MATRIX FEJÉR-RIESZ THEOREM WITH GAPS

ALJAŽ ZALAR

ABSTRACT. Two equivalent versions of the matrix Fejér-Riesz theorem characterize positive semidefinite matrix polynomials on the complex unit circle $\mathbb{T}$ and on the real line $\mathbb{R}$. We extend the characterization to arbitrary closed basic semialgebraic sets $\mathscr{K} \subseteq \mathbb{T}$ and $K \subseteq \mathbb{R}$ by the use of matrix preorderings from real algebraic geometry. In the $\mathbb{T}$-case the characterization is the same for all sets $\mathscr{K}$, while in the $\mathbb{R}$-case the characterizations for compact and non-compact sets K are different. Furthermore, we study a complexity of the characterizations in terms of a bound on the degrees of the summands needed. We prove, for which sets $\mathscr{K}$, K the degrees can be bounded by the degree of the given matrix polynomial and provide counterexamples for the sets, where this is not possible. At the end we give an application of results to a matrix moment problem.

1. INTRODUCTION

1.1. **Motivation.** The name Matrix Fejér-Riesz theorem refers to the following two results.

Theorem 1.1 (Fejér-Riesz theorem on $\mathbb{T}$)**.** *Let*

$$A(z) = \sum_{m=-N}^{N} A_m z^m$$

be a $n \times n$ matrix Laurent polynomial from $M_n\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$, which is positive semidefinite on $\mathbb{T}$. Then there exists a matrix polynomial $B(z) = \sum_{m=0}^{N} B_m z^m$ from $M_n(\mathbb{C}[z])$, such that

$$A(z) = B(z)^* B(z),$$

where $B(z)^* = \overline{B\left(\frac{1}{\bar{z}}\right)}^T$.

Theorem 1.2 (Fejér-Riesz theorem on $\mathbb{R}$)**.** *Let*

$$F(x) = \sum_{m=0}^{2N} F_m x^m$$

be a $n \times n$ matrix polynomial from $M_n(\mathbb{C}[x])$, which is positive semidefinite on $\mathbb{R}$. Then there exists a matrix polynomial $G(x) = \sum_{m=0}^{N} G_m x^m \in M_n(\mathbb{C}[x])$, such that

$$F(x) = G(x)^* G(x),$$

where $G(x)^* = \overline{G\left(x\right)}^T$.

The first proof of Theorem 1.2 that we are aware of is probably [11, Theorem 8.2] from 1950s. The problem appears in the study of systems of integral equations and they provide a complex analytical proof of the result. Due to an importance of the factorization in linear systems (see [23], [17]), many different proofs have appeared in literature. The factorization is called the *continuous spectral factorization*. Under a conformal mapping of the upper half plane into the unit disk the factorization is equivalent to the factorization of a matrix polynomial, positive semidefinite on a unit complex circle, called the *discrete spectral factorization* (see Theorem 1.1 above; for an equivalence of the factorizations see Subsections 2.2, 3.2). Some of the proofs of either of the factorizations can be found in [26], [23, Appendix B], [16], [7], [4], [25, Theorem 12.8], [20], [8], [9], [30], [10], [13] etc. The main problems of our paper are the following.

Problem 1. *Characterize univariate matrix Laurent polynomials, which are positive semidefinite on a union of points and arcs in* $\mathbb{T}$.

Problem 2. *Characterize univariate matrix polynomials, which are positive semidefinite on a union of points and intervals (not necessarily bounded) in* $\mathbb{R}$.

1.2. **Problem 1 - notation and known results.** Let $\mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$ be the complex unit circle. Let $\mathbb{C}\left[z, \frac{1}{z}\right]$ be the set of complex Laurent polynomials with conjugation and $z^* = \frac{1}{z}$ as the involution. Let $M_n\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$ be the set of $n \times n$ complex Laurent polynomials over $\mathbb{C}\left[z, \frac{1}{z}\right]$ with conjugated transpose as the *involution*. We say $A(z) \in M_n\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$ is *hermitian*, if $A(z) = A(z)^*$. We write $H_n\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$ for the set of all hermitian matrix Laurent polynomials from $M_n\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$. The *degree* of the hermitian matrix Laurent polynomial

$$A(z) = \sum_{m=-N}^{N} A_m z^m \in H_n\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$$

is N, if $A_N \neq 0$ (and hence $A_{-N} = A_N^* \neq 0$), where $N \in \mathbb{N} \cup \{0\}$. We write $\deg(A) = N$. $A(z) \in H_n\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$ is *positive definite* (resp. *positive semidefinite*) in $z_0 \in \mathbb{T}$ if $v^* A(z_0)v > 0$ (resp. $v^* A(z_0)v \geq 0$) for every nonzero $v \in \mathbb{C}^n$. We write $\sum M_n\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)^2$ for the set of all finite sums of the expressions of the form $B(z)^* B(z)$ where $B(z) \in M_n\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$. We call such expressions *hermitian squares* of matrix Laurent polynomials.

A *basic closed semialgebraic set* $\mathscr{K}_{\mathscr{S}} \subseteq \mathbb{T}$ associated to a finite subset

$$\mathscr{S} = \{b_1, \ldots, b_s\} \subset H_1\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$$

is given by

$$\mathscr{K} := \mathscr{K}_{\mathscr{S}} = \{z \in \mathbb{T} \colon b_j(z) \geq 0, \ j = 1, \ldots, s\}.$$

The set $\mathscr{K}$ is *regular*, if it is equal to the closure of its interior. We define a *n-th matrix preordering* $\mathscr{T}_{\mathscr{S}}^n \subseteq H_n\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$ by

$$\mathscr{T}_{\mathscr{S}}^n = \left\{ \sum_{e \in \{0,1\}^s} \tau_e \underline{b}^e \colon \tau_e \in \sum M_n\left(\mathbb{C}\left[z\right]\right)^2 \text{ for all } e \in \{0,1\}^s \right\}$$

where $e = (e_1, \ldots, e_s)$ and $\underline{b}^e$ stands for $b_1^{e_1} \cdots b_s^{e_s}$.

Remark 1.3. (1) If $A(z) = \sum_{m=-M_1}^{M_2} A_m z^m$, where $M_1, M_2 \in \mathbb{N} \cup \{0\}$ and $A_m \in M_n(\mathbb{C})$, then we can write

$$A(z)^* A(z) = \left(z^{-M_1} A_1(z)\right)^* \left(z^{-M_1} A_1(z)\right) = A_1(z)^* A_1(z) \in \sum M_n \left(\mathbb{C}[z]\right)^2.$$

Therefore we have the equality $\sum M_n \left(\mathbb{C}[z]\right)^2 = \sum M_n \left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)^2$.
(2) The set $\mathscr{T}_{\mathscr{S}}^n$ is the set of all finite sums of the elements from the set $\mathscr{T}_{\mathscr{S}}^1$ · $\sum M_n \left(\mathbb{C}[z]\right)^2$.

We write $\text{Pos}_{\succeq 0}^n(\mathscr{K}_{\mathscr{S}})$ (resp. $\text{Pos}_{\succ 0}^n(\mathscr{K}_{\mathscr{S}})$) for the set of all $n \times n$ hermitian matrix Laurent polynomials, which are positive semidefinite (resp. positive definite) on $\mathscr{K}_{\mathscr{S}}$. We say $\mathscr{T}_{\mathscr{S}}^n$ is *saturated* if $\mathscr{T}_{\mathscr{S}}^n = \text{Pos}_{\succeq 0}^n(\mathscr{K}_{\mathscr{S}})$. Saturated matrix pre-ordering $\mathscr{T}_{\mathscr{S}}^n$ is *boundedly saturated* (resp. *boundedly weakly saturated*), if every $A \in \text{Pos}_{\succeq 0}^n(\mathscr{K}_{\mathscr{S}})$ (resp. $A \in \text{Pos}_{\succ 0}^n(\mathscr{K}_{\mathscr{S}})$) is of the form $\sum_{e \in \{0,1\}^s} \tau_e \underline{b}^e$, where $\deg(\tau_e \underline{b}^e) \leq \deg(A)$ holds for every $e \in \{0,1\}^s$.

Theorem 1.1 can be restated in the following form.

Theorem 1.1'. *Assume the notation as above. The set $\mathscr{T}_{\emptyset}^n$ is boundedly saturated for every $n \in \mathbb{N}$.*

The aim of this article is to study matrix generalizations of Theorem 1.1' to an arbitrary basic closed semialgebraic set $\mathscr{K} \subseteq \mathbb{T}$. The problem is the following.

Problem 1'. *Assume $\mathscr{K} \subseteq \mathbb{T}$ is a basic closed semialgebraic set. Does there exist a finite set $\mathscr{S} \subset H_1(\mathbb{C}[z])$, such that $\mathscr{K} = \mathscr{K}_{\mathscr{S}}$ and the n-th matrix preordering $\mathscr{T}_{\mathscr{S}}^n$ is saturated for every $n \in \mathbb{N}$?*

If the answer to Problem 1' is yes, another problem appears.

Problem 1". *Assume $\mathscr{K} \subseteq \mathbb{T}$ is a basic closed semialgebraic set. Suppose that for a finite set $\mathscr{S} \subset H_1(\mathbb{C}[z])$, such that $\mathscr{K} = \mathscr{K}_{\mathscr{S}}$, the n-th matrix preordering $\mathscr{T}_{\mathscr{S}}^n$ is saturated for every $n \in \mathbb{N}$. Is $\mathscr{T}_{\mathscr{S}}^n$ boundedly saturated for every $n \in \mathbb{N}$?*

Now we define two descriptions of the set $\mathscr{K}$, which answer Problems 1' and 1".
Let $\mathscr{K} \subseteq \mathbb{T}$ be a basic closed semialgebraic set. A set $\mathscr{S} = \{b_1, \ldots, b_s\} \subset H_1\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$ is a *saturated description* of $\mathscr{K}$, if the following conditions hold:

(a) $\mathscr{K} = \mathscr{K}_S$.
(b) For every boundary point $a \in \mathscr{K}$, which is not isolated, there exists $k \in \{1, \ldots, s\}$, such that $b_k(a) = 0$ and $\frac{db_k}{dz}(a) \neq 0$.
(c) For every isolated point $a \in \mathscr{K}$, there exist $k, l \in \{1, \ldots, s\}$, such that $b_k(a) = b_l(a) = 0$, $\frac{db_k}{dz}(a) \neq 0$, $\frac{db_l}{dz}(a) \neq 0$ and $b_k b_l \neq 0$ on some neighborhood of a.

Let $\mathbb{T}$ be positively oriented. For $z, w \in \mathbb{T}$, $z \neq w$, let $[z, w]$ (resp. (z, w)) denote a closed (resp. open) arc on $\mathbb{T}$ with endpoints z and w. A set $\mathscr{S} \subset H_1\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$ is the *natural description* of $\mathscr{K}$, if the following conditions hold:

(a) For every $z_1, z_2 \in \mathscr{K}$, $z_1 \neq z_2$ and $(z_1, z_2) \cap \mathscr{K} = \emptyset$,

$$b(z) := k \cdot \frac{(z - z_1)(z - z_2)}{z} \in \mathscr{S},$$

where $k = \sqrt{z_1 z_2} \in \mathbb{T}$ is such that $b(z) \in \text{Pos}_{\succeq 0}^1([z_2, z_1])$.

(b) If $\mathscr{K} = \{z_0\}$, then

$$b_1(z) = k_1 \cdot \frac{(z - z_0)(z - iz_0)}{z} \in \mathscr{S} \quad \text{and} \quad b_2(z) = k_2 \cdot \frac{(z - z_0)(z + iz_0)}{z} \in \mathscr{S},$$

where $k_1 = \sqrt{iz_0^2} \in \mathbb{T}, k_2 = \sqrt{-iz_0^2} \in \mathbb{T}$ are such that

$$b_1(z) \in \mathrm{Pos}_{\geq 0}^1([z_0, iz_0]) \quad \text{and} \quad b_2(z) \in \mathrm{Pos}_{\geq 0}^1([-iz_0, z_0]).$$

(c) These are the only elements of $\mathscr{S}$.

Convention 1. An arc always has a non-empty interior. Therefore it is a regular set.

1.3. **Problem 1 - new results.** One of the main results of the paper, which solves Problem 1', is the following.

Theorem A. *The n-th matrix preordering $\mathscr{T}_{\mathscr{G}}^n$ is saturated for every integer $n \in \mathbb{N}$ if and only if $\mathscr{S}$ is a saturated description of $\mathscr{K}$ (see Theorem 2.1).*

The answer to Problem 1" (except for a union of an arc and a point) is the following.

Theorem B. *Let $\mathscr{K}$ be a basic closed semialgebraic set.*
The n-th matrix preordering $\mathscr{T}_{\mathscr{G}}^n$ is boundedly saturated for the natural description $\mathscr{S}$ of $\mathscr{K}$ in either of the following cases:

- *$n = 1$ and $\mathscr{K}$ is arbitrary,*
- *$n \in \mathbb{N}$ is arbitrary and $\mathscr{K}$ is an arc,*
- *$n \in \mathbb{N}$ is arbitrary and $\mathscr{K}$ is a union of at most three points,*

(see Theorem 3.6).
The n-th matrix preordering $\mathscr{T}_{\mathscr{G}}^n$ is not boundedly saturated for any set finite set $\mathscr{S} \subseteq H_1\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$ such that $\mathscr{K} = \mathscr{K}_{\mathscr{S}}$ in the following cases:

- *$n \geq 2$ and $\mathscr{K}$ contains at least two arcs,*
- *$n \geq 2$ and $\mathscr{K}$ is a union of m points with $m \geq 4$,*
- *$n \geq 2$ and $\mathscr{K}$ is a union of an arc an m isolated points with $m \geq 2$,*

(see Theorem 4.2).

Theorem B solves Problem 1" for every closed semialgebraic set $\mathscr{K} \subseteq \mathbb{T}$, different from a union of an arc and a point. We formulate the remaining case as a conjecture.

Conjecture 1. *Let $\mathscr{K} \subseteq \mathbb{T}$ be a union of an arc and a point. Suppose $\mathscr{S}$ is the natural description of $\mathscr{K}$. Then the n-th matrix preordering $\mathscr{T}_{\mathscr{G}}^n$ is boundedly saturated for every integer $n \in \mathbb{N}$.*

The following table summarizes Theorems A and B and Conjecture 1.

$\mathscr{K}$	A	B	C
a union of at most three points	Yes	Yes	Yes
a union of m points, where $m \geq 4$	Yes	Yes	No
an arc	Yes	Yes	Yes
a union of an arc and an isolated point	Yes	Yes	D
a union of an arc and m isolated points, where $m \geq 2$	Yes	Yes	No
includes at least two arcs	Yes	Yes	No

$A \quad := \quad$ The n-th matrix preordering $\mathcal{T}_{\mathscr{S}}^n$ is saturated for every saturated

description $\mathscr{S}$ of $\mathscr{K}$ and every integer $n \in \mathbb{N}$.

$B \quad := \quad$ The preordering $\mathcal{T}_{\mathscr{S}}^1$ is boundedly saturated for the natural description

$\mathscr{S}$ of $\mathscr{K}$.

$C \quad := \quad$ The n-th matrix preordering $\mathcal{T}_{\mathscr{S}}^n$ is boundedly saturated for the natural

description $\mathscr{S}$ of $\mathscr{K}$ and every integer $n \in \mathbb{N}$.

$D \quad := \quad$ See Conjecture 1.

The classification covers all closed semialgebraic sets $\mathscr{K} \subseteq \mathbb{T}$. Except for the conjectured case it is complete. For example, the classification is complete for regular sets.

1.4. **Problem 2 - notation and known results.** Let $M_n(\mathbb{C}[x])$ be a set of all $n \times n$ complex matrix polynomials over $\mathbb{C}[x]$ with conjugated transpose as the involution. The *degree* of a matrix polynomial

$$F(x) = \sum_{m=0}^{N} F_m x^m \in M_n(\mathbb{C}[x])$$

is N if $F_N \neq 0$. We write $\deg(F) = N$. We say $F(x) \in M_n(\mathbb{C}[x])$ is *hermitian*, if $F(x) = F(x)^*$. We write $H_n(\mathbb{C}[x])$ for the set of all hermitian matrix polynomials from $M_n(\mathbb{C}[x])$. $F(x) \in H_n(\mathbb{C}[x])$ is *positive definite* (resp. *positive semidefinite*) in $x_0 \in \mathbb{C}$ if $v^* F(x_0)v > 0$ (resp. $v^* F(x_0)v \geq 0$) for every nonzero $v \in \mathbb{C}^n$. We write $\sum M_n(\mathbb{C}[x])^2$ for the set of all finite sums of the expressions of the form $G(x)^* G(x)$ where $G(x) \in M_n(\mathbb{C}[x])$. We call such expressions *hermitian squares* of matrix polynomials.

A *basic closed semialgebraic set* $K_S \subseteq \mathbb{R}$ associated to a finite subset

$$S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$$

is given by

$$K := K_S = \{x \in \mathbb{R} \colon g_j(x) \geq 0, \ j = 1, \ldots, s\}.$$

The set K is *regular*, if it is equal to the closure of its interior. We define the *n-th matrix preordering* $T_S^n \subseteq H_n(\mathbb{C}[x])$ by

$$T_S^n \quad := \quad \left\{ \sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e \colon \sigma_e \in \sum M_n(\mathbb{C}[x])^2 \text{ for all } e \in \{0,1\}^s \right\},$$

where $e = (e_1, \ldots, e_s)$ and $\underline{g}^e$ stands for $g_1^{e_1} \cdots g_s^{e_s}$.

Remark 1.4. Note that T_S^n is the set of all finite sums of elements from the set $T_S^1 \cdot \sum M_n(\mathbb{C}[x])^2$.

We write $\mathrm{Pos}_{\succeq 0}^n(K)$ (resp. $\mathrm{Pos}_{\succ 0}^n(K)$) for the set of all $n \times n$ hermitian matrix polynomials, which are positive semidefinite (resp. definite) on K_S. We say T_S^n is *saturated* if $T_S^n = \mathrm{Pos}_{\succeq 0}^n(K_S)$. Saturated matrix preordering T_S^n is *boundedly saturated* (resp. *boundedly weakly saturated*), if every $F \in \mathrm{Pos}_{\succeq 0}^n(K_S)$ (resp. $F \in \mathrm{Pos}_{\succ 0}^n(K_S)$) is of the form $\sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e$, where $\deg(\sigma_e \underline{g}^e) \leq \deg(F)$ holds for every $e \in \{0,1\}^s$. Theorem 1.2 can be restated in the following form.

Theorem 1.2'. *Assume the notation as above. The set $T_\emptyset^n$ is boundedly saturated for every $n \in \mathbb{N}$.*

The aim of this article is to study matrix generalizations of Theorem 1.2' to an arbitrary basic closed semialgebraic set $K \subseteq \mathbb{R}$. The problem is the following.

Problem 2'. *Assume $K \subseteq \mathbb{R}$ is a basic closed semialgebraic set. Does there exist a finite set $S \subset \mathbb{R}[x]$, such that $K = K_S$ and the n-th matrix preordering T_S^n is saturated for every $n \in \mathbb{N}$?*

If the answer to Problem 2' is yes, another problem appears.

Problem 2". *Assume $K \subseteq \mathbb{R}$ is a basic closed semialgebraic set. Suppose that for a finite set $S \subset \mathbb{R}[x]$, such that $K = K_S$, the n-th matrix preordering T_S^n is saturated for every $n \in \mathbb{N}$. Is T_S^n boundedly saturated for every $n \in \mathbb{N}$?*

Let $K \subseteq \mathbb{R}$ be a basic closed semialgebraic set. A set $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ is the *natural description* of K, if it satisfies the following conditions:

   (a) If K has the least element a, then $x - a \in S$.
   (b) If K has the greatest element a, then $a - x \in S$.
   (c) For every $a \neq b \in K$, if $(a, b) \cap K = \emptyset$, then $(x - a)(x - b) \in S$.
   (d) These are the only elements of S.

Problems 2' and 2" have already been solved in the following cases:

   (1) T_S^1 is boundedly saturated for the natural description S of K (see [18, Theorem 2.2] or [22, 2.7.3 Proposition] and [19, Theorem 4.1]).
   (2) For $K = K_{\{x, 1-x\}} = [0, 1]$, $T_{\{x, 1-x\}}^n$ is boundedly saturated for every $n \in \mathbb{N}$ (see [6] and [30]).
   (3) For $K = K_{\{x\}} = [0, \infty)$, $T_{\{x\}}^n$ is boundedly saturated for every $n \in \mathbb{N}$ (see [30] and [3]).

Even more can be said in the case $n = 1$. There is a characterization of finite sets $S = \{g_1, \ldots, g_S\} \subset \mathbb{R}[x]$ such that the preordering T_S^1 is saturated, which we now explain. If the set K_S is not compact, then T_S^1 is saturated iff S contains each of the polynomials in the natural description of K_S up to scaling by positive constants (see [18, Theorem 2.2]). Let now K_S be a compact set. Write K_S as the union of pairwise disjoint points and intervals, i.e. $K_S = [x_j, y_j]$, where $x_j \leq y_j$ for every $j = 1, \ldots, t$. Then T_S^1 is saturated if and only if the following two conditions hold:

   (a) For every left endpoint x_j there exists $k \in \{1, \ldots, s\}$, such that $g_k(x_j) = 0$ and $g_k'(x_j) > 0$.
   (b) For every right endpoint y_j there exists $k \in \{1, \ldots, s\}$, such that $g_k(y_j) = 0$ and $g_k'(y_j) < 0$.

(See [22, 9.3.3 Theorem] or [19, Theorem 3.2]. For the extension to curves in $\mathbb{R}^n$, see [28, Theorem 5.17] or [22, 9.3.5 Theorem].). We call every set $S \subset \mathbb{R}[x]$, that satisfies the two conditions above, a *saturated description* of K_S.

Convention 2. An interval always has a non-empty interior.

1.5. **Problem 2 - new results.** One of the main results of the paper, which solves Problem 2' for compact sets K, is the following.

Theorem C. *Let K be compact. The n-th matrix preordering T_S^n is saturated for every $n \in \mathbb{N}$ if and only if S is a saturated description of K (see Theorem 2.2).*

The answers to Problem 2' for unbounded sets K (except for a union of one or two unbounded intervals and a point) and to Problem 2" (except for a union of a regular component and a point or a union of two unbounded intervals and a point), are given by Theorem D below.

Theorem D. *Let K be a basic closed semialgebraic set.*

The n-th matrix preordering T_S^n is boundedly saturated for the natural description S of K and every $n \in \mathbb{N}$ if K is either of the following:

- *an interval,*
- *a union of two unbounded intervals,*
- *a union of at most three points,*

(see Theorem 3.2).

The n-th matrix preordering T_S^n is not boundedly saturated for any finite set $S \subset \mathbb{R}[x]$ such that $K = K_S$ in the following cases:

- *$n \geq 2$ and K contains at least two intervals with at least one of them bounded,*
- *$n \geq 2$ and K is a union of m points with $m \geq 4$,*
- *$n \geq 2$ and K is a union of an interval (bounded or unbounded) and m isolated points with $m \geq 2$.*
- *$n \geq 2$ and K is a union of two unbounded intervals and m isolated points with $m \geq 2$.*

Moreover, T_S^n is not even boundedly weakly saturated, if K is regular and has at least two components, one of which is unbounded and the others are bounded (see Theorem 4.1).

Theorem D solves Problem 2" for every closed semialgebraic set $K \subseteq \mathbb{R}$, different from the ones covered by Conjecture 2 below. Conjecture 2 is based on the investigation of some examples and is the following.

Conjecture 2. *Let $K \subseteq \mathbb{R}$ be either of the following:*

- *A union of a bounded interval and a point.*
- *A union of an unbounded interval and a point.*
- *A union of two unbounded intervals and a point.*

Suppose S is the natural description of K. Then the n-th matrix preordering T_S^n is boundedly saturated for every integer $n > 1$.

Conjecture 2 is true for all three cases covered if and only if it is true for at least one case covered. Let us explain. Suppose F is a matrix polynomials with a 'bounded description' on a union of a bounded interval and a point. Then for an appropriate $d \in \mathbb{R}$, $G(x) = (\pm(x-d))^{\deg(F)} F\left(\pm\frac{1}{x-d}\right)$ is a matrix polynomial with a 'bounded description' on a union of an unbounded interval and a point or a 'bounded description' on a union of two unbounded intervals and a point. Vice versa, by the equality $F(x) = x^{\deg(F)} G\left(\pm\frac{1}{x} + d\right)$, the reverse statements also hold. Furthermore, by the results of Subsections 2.2 and 3.2, Conjecture 2 is true if and only if Conjecture 1 is true.

If the set K is unbounded, then by the form of the polynomials in the natural description S of K, the n-th matrix preordering T_S^n is saturated if and only if T_S^n is boundedly saturated. Therefore, unbounded sets K without boundedly saturated T_S^n (see Theorem D above), also do not have saturated T_S^n. Since for the natural

description S of K, the preordering T_S^1 is always boundedly saturated, the same is true for an arbitrary finite set $S \subset \mathbb{R}[x]$ such that $K = K_S$. However, the characterization of the set $\mathrm{Pos}_{\succeq 0}^n(K)$ for those sets K is the following.

Theorem E. *Let K be an unbounded basic closed semialgebraic set with a saturated description S and $n \in \mathbb{N}$. Then $F \in M_n(\mathbb{C}[x])$ belongs to $\mathrm{Pos}_{\succeq 0}^n(K)$ if and only if for every $w \in \mathbb{C}$ there exists $h \in \mathbb{R}[x]$, such that $h(w) \neq 0, h(\overline{w}) \neq 0$ and $h^2 F \in T_S^n$ (see Theorem 5.1).*

The following table summarizes [19, Theorem 4.1], Theorems C and D and Conjecture 2.

K	A	B	C
a union of at most three points	Yes	Yes	Yes
a union of m points with $m \geq 4$	Yes	Yes	No
a bounded interval	Yes	Yes	Yes
a union of a bounded interval and an isolated point	Yes	Yes	D
a union of a bounded interval and m isolated points with $m \geq 2$	Yes	Yes	No
a compact set containing at least two intervals	Yes	Yes	No
an unbounded interval	Yes	Yes	Yes
a union of an unbounded interval and an isolated point	Yes	D	D
a union of an unbounded interval and m isolated points with $m \geq 2$	Yes	No	No
a union of two unbounded intervals	Yes	Yes	Yes
a union of two unbounded intervals and an isolated point	Yes	D	D
a union of two unbounded intervals and m isolated points with $m \geq 2$	Yes	No	No
includes a bounded and an unbounded interval	Yes	No	No

A := The preordering T_S^1 is boundedly saturated for the natural description S of K.

B := The n-th matrix preordering T_S^n is saturated for some finite set S such that $K = K_S$ and every integer $n \in \mathbb{N}$.

C := The n-th matrix preordering T_S^n is boundedly saturated for the natural description S of K and every integer $n \in \mathbb{N}$.

D := See Conjecture 2.

Note that the classification covers all closed semialgebraic sets $K \subseteq \mathbb{R}$. If K includes at least one unbounded interval, then by the form of the polynomials in the natural description S of K, T_S^n is saturated if and only if T_S^n is boundedly saturated. By the paragraph after Conjecture 2 above, the value of all D-s in the table is the same. It is also the same to the value of D in the table classifying sets $\mathscr{K} \subseteq \mathbb{T}$ above. However, for regular sets $K \subseteq \mathbb{R}$, the classification is complete.

At the end we solve a matrix moment problem, which we studied in [3], for the union of two unbounded intervals (see Theorem 6.1).

1.6. **Facts about higher dimensions.** The motivation for our research comes from noncommutative real algebraic geometry for matrix polynomials. The problem is the following: *For a given semialgebraic set K in $\mathbb{R}^n$ characterize matrix polynomials which are positive semidefinite on K.* Let us briefly survey what is known about the characterizations in higher dimensions. Positive definite polynomials on $\mathbb{R}^n$ where first characterized in [12]. They generalize the characterization of positive semidefinite polynomial from $M_n(Q)$, where Q is a finite dimensional extension of rational numbers (see [5]). For other proofs see [24], [14]. Generally, for an arbitrary semialgebraic set $K \subseteq \mathbb{R}^n$ and multivariate polynomials the Positivstellensatz in the sense of Krivine-Stengle result was obtained by Cimprič (see [2]). For a compact set K, a denominator free characterization of positive definite matrix polynomials is a matrix version of Schmüdgen's Positivstellensatz (see [3, Theorem 6]).

2. Saturated descriptions of an arbitrary $\mathscr{K} \subseteq \mathbb{T}$ and a compact $K \subset \mathbb{R}$ generate saturated n-th matrix preorderings

The solutions to Problems 1' for an arbitrary $\mathscr{K}$ and 2' for a compact set K from the Introduction, are the main results of this section (see Theorems 2.1 and 2.2 below). They also characterize all finite sets $\mathscr{S}$ and S, such that the preorderings $\mathscr{T}_{\mathscr{S}}^n$ and T_S^n are saturated for every integer $n \in \mathbb{N}$.

Theorem 2.1. *Suppose $\mathscr{K}$ is a non-empty basic closed semialgebraic set in $\mathbb{T}$. The n-th matrix preordering $\mathscr{T}_{\mathscr{S}}^n$ is saturated for every $n \in \mathbb{N}$ if and only if $\mathscr{S}$ a saturated description of $\mathscr{K}$.*

Theorem 2.2. *Suppose K is a non-empty basic compact semialgebraic set in $\mathbb{R}$. The n-th matrix preordering T_S^n is saturated for every $n \in \mathbb{N}$ if and only if S a saturated description of K.*

Note that by [15, Theorem 2], T_S^n is weakly saturated for every finite set $S \subset \mathbb{R}[x]$ satisfying $K = K_S$. By Theorem 2.2, T_S^n is even saturated exactly for every saturated description S of K.

Theorems 2.1 and 2.2 can be proved independently from each other by the induction on the size of matrix polynomials n using exactly the same methods. However, to avoid repetition and to establish the connection between Problems 1 and 2 (see Subsection 2.2), we choose to prove Theorem 2.1 independently (see Subsection 2.1) and then derive Theorem 2.2 from it (see Subsection 2.3). The advantage of this choice is also the fact, that we will need the connection between Problems 1 and 2 in the subsequent sections. The main ingredients in the proof of Theorems 2.1 and 2.2 are:

(1) The $n = 1$ case (For Problem 1' it is derived from [28, Theorem 5.17] - see Proposition 2.4 below. For Problem 2' this is [22, 9.3.3 Theorem].).
(2) Proposition 2.7 for Problem 1' and Corollary 2.8 for Problem 2' (The proofs use the idea of diagonalizing matrix polynomials. See [29, 4.3].).
(3) Getting rid of the denominators in Proposition 2.7 and Corollary 2.8 with the use of Proposition 2.3 below, which is [27, Proposition 2.7] or [22, 9.6.1 Lemma].
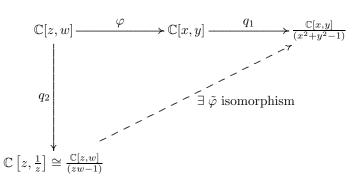
Proposition 2.3. *Suppose R is a commutative ring with 1 and $\mathbb{Q} \subseteq R$. Let $\Phi : R \to C(K, \mathbb{R})$ be a ring homomorphism, where K is a topological space which is*

compact and Hausdorff. Suppose $\Phi(R)$ separates points in K. Suppose $f_1, \ldots, f_k \in R$ are such that $\Phi(f_j) \geq 0$, $j = 1, \ldots, k$ and $(f_1, \ldots, f_k) = (1)$. Then there exist $s_1, \ldots, s_k \in R$ such that $s_1 f_1 + \ldots + s_k f_k = 1$ and such that each $\Phi(s_j)$ is strictly positive.

2.1. **Proof for $\mathscr{T}_{\mathscr{S}}^n$.** The $n = 1$ case of Theorem 2.1 is the following.

Proposition 2.4. *Suppose $\mathscr{K}$ is a non-empty basic closed semialgebraic set in $\mathbb{T}$. The preordering $\mathscr{T}_{\mathscr{S}}^1$ is saturated if and only if $\mathscr{S}$ is a saturated description of $\mathscr{K}$.*

Proof. We have the following diagram:

$$
\begin{array}{ccc}
\mathbb{C}[z,w] & \xrightarrow{\ \ \varphi\ \ } & \mathbb{C}[x,y] \xrightarrow{\ \ q_1\ \ } \frac{\mathbb{C}[x,y]}{(x^2+y^2-1)} \\
\Big\downarrow{\scriptstyle q_2} & & \\
\mathbb{C}\left[z, \frac{1}{z}\right] \cong \frac{\mathbb{C}[z,w]}{(zw-1)} & &
\end{array}
$$

$\exists\, \tilde{\varphi}$ isomorphism

where φ is a unital ring homomorphism with $\varphi(z) := x + iy$, $\varphi(w) := x - iy$ and q_1, q_2 are quotient projections. Define $f(x,y) := x^2 + y^2 - 1$. By the diagram above, $\tilde{\varphi}\left(H_1\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)\right) = \frac{\mathbb{R}[x,y]}{(f)}$. Let us define the set $S_1 := \tilde{\varphi}(\mathscr{S}) \subseteq \frac{\mathbb{R}[x,y]}{(f)}$. We write $S_1 = \{g_1 + (f), \ldots, g_s + (f)\}$ and $S := \{g_1, \ldots, g_s\} \subset \mathbb{R}[x,y]$. Suppose $\mathcal{Z} := \{(x,y) \in \mathbb{R}^2 \colon f(x,y) = 0\}$. Let v_p be the natural valuation on the completion of $\frac{\mathbb{R}[x,y]}{(f)}$ at the point $p \in \mathcal{Z}$. By [28, Theorem 5.17] or [22, 9.3.5 Theorem], $T_S^1 + I$ is saturated if and only if the following conditions hold:

  (1) For each boundary point $p \in K_S \cap \mathcal{Z}$, which is not an isolated point of $K_S \cap \mathcal{Z}$, there exists $k \in \{1, \ldots, s\}$, such that $v_p(g_k) = 1$.
  (2) For each isolated point $p \in K_S \cap \mathcal{Z}$, there exist $k, l \in \{1, \ldots, s\}$, such that $v_p(g_k) = v_p(g_l) = 1$ and $g_k g_l \leq 0$ in some neighbourhood of p in $\mathcal{Z}$.

Let us prove, that this is fulfilled exactly when $\mathscr{S}$ is a saturated description of $\mathscr{K}$. First notice that $p := (x_0, y_0)$ is a boundary (and isolated) point of $K_S \cap \mathcal{Z}$ iff $x_0 + iy_0$ is a boundary (and isolated) point of $\mathscr{K}$. Take $g \in \mathbb{R}[x,y]$, such that $g(p) = 0$. Write g in the form

$$k_1(x - x_0) + k_2(y - y_0) + (x - x_0)(y - y_0)p(x,y) + (x - x_0)^2 q(x) + (y - y_0)^2 r(y),$$

where $k_1, k_2 \in \mathbb{R}, p(x,y) \in \mathbb{R}[x,y], q(x) \in \mathbb{R}[x], r(y) \in \mathbb{R}[y]$. By the use of $x_0^2 + y_0^2 = 1$, we write

$$f(x,y) = (x - x_0)^2 + (y - y_0)^2 + 2x_0(x - x_0) + 2y_0(y - y_0).$$

We may assume $x_0 \neq 0$ (For $y_0 \neq 0$ the proof is analogous.). Then we can write

$$g(x,y) = \frac{k_1}{2x_0} f(x,y) + \left(k_2 - k_1 \frac{y_0}{x_0}\right)(y - y_0) + \sum_{l+m \geq 2} k_{lm} f(x,y)^l (y - y_0)^m,$$

with $k_{lm} \in \mathbb{R}$. By the proof of [22, 12.2.2 Theorem],

$$\frac{\mathbb{R}\left[\hat{x}, y\right]}{(f)}_p = \frac{\mathbb{R}\left[\left[x - x_0, y - y_0\right]\right]}{(\hat{f})_p} = \frac{\mathbb{R}\left[\left[f, y - y_0\right]\right]}{(\hat{f})_p} = \mathbb{R}\left[\left[y - y_0\right]\right],$$

where $\frac{\mathbb{R}[\hat{x},y]}{(f)}_p$ (resp. $(\hat{f})_p$) is the completion of $\frac{\mathbb{R}[x,y]}{(f)}$ (resp. (f)) at the point p. Therefore $v_p(g) = 1$ iff $k_2 \neq k_1 \frac{y_0}{x_0}$. Now,

$$b(z) = \tilde{\varphi}^{-1}(g + (f)) = \frac{z - (x_0 + iy_0)}{2z} \cdot c(z) + (z - (x_0 + iy_0))^2 \cdot d(z),$$

where

$$c(z) = k_1(z - (x_0 - iy_0)) - ik_2(z + (x_0 - iy_0))$$

and $d(z) \in \mathbb{C}\left[z, \frac{1}{z}\right]$. Furthermore, $c(x_0 + iy_0) = 2ik_1y_0 - 2ik_2x_0$. Hence,

$$c(x_0 + iy_0) \neq 0 \quad \Leftrightarrow \quad k_2 \neq k_1 \frac{y_0}{x_0} \quad \Leftrightarrow \quad \frac{db}{dz}(x_0 + iy_0) \neq 0.$$

Finally,

$$v_p(g) = 1 \quad \Leftrightarrow \quad \frac{db}{dz}(x_0 + iy_0) \neq 0.$$

Therefore, from the necessary and sufficient conditions for $T_S^1 + I$ being saturated above we conclude that $\mathcal{T}_{\mathscr{S}}^1$ is saturated if and only if $\mathscr{S}$ is a saturated description of $\mathscr{K}$. $\qquad\square$

To prove Proposition 2.7 below, which is the second main step in the proof of Theorem 2.1, we need Lemmas 2.5 and 2.6 below.

Lemma 2.5. *Let $B = [b_{kl}]_{kl} \in M_n\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$. For every $1 \leq k \leq l \leq n$ there exist unitary matrices $U_{kl} \in M_n(\mathbb{R})$ and $V_{kl} \in M_n(\mathbb{C})$, such that*

$$U_{kl}BU_{kl}^* = \begin{bmatrix} c_{kl} & * \\ * & * \end{bmatrix}, \quad V_{kl}BV_{kl}^* = \begin{bmatrix} d_{kl} & * \\ * & * \end{bmatrix},$$

where

$$c_{kl} = \begin{cases} b_{kl}, & \text{for } 1 \leq k = l \leq n \\ \frac{1}{2}(b_{kl} + b_{lk} + b_{kk} + b_{ll}), & \text{for } 1 \leq k < l \leq n \end{cases},$$

$$d_{kl} = \begin{cases} b_{kl}, & \text{for } 1 \leq k = l \leq n \\ \frac{i}{2}(-b_{kl} + b_{lk}) + \frac{1}{2}(b_{kk} + b_{ll}), & \text{for } 1 \leq k < l \leq n \end{cases}.$$

Proof. We define $U_{11} = V_{11} := I_n$, $U_{kk} = V_{kk} := P_k$ for $k = 2, \ldots, n$, where P_k denotes the permutation matrix which permutes the first row and the k-th row.

For $1 \leq k < l \leq n$, define $U_{kl} := P_k S_{kl}$, where $S_{kl} = \left(s_{pr}^{(kl)}\right)_{pr} \in M_n(\mathbb{R})$ be the matrix with $s_{kk}^{(kl)} = s_{kl}^{(kl)} = s_{lk}^{(kl)} = \frac{1}{\sqrt{2}}$, $s_{ll}^{(kl)} = -\frac{1}{\sqrt{2}}$, $s_{pp}^{(kl)} = 1$ if $p \notin \{k, l\}$ and $s_{pr}^{(kl)} = 0$ otherwise.

For $1 \leq k < l \leq n$, define $V_{kl} := P_k \tilde{S}_{kl}$, where $\tilde{S}_{kl} = \left(\tilde{s}_{pr}^{(kl)}\right)_{pr} \in M_n(\mathbb{C})$ be the matrix with $\tilde{s}_{kk}^{(kl)} = \tilde{s}_{lk}^{(kl)} = \frac{1}{\sqrt{2}}$, $\tilde{s}_{kl}^{(kl)} = \frac{i}{\sqrt{2}}$, $\tilde{s}_{ll}^{(kl)} = -\frac{i}{\sqrt{2}}$, $\tilde{s}_{pp}^{(kl)} = 1$ if $p \notin \{k, l\}$ and $\tilde{s}_{pr}^{(kl)} = 0$ otherwise. $\qquad\square$

Lemma 2.6. *For $A = \begin{bmatrix} a & \beta \\ \beta^* & C \end{bmatrix} \in H_n\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$, where $a = a^* \in \mathbb{C}\left[z, \frac{1}{z}\right]$, $\beta \in M_{1,n-1}\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$ and $C \in H_{n-1}\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$ it holds*

$$(i) \quad a^4 \cdot A = \begin{bmatrix} a^* & 0 \\ \beta^* & a^* I_{n-1} \end{bmatrix} \begin{bmatrix} a^3 & 0 \\ 0 & a(aC - \beta^*\beta) \end{bmatrix} \begin{bmatrix} a & \beta \\ 0 & a I_{n-1} \end{bmatrix}$$

$$(ii) \quad \begin{bmatrix} a^3 & 0 \\ 0 & a(aC - \beta^*\beta) \end{bmatrix} = \begin{bmatrix} a^* & 0 \\ -\beta^* & a^* I_{n-1} \end{bmatrix} \cdot A \cdot \begin{bmatrix} a & -\beta \\ 0 & a I_{n-1} \end{bmatrix}.$$

Proof. Easy computation. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we come to the second main step in the proof of Theorem 2.1.

Proposition 2.7. *Suppose $\mathscr{K}$ is a non-empty basic closed semialgebraic set in $\mathbb{T}$ and $\mathscr{S}$ a saturated description of $\mathscr{K}$. Then for every $A \in \mathrm{Pos}^n_{\succeq 0}(\mathscr{K})$ and every $w \in \mathbb{C} \setminus \{0\}$ there exists $b \in \mathbb{C}[z]$, such that $b(w), b\left(\frac{1}{\overline{w}}\right) \neq 0$ and $(\overline{b^*b}) \cdot A \in \mathscr{T}^n_{\mathscr{S}}$.*

Proof. We prove by the induction on the size n of the matrix polynomials. For $n = 1$ we can take $b = 1$ by Proposition 2.4. Suppose the proposition holds for $n - 1$. We will prove, that it holds for n. Let us take $A \in \mathrm{Pos}^n_{\succeq 0}(\mathscr{K})$. For $\mathscr{K} = \mathbb{T}$ we can take $b = 1$ by Theorem 1.1. Suppose now $\mathscr{K} \neq \mathbb{T}$. Take $w \in \mathbb{C} \setminus \{0\}$. We separate two cases. If $w \notin \mathbb{T}$, then we define $c(z) = (z - w)^*(z - w)$. Else $w \in \mathbb{T}$ and we define $c(z) = z - w$. If $A = 0$, we can take $b = 1$. Otherwise $A \neq 0$ and we can write $A = c^m B$, where $m \in \mathbb{N} \cup \{0\}$, $B = [b_{kl}]_{kl} \in M_n\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$, and $B(w) = B\left(\frac{1}{\overline{w}}\right) \neq 0$. Let $U_{kl}, V_{kl}, c_{kl}, d_{kl}$ be as in Lemma 2.5. If for some $k_0 \in \{1, \ldots, n\}$, it holds $b_{k_0 k_0}(w) = b_{k_0 k_0}\left(\frac{1}{\overline{w}}\right) \neq 0$, then we define $k_0 = l_0$, $T_{k_0 k_0} := U_{k_0 k_0}$ and $\tilde{b}_{k_0 k_0} := c_{k_0 k_0}$. Otherwise there exist $k_0 < l_0$, such that $b_{k_0 l_0}(w) \neq 0$ or $b_{k_0 l_0}\left(\frac{1}{\overline{w}}\right) \neq 0$.

Case 1: $w \in \mathbb{T}$. From $c_{k_0 l_0}(w) = d_{k_0 l_0}(w) = 0$ we get $b_{k_0 l_0}(w) = 0$, which is a contradiction. Hence either $c_{k_0 l_0}(w) \neq 0$ or $d_{k_0 l_0}(w) \neq 0$.

Case 2: $w \notin \mathbb{T}$. We have $c_{k_0 l_0} = \mathrm{Re}(b_{k_0 l_0}) \in H_1\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$ and $d_{k_0 l_0} = \mathrm{Im}(b_{k_0 l_0}) \in H_1\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$. Since for $c \in H_1\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$, we know that $\overline{c(w)} = c\left(\frac{1}{\overline{w}}\right)$, it follows that $c_{k_0 l_0}(w) \neq 0$ and $c_{k_0 l_0}\left(\frac{1}{\overline{w}}\right) \neq 0$ or $d_{k_0 l_0}(w) \neq 0$ and $d_{k_0 l_0}\left(\frac{1}{\overline{w}}\right) \neq 0$.

If $c_{k_0 l_0}(w) \neq 0$ and $c_{k_0 l_0}\left(\frac{1}{\overline{w}}\right) \neq 0$, we define $T_{k_0 l_0} := U_{k_0 l_0}$ and $\tilde{b}_{k_0 l_0} := c_{k_0 l_0}$. Else $d_{k_0 l_0}(w) \neq 0$ and $d_{k_0 l_0}\left(\frac{1}{\overline{w}}\right) \neq 0$ and we define $T_{k_0 l_0} := V_{k_0 l_0}$, $\tilde{b}_{k_0 l_0} := d_{k_0 l_0}$.

If we write $T_{k_0 l_0} B T^*_{k_0 l_0} = \begin{bmatrix} \tilde{b}_{k_0 l_0} & \tilde{\beta} \\ * & \tilde{C} \end{bmatrix}$ with $\tilde{\beta} \in M_{1,n-1}\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$ and $\tilde{C} \in M_{n-1}\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$, then $T_{k_0 l_0} A T^*_{k_0 l_0} = \begin{bmatrix} c^m \tilde{b}_{k_0 l_0} & c^m \tilde{\beta} \\ (c^m \tilde{\beta})^* & c^m \tilde{C} \end{bmatrix} =: \begin{bmatrix} a & \beta \\ \beta^* & C \end{bmatrix}$. Therefore by Lemma 2.6.(i) and dividing by $(c^* c)^{2m}$, it follows that

$$\tilde{b}^2 A = T^*_{k_0 l_0} \begin{bmatrix} \tilde{b}^*_{k_0 l_0} & 0 \\ \beta^* & \tilde{b}^*_{k_0 l_0} I_{n-1} \end{bmatrix} \begin{bmatrix} d & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \tilde{b}_{k_0 l_0} & \tilde{\beta} \\ 0 & \tilde{b}_{k_0 l_0} I_{n-1} \end{bmatrix} T_{k_0 l_0},$$

where

$$\tilde{b} = \left(\frac{c}{c^*}\right)^m \tilde{b}^2_{k_0 l_0} \in H_1\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$$

$$d = \left(c\frac{c}{c^*}\right)^m \tilde{b}^3_{k_0 l_0} \in H_1\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right),$$

$$D = \left(c\frac{c}{c^*}\right)^m \tilde{b}_{k_0 l_0}\left(\tilde{b}_{k_0 l_0}\tilde{C} - \tilde{\beta}^*\tilde{\beta}\right) \in H_{n-1}\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right).$$

By Lemma 2.6.(ii) and dividing by $(c^*c)^m$, we have also

$$
\begin{bmatrix} d & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} \tilde{b}^*_{k_0 l_0} & 0 \\ -\tilde{\beta}^* & \tilde{b}^*_{k_0 l_0} I_{n-1} \end{bmatrix} T_{k_0 l_0} A T^*_{k_0 l_0} \begin{bmatrix} \tilde{b}_{k_0 l_0} & -\tilde{\beta} \\ 0 & \tilde{b}_{k_0 l_0} I_{n-1} \end{bmatrix}
$$

It follows that $d \geq 0$, $D \succeq 0$ on $\mathscr{K}$. By the induction hypothesis, used for the polynomial $D \in H_{n-1}\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$, there exists $b_1 \in \mathbb{C}[z]$, such that $b_1(w) \neq 0$, $b_1\left(\frac{1}{\bar{w}}\right) \neq 0$ and $b_1^* b_1 D \in \mathcal{T}_{\mathscr{S}}^{n-1}$. By Proposition 2.4, $b_1^* b_1 d \in \mathcal{T}_{\mathscr{S}}^1$. Hence, $b^* b A \in \mathcal{T}_{\mathscr{S}}^n$, where $b = b_1 \cdot \tilde{b}$ and $b(w) \neq 0$, $b\left(\frac{1}{\bar{w}}\right) \neq 0$. This concludes the proof. $\qquad \square$

Finally, we can prove Theorem 2.1. We will use Proposition 2.3 to get rid of the denominators in Proposition 2.7.

Proof of Theorem 2.1. By Proposition 2.4, $\mathcal{T}_{\mathscr{S}}^1$ is saturated if and only if $\mathscr{S}$ is a saturated description of $\mathscr{K}$. Therefore we have to prove only the if part. Let $\mathscr{S}$ be a saturated description of $\mathscr{K}$. We will prove that the set $\mathcal{T}_{\mathscr{S}}^n$ is saturated for every $n \in \mathbb{N}$. Let R be the ring $H_1\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$ and $\Phi : R \to C(\mathbb{T}, \mathbb{R})$ the natural map, i.e. $\Phi(a) = a|_{\mathbb{T}}$. Φ is a ring homomorphism and $\mathbb{T}$ is a compact and Hausdorff topological space. $\Phi(R)$ separates points in $\mathbb{T}$. Indeed, define $a_1(z) = \frac{1}{z} + z$ and $a_2(z) = i(\frac{1}{z} - z)$. Notice that $a_1, a_2 \in H_1\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$. The equality $a_1(z_1) = a_2(z_2)$ holds if

$$
\frac{1}{z_1} + z_1 = \frac{1}{z_2} + z_2 \quad \Leftrightarrow \quad \frac{1 + z_1^2}{z_1} = \frac{1 + z_2^2}{z_2} \Leftrightarrow (1 + z_1^2)z_2 = (1 + z_2^2)z_1
$$
$$
\Leftrightarrow \quad z_2 - z_1 = z_1 z_2 (z_2 - z_1).
$$

The latter is true in $\mathbb{T}$ if and only if $z_2 \in \{z_1, \overline{z_1}\}$. So a_1 separates all non-conjugate pairs z_1, z_2. Similarly, $a_2(z_1) = a_2(z_2)$ if and only if $z_1 z_2 (z_2 - z_1) = z_1 - z_2$. The latter is true in $\mathbb{T}$ if and only if $z_2 \in \{z_1, -\overline{z_1}\}$. So a_2 separates all conjugate pairs z_1, z_2.

Let $A \in \text{Pos}_{\succeq 0}(\mathscr{K})$. We will prove that $A \in \mathcal{T}_{\mathscr{S}}^n$. We define the ideal I' in $\mathbb{C}\left[z, \frac{1}{z}\right]$ by

$$
I' := (b^* b \colon b \in \mathbb{C}[z], b^* b \cdot A \in \mathcal{T}_{\mathscr{S}}^n) \subseteq \mathbb{C}\left[z, \frac{1}{z}\right].
$$

Every ideal $J' \subseteq \mathbb{C}\left[z, \frac{1}{z}\right]$ determines the ideal $J := J' \cap \mathbb{C}[z] \subseteq \mathbb{C}[z]$. It holds that $J' = \left\{\frac{J}{z^n} \colon n \in \mathbb{N} \cup \{0\}\right\}$. Since the maximal ideals in $\mathbb{C}[z]$ are precisely $(z - w)$, where $w \in \mathbb{C}$, the maximal ideals in $\mathbb{C}\left[z, \frac{1}{z}\right]$ are precisely $(z - w)$, where $w \in \mathbb{C} \setminus \{0\}$.

By Proposition 2.7, for every $w \in \mathbb{C} \setminus \{0\}$ there exists $b \in \mathbb{C}[z]$, such that $b(w), b\left(\frac{1}{\bar{w}}\right) \neq 0$ and $b^* b \cdot A \in \mathcal{T}_{\mathscr{S}}^n$. Therefore $I' = \mathbb{C}[z, \frac{1}{z}]$.

Now we define the ideal I in R by

$$
I = (b^* b \colon b \in \mathbb{C}[z], b^* b \cdot A \in \mathcal{T}_{\mathscr{S}}^n) \subseteq R.
$$

We claim that $I = R$. Since $I = \mathbb{C}\left[z, \frac{1}{z}\right]$ there exist $c_1, \ldots, c_m \in \mathbb{C}\left[z, \frac{1}{z}\right]$ and $b_1, \ldots, b_m \in I'$, such that $\sum_{j=1}^m c_j(b_j^* b_j) = 1$. Then also $\sum_{j=1}^m c_j^*(b_j^* b_j) = 1$. Hence, $\sum_{j=1}^m \frac{c_j^* + c_j}{2}(b_j^* b_j) = 1$. Therefore $I = R$.

By Proposition 2.3, there exist $d_1, \ldots, d_m \in \text{Pos}_{\succ 0}^1(\mathbb{T})$, such that $\sum_{j=1}^m d_j(b_j^* b_j) = 1$. Therefore $\sum_{j=1}^m d_j(b_j^* b_j)A = A \in \mathcal{T}_{\mathscr{S}}^n$, which concludes the proof. $\qquad \square$

2.2. **Connection between Problems 1 and 2.** In this subsection we link, by the use of Möbius transformations, closed semialgebraic set in $\mathbb{R}$ with closed semialgebraic sets in $\mathbb{T}$. To every matrix polynomial, positive semidefinite on a given semialgebraic set in $\mathbb{R}$, and to each linked semialgebraic set in $\mathbb{T}$, we assign a matrix polynomial, positive semidefinite on the linked set.

Möbius transformations that map $\mathbb{R} \cup \{\infty\}$ bijectively into $\mathbb{T}$ are exactly the maps of the form

$$\lambda_{z_0, w_0} : \mathbb{R} \cup \{\infty\} \to \mathbb{T}, \quad \lambda_{z_0, w_0}(x) := z_0 \frac{x - w_0}{x - \overline{w_0}},$$

where $z_0 \in \mathbb{T}$ and $w_0 \in \mathbb{C} \setminus \mathbb{R}$. Therefore, we connect a closed semialgebraic set $K \subseteq \mathbb{R}$ with a closed semialgebraic set

$$\mathscr{K}_{z_0, w_0} := \mathrm{Cl}\left(\lambda_{z_0, w_0}(K)\right),$$

where $\mathrm{Cl}(\cdot)$ is the closure operator. Let $F(x)$ be a matrix polynomial from the set $\mathrm{Pos}^n_{\succeq 0}(K)$. A matrix polynomial $\Lambda_{z_0, w_0, F}(z) \in \mathrm{Pos}^n_{\succeq 0}(\mathscr{K}_{z_0, w_0})$ is defined by the rule

$$\Lambda_{z_0, w_0, F}(z) := \left((z - z_0)^*(z - z_0)\right)^{\left\lceil \frac{\deg(F)}{2} \right\rceil} \cdot F\left(\lambda^{-1}_{z_0, w_0}(z)\right),$$

where $\lceil \cdot \rceil$ is the ceiling function. Note that $\Lambda_{z_0, w_0, F}(z)$ is well defined, since

$$\lambda^{-1}_{z_0, w_0}(z) = \frac{z \overline{w_0} - z_0 w_0}{z - z_0}.$$

Note also, that the degree of $\Lambda_{z_0, w_0, F}(z)$ is at most $\left\lceil \frac{\deg(F)}{2} \right\rceil$. We also have

$$(*) \qquad F(x) = \left(\frac{(x - \overline{w_0})(x - w_0)}{4 \cdot \mathrm{Im}(w_0)^2}\right)^{\left\lceil \frac{\deg(F)}{2} \right\rceil} \Lambda_{z_0, w_0, F}(\lambda_{z_0, w_0}(x)),$$

where $\mathrm{Im}(\cdot)$ is the imaginary part of $\cdot$.

2.3. **Proof for** T^n_S**.** The second main step in the proof of Theorem 2.2, which we prove by the use of Theorem 2.1 and the correspondence from Subsection 2.2, is the following.

Corollary 2.8. *Suppose K is a non-empty closed semialgebraic set in $\mathbb{R}$ and S a saturated description of K. Then, for any $F \in H_n(\mathbb{C}[x])$, the following are equivalent:*

(1) *$F \succeq 0$ on K.*
(2) *For $w_0 \in \mathbb{C} \setminus \mathbb{R}$ there is $k_{w_0} \in \mathbb{N} \cup \{0\}$, such that*

$$\left((x - \overline{w_0})(x - w_0)\right)^{k_{w_0}} F \in T^n_S.$$

Proof. The non-trivial direction is $(1) \Rightarrow (2)$. Choose $w_0 \in \mathbb{C} \setminus \mathbb{R}$. $\Lambda_{1, w_0, F}(z)$ belongs to $\mathrm{Pos}^n_{\succeq 0}(\mathscr{K}_{1, w_0})$. The set $\mathscr{S} := \{\Lambda_{1, w_0, g_1}(z), \ldots, \Lambda_{1, w_0, g_s}(z)\}$ is a saturated description of $\overline{\mathscr{K}}$ and by Theorem 2.1, we have $\Lambda_{1, w_0, F}(z) \in \mathscr{T}^n_{\mathscr{S}}$. By the equality $(*)$,

$$\left(\frac{(x - \overline{w_0})(x - w_0)}{4 \cdot \mathrm{Im}(w_0)^2}\right)^{k_{w_0}} \cdot F(x) \in T^n_S,$$

where $k_{w_0} \in \mathbb{N} \cup \{0\}$ equals $k - \left\lceil \frac{\deg(F)}{2} \right\rceil$, where k is the degree of the summand of the highest degree in one of the expression of $\Lambda_{1, w_0, F}(z)$ as the element of $\mathscr{T}^n_{\mathscr{S}}$. This concludes the proof of $(1) \Rightarrow (2)$. $\qquad \square$

Finally, we can prove Theorem 2.2. We will use Proposition 2.3 to get rid of the denominators in Corollary 2.8.

Proof of Theorem 2.2. By [22, 9.3.3 Theorem], T_S^1 is saturated if and only if S is a saturated description of K. Therefore we have to prove only the if part. Let S be a saturated description of K. We will prove that T_S^n is saturated for every $n \in \mathbb{N}$. Let $R := \mathbb{R}[x]$ and $\Phi : R \to C(K, \mathbb{R})$ be the natural map, i.e. $\Phi(f) = f|_K$. Let $F \in \mathrm{Pos}_{\succeq 0}^n(K)$. We will prove that $F \in T_S^n$. We define the ideal I in $\mathbb{R}[x]$ by

$$ I := (h^*h \colon h \in \mathbb{C}[x], h^*hF \in T_S^n) . $$

By Corollary 2.8, for every $w \in \mathbb{C}$ there exists $h \in \mathbb{C}[x]$, such that $h(w), h(\overline{w}) \neq 0$ and $h^*hF \in T_S^n$. Indeed, for $w \neq \{i, -i\}$ take $w_0 = i$ in Corollary 2.8, while for $w \in \{i, -i\}$ take $w_0 = 2i$. Therefore $I = \mathbb{R}[x]$. By Proposition 2.3, there exist $s_1, \dots, s_m \in \mathrm{Pos}_{\succ 0}^1(K)$ and $h_1, \dots, h_m \in I$, such that $\sum_{j=1}^m s_j(h_j^*h_j) = 1$. Hence, $\sum_{j=1}^m s_j(h_j^*h_j)F = F \in T_S^n$, which concludes the proof. □

3. Natural descriptions and boundedly saturated n-th preorderings

In this section we study Problems 1" and 2". In Subsection 3.1 we work with Problem 2". In Subsection 3.2 we continue to study connection between Problems 1 and 2 from Subsection 2.2. Finally, in Subsection 3.3, we use the results for Problem 2" to derive the results for Problem 1".

3.1. **Problem 2".** By the following proposition, it suffices to study the natural description S of a given set $K \subseteq \mathbb{R}$, in Problem 2".

Proposition 3.1. *Let $K \subseteq \mathbb{R}$ be a non-empty basic closed semialgebraic set with natural description S. Let $S_1 \subset \mathbb{R}[x]$ be a finite set, such that $K_{S_1} = K$. If T_S^n is not boundedly saturated, then $T_{S_1}^n$ is not boundedly saturated.*

Proof. Let us write $S = \{g_1, \dots, g_s\}$ and $S_1 = \{f_1, \dots, f_t\}$, $s, t \in \mathbb{N}$. By [19, Theorem 4.1], the preordering T_S^1 is boundedly saturated. Therefore for every $j = 1, \dots, t$, $f_j = \sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e$, where $\sigma_e \in \sum \mathbb{R}[x]^2$ and $\deg\left(\sigma_e \underline{g}^e\right) \leq \deg\left(f_j\right)$ for each e. Hence, if $F \in T_{S_1}^n$ is of the form $F = \sum_{e' \in \{0,1\}^t} \tau_{e'} \underline{f}^{e'}$, where $\tau_{e'} \in \sum M_n\left(\mathbb{C}[x]\right)^2$ and $\deg\left(\tau_{e'} \underline{f}^{e'}\right) \leq \deg\left(F\right)$ for each e', then also $F = \sum_{e \in \{0,1\}^s} \tau_e \underline{g}^e$, where $\tau_e \in \sum M_n\left(\mathbb{C}[x]\right)^2$ and $\deg\left(\tau_e \underline{g}^e\right) \leq \deg\left(F\right)$ for each e. Hence, if T_S^n is not boundedly saturated, also $T_{S_1}^n$ is not boundedly saturated. □

The affirmative answer to the question of Problem 2" for some sets $K \subseteq \mathbb{R}$ and every $n \in \mathbb{N}$ is the following.

Theorem 3.2. *Let $K \subseteq \mathbb{R}$ be either of the following:*
- *an interval,*
- *a union of two unbounded intervals,*
- *a union of at most three points.*

Let S be the natural description of K. The n-th matrix preordering T_S^n is boundedly saturated for every $n \in \mathbb{N}$.

Let $K \subseteq \mathbb{R}$ be a semialgebraic set and $S := \{g_1, \dots, g_s\} \subset \mathbb{R}[x]$ with $K = K_S$. We say that F belongs to the *bounded part* $T_{S,b}^n$ of a n-th matrix preordering T_S^n,

if it can be written in the form $F = \sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e$ with $\sigma_e \in \sum M_n(\mathbb{C}[x])^2$ and $\deg(\sigma_e \underline{g}^e) \leq \deg(F)$ for each e.

For the proof of the case of a union of at most three points we need the following.

Proposition 3.3. *Let* $K = \cup_{j=1}^m \{x_j\} \subseteq \mathbb{R}$ *be a union of points, where* $m \in \mathbb{N}$. *Suppose* S *is the natural description of* K *and* $n \in \mathbb{N}$. *Then every* $F \in Pos_{\succeq 0}^n(K)$ *with* $\deg(F) \geq m - 1$ *belongs to* $T_{S,b}^n$.

Proof. If we divide F with $\prod_{j=1}^m (x - x_j)$, we get $F = \prod_{j=1}^m (x - x_j)G(x) + R(x)$, where $G(x), R(x) \in M_n(\mathbb{C}[x])$ and $\deg(R) < m$. Let us expand a vector R in the basis $f_j(x) := \prod_{\ell \neq j}(x - x_\ell)$ to get

$$F(x) = \sum_{j=1}^m \left((-1)^{k_j} \prod_{\ell \neq j}(x - x_\ell) \right) F_j + \prod_{j=1}^m (x - x_j)G(x),$$

where $k_j \in \{0,1\}$ is such that $\prod_{\ell \neq j}(-1)^{k_j}(x - x_\ell) \in Pos_{\succeq 0}^1(K)$ and $F_j \in M_n(\mathbb{C})$. Since $F \in Pos_{\succeq 0}^n(K)$, it follows that $F_j \succeq 0$ for each j. Write $G(x)$ in the form

$$G(x) := G_1\left((x - x_1)^2\right) + (x - x_1) \cdot G_2\left((x - x_1)^2\right),$$

where $G_1, G_2 \in M_n(\mathbb{C}[x])$, and $2 \deg(G_1) \leq \deg(G), 2 \deg(G_2) \leq \deg(G) - 1$. Using the identity

$$A = \frac{(A+1)^*(A+1)}{4} - \frac{(A-1)^*(A-1)}{4}$$

for the matrix coefficients of $G_j\left((x - x_1)^2\right)$, note that $G_j\left((x - x_1)^2\right)$ can be writen as

$$G_j\left((x - x_1)^2\right) = G_{j1}(x) - G_{j2}(x),$$

where $G_{j1}, G_{j2} \in \sum M_n(\mathbb{C}[x])^2$ and $\deg(G_{j1}), \deg(G_{j2}) \leq 2 \deg(G_j)$ for $j = 1, 2$. By [19, Theorem 4.1], it is also true that

$$(-1)^{k_j} \prod_{\ell \neq j}(x - x_\ell) \in T_{S,b}^1, \quad \pm \prod_{j=1}^m (x - x_j) \in T_{S,b}^1, \quad \pm(x - x_1) \prod_{j=1}^m (x - x_j) \in T_{S,b}^1,$$

and hence $F \in T_{S,b}^n$. $\qquad\qquad\square$

Proof of Theorem 3.2. We separate two cases:

Case 1: *K has non-empty interior:*

- $K = \mathbb{R}$: The statement follows by Theorem 1.2.
- $K = [a, \infty)$ or $K = (-\infty, b]$ or $K = [c, d]$, $a, b, c, d \in \mathbb{R}$, $c < d$: The statement follows by the known results for $[0, 1]$ and $[0, \infty)$ (see Subsection 1.4) with the use of linear transformations $x \mapsto kx + n$, $k, n \in \mathbb{R}$.
- $K = (-\infty, a] \cup [b, \infty)$, $a, b \in \mathbb{R}$, $a < b$: By a linear change of variable, we may assume that $K = (-\infty, -1] \cup [1, \infty)$. Every $F \in Pos_{\succeq 0}^n(K)$ is of even degree. We define $F_1(x) = x^{\deg(F)}F\left(\frac{1}{x}\right)$ and observe that $F_1 \succeq 0$ on $[-1, 1]$. By the above and by the identity

$$x \pm 1 = \frac{(x \pm 1)^2 + (x + 1)(1 - x)}{2},$$

there exist matrix polynomials G_1, H_1 with $\deg(G_1) \leq \frac{\deg(F)}{2}, \deg(H_1) \leq \frac{\deg(F)}{2} - 1$, such that

$$F_1(x) = G_1(x)^*G_1(x) + H_1(x)^*H_1(x)(x + 1)(1 - x).$$

Therefore

$$F(x) \; = \; x^{\deg(F)} F_1\left(\frac{1}{x}\right)$$

$$= \; x^{\deg(F)}\left(G_1\left(\frac{1}{x}\right)^* G_1\left(\frac{1}{x}\right) + H_1\left(\frac{1}{x}\right)^* H_1\left(\frac{1}{x}\right)\left(\frac{1}{x}+1\right)\left(1-\frac{1}{x}\right)\right)$$

$$=: \; G(x)^* G(x) + H(x)^* H(x)(1+x)(x-1),$$

where

$$G(x) := x^{\frac{\deg(F)}{2}} G_1\left(\frac{1}{x}\right), \quad H := x^{\frac{\deg(F)}{2}-1} H_1\left(\frac{1}{x}\right)$$

are matrix polynomials with $\deg(G) \le \frac{\deg(F)}{2}$, $\deg(H) \le \frac{\deg(F)}{2} - 1$.

Case 2: K *is a union of at most three points.*

- $|K| = 1$: The statement follows by Proposition 3.3.
- $|K| = 2$: Let $F \in \mathrm{Pos}^n_{\succeq 0}(K)$. If F is of degree 0, then it is of the form $G^* G$ for some $G \in M_n(\mathbb{C})$. Hence, $F \in T^n_{S,b}$. Otherwise, F is of degree ≥ 1 and $F \in T^n_{S,b}$ by Proposition 3.3. Hence, $T^n_{S,b} = T^n_S$ and T^n_S is boundedly saturated for every $n \in \mathbb{N}$.
- $|K| = 3$: Let $K := \{x_1\} \cup \{x_2\} \cup \{x_3\}$ with $x_1, x_2, x_3 \in \mathbb{R}$, $x_1 < x_2 < x_3$. Let $F \in \mathrm{Pos}^n_{\succeq 0}(K)$. If F is of degree 0, then $F \in T^n_{S,b}$ by the same argument as above for $|K| = 2$. If F is of degree 1, then by the convexity of the set $\{x \in \mathbb{R}\colon F(x) \succeq 0\}$, it follows that $F \in \mathrm{Pos}^n_{\succeq 0}([x_1, x_3])$. By Case 1 above, F is of the form $F_0^* F_0 + (x-x_1)F_1^* F_1 + (x_3-x)F_2^* F_2$ with $F_0, F_1, F_2 \in M_n(\mathbb{C})$. Hence, $F \in T^n_{S,b}$. Finally, if F is of degree ≥ 2, then $F \in T^n_{S,b}$ by Proposition 3.3. Hence, $T^n_{S,b} = T^n_S$ and T^n_S is boundedly saturated for every $n \in \mathbb{N}$.

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2. **Further connection between Problems 1 and 2.** Assume the notation as in Subsection 2.2. In this subsection we link, by the use of Möbius transformations, closed semialgebraic set in $\mathbb{T}$ with closed with closed semialgebraic sets in $\mathbb{R}$. To every matrix polynomial, positive semidefinite on a given semialgebraic set in $\mathbb{T}$, and to each linked semialgebraic set in $\mathbb{R}$, we assign a matrix polynomial, positive semidefinite on the linked set. Finally, in Proposition 3.5, a connection between natural descriptions of a given semialgebraic set and each linked set is established.

Recall that a map $\lambda_{z_0, w_0}(x) : \mathbb{R} \cup \{\infty\} \to \mathbb{T}$ is defined by $\lambda_{z_0, w_0}(x) := z_0 \frac{x - w_0}{x - \overline{w_0}}$, where $z_0 \in \mathbb{T}$ and $w_0 \in \mathbb{C} \setminus \mathbb{R}$ (see Subsection 2.2). We link a closed semialgebraic set $\mathscr{K} \subseteq \mathbb{T}$ with a closed semialgebraic set

$$K_{z_0, w_0} := \lambda^{-1}_{z_0, w_0}(\mathscr{K}) \setminus \{\infty\}.$$

To each polynomial $A(z) \in \mathrm{Pos}^n_{\succeq 0}(\mathscr{K})$ we assign a polynomial $\Gamma_{z_0, w_0, A}(x) \in \mathrm{Pos}^n_{\succeq 0}(K_{z_0, w_0})$ by the rule

$$\Gamma_{z_0, w_0, A}(x) = \left(\frac{(x - \overline{w_0})(x - w_0)}{4 \cdot \mathrm{Im}(w_0)^2}\right)^{\deg(A)} \cdot A\left(\lambda_{z_0, w_0}(x)\right),$$

where $\mathrm{Im}(\cdot)$ is the imaginary part of $\cdot$. Note that $\Gamma_{z_0, w_0, A}(x)$ is well defined by the definition of $\lambda_{z_0, w_0}(x)$, and that the degree of $\Gamma_{z_0, w_0, A}(x)$ is at most $2\deg(A)$. We also have the identity

$$(\circ) \qquad\qquad A(z) = ((z - z_0)^*(z - z_0))^{\deg(A)} \cdot \Gamma_{z_0, w_0, A}(\lambda^{-1}_{z_0, w_0}(z)).$$

Now we connect natural descriptions of $K_{z_0,w_0} \subseteq \mathbb{R}$ and $\mathscr{K} \subseteq \mathbb{T}$. For tehnical reasons, we introduce the notion of the even natural description of a semialgebraic set. We call a set S' the *even natural description* of a basic closed semialgebraic set $K \subseteq \mathbb{R}$ if it satisfies $(a) - (d)$ in the definition of the natural description of K and in addition:

(e) If S' includes both elements of the form $x - a$ and $b - x$ with $a < b$, then we replace them by the element $(x - a)(b - x)$.

Remark 3.4. By the equalities

$$x - a = \frac{(x-a)(b-x) + (x-a)^2}{b-a}, \quad b - x = \frac{(x-a)(b-x) + (b-x)^2}{b-a},$$

we have $T_S^1 = T_{S'}^1$.

The connection between the natural description of $\mathscr{K} \subseteq \mathbb{T}$ and the even natural description of $K_{z_0,w_0} \subseteq \mathbb{R}$ is the following.

Proposition 3.5. *Let $\mathscr{K} \subseteq \mathbb{T}$ be the closed semialgebraic set with $|\mathscr{K}| > 1$ and $K_{z_0,w_0} \subseteq \mathbb{R}$ the corresponding closed semialgebraic set with $|K_{z_0,w_0}| > 1$. Let $S' := \{g_1(x), \ldots, g_s(x)\}$ be the even natural description of K_{z_0,w_0}. Then the set*

$$\Lambda_{z_0,w_0,S'} := \{\Lambda_{z_0,w_0,g_1}, \ldots, \Lambda_{z_0,w_0,g_s}\}$$

is exactly the set of polynomials from the natural description of the set $\mathscr{K}$, up to multiplying each member by some positive constant. Moreover,

$$\Gamma_{z_0,w_0,\Lambda_{z_0,w_0,S'}} = S'.$$

Proof. Note that

$$K_{z_0,w_0} = \cap_{j=1}^s \{x \in \mathbb{R} \colon g_j(x) \geq 0\} \text{ and } \mathscr{K} = \cap_{j=1}^s \{z \in \mathbb{T} \colon \Lambda_{z_0,w_0}(g_j) \geq 0\}.$$

Therefore it remains to show only, that every polynomial $\Lambda_{z_0,w_0}(g_j)$ is a polynomial from the natural description of $\mathscr{K}$, multiplied by some positive constant. We separate two cases:

Case 1: $g_j(x) = \pm(x - a)(x - b)$. Then

$$\Lambda_{z_0,w_0,g_j} = k \cdot \frac{(z - \lambda_{z_0,w_0}(a))\,(z - \lambda_{z_0,w_0}(b))}{z}$$

where $k \in \mathbb{C} \setminus \{0\}$ is such that $\Lambda_{z_0,w_0,g_j} \in \text{Pos}_{\succeq 0}^1(\mathscr{K})$.

Case 2: $g_j(x) = \pm(x - a)$. Then

$$\Lambda_{z_0,w_0,g_j} = k \cdot \frac{(z - \lambda_{z_0,w_0}(a))\,(z - z_0)}{z}$$

where $k \in \mathbb{C} \setminus \{0\}$ is such that $\Lambda_{z_0,w_0,g_j} \in \text{Pos}_{\succeq 0}^1(\mathscr{K})$.

The equality $\Gamma_{z_0,w_0,\Lambda_{z_0,w_0,S'}} = S'$ is easily verified. $\square$

3.3. **Problem 1".** By the use of previous two Subsections we come to the following affirmative answer to the question of Problem 1".

Theorem 3.6. *Let $\mathscr{K} \subseteq \mathbb{T}$ be a non-empty basic closed semialgebraic set with natural description $\mathscr{S}$. The n-th matrix preordering $\mathscr{T}_{\mathscr{S}}^n$ is boundedly saturated in either of the following cases:*

- *$n = 1$ and $\mathscr{K}$ is arbitrary.*
- *$n \in \mathbb{N}$ is arbitrary and $\mathscr{K}$ is an arc,*

- $n \in \mathbb{N}$ *is arbitrary and $\mathscr{K}$ is a union of at most three points.*

Proof of Theorem 3.6. We separate two cases:

Case 1: $|\mathscr{K}| > 1$. Take $A(z) \in \mathrm{Pos}_{\succeq 0}^n(\mathscr{K})$. If $\mathscr{K} \neq \mathbb{T}$, choose $z_0 \in \mathbb{T} \setminus \mathscr{K}$. Otherwise choose an arbitrary $z_0 \in \mathscr{K}$. Choose also $w_0 \in \mathbb{C} \setminus \mathbb{R}$. Then $\Gamma_{z_0,w_0,A}(x) \in \mathrm{Pos}_{\succeq 0}^n(K_{z_0,w_0})$ and $\deg(\Gamma_{z_0,w_0,A}(x)) \leq 2\deg(A)$. By [19, Theorem 4.1] and Theorem 3.2, it follows that that $\Gamma_{z_0,w_0,A}(x) \in T_{S,b}^n$, where S is the natural description of K_{z_0,w_0}. By replacing the natural description S with the even natural description $S' := \{g_1, \ldots, g_s\}$, we get $\Gamma_{z_0,w_0,A}(x) = \sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e$, where $\deg\left(\sigma_e \underline{g}^e\right) \leq 2\deg(A)$ for each e. By the identity $(\circ)$ above and by Proposition 3.5, it follows that $A(z) \in \mathscr{T}_{\mathscr{S}}^n$.

Case 2: $|\mathscr{K}| = 1$. By the definition, $\mathscr{S} = \{b_1(z), b_2(z)\}$, where $b_1(z) = k_1 \cdot \frac{(z-z_0)(z-iz_0)}{z}$, $b_2(z) = k_2 \cdot \frac{(z-z_0)(z+iz_0)}{z}$, $k_1 = \sqrt{iz_0^2}, k_2 = \sqrt{-iz_0^2}$, such that $b_1 \in \mathrm{Pos}_{\succeq 0}^1([z_0, iz_0])$, $b_2 \in \mathrm{Pos}_{\succeq 0}^1([-iz_0, z_0])$. Choose $z_3 \in (iz_0, -iz_0)$. Then

$$
\begin{aligned}
g_1(x) := \Gamma_{z_3,i,b_1}(x) &= k_3(x-x_0)(x-x_1), \\
g_2(x) := \Gamma_{z_3,i,b_2}(x) &= k_4(x-x_0)(x-x_2),
\end{aligned}
$$

where $k_3, k_4 < 0$ are negative constants, $x_0 = \lambda_{z_3,i}^{-1}(z_0)$, $x_1 = \lambda_{z_3,i}^{-1}(iz_0)$, $x_2 = \lambda_{z_3,i}^{-1}(-iz_0)$ and $x_2 < x_0 < x_1$. Define $S' := \{g_1, g_2\}$. Then $K := K_{S'} = \{x_0\}$. Let S be the natural description of K. Choose $A(z) \in \mathrm{Pos}_{\succeq 0}^n(\mathscr{K})$. Then $\Gamma_{z_3,i,A}(x) \in \mathrm{Pos}_{\succeq 0}^n(K)$ and $\deg(\Gamma_{z_3,i,A}) \leq 2\deg(A)$. We know that

$$
\begin{aligned}
x - x_0 &= \frac{-k_3(x-x_0)^2 + g_1(x)}{(-k_3)(x_1-x_0)} = \sigma_0 + \sigma_1 g_1 \in T_{S'}^1, \\
-(x-x_0) &= \frac{-k_4(x-x_0)^2 + g_2(x)}{(-k_4)(x_0-x_2)} = \sigma_2 + \sigma_3 g_2 \in T_{S'}^1, \\
-(x-x_0)^2 &= \frac{g_1(x) + g_2(x)}{-(k_3+k_4)} + c(x-x_0) = \sigma_4 + \sigma_5 g_1 + \sigma_6 g_2 \in T_{S'}^1,
\end{aligned}
$$

where $c \in \mathbb{R}$, $\sigma_j \in \sum \mathbb{R}[x]^2$ for $j = 1, \ldots, 6$ and $\deg(\sigma_j) \leq 2$ for $j = 0, 2, 4$ and $\deg(\sigma_j) = 0$ for $j = 1, 3, 5, 6$. Since $\deg(\Gamma_{z_3,i,A}) \leq 2\deg(A)$. Then

$$
\Gamma_{z_3,i,A}(x) = \tau_0 + \tau_1(x-x_0) - \tau_2(x-x_0) - \tau_3(x-x_0)^2,
$$

where $\tau_j \in \sum M_n(\mathbb{C}[x])^2$ for each j and $\deg(\tau_0) \leq 2\deg(A), \deg(\tau_j) \leq 2\deg(A) - 2$ for $j = 1, 2, 3$. Therefore

$$
\Gamma_{z_3,i,A}(x) = \hat{\tau}_0 + \hat{\tau}_1 g_1 + \hat{\tau}_2 g_2,
$$

where $\hat{\tau}_j \in \sum M_n(\mathbb{C}[x])^2$ for each j and $\deg(\hat{\tau}_0), \deg(\hat{\tau}_1 g_1), \deg(\hat{\tau}_2 g_2) \leq 2\deg(A)$. Hence,

$$
A(z) = ((z-z_3)^*(z-z_3))^{\deg(A)} \Gamma_{z_3,i,A}(\lambda_{z_3,i}^{-1}(z)) \in \mathscr{T}_{\mathscr{S},b}^n.
$$

This concludes the proof. $\qquad \square$

4. Sets $\mathscr{K}$, K without boundedly saturated $\mathscr{T}_{\mathscr{S}}^2$, T_S^2 for any finite sets $\mathscr{S}$, S with $\mathscr{K}_{\mathscr{S}} = \mathscr{K}$, $K_S = K$

The negative answers to the questions of Problems 1" and 2" for almost all remaining sets K, $\mathscr{K}$ not covered by Theorems 3.2 and 3.6 (except for a union of an interval and a point or a union of two unbounded intervals and a point) and all $n \geq 2$ are the main results of this section (see Theorems 4.1 and 4.2 below). By Propositions 3.1 and 4.5 below, it suffices to study natural descriptions.

Theorem 4.1. *Let a non-empty basic closed semialgebraic set $K \subseteq \mathbb{R}$ satisfy either of the following:*

    (1) *K contains at least two intervals with at least one of them bounded,*
    (2) *K is a union of m points with $m \geq 4$,*
    (3) *K is a union of an interval (bounded or unbounded) and m isolated points with $m \geq 2$.*
    (4) *K is a union of two unbounded intervals and m isolated points with $m \geq 2$.*

If $S \subset \mathbb{R}[x]$ is a finite set with $K_S = K$, then the 2-nd matrix preordering T_S^2 is not boundedly saturated. Moreover,

    (5) *If K is regular with at least two components, one of which is unbounded and the others are bounded, then T_S^2 is not even boundedly weakly saturated.*

Theorem 4.2. *Let a non-empty basic closed semialgebraic set $\mathscr{K} \subseteq \mathbb{T}$ satisfy either of the following:*

    (1) *$\mathscr{K}$ contains at least two arcs,*
    (2) *$\mathscr{K}$ is a union of m points with $m \geq 4$,*
    (3) *$\mathscr{K}$ is a union of an arc and m isolated points with $m \geq 2$.*

If $\mathscr{S} \subset H_1\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$ is a finite set with $\mathscr{K}_S = \mathscr{K}$, then the 2-nd matrix preordering $\mathscr{T}_{\mathscr{S}}^2$ is not boundedly saturated.

Let $K \subseteq \mathbb{R}$ be a semialgebraic set with natural description $S := \{g_1, \ldots, g_s\}$. Recall that F belongs to the bounded part $T_{S,b}^n$ of a n-th matrix preordering T_S^n, if it can be written in the form $F = \sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e$ with $\sigma_e \in \sum M_n(\mathbb{C}[x])^2$ and $\deg(\sigma_e \underline{g}^e) \leq \deg(F)$ for each e.

4.1. **Proof of (1) and (5) of Theorem 4.1.**

Proposition 4.3. *Let $K = [x_1, x_2] \cup [x_3, \infty)$ be a union of a bounded and an unbounded interval, where $x_1 < x_2 < x_3$. Let us define the polynomial*

$$F_k(x) := \left[\begin{array}{cc} x + A(k) & D(k) \\ D(k) & x^2 + B(k)x + C(k) \end{array} \right],$$

where

$$
\begin{aligned}
A(k) &:= k - x_1, \\
B(k) &:= -k - x_2 - x_3, \\
C(k) &:= k^2 + k(-x_1 + x_2 + x_3) + x_2 x_3, \\
D(k) &:= \sqrt{A(k)C(k) + x_1 x_2 x_3} = \\
&= \sqrt{k^3 + k^2(-2x_1 + x_2 + x_3) + k(x_2 x_3 + x_1^2 - x_1 x_2 - x_1 x_3)}.
\end{aligned}
$$

For every k, which satisfies

$$k^3 + k^2(-2x_1 + x_2 + x_3) + k(x_2 x_3 + x_1^2 - x_1 x_2 - x_1 x_3) > 0,$$

$$\frac{3}{4}k^2 + k\left(-x_1 + \frac{x_2 + x_3}{2}\right) - \left(\frac{x_2 - x_3}{2}\right)^2 > 0,$$

$F_k(x)$ belongs to $Pos_{\succeq 0}^2(K)$ and does not belong to $T_{S_1}^2$, where S_1 is the natural description of any set K_1 of the form

$$[x_1, x_2] \cup \cup_{j=1}^m [x_{2j+1}, x_{2j+2}] \cup [x_{2m+3}, \infty) \subseteq K$$

with $m \geq 0$ and $x_j \leq x_{j+1}$ for each j. In particular, $F_k(x) \notin T_S^2$.

Moreover, for every fixed k above and every $\epsilon > 0$ sufficiently small, $F_k(x) + \epsilon I_2 \notin T_{S_1}^2$.

Proof. First we will prove, that $F_k(x) \in \text{Pos}_{\succeq 0}^2(K)$ for k satisfying the conditions in the statement of the proposition. The determinant of $F_k(x)$ is $(x - x_1)(x - x_2)(x - x_3) \in \text{Pos}_{\succeq 0}^1(K)$. The upper left corner of F is non-negative for $x \geq x_1 - k$; hence it belongs to $\text{Pos}_{\succeq 0}^1(K)$. The lower right corner is a quadratic polynomial $p(x) := x^2 + Bx + C$ with a vertex in $x = \frac{-B}{2}$. By the choice of k,

$$p\left(\frac{-B}{2}\right) = \frac{3}{4}k^2 + k\left(-x_1 + \frac{x_2 + x_3}{2}\right) - \left(\frac{x_2 - x_3}{2}\right)^2 > 0.$$

So $p(x)$ is positive on $\mathbb{R}$ and hence $p \in \text{Pos}_{\succeq 0}^1(K)$. Since all principal minors of $F_k(x)$ are non-negative on K, $F_k(x) \in \text{Pos}_{\succeq 0}^2(K)$.

Now we will prove that $F_k(x) \notin T_{S_1}^2$. We know that

$$
\begin{aligned}
S_1 &= \{x - x_1, (x - x_2)(x - x_3), \ldots, (x - x_{2m+2})(x - x_{2m+3})\} \\
&=: \{g_1, g_2, \ldots, g_{m+2}\}.
\end{aligned}
$$

If $F_k \in T_{S_1}^2$, then it can be written in the form $F_k = \sum_{e \in \{0,1\}^{m+2}} \sigma_e \underline{g}^e$, where $\sigma_e \in \sum M_n(\mathbb{C}[x])^2$ for every $\{0,1\}^{m+2}$ and $\underline{g}^e = g_1^{e_1} \cdots g_{m+2}^{e_{m+2}}$ for $e = (e_1, \ldots, e_{m+2})$. By the degree comparison we conclude, that the non-zero part can be just

$$(\bullet) \qquad F_k(x) = \sigma_0 + \sigma_1(x - x_1) + \sum_{j=1}^{m+1} \sigma_{j+1}(x - x_{2j})(x - x_{2j+1}),$$

where $\sigma_j \in \sum M_2(\mathbb{C}[x])^2$ for each j and $\deg(\sigma_0) \leq 2$, $\deg(\sigma_j) = 0$ for $j = 1, \ldots, m + 2$. By observing the monomial x^2 on both sides, it follows that $\sigma_2 = \begin{bmatrix} 0 & 0 \\ 0 & k_0 \end{bmatrix}$ for some $k_0 \in [0,1]$. Equivalently $(\bullet)$ can be written as

$$F_k(x) - \sigma_2(x - x_2)(x - x_3) = \sigma_0 + \sigma_1(x - x_1) + \sum_{i=2}^{m} \sigma_i(x - x_{2i})(x - x_{2i+1}).$$

The right-hand side belongs to $\text{Pos}_{\succeq 0}^2(\hat{K}_1)$, where $\hat{K}_1 = K_1 \cup [x_2, x_3]$. But the determinant of the left-hand side

$$q(x) = (x - x_2)(x - x_3)(x(1 - k_0) - (x_1 - x_1 k_0 + k k_0))$$

is a non-zero polynomial of degree 3 with zeroes $x = x_2$ and $x = x_3$. q is indeed non-zero, since otherwise $1 - k_0 = x_1 - x_1 k_0 + k k_0 = 0$, so $k_0 = 1$ and $x_1 - x_1 + k = k = 0$, which is a contradiction. Since q cannot have double zeroes at $x = x_2$ and $x = x_3$, $q \notin \text{Pos}(\hat{K}_1)$. Hence $F_k(x) - \sigma_2(x - x_2)(x - x_3) \notin \text{Pos}_{\succeq 0}^2(\hat{K}_1)$, which is a contradiction. Therefore F_k cannot be expressed in the form (*) and so $F_k \notin T_{S_1}^2$.

Finally we will prove, that for a fixed k there and $\epsilon > 0$ sufficiently small, $F_k(x) + \epsilon I_2 \notin T_{S_1}^2$. With the same arguments as above, $F_k + \epsilon I_2 \in T_{S_1}^2$ would imply

$$F_k(x) + \epsilon I_2 - \sigma_2(x - x_2)(x - x_3) = \sigma_0 + \sigma_1(x - x_1) + \sum_{i=2}^{m} \sigma_{i+1}(x - x_{2i})(x - x_{2i+1}),$$

where $\sigma_2 = \begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix}$ for some $k \in [0,1]$. The right-hand side again belongs to $\text{Pos}^2_{\succeq 0}(\hat{K})$, where

$$\hat{K} = \begin{cases} [x_1, \infty), & \text{if } m = 1 \\ [x_1, x_4] \cup \cup_{i=3}^m [x_{2i-1}, x_{2i}] \cup [x_{2m+1}, \infty), & \text{otherwise} \end{cases}.$$

On computing the determinant of the left-hand side in the point $x = x_1$ we get

$$-k(x_1 - x_2)(x_1 - x_3)(x_1 + A(k) + \epsilon) + \epsilon(x_1^2 + B(k)x_1 + C(k) + x_1 + A(k) + \epsilon),$$

which is non-negative for

$$k \leq \frac{\epsilon(x_1^2 + B(k)x_1 + C(k) + x_1 + A(k) + \epsilon)}{(x_1 - x_2)(x_1 - x_3)(x_1 + A(k) + \epsilon)}.$$

On computing the determinant of the left-hand side in the point $x = \frac{x_2+x_3}{2}$ we get

$$k\frac{(x_3 - x_2)^2}{4}\left(\frac{x_2 + x_3}{2} + A(k) + \epsilon\right) + \det\left(F_1\left(\frac{x_2 + x_3}{2}\right) + \epsilon I_2\right).$$

Since $\frac{x_2+x_3}{2} + A + \epsilon > x_1 + A + \epsilon = N + \epsilon > 0$, this is equivalent to

$$k \geq \frac{-\det\left(F_1\left(\frac{x_2+x_3}{2}\right) + \epsilon I_2\right)}{\frac{(x_3-x_2)^2}{4}\left(\frac{x_2+x_3}{2} + A + \epsilon\right)}.$$

Since $\det\left(F_1\left(\frac{x_2+x_3}{2}\right)\right) < 0$, for $\epsilon > 0$ small enough we have

$$-\det\left(F_1\left(\frac{x_2 + x_3}{2}\right) + \epsilon I_2\right) > \frac{-\det\left(F_1\left(\frac{x_2+x_3}{2}\right)\right)}{2} > 0.$$

Hence, for $\epsilon > 0$ small enough,

$$\epsilon \cdot \frac{(x_1^2 + Bx_1 + C + x_1 + A + \epsilon)}{(x_1 - x_2)(x_1 - x_3)(x_1 + A + \epsilon)} < \frac{-\det\left(F_1\left(\frac{x_2+x_3}{2}\right) + \epsilon I_2\right)}{\frac{(x_3-x_2)^2}{4}\left(\frac{x_2+x_3}{2} + A + \epsilon\right)}.$$

For those ϵ, the determinant of $F_1(x) + \epsilon I_2 - \sigma_2(x - x_2)(x - x_3)$ cannot be positive semidefinite in x_1 and $\frac{x_2+x_3}{2}$ simultaneously, which is a contradiction. Therefore $F_k(x) + \epsilon I_2 \notin T^2_{S_1}$ for $\epsilon > 0$ sufficiently small. $\square$

Proof of (5) of Theorem 4.1. By Proposition 3.1, we may assume S is the natural description of K. We separate two cases, depending on the form of K.

Case 1: *K is bounded from below and unbounded from above.* K is of the form $\cup_{j=1}^m [x_{2j-1}, x_{2j}] \cup [x_{2m+1}, \infty)$, where $m \in \mathbb{N}$ and $x_j < x_{j+1}$ for $j = 1, \ldots, 2m$. By Proposition 3.1, we may assume S is the natural description of K. Let us define the set $K_1 := [x_1, x_2] \cup [x_3, \infty)$. By Proposition 4.3, there exists a polynomial $F \in \text{Pos}^2_{\succ 0}(K_1)$, such that $F \notin T^2_S = T^2_{S,b}$. Hence, T^2_S is not weakly saturated.

Case 2: *K is unbounded from below and bounded from above.* K is of the form $(-\infty, x_1] \cup_{j=1}^m [x_{2j}, x_{2j+1}]$, where $m \in \mathbb{N}$ and $x_j < x_{j+1}$ for $j = 1, \ldots, 2m+1$. By Case 1, $T^2_{S_1}$ is not saturated, where S_1 is the natural description of $-K$. Hence, T^2_S is not weakly saturated. $\square$

Proof of (1) of Theorem 4.1. By Proposition 3.1, we may assume S is the natural description of K. Let us write K in the form $K := \cup_{j=1}^m \{x_j\} \cup K_1$, where $m \in \mathbb{N}$ and K_1 is the regular part of K. Let S_1 be the natural description of K_1. We separate two cases, depending on the form of K_1.

Case 1: K_1 *is bounded from one side and unbounded from the other.* By (5) of Theorem 4.1, there is a polynomial $F_1 \in \mathrm{Pos}^2_{\succ 0}(K_1)$ and $F_1 \notin T^2_{S_1} = T^2_{S_1,b}$. Write $S := \{g_1, \ldots, g_s\}$. The polynomial $F(x) := \prod_{j=1}^m (x - x_j) \cdot F_1(x)$ belongs to $\mathrm{Pos}^2_{\succeq 0}(K)$. If $F \in T^2_S = T^2_{S,b}$, then $F = \sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e$, where each $\sigma_e \in \sum M_n(\mathbb{C}[x])^2$. Since $F(x_j) = 0$ and $\sigma_e \underline{g}^e(x_j) \succeq 0$, we conclude that $\sigma_e \underline{g}^e(x_j) = 0$ for $j = 1, \ldots, m$ and each e. Therefore $\prod_{j=1}^{\overline{m}} (x - x_j)$ divides each $\sigma_e \underline{g}^e$. Hence, $\sigma_e \underline{g}^e = \prod_{j=1}^m (x - x_j) \cdot \tau_e h_e$, where $\tau_e \in \sum M_n(\mathbb{C}[x])^2$, $h_e \in \mathrm{Pos}^1_{\succeq 0}(K_1)$ and $\deg(\tau_e h_e) \leq \deg(F_1)$. By [19, Theorem 4.1], $h_e \in T^1_{S_1,b}$. It follows that $F_1 \in T^2_{S_1,b}$, which is a contradiction. Therefore T^2_S is not saturated.

Case 2: *Other* K_1. Let $d \in \mathbb{R}$ be the maximum of K_1. Define the map $\lambda_d : \mathbb{R} \setminus \{d\} \to \mathbb{R}$ with $\lambda_d(x) := \frac{1}{d-x}$. Observe that $\lambda_d(K_1) =: K_2$ is the set of the form $\cup_{j=1}^{m_1} [\hat{x}_{2j-1}, \hat{x}_{2j}] \cup [\hat{x}_{2m_1+1}, \infty)$, where $m_1 \in \mathbb{N}$ and $\hat{x}_j < \hat{x}_{j+1}$ for $j = 1, \ldots, 2m_1$. By Proposition 4.3, there is a polynomial $F_2 \in \mathrm{Pos}^2_{\succeq 0}(K_2)$ of degree 2 with $F_2 \notin T^2_{S_2} = T^2_{S_2,b}$, where S_2 is the natural description of K_2. Therefore $F_1(x) = x^2 F_2\left(d - \frac{1}{x}\right) \in \mathrm{Pos}^2_{\succeq 0}(K_1)$ and $F_1 \notin T^2_{S_1,b}$, To construct $F(x) \in \mathrm{Pos}^2_{\succeq 0}(K)$ with $F \notin T^2_{S,b}$, proceed as in Case 1. Therefore T^2_S is not boundedly saturated. $\square$

Concrete examples for the statement of (5) and (1) of Theorem 4.1 in the cases $K_1 := [-1, 0] \cup [1, \infty)$ and $K_2 = (-\infty, -2] \cup \left[0, \frac{2}{3}\right] \cup [2, \infty)$ respectively, are the following.

Example 1. Let us take $a \in (1, \infty)$ and $\varphi \in [0, 2\pi)$. The matrix polynomial

$$F_{a,\varphi}(x) := \begin{bmatrix} x + a & e^{i\varphi}\sqrt{a^3 - a} \\ e^{-i\varphi}\sqrt{a^3 - a} & x^2 - a \cdot x + (a^2 - 1) \end{bmatrix}$$

is positive semidefinite on $K_1 := [-1, 0] \cup [1, \infty)$, but

$$F_{a,\varphi} \notin T^2_{S_1},$$

where S_1 is the natural description of K_1. Moreover, for $\epsilon > 0$ sufficiently small also

$$F_{a,\varphi} + \epsilon I_2 \notin T^2_{S_1},$$

where I_2 is the 2×2 identity matrix.

Proof. The arguments are the same as for the matrix polynomial $F_k(x)$ in Proposition 4.3. $\square$

Example 2. Let us take $a \in (1, \infty)$ and $\varphi \in [0, 2\pi)$. The matrix polynomial

$$\begin{aligned} G_{a,\varphi}(x) &:= x^2 F_{a,\varphi}\left(\frac{1}{x} - \frac{1}{2}\right) \\ &= \begin{bmatrix} x^2\left(a - \frac{1}{2}\right) + x & x^2 \cdot e^{i\varphi}\sqrt{a^3 - a} \\ x^2 \cdot e^{-i\varphi}\sqrt{a^3 - a} & x^2\left(a^2 + \frac{a}{2} - \frac{3}{4}\right) + x(1 - a) + 1 \end{bmatrix} \end{aligned}$$

is positive semidefinite on $K_2 := (-\infty, -2] \cup \left[0, \frac{2}{3}\right] \cup [2, \infty)$, but

$$G_{a,\varphi} \notin T^2_{S_2},$$

where S_2 is the natural description of K_2.

Proof. From $G_{a,\varphi}(x) = x^2 F_{a,\varphi}\left(\frac{1}{x} - \frac{1}{2}\right)$, it follows that $G_{a,\varphi}(x) \succeq 0$ for $\frac{1}{x} - \frac{1}{2} \in K_1$. Hence, $G_{a,\varphi}(x) \succeq 0$ for $x \in K_2$. If $G_{a,\varphi} \in T^2_{\tilde{S}_2}$, then

$$G_{a,\varphi} = \sigma_0 + \sigma_1(x+2)x + \sigma_2\left(x - \frac{2}{3}\right)(x-2) + \sigma_3(x+2)x\left(x - \frac{2}{3}\right)(x-2),$$

where $\sigma_j \in \sum M_2\left(\mathbb{C}[x]\right)$ for $j = 0, 1, 2, 3$. From the degree comparison we conclude that $\sigma_3 = 0$, $\deg(\sigma_0) \leq 2$, and $\deg(\sigma_1) = \deg(\sigma_2) = 0$. But then $F_{a,\varphi}(x) = \tilde{\sigma}_0 + \tilde{\sigma}_1(x+1) + \tilde{\sigma}_2 x(x-1) \in T^2_{\tilde{S}_1}$, where $\tilde{\sigma}_0(x) = \left(x + \frac{1}{2}\right)^2 \sigma_0\left(\frac{1}{x+\frac{1}{2}}\right)$, $\tilde{\sigma}_1 = 2\sigma_1$ and $\tilde{\sigma}_2 = \frac{4}{3}\sigma_2$. This is in contradiction with Example 1. Hence, $G_{a,\varphi} \notin T^2_{\tilde{S}_2}$. $\square$

4.2. **Proof of (2) and (3) of Theorem 4.1.** The following Proposition and the construction of the counterexample for the statements (2) and (3) of Theorem 4.1 is due to Jaka Cimprič. I thank him for allowing me to include his result here.

Proposition 4.4. *Let* $K = \{x_1, x_2, x_3, x_4\}$ *be the 4 element set with* $x_1 < x_2 < x_3 < x_4$. *The polynomial* $F(x) := F_2 x^2 + F_1 x + F \in H_n(\mathbb{C}[x])$, *which belongs to* $Pos^2_{\succeq 0}(K)$ *and satisfies*

(1) $\ker(F(x_2)) \oplus \ker(F(x_3)) = \mathbb{C}^n$,
(2) $F_2 \not\succeq 0$,

does not belong to the bounded part $T^n_{S,b}$ *of the* n-*th matrix preordering* T^n_S, *where* S *is the natural description of* K.

Proof. Write $e_{ij}(x) = (x - x_i)(x - x_j)$ for $i, j = 1, 2, 3, 4$. If $F(x)$ belongs to $T^n_{S,b}$, then we can write $F(x)$ in the form

$$(*) \qquad F(x) = Ae_{12}(x) + Be_{23}(x) + Ce_{34}(x) + D(-e_{14}(x)) + G(x)$$

with $A, B, C, D \in \sum M_n(\mathbb{C})^2$ and $G(x) := G_2 x^2 + G_1 x + G_0 \in \sum M_n(\mathbb{C}[x])^2$. We have $F(x_2) = Ce_{34}(x_2) + D(-e_{14}(x_2)) + G(x_2)$ and $F(x_3) = Ae_{12}(x_3) + D(-e_{14}(x_3)) + G(x_3)$. Therefore $\ker(F(x_2)) \subseteq \ker(D)$ and $\ker(F(x_3)) \subseteq \ker(D)$. Hence $\mathbb{C}^n = \ker(F(x_2)) \oplus \ker(F(x_3)) \subseteq \ker(D)$. So $D = 0$. Comparing the leading coefficients in $(*)$ we get $F_2 = A + B + C + G_2 \succeq 0$, which is a contradiction. $\square$

Proof of (2) and (3) of Theorem 4.1. The set K has one of the following forms:

- $K = \cup_{j=1}^m \{x_j\}$, where $m \geq 4$ and $x_j < x_{j+1}$ for $j = 1, \ldots, m-1$.
- $K = [x_1, x_2] \cup \cup_{j=3}^m \{x_j\}$, where $m \geq 4$, $x_{j_1} \neq x_{j_2}$ for $j_1 \neq j_2$ and $x_j \notin [x_1, x_2]$ for $j = 3, \ldots, m$.

By an appropriate substitution (see Case 2 in the proof of (1) of Theorem 4.1), we may assume $x_1 < x_2 < x_3 < x_4$ for either of the forms above. Define the polynomials $e_{j\ell}(x) := (x - x_j)(x - x_\ell)$, where $j, \ell = 1, 2, 3, 4$. Let us define the matrix polynomial $F_k(x) = A_k e_{12}(x) + B_k e_{23}(x) + C_k e_{34}(x)$, where

$$A_k = (x_4 - x_3)\begin{bmatrix} 1 & k \\ k & k^2 \end{bmatrix}, \; B_k = (x_1 - x_4)\begin{bmatrix} 1 & k \\ k & 1 \end{bmatrix}, \; C_k = (x_2 - x_1)\begin{bmatrix} k^2 & k \\ k & 1 \end{bmatrix}.$$

For $k > 1$ we have

$$F_k(x_1) = (x_2 - x_1)(x_3 - x_1)(x_4 - x_1) \begin{bmatrix} k^2 - 1 & 0 \\ 0 & 0 \end{bmatrix} \succeq 0,$$

$$F_k(x_2) = (x_2 - x_1)(x_3 - x_2)(x_4 - x_2) \begin{bmatrix} k^2 & k \\ k & 1 \end{bmatrix} \succeq 0,$$

$$F_k(x_3) = (x_3 - x_1)(x_3 - x_2)(x_4 - x_3) \begin{bmatrix} 1 & k \\ k & k^2 \end{bmatrix} \succeq 0,$$

$$F_k(x_4) = (x_4 - x_1)(x_4 - x_2)(x_4 - x_3) \begin{bmatrix} 0 & 0 \\ 0 & k^2 - 1 \end{bmatrix} \succeq 0.$$

Since $k > 1$, it holds also that $F_k(x_2) \oplus F_k(x_3) = \mathbb{C}^2$.

Since $\det(F_k(x_j)) = 0$ for $j = 1, 2, 3, 4$ and $\deg(\det(F_k)) \leq 4$, it follows that $\det(F_k(x_j)) = p(k) \prod_{j=1}^{4}(x - x_j)$, where

$$p(k) := \frac{(k^2 - 1)}{k} \left(k^2(x_2 - x_1)(x_4 - x_3) + (x_3 - x_1)(x_2 - x_4) \right) \in \mathbb{R}[k].$$

Let us define the interval $I := \left(1, \sqrt{\frac{(x_3-x_1)(x_4-x_2)}{(x_2-x_1)(x_4-x_3)}}\right)$. For $k \in I$, $p(k) < 0$ and $\det(F_k) \in \mathrm{Pos}^1_{\succeq 0}([x_1, x_2] \cup [x_3, x_4])$. Let us write $F_k(x) := \begin{bmatrix} a_k(x) & b_k(x) \\ b_k(x) & c_k(x) \end{bmatrix}$ with $a_k(x), b_k(x), c_k(x) \in \mathbb{R}[x, k]$. By the calculations of $F_k(x_j)$ above, $a(x_1) > 0$, $a(x_2) > 0$, $a(x_3) > 0$ and $a(x_4) = 0$. Since $\deg_x(a_k) \leq 2$, if a_k has another zero x_0 on the interval (x_1, x_4), then it lies on the interval (x_3, x_4). But then $\det(F_k(x_0)) \leq 0$, which is contradiction. Thus, $(a_k)|_{[x_1,x_4]} \geq 0$. Similarly also $(c_k)|_{[x_1,x_4]} \geq 0$. Hence, $F_k \in \mathrm{Pos}^2_{\succeq 0}([x_1, x_2] \cup [x_3, x_4])$. Let S_1 be the natural description of the set $K_1 := \{x_1, x_2, x_3, x_4\}$. By Proposition 4.4, $F_k(x) \notin T^2_{S_1, b}$ for $k \in I$. Suppose S_2 is the natural description of $K_2 := [x_1, x_2] \cup \{x_3, x_4\}$. Since $S_2 \subset S_1$, it follows that $F_k(x) \notin T^2_{S_2, b}$. Define the polynomial $r(x) = \prod_{j=5}^{m}(x - x_j)$ and the matrix polynomial $G_k(x) = r(x) F_k(x)$. Let us assume $G_k \in T^2_{S, b}$, where S is the natural description of K. Then all the summands in the sum are divisible by $r(x)$. Hence, $F_k(x) = \frac{G_k(x)}{r(x)}$ belongs to $T^2_{S_1, b}$. This is a contradiction. $\square$

4.3. **Proof of Theorem 4.2.** It is enough to prove Theorem 4.2 for the natural description $\mathscr{S}$ of $\mathscr{K}$ by the following.

Proposition 4.5. *Let $\mathscr{K} \subseteq \mathbb{T}$ be a non-empty basic closed semialgebraic set with natural description $\mathscr{S}$. Let $\mathscr{S}_1 \subset H_1\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$ be a finite set, such that $\mathscr{K}_{\mathscr{S}_1} = \mathscr{K}$. If $\mathscr{T}^n_{\mathscr{S}}$ is not boundedly saturated, then $\mathscr{T}^n_{\mathscr{S}_1}$ is not boundedly saturated.*

Proof. The proof is analogous to the proof of Proposition 3.1, just that we use Theorem 3.6 instead of [18, Theorem 4.1]. $\square$

Proof of Theorem 4.2. Assume the notation from Subsections 2.2 and 3.2. Choose $z_0 \notin \mathscr{K}$. The set $K_{z_0, i}$ has one of the forms from Theorem 4.1. Hence, there is a polynomial $F \in \mathrm{Pos}^2_{\succeq 0}(K_{z_0, i})$ (or $F \in \mathrm{Pos}^2_{\succ 0}(K_{z_0, i})$) of even degree, such that $F \notin T^2_{S', b}$, where S' is the even natural description of $K_{z_0, i}$. But then $\Lambda_{z_0, i, F}(z) \in \mathrm{Pos}^2_{\succeq 0}(\mathscr{K})$ (or $\Lambda_{z_0, i, F}(z) \in \mathrm{Pos}^2_{\succ 0}(\mathscr{K})$) and $\Lambda_{z_0, i, F}(z) \notin \mathscr{T}^2_{\mathscr{S}, b}$ (Here we used Proposition 3.5.). $\square$

5. Nichtnegativstellensatz for an arbitrary K

By the results of Section 3, we see, that for almost all unbounded sets K, there are polynomials from $\mathrm{Pos}_{\succeq 0}^n(K)$ that do not belong to T_S^n, where S is any finite set $S \subset \mathbb{R}[x]$ such that $K = K_S$. The main result of this section is the characterization of $\mathrm{Pos}_{\succeq 0}^n(K)$ for unbounded sets K.

Theorem 5.1. *Suppose K is an unbounded basic closed semialgebraic set in $\mathbb{R}$ and S a saturated description of K. Then, for any $F \in H_n(\mathbb{C}[x])$, the following are equivalent:*

(1) *$F \in \mathrm{Pos}_{\succeq 0}^n(K)$.*
(2) *$(1 + x^2)^k F \in T_S^n$ for some $k \in \mathbb{N} \cup \{0\}$.*
(3) *For $w \in \mathbb{C} \setminus K$ there exists $k_w \in \mathbb{N} \cup \{0\}$, such that $((x - w)^*(x - w))^{k_w} F \in T_S^n$.*
(4) *For every $w \in \mathbb{C}$ there exists $h \in \mathbb{R}[x]$, such that $h(w) \neq 0$, $h(\overline{w}) \neq 0$ and $h^2 F \in T_S^n$.*
(5) *For every $p \in \mathbb{N}$ there exists $h \in \sum \mathbb{R}[x]^2$, such that $hF = F^{2p} + F'$, where $F' \in T_S^n$.*

The characterization of $\mathrm{Pos}_{\succeq 0}^n(K)$ in the case of multivariate real matrix polynomials is [2, Theorem B]. The improvement of [2, Theorem B] in the univariate case is the fact, that h in (5) of Theorem 5.1 above can be taken from $\mathbb{R}[x]$ instead of $M_n(\mathbb{R}[x])$.

Proof of equivalence $(1) \Leftrightarrow (2)$ **of Theorem 5.1.** It follows by Corollary 2.8 for $w_0 = i$.

Proof of equivalence $(1) \Leftrightarrow (3)$ **of Theorem 5.1.** We will need an additional lemma to prove $(1) \Rightarrow (3)$ for the case $w \in \mathbb{R} \setminus K$. Let $\mathscr{K} \subseteq \mathbb{T}$ be a semialgebraic set with a saturated description $\mathscr{S} := \{b_1, \ldots, b_s\}$. We say that A lies in the *bounded part* $\mathscr{T}_{\mathscr{S},b}^n$ of $\mathscr{T}_{\mathscr{S}}^n$, if it can be writen in the form $A = \sum_{e \in \{0,1\}^s} \tau_e \underline{b}^e$ with $\tau_e \in \sum M_n(\mathbb{C}[z])^2$ and $\deg(\tau_e \underline{b}^e) \leq \deg(A)$ for each e.

Assume the notation from Subsections 2.2 and 3.2.

Lemma 5.2. *Let $\mathscr{K} \subseteq \mathbb{T}$ be a non-empty basic closed semialgebraic set and $\mathscr{S}$ a saturated description of $\mathscr{K}$. Then for every $A \in \mathrm{Pos}_{\succeq 0}^n(\mathscr{K})$ and every*

(1) *$z_0 \in \mathbb{T}$ if $\mathscr{K} = \mathbb{T}$,*
(2) *$z_0 \in \mathbb{T} \setminus \mathscr{K}$ if $\mathscr{K} \neq \mathbb{T}$,*

there exists $\ell_{z_0} \in \mathbb{N} \cup \{0\}$, such that $((z - z_0)^*(z - z_0))^{\ell_{z_0}} A \in \mathscr{T}_{\mathscr{S},b}^n$.

Proof. If $\mathscr{K} = \mathbb{T}$, we choose $k_{z_0} = 0$ and the result follows by Theorem 1.1. Otherwise let $z_0 \in \mathbb{T} \setminus \mathscr{K}$. We have $\Gamma_{z_0,i,A}(x) \in \mathrm{Pos}_{\succeq 0}^n(K_{z_0,i})$. Note that $K_{z_0,i}$ is a non-empty basic compact semialgebraic set with a saturated description $\Gamma_{z_0,i,\mathscr{S}}$. By Theorem 2.2, $\Gamma_{z_0,i,A}(x) \in T_{\Gamma_{z_0,i,\mathscr{S}}}^n$. But then by $(\circ)$,

$$((z - z_0)^* \cdot (z - z_0))^{\ell_{z_0}} \cdot A(z) \in \mathscr{T}_{\mathscr{S},b}^n$$

for $\ell_{z_0} \in \mathbb{N} \cup \{0\}$ great enough. $\square$

Proof of equivalence $(1) \Leftrightarrow (3)$ *of Theorem 5.1.* For $w \in \mathbb{C} \setminus \mathbb{R}$ the statement follows by Corollary 2.8. Let now $w \in \mathbb{R} \setminus K$. If $K = \mathbb{R}$, there is nothing to prove. Otherwise

$K \neq \mathbb{R}$. $\Lambda_{1,i,F}(z) \in \mathrm{Pos}^n_{\succeq 0}(\mathscr{K}_{1,i})$. By Lemma 5.2, for every $z_0 \in \mathbb{T} \setminus \mathscr{K}$ there exists $\ell_{z_0} \in \mathbb{N} \cup \{0\}$, such that

$$((z-z_0)^*(z-z_0))^{\ell_{z_0}} \Lambda_{1,i,F}(z) \in \mathscr{T}^n_{\Lambda_{1,i,S},b}.$$

By $(*)$ and the fact that $z_0 \neq 1$ (Since K is unbounded, $1 \in \mathscr{K}_{1,i}$.), it follows that

$$\left(\frac{(x-i)(x+i)}{4}\right)^{\lceil \frac{\deg(F)}{2}\rceil + \ell_{z_0}} \cdot \left(\left((z-z_0)^*(z-z_0)\right)^{\ell_{z_0}} \Lambda_{1,i,F}\right)(\lambda_{1,i}(x)) =$$

$$= \left(\frac{(x-i)(x+i)}{4}\right)^{\lceil \frac{\deg(F)}{2}\rceil + \ell_{z_0}} \left(\left(\frac{(\lambda_{1,i}(x)-z_0)^2}{-z_0 \lambda_{1,i}(x)}\right)^{\ell_{z_0}} \cdot \Lambda_{1,i,F}(\lambda_{1,i}(x))\right) =$$

$$= \left(\frac{(2-(z_0+\overline{z_0}))}{4}\right)^{\ell_{z_0}} \cdot (x - \lambda_{1,i}(z_0))^{\ell_{z_0}} \cdot F(x) \in T^n_S.$$

Hence,

$$(x - \lambda_{1,i}(z_0))^{\ell_{z_0}} \cdot F(x) \in T^n_S$$

for some $\ell_{z_0} \in \mathbb{N} \cup \{0\}$. When z_0 runs over $\mathbb{T} \setminus \mathscr{K}$, $\lambda_{1,i}(z_0)$ runs over $\mathbb{R} \setminus K$, which is exactly (3) for $w \in \mathbb{R} \setminus K$ (For $w = \lambda_{1,i}(z_0)$, $k_w := \ell_{z_0}$.) This concludes the proof. $\qquad\square$

Proof of equivalence $(1) \Leftrightarrow (4)$ **of Theorem 5.1.** It follows from the equivalence $(1) \Leftrightarrow (3)$. If $K = \mathbb{R}$, we can take $h = 1$, by Theorem 1.2. Otherwise there are $x_1, x_2 \in \mathbb{R} \setminus K$, $x_1 \neq x_2$. For $w \neq x_1$ take $h(x) = (x-x_1)^{k_{x_1}}$ from (3), while for $w = x_1$ take $h(x) = (x-x_2)^{k_{x_2}}$ from (3).

Proof of equivalence $(1) \Leftrightarrow (5)$ **of Theorem 5.1.** The non-trivial direction is $(1) \Rightarrow (5)$. Let us write

$$F^{2p-1} = [f_{jl}]_{jl} = \sum_{j=1}^n f_{jj} E_{jj} + \sum_{j<l} f_{jl} (E_{jl} + E_{ji}).$$

For every $j = 1, \dots, n$ we have

$$f_{jj} E_{jj} \preceq \left(1 + f_{jj}^2\right) E_{jj} \preceq \left(1 + f_{jj}^2\right) I_n.$$

For every $1 \leq j < l \leq n$ we have

$$f_{jl}(E_{jl} + E_{ji}) \preceq \left(1 + f_{jl}\overline{f_{jl}}\right)(E_{jj} + E_{ll}) \preceq \left(1 + f_{jl}\overline{f_{jl}}\right) I_n.$$

We use those inequalities and obtain $F^{2p-1} \preceq \tilde{h} I_n$, where $\tilde{h} := \sum_{j,l} \left(1 + f_{jl}\overline{f_{jl}}\right) I_n$.

We will argue that $\tilde{h} F - F^{2p} \in \mathrm{Pos}^n_{\succeq 0}(K)$. By the equality $\tilde{h} F - F^{2p} = F(\tilde{h} I_n - F^{2p-1})$, the inclusions $F \in \mathrm{Pos}^n_{\succeq 0}(K)$ and $\tilde{h} I_n - F^{2p-1} \in \mathrm{Pos}^n_{\succeq 0}(\mathbb{R}) \subseteq \mathrm{Pos}^n_{\succeq 0}(K)$, and the fact that the matrix polynomials $F, \tilde{h} I_n - F^{2p-1}$ commute, it follows that $\tilde{h} F - F^{2p} \in \mathrm{Pos}^n_{\succeq 0}(K)$.

Now we will prove, that there is $h \in \sum \mathbb{R}[x]^2$ and $F' \in T^n_S$, such that $hF = F^{2p} + F'$. By the equivalence $(1) \Leftrightarrow (2)$ of Theorem 5.1, there exists $k \in \mathbb{N} \cup \{0\}$, such that $(1+x^2)^k(\tilde{h} F - F^{2p}) \in T^n_S$. It follows that $(1+x^2)^{2k}\tilde{h} F = (1+x^2)^{2k} F^{2p} + \tilde{F}$, where $\tilde{F} \in T^n_S$. We can write $(1 + x^2)^{2k} = 1 + r(x)$, where $r \in \sum \mathbb{R}[x]^2$. It follows that

$$(1+x^2)^{2k}\tilde{h} F = (1+x^2)^{2k} F^{2p} + \tilde{F} = F^{2p} + rF^{2k} + \tilde{F}.$$

Therefore $hF = F^{2p} + F'$ for $h := (1+x^2)^{2k}\tilde{h} \in \sum \mathbb{R}[x]^2$ and $F' := rF^{2k} + \tilde{F} \in T^n_S$.

6. Application to the matrix moment problem

Given a closed subset K of $\mathbb{R}^n$, one of the matrix versions of the K-moment problem asks for a characterization of the linear functionals $L : M_n(\mathbb{R}[x_1, \ldots, x_n]) \to \mathbb{R}$ which arise by integration with respect to a suitable operator Borel measure on K. See [3] for details and the basic terminology. [3, Theorem 3] is such a characterization. With the use of results of this paper we obtain Švecov's theorem for matrix polynomials (see [3, Corollary 1] for Hamburger's, Stieltjes' and Hausdorff's theorems for matrix polynomials):

Theorem 6.1. *Let L be a linear functional on $H_n(\mathbb{R}[x])$. For each $p \in \mathbb{N}_0$ write $S_p := [L(x^p E_{k,l})]_{k,l=1,\ldots,n}$ where $E_{k,l}$ are coordinate matrices. Then L has an integral representation with a positive operator-valued measure E whose support is contained in $(-1, 0] \cup [1, \infty)$ iff $[S_{i+j}]_{i,j=0,\ldots,m}$ and $[S_{i+j+2} - S_{i+j+1}]_{i,j=0,\ldots,m}$ are positive semidefinite for every $m \in \mathbb{N}_0$.*

Proof. Use [3, Theorem 3] and Theorem 3.2 for a union of two unbounded intervals. □

References

[1] J. Cimprič, Strict positivstellensätze for matrix polynomials with scalar constraints, Linear algebra appl. 434 (2011) 1879–1883.

[2] J. Cimprič, Real algebraic geometry for matrices over commutative rings, J. Algebra 359 (2012) 89–103.

[3] J. Cimprič, A. Zalar, Moment problems for operator polynomials. J. Math. Anal. Appl. 401 (2013) 307–316.

[4] M. D. Choi, T. Y. Lam, B. Reznick, Real zeros of positive semidefinite forms I, Math. Z. 171 (1980) 1–26.

[5] R. Ciampi, Characterization of a class of matrices as sums of squares, Linear Algebra and its Applications 3 (1970) 45–50.

[6] H. Dette, W. J. Studden, Matrix measures, moment spaces and Favard's theorem for the interval [0, 1] and [0, ∞), Linear Algebra Appl. 345 (2002) 169–193.

[7] D. Z. Djoković, Hermitian matrices over polynomial rings. J. Algebra 43 (1976) 359–374.

[8] M. Dritschel, On factorization of trigonometric polynomials, Integr. equ. oper. theory 49 (2004) 11–42.

[9] L. Ephremidze, G. Janashia, E. Lagvilava, A simple proof of the matrix-valued Fejér-Riesz Theorem, J. Fourier Anal. Appl. 15 (2009) 124–127.

[10] L. Ephremidze, An elementary proof of the polynomial matrix spectral factorization theorem. Proc. Roy. Soc. Edinburgh Sect. A 144 (2014) 747–751.

[11] I. T. Gohberg, M. G. Krein, A system of integral equation on a semiaxis with kernels depending on different arguments. Uspekhi matemat. nauk 13 (1958) 3–72.

[12] D. Gondard, P. Ribenboim P., Le 17e problème de Hilbert pour les matrices, Bull. Sci. Math. 98 (1974) 49–56.

[13] C. Hanselka, M. Schweighofer M., Positive semidefinite matrix polynomials, preprint.

[14] C. J. Hillar, J. Nie, An elementary and constructive solution to Hilbert's 17th problem for matrices, Proc. Amer. Math. Soc. 136 (2008) 73–76.

[15] C. W. J. Hol, C. W. Scherer, Matrix sum-of-squares relaxations for robust semi-definite programs. Math. Program. 107 (2006) 189–211.

[16] V. A. Jakubovič, Factorization of symmetric matrix polynomials, Dokl. Akad. Nauk 194 (1970) 532–535.

[17] T. Kailath, H. Sayed, B. Hassibi, Linear estimation, Prentice Hall, New Jersey, 2000.

[18] S. Kuhlmann, M. Marshall, Positivity, sums of squares and the multidimensional moment problem, Trans. Amer. Math. Soc. 354 (2002) 4285–4301.

[19] S. Kuhlmann, M. Marshall, N. Schwartz, Positivity, sums of squares and the multidimensional moment problemII, Adv. Geom. 5 (2005) 583–607.

[20] A. N. Malyshev, Factorization of matrix polynomials, Sibirsk. Mat. Zh. 23 (1982) 136–146.

[21] B. Mangold, Quadratsummen von Matrixpolynomen in einer Variable, Bachelorarbeit, University of Konstanz, 2013.

[22] M. Marshall, Positive polynomials and sums of squares, American Mathematical Society, Providence, 2008.

[23] V. M. Popov, Hyperstability of control systems, Springer-Verlag, Berlin, 1973.

[24] C. Procesi, M. Schacher M., A non-commutative real Nullstellensatz and Hilbert's 17th problem, Ann. of Math. 104 (1976) 395–406.

[25] I. Gohberg, P. Lancaster, L. Rodman, Matrix polynomials, Computer Science and Applied Mathematics. Academic Press, Inc., New York-London, 1982.

[26] M. Rosenblatt, A multidimensional prediction problem, Ark. Mat. vol. 3 (1958) 407–424.

[27] K. Scheiderer, Sums of squares on real algebraic surfaces, Manuscr. Math. 119 (2006) 395–410.

[28] K. Scheiderer, Sums of squares on real algebraic curves, Math. Z. 245 (2003) 725–760.

[29] K. Schmüdgen, Noncommutative real algebraic geometry - some basic concepts and first ideas. in: Emerging applications of algebraic geometry, IMA Vol. Math. Appl., 149, Springer, New York, 2009, pp. 325–350.

[30] Y. Savchuk, K. Schmüdgen K., Positivstellensätze for algebras of matrices. Linear Algebra Appl. 436 (2012) 758–788.

University of Ljubljana, Faculty of Math. and Phys., Dept. of Math., Jadranska 19, SI-1000 Ljubljana, Slovenia

E-mail address: `aljaz.zalar@imfm.s`