same rule. Note that: $\Phi(h) = (s, 0, 0) \in (\widetilde{\mathrm{ILines}}_1)_{\le i}$. Replace 1 by h to obtain $\widetilde{\mathcal{B}}_{\le i} = \mathcal{B}_{\le i} \setminus \{1\} \cup \{h\}$. Note that $\Phi(\widetilde{\mathcal{B}}_{\le i})$ is a basis for $(\widetilde{\mathrm{ILines}}_1)_{\le i}$. ☐

8.15. **Intersecting lines type 2.** Let P_{28} be as in Proposition 8.1 above. Let

$$\begin{aligned}
\mathrm{ILines}_2 &:= \{(f, g, h) \in \mathbb{R}[s] \times \mathbb{R}[t] \times \mathbb{R}[u] \colon f(0) = h(0),\ g(0) = h(-1)\}, \\
(\mathrm{ILines}_2)_{\le i} &:= \{(f, g, h) \in \mathbb{R}[s]_{\le i} \times \mathbb{R}[t]_{\le i} \times \mathbb{R}[u]_{\le i} \colon f(0) = h(0), \\
&\qquad\qquad g(0) = h(-1),\ f_i = g_i\}, \\
\mathrm{Pos}((\mathrm{ILines}_2)_{\le i}) &:= \{(f, g, h) \in (\mathrm{ILines}_2)_{\le i} \colon f(s) \ge 0,\ g(t) \ge 0,\ h(u) \ge 0 \\
&\qquad\qquad \text{for every } (s, t, u) \in \mathbb{R}^3\}, \\
(\widetilde{\mathrm{ILines}}_2)_{\le i} &:= \{(f, g, h) \in \mathbb{R}[s]_{\le i} \times \mathbb{R}[t]_{\le i} \times \mathbb{R}[u]_{\le i} \colon f(0) = h(0), \\
&\qquad\qquad g(0) = h(-1),\ f_i = -g_i\}.
\end{aligned}$$

Theorem 8.16. *Let $(p_1, p_2, p_3) \in \mathrm{Pos}((\mathrm{ILines}_2)_{\le 2k})$. Then there exist finitely many $(f_{1;i}, f_{2;i}, f_{3;i}) \in (\mathrm{ILines}_2)_{\le k}$ and $(g_{1;j}, g_{2;j}, g_{3;j}) \in (\widetilde{\mathrm{ILines}}_2)_{\le k}$ such that*

$$(p_1, p_2, p_3) = \sum_i (f_{1;i}^2, f_{2;i}^2, f_{3;i}^2) + \sum_j (g_{1;j}^2, g_{2;j}^2, g_{3;j}^2).$$

Moreover, for $C = \mathcal{Z}(P_{28})$ the appropriate choices of f and $V^{(k)}$ in Theorem 2.3 are as stated in Table 4.

Proof. Let $\Phi : \mathbb{R}[\mathcal{Z}(P_{28})] \to \mathrm{ILines}_2$ be a map with

$$\Phi(p(x, y)) = (p(s, 0), p(t, -1), p(0, u)).$$

Analogously as in the proof of Theorem 8.14 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_{28})]_{\le i}$ is in one-to-one correspondence with the set $(\mathrm{ILines}_2)_{\le 3i}$ under Φ.

Let $p = (p_1, p_2, p_3)$ be an extreme ray the cone $\mathrm{Pos}((\mathrm{ILines}_2)_{\le 2k})$. Using Corollary 3.4, $(p_1, p_2, p_3) = (u_1^2, u_2^2, u_3^2)$ for some $u_1 \in \mathbb{R}[s]_{\le k}$, $u_2 \in \mathbb{R}[t]_{\le k}$, $u_3 \in \mathbb{R}[u]_{\le k}$ such that $(u_1^2)(0) = (u_3^2)(0)$, $(u_2^2)(0) = (u_3^2)(-1)$ and $(u_1^2)_{2k} = (u_2^2)_{2k}$. Hence, $u_1(0) = \pm u_3(0)$,

$u_2(0) = \pm u_3(-1)$ and $(u_1)_k = \pm(u_2)_k$. Upon multiplying u_1, u_2 by -1 if necessary, we may assume that $u_1(0) = u_3(0)$, $u_2(0) = u_3(-1)$. If $(u_1)_k = (u_2)_k$, then $(u_1, u_2, u_3) \in (\mathrm{ILines}_2)_{\leq k}$. Else $(u_1)_k = -(u_2)_k$ and $(u_1, u_2, u_3) \in \widetilde{(\mathrm{ILines}_2)}_{\leq k}$.

It remains to prove the moreover part. Let

$$\mathcal{B}_{\leq i} = \{1, x, y, x^2, xy, y^2, \ldots, x^j, x^{j-1}y, y^j, \ldots x^i, x^{i-1}y, y^i\}$$

be a basis for $\mathbb{R}[C]_{\leq i}$. Note that $\Phi(x^i + 2yx^i) = (s^i, -t^i, 0) \in \widetilde{(\mathrm{ILines}_2)}_{\leq i}$. Replace x^k by $x^i + 2yx^i$ to obtain $\widetilde{\mathcal{B}}_{\leq i} = \mathcal{B}_{\leq i} \setminus \{x^i\} \cup \{x^i + 2yx^i\}$. Note that $\Phi(\widetilde{\mathcal{B}}_{\leq i})$ is a basis for $\widetilde{(\mathrm{ILines}_2)}_{\leq i}$. □

8.16. **Intersecting lines type 3.** Let P_{29} be as in Proposition 8.1 above. Let

$$\begin{aligned}
\mathrm{ILines}_3 &:= \{(f, g, h) \in \mathbb{R}[s] \times \mathbb{R}[t] \times \mathbb{R}[u] \colon f(-1) = g(-1),\\
&\qquad\qquad f(1) = h(1),\ g(0) = h(0)\},\\
(\mathrm{ILines}_3)_{\leq i} &:= \{(f, g, h) \in \mathbb{R}[s]_{\leq i} \times \mathbb{R}[t]_{\leq i} \times \mathbb{R}[u]_{\leq i} \colon f(-1) = g(-1),\\
&\qquad\qquad f(1) = h(1),\ g(0) = h(0)\},\\
\mathrm{Pos}((\mathrm{ILines}_3)_{\leq i}) &:= \{(f, g, h) \in (\mathrm{ILines}_3)_{\leq i} \colon f(s) \geq 0,\ g(t) \geq 0,\ h(u) \geq 0\\
&\qquad\qquad \text{for every } (s, t, u) \in \mathbb{R}^3\},\\
\widetilde{(\mathrm{ILines}_3)}_{\leq i} &:= \{(f, g, h) \in \mathbb{R}[s]_{\leq i} \times \mathbb{R}[t]_{\leq i} \times \mathbb{R}[u]_{\leq i} \colon f(-1) = g(-1),\\
&\qquad\qquad f(1) = h(1),\ g(0) = -h(0)\}.
\end{aligned}$$

Theorem 8.17. *Let* $(p_1, p_2, p_3) \in \mathrm{Pos}((\mathrm{ILines}_3)_{\leq 2k})$. *Then there exist finitely many* $(f_{1;i}, f_{2;i}, f_{3;i}) \in (\mathrm{ILines}_3)_{\leq k}$ *and* $(g_{1;j}, g_{2;j}, g_{3;j}) \in \widetilde{(\mathrm{ILines}_3)}_{\leq k}$ *such that*

$$(p_1, p_2, p_3) = \sum\nolimits_i (f_{1;i}^2, f_{2;i}^2, f_{3;i}^2) + \sum\nolimits_j (g_{1;j}^2, g_{2;j}^2, g_{3;j}^2).$$

Moreover, for $C = \mathcal{Z}(P_{29})$ *the appropriate choices of* f *and* $V^{(k)}$ *in Theorem 2.3 are as stated in Table 4.*

Proof. Let $\Phi : \mathbb{R}[\mathcal{Z}(P_{29})] \to \mathrm{ILines}_3$ be a map with

$$\Phi(p(x, y)) = (p(s, 0), p(t, 1 + t), p(u, 1 - u))\,.$$

Analogously as in the proof of Theorem 8.14 we see that Φ is a ring isomorphism and that the vector subspace $\mathbb{R}[\mathcal{Z}(P_{29})]_{\leq i}$ is in one-to-one correspondence with the set $(\mathrm{ILines}_3)_{\leq 3i}$ under Φ.

Let $p = (p_1, p_2, p_3)$ be an extreme ray the cone $\mathrm{Pos}((\mathrm{ILines}_3)_{\leq 2k})$. Using Corollary 3.4, $(p_1, p_2, p_3) = (u_1^2, u_2^2, u_3^2)$ for some $u_1 \in \mathbb{R}[s]_{\leq k}$, $u_2 \in \mathbb{R}[t]_{\leq k}$, $u_3 \in \mathbb{R}[u]_{\leq k}$ such that $(u_1^2)(-1) = (u_2^2)(-1)$, $(u_1^2)(1) = (u_3^2)(1)$ and $(u_2^2)(0) = (u_3^2)(0)$. Hence, $u_1(-1) = \pm u_2(-1)$, $u_1(1) = \pm u_3(1)$ and $u_2(0) = \pm u_3(0)$. Upon multiplying u_2, u_3 by -1 if necessary, we may assume that $u_1(-1) = u_2(-1)$ and $u_1(1) = u_3(1)$. If $u_2(0) = u_3(0)$, then $(u_1, u_2, u_3) \in (\mathrm{ILines}_3)_{\leq k}$. Else $u_2(0) = -u_3(0)$ and $(u_1, u_2, u_3) \in \widetilde{(\mathrm{ILines}_3)}_{\leq k}$.

It remains to prove the moreover part. Let

$$\mathcal{B}_{\leq i} = \{1, x, y, x^2, xy, y^2, \ldots, x^j, x^{j-1}y, x^{j-2}y^2, \ldots x^i, x^{i-1}y, x^{i-2}y^2\}$$

be a basis for $\mathbb{R}[C]_{\leq i}$. Let $h(x,y)=(x-1)(x+1)+\frac{y(1+x-y)}{x}$. Note that:

$$\Phi(h)=(s^2-1,t^2-1,(u-1)^2)\in(\widetilde{\mathrm{ILines}_3})_{\leq i}.$$

Replace 1 by h to obtain $\widetilde{\mathcal{B}}_{\leq i}=\mathcal{B}_{\leq i}\setminus\{1\}\cup\{h\}$. Note that $\Phi(\widetilde{\mathcal{B}}_{\leq i})$ is a basis for $(\widetilde{\mathrm{ILines}_3})_{\leq i}$. □

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