Matrix Fejér-Riesz Theorem with gaps

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Abstract

The matrix Fejér-Riesz theorem characterizes positive semidefinite matrix polynomials on the real line $\mathbb{R}$. We extend a characterization to arbitrary closed semialgebraic sets $K \subseteq \mathbb{R}$ by the use of matrix preorderings from real algebraic geometry. In the compact case a denominator-free characterization exists, while in the non-compact case there are counterexamples. However, there is a weaker characterization with denominators in the non-compact case. At the end we extend the results to algebraic curves.

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1. Introduction

1.1. Motivation

The matrix Fejér-Riesz theorem is the following result (For the proof see either of [8], [18], [10], [6], [4], [17], [7]).

Theorem 1.1. *Let $F(x) = \sum_{m=0}^{2N} F_m x^m$ be a $n \times n$ matrix polynomial from $M_n(\mathbb{C}[x])$ which is positive semidefinite on $\mathbb{R}$. Then there exists a matrix polynomial $G(x) = \sum_{m=0}^{N} G_m x^m \in M_n(\mathbb{C}[x])$ such that $F(x) = G(x)^* G(x)$ where $G(x)^* = \sum_{m=0}^{N} G_m^* x^m = \sum_{m=0}^{N} \overline{G_m}^T x^m = \overline{G(x)}^T$.*

In the scalar case ($n = 1$) Theorem 1.1 has already been extended to a finite union of points and intervals (not necessarily bounded) in $\mathbb{R}$ by S. Kuhlmann and Marshall [11, Theorem 2.2]. The main problem of our paper is the following.

Problem. *Characterize univariate matrix polynomials which are positive semidefinite on a finite union of points and intervals (not necessarily bounded) in $\mathbb{R}$.*

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Our main results, which will be explicitly stated in Subsection 1.3, are a denominator-free generalization of Theorem 1.1 to a finite union of compact intervals in $\mathbb{R}$, a classification of counterexamples for a denominator-free generalization to an unbounded finite union of closed intervals in $\mathbb{R}$ and a generalization with denominators in this case.

1.2. Notation and known results

Let $M_n(\mathbb{C}[x])$ be a set of all $n \times n$ matrix polynomials over $\mathbb{C}[x]$ equipped with the *involution* $F(x)^* = \overline{F(x)}^T$ where $\overline{x} = x$.

Remark 1.2. For $n = 1$ and $p(x) := \sum_{i=0}^m a_i x^i \in \mathbb{C}[x]$, the involution is $p(x)^* = \sum_{i=0}^m \overline{a_i} x^i$.

We say $F(x) \in M_n(\mathbb{C}[x])$ is *hermitian* if $F(x) = F(x)^*$. We write $\mathbb{H}_n(\mathbb{C}[x])$ for the set of all hermitian matrix polynomials from $M_n(\mathbb{C}[x])$. A matrix polynomial $F(x) \in \mathbb{H}_n(\mathbb{C}[x])$ is *positive semidefinite* in $x_0 \in \mathbb{C}$ if $v^* F(x_0) v \geq 0$ for every nonzero $v \in \mathbb{C}^n$. We denote by $\sum M_n(\mathbb{C}[x])^2$ the set of all finite sums of the expressions of the form $G(x)^* G(x)$ where $G(x) \in M_n(\mathbb{C}[x])$. We call such expressions *hermitian squares* of matrix polynomials.

The *closed semialgebraic set* associated to a finite subset $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ is given by $K_S = \{x \in \mathbb{R} \colon g_j(x) \geq 0,\ j = 1, \ldots, s\}$. We define the *n-th matrix quadratic module generated by* S in $\mathbb{H}_n(\mathbb{C}[x])$ by

$$M_S^n \quad := \quad \left\{\sigma_0 + \sigma_1 g_1 + \ldots + \sigma_s g_s \colon \sigma_j \in \sum M_n(\mathbb{C}[x])^2,\ j = 0, \ldots, s\right\},$$

and the *n-th matrix preordering generated by* S in $\mathbb{H}_n(\mathbb{C}[x])$ by

$$T_S^n \quad := \quad \left\{\sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e \colon \sigma_e \in \sum M_n(\mathbb{C}[x])^2 \text{ for all } e \in \{0,1\}^s\right\},$$

where $e := (e_1, \ldots, e_s)$ and $\underline{g}^e$ stands for $g_1^{e_1} \cdots g_s^{e_s}$.

Remark 1.3. Note that T_S^n is the quadratic module generated by all products $\underline{g}^e$, $e \in \{0,1\}^s$.

We write $\mathrm{Pos}_{\succeq 0}^n(K_S)$ for the set of all $n \times n$ hermitian matrix polynomials which are positive semidefinite on K_S. We say M_S^n (resp. T_S^n) is *saturated* if $M_S^n = \mathrm{Pos}_{\succeq 0}^n(K_S)$ (resp. $T_S^n = \mathrm{Pos}_{\succeq 0}^n(K_S)$).

Theorem 1.1 can be restated in the following form.

Theorem 1.1'. *Assume the notation as above. The set* $M_\emptyset^n = T_\emptyset^n$ *is saturated for every* $n \in \mathbb{N}$.

The aim of this article is to study matrix generalizations of Theorem 1.1' to an arbitrary closed semialgebraic set $K \subseteq \mathbb{R}$. In this notation Problem becomes the following.

Problem'. *Assume $K \subseteq \mathbb{R}$ is a closed semialgebraic set. Does there exist a finite set $S \subset \mathbb{R}[x]$ such that $K = K_S$ and the n-th matrix quadratic module M_S^n or preordering T_S^n is saturated for every $n \in \mathbb{N}$?*

Now we recall a description of a closed semialgebraic set $K \subseteq \mathbb{R}$, introduced in [11], which solves Problem' for $n = 1$. A set $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ is the *natural description* of K if it satisfies the following conditions:

(a) If K has the least element a, then $x - a \in S$.
(b) If K has the greatest element a, then $a - x \in S$.
(c) For every $a \neq b \in K$, if $(a, b) \cap K = \emptyset$, then $(x - a)(x - b) \in S$.
(d) These are the only elements of S.

Problem' has already been solved in the following cases:

1. The preordering T_S^1 is saturated for the natural description S of K (see [11, Theorem 2.2]).
2. For $K = K_{\{x,1-x\}} = [0, 1]$, $M_{\{x,1-x\}}^n$ is saturated for every $n \in \mathbb{N}$ (see [5, Theorem 2.5] or [24, Theorem 7]).
3. For $K = K_{\{x\}} = [0, \infty)$, $M_{\{x\}}^n$ is saturated for every $n \in \mathbb{N}$ (see [24, Theorem 8] or [3, Proposition 3]).

Even more can be said in the case $n = 1$. There is a characterization of finite sets $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ such that the preordering T_S^1 is saturated, which we now explain. We separate two possibilities according to the compactness of K_S.

1. *K_S is not compact:* By [11, Theorem 2.2], T_S^1 is saturated iff S contains each of the polynomials in the natural description of K_S up to scaling by positive constants.
2. *K_S is compact:* Write K_S as the union of pairwise disjoint points and intervals, i.e., $K_S = \cup_{j=1}^t [x_j, y_j]$ where $x_j \leq y_j$ for every $j = 1, \ldots, t$. By a special case of Scheiderer's results [22, Corollary 4.4], [21, Theorem 5.17] (which cover non-singular curves in $\mathbb{R}^n$), $M_S^1 = T_S^1$ and M_S^1 is saturated iff the following two conditions hold:
   (a) For every left endpoint x_j there exists $k \in \{1, \ldots, s\}$ such that $g_k(x_j) = 0$ and $g_k'(x_j) > 0$.
   (b) For every right endpoint y_j there exists $k \in \{1, \ldots, s\}$ such that $g_k(y_j) = 0$ and $g_k'(y_j) < 0$.

   (For another proof see [12, Theorem 3.2].). We call every set $S \subset \mathbb{R}[x]$ which satisfies the two conditions above a *saturated description* of K_S.

Convention. An interval always has a non-empty interior.

1.3. New results

One of the main results of the paper which solves Problem' for compact sets K is the following.

Theorem C. *Let K be a compact semialgebraic set. The n-th matrix quadratic module M_S^n is saturated for every $n \in \mathbb{N}$ iff S is a saturated description of K (see Theorem 2.1).*

The answers to Problem' for unbounded sets K (except for a union of one or two unbounded intervals and a point) are given by the following result.

Theorem D. *Let K be an unbounded closed semialgebraic set.*

The n-th matrix quadratic module M_S^n is saturated for the natural description S of K and every $n \in \mathbb{N}$ if K is either of the following:

1. *An unbounded interval (by Theorem 1.1' and [24, Theorem 8]).*
2. *A union of two unbounded intervals (see Proposition 3.1).*

The n-th matrix preordering T_S^n is not saturated for any finite set $S \subset \mathbb{R}[x]$ such that $K = K_S$ in the following cases (see Theorem 3.2):

1. *$n \geq 2$ and K contains at least two intervals with at least one of them bounded.*
2. *$n \geq 2$ and K is a union of an unbounded interval and m isolated points with $m \geq 2$.*
3. *$n \geq 2$ and K is a union of two unbounded intervals and m isolated points with $m \geq 2$.*

In the remaining cases of a union of one or two unbounded intervals and a point not covered by Theorems C and D we state the following conjecture based on the investigation of some examples.

Conjecture. *Let $K \subseteq \mathbb{R}$ be either of the following:*

1. *A union of an unbounded interval and a point.*
2. *A union of two unbounded intervals and a point.*

Suppose S is the natural description of K. Then the n-th matrix preordering T_S^n is saturated for every natural number $n > 1$.

Note that by an appropriate substitution of variables both cases covered by Conjecture are equivalent.

For the unbounded sets K with a negative answer to Problem' we obtain the following characterization of the set $\mathrm{Pos}_{\succeq 0}^n(K)$.

Theorem E. *Let K be an unbounded closed semialgebraic set with a natural description S and $n \in \mathbb{N}$. Then the following statements are equivalent:*

1. *$F \in Pos_{\succeq 0}^n(K)$.*
2. *For every $w \in \mathbb{C}$ there exists $h \in \mathbb{R}[x]$ such that $h(w) \neq 0$ and $h^2 F \in T_S^n$ (see Theorem 3.5).*
3. *For every $w \in \mathbb{C} \setminus K$ there exists $k_w \in \mathbb{N} \cup \{0\}$ such that*

$$((x - \overline{w})(x - w))^{k_w} F \in T_S^n$$

(see Corollary 4.3 and Remark 4.4).

4. $(1+x^2)^k F \in T_S^n$ *for some* $k \in \mathbb{N} \cup \{0\}$ *(Take* $w = i$ *in 3.).*

The following table summarizes [11, Theorem 2.2], Theorems C, D and Conjecture.

K	A	B
a bounded set	Yes	Yes
an unbounded interval	Yes	Yes
a union of an unbounded interval and an isolated point	Yes	C
a union of an unbounded interval and m isolated points with $m \geq 2$	Yes	No
a union of two unbounded intervals	Yes	Yes
a union of two unbounded intervals and an isolated point	Yes	C
a union of two unbounded intervals and m isolated points with $m \geq 2$	Yes	No
includes a bounded and an unbounded interval	Yes	No

A := The preordering T_S^1 is saturated for the natural description S of K.

B := The n-th matrix preordering T_S^n is saturated for the natural description S of K and every integer $n \in \mathbb{N}$.

C := See Conjecture.

Remark 1.4. 1. Since T_S^1 is saturated for the natural description S of K, it follows that if T_S^n is not saturated for some $n \in \mathbb{N}$, then $T_{S_1}^n$ is not saturated for any finite set S_1 satisfying $K_{S_1} = K$.

2. The classification covers all closed semialgebraic sets $K \subseteq \mathbb{R}$. A set K is *regular* if it is equal to the closure of its interior. For regular sets $K \subseteq \mathbb{R}$ the classification is complete.

2. Saturated descriptions of a compact set $K \subset \mathbb{R}$ generate saturated n-th matrix quadratic modules

The solution to Problem' from the Introduction for a compact set K is the main result of this section (see Theorem 2.1 below). It also characterizes all finite sets S such that the quadratic module M_S^n is saturated for every natural number $n \in \mathbb{N}$.

Theorem 2.1. *Suppose K is a non-empty compact semialgebraic set in $\mathbb{R}$. The n-th matrix quadratic module M_S^n is saturated for every $n \in \mathbb{N}$ iff S a saturated description of K.*

The main ingredients in the proof of Theorem 2.1 are:

1. The $n = 1$ case [21, Theorem 5.17].

2. The "h^2F-proposition" (See Proposition 2.2 below. The proof uses the idea of diagonalizing matrix polynomials from [23, 4.3].).
3. Getting rid of h^2 in "h^2F-proposition" (The proof uses [20, Proposition 2.7], which is Proposition 2.6 below.).

2.1. "h^2F-proposition"

We call the following result "h^2F-proposition".

Proposition 2.2. *Suppose K is a non-empty compact semialgebraic set in $\mathbb{R}$ with a saturated description S. Then, for any $F \in \mathbb{H}_n(\mathbb{C}[x])$ such that $F \succeq 0$ on K and every point $x_0 \in \mathbb{C}$, there exists $h \in \mathbb{R}[x]$ such that $h(x_0) \neq 0$ and $h^2F \in M_S^n$.*

To prove Proposition 2.2 we need Lemmas 2.3 and 2.4 below.

Lemma 2.3. *Let $G = [g_{kl}]_{kl} \in M_n(\mathbb{C}[x])$. For every $1 \leq k \leq l \leq n$ there exist unitary matrices $U_{kl} \in M_n(\mathbb{R})$ and $V_{kl} \in M_n(\mathbb{C})$ such that*

$$U_{kl} G U_{kl}^* = \begin{bmatrix} p_{kl} & * \\ * & * \end{bmatrix}, \quad V_{kl} G V_{kl}^* = \begin{bmatrix} r_{kl} & * \\ * & * \end{bmatrix},$$

where

$$p_{kl} = \begin{cases} g_{kl}, & \text{for } 1 \leq k = l \leq n \\ \frac{1}{2}(g_{kl} + g_{lk} + g_{kk} + g_{ll}), & \text{for } 1 \leq k < l \leq n \end{cases},$$

$$r_{kl} = \begin{cases} g_{kl}, & \text{for } 1 \leq k = l \leq n \\ \frac{i}{2}(-g_{kl} + g_{lk}) + \frac{1}{2}(g_{kk} + g_{ll}), & \text{for } 1 \leq k < l \leq n \end{cases}.$$

Proof. We define $U_{11} = V_{11} := I_n$, $U_{kk} = V_{kk} := P_k$ for $k = 2, \ldots, n$, where P_k denotes the permutation matrix which permutes the first row and the k-th row.

For $1 \leq k < l \leq n$, define $U_{kl} := P_k S_{kl}$ where $S_{kl} = \left(s_{pr}^{(kl)}\right)_{pr} \in M_n(\mathbb{R})$ is the matrix with $s_{kk}^{(kl)} = s_{kl}^{(kl)} = s_{lk}^{(kl)} = \frac{1}{\sqrt{2}}$, $s_{ll}^{(kl)} = -\frac{1}{\sqrt{2}}$, $s_{pp}^{(kl)} = 1$ if $p \notin \{k, l\}$ and $s_{pr}^{(kl)} = 0$ otherwise.

For $1 \leq k < l \leq n$, define $V_{kl} := P_k \tilde{S}_{kl}$ where $\tilde{S}_{kl} = \left(\tilde{s}_{pr}^{(kl)}\right)_{pr} \in M_n(\mathbb{C})$ is the matrix with $\tilde{s}_{kk}^{(kl)} = \tilde{s}_{lk}^{(kl)} = \frac{1}{\sqrt{2}}$, $\tilde{s}_{kl}^{(kl)} = \frac{i}{\sqrt{2}}$, $\tilde{s}_{ll}^{(kl)} = -\frac{i}{\sqrt{2}}$, $\tilde{s}_{pp}^{(kl)} = 1$ if $p \notin \{k, l\}$ and $\tilde{s}_{pr}^{(kl)} = 0$ otherwise. □

Lemma 2.4. *For $F = \begin{bmatrix} a & \beta \\ \beta^* & C \end{bmatrix} \in \mathbb{H}_n(\mathbb{C}[x])$ where $a = a^* \in \mathbb{R}[x]$, $\beta \in M_{1,n-1}(\mathbb{C}[x])$ and $C \in \mathbb{H}_{n-1}(\mathbb{C}[x])$ it holds that*

(i) $$a^4 \cdot F = \begin{bmatrix} a^* & 0 \\ \beta^* & a^* I_{n-1} \end{bmatrix} \begin{bmatrix} a^3 & 0 \\ 0 & a(aC - \beta^*\beta) \end{bmatrix} \begin{bmatrix} a & \beta \\ 0 & aI_{n-1} \end{bmatrix}.$$

(ii) $$\begin{bmatrix} a^3 & 0 \\ 0 & a(aC - \beta^*\beta) \end{bmatrix} = \begin{bmatrix} a^* & 0 \\ -\beta^* & a^* I_{n-1} \end{bmatrix} \cdot F \cdot \begin{bmatrix} a & -\beta \\ 0 & aI_{n-1} \end{bmatrix}.$$

Proof. Easy computation. □

Proof of Proposition 2.2. The proof is by induction on the size n of the matrix polynomials. For $n = 1$ the proposition holds by the scalar case (We take $h = 1$ and use [21, Theorem 5.17] and [22, Corollary 4.4].). Suppose the proposition holds for $n-1$. We will prove that it holds for n. Let us take $F := [f_{kl}]_{kl} \in \mathbb{H}_n(\mathbb{C}[x])$ where $F \succeq 0$ on K. Let us define

$$c(x) := \begin{cases} x - x_0, & x_0 \in \mathbb{R} \\ (x-x_0)(x-\overline{x_0}), & x_0 \in \mathbb{C} \setminus \mathbb{R} \end{cases}.$$

If $F \equiv 0$, we can take $h = 1$. Otherwise $F \not\equiv 0$ and we write

$$F = c^m G,$$

where $m \in \mathbb{N} \cup \{0\}$, $G = [g_{kl}]_{kl} \in \mathbb{H}_n(\mathbb{C}[x])$ and

$$G(x_0) = [g_{kl}(x_0)]_{kl} \neq 0. \tag{1}$$

Claim. One of the following two cases applies:

Case 1: $g_{k_0k_0}(x_0) \neq 0$ for some $k_0 \in \{1, \ldots, n\}$.

Case 2: $g_{kk}(x_0) = 0$ for all $k \in \{1, \ldots, n\}$ and for some $1 \leq k_0 < l_0 \leq n$ we have

$$\Re(g_{k_0l_0})(x_0) \neq 0 \quad \text{or} \quad \Im(g_{k_0l_0})(x_0) \neq 0,$$

where $\Re(g_{k_0l_0}) := \frac{g_{k_0l_0} + \overline{g_{k_0l_0}}}{2} \in \mathbb{R}[x]$ and $\Im(g_{k_0l_0}) := \frac{g_{k_0l_0} - \overline{g_{k_0l_0}}}{2i} \in \mathbb{R}[x]$.

Proof of Claim. Let us assume that none of the two cases applies. Then $\Re(g_{kl})(x_0) = \Im(g_{kl})(x_0) = 0$ for all $1 \leq k \leq l \leq n$. Let us take $l < k$. Since $G \in \mathbb{H}_n(\mathbb{C}[x])$ is hermitian, it follows that $g_{lk} = \overline{g_{kl}} = \Re g_{kl} - i \cdot \Im g_{kl}$. Therefore $g_{lk}(x_0) = \Re g_{kl}(x_0) - i \cdot \Im g_{kl}(x_0) = 0$. Hence $g_{kl}(x_0) = 0$ for all $k, l \in \{1, \ldots, n\}$. This is a contradiction with (1) and proves Claim.

Let U_{kl}, V_{kl}, p_{kl}, r_{kl} be as in Lemma 2.3. We study each case from Claim separately:

Case 1: We define $T_{k_0k_0} := U_{k_0k_0}$, $\tilde{g}_{k_0k_0} := g_{k_0k_0}$. Notice that $\tilde{g}_{k_0k_0}(x_0) = g_{k_0k_0}(x_0) \neq 0$.

Case 2: We will separate three subcases:

Subcase 2.1. $p_{k_0l_0}(x_0) \neq 0$: We define $T_{k_0l_0} := U_{k_0l_0}, \tilde{g}_{k_0l_0} := p_{k_0l_0}$. Notice that $\tilde{g}_{k_0l_0}(x_0) \neq 0$.

Subcase 2.2. $r_{k_0l_0}(x_0) \neq 0$: We define $T_{k_0l_0} := V_{k_0l_0}, \tilde{g}_{k_0l_0} := r_{k_0l_0}$. Notice that $\tilde{g}_{k_0l_0}(x_0) \neq 0$.

Subcase 2.3. $p_{k_0l_0}(x_0) = r_{k_0l_0}(x_0) = 0$: We will prove that this subcase does not happen. By definition and assumptions we have

$$\begin{aligned} p_{k_0l_0}(x_0) &= \frac{1}{2}(g_{k_0l_0} + g_{l_0k_0} + g_{k_0k_0} + g_{l_0l_0})(x_0) = \frac{1}{2}(g_{k_0l_0} + g_{l_0k_0})(x_0) = \\ &= (\Re g_{k_0l_0})(x_0) \\ r_{k_0l_0}(x_0) &= \frac{i}{2}(-g_{k_0l_0} + g_{l_0k_0})(x_0) + \frac{1}{2}(g_{k_0k_0} + g_{l_0l_0})(x_0) = \frac{i}{2}(-g_{k_0l_0} + g_{l_0k_0})(x_0) = \\ &= (\Im g_{k_0l_0})(x_0) \end{aligned}$$

Since we are in Case 2, $(\Re g_{k_0l_0})(x_0) \neq 0$ or $(\Im g_{k_0l_0})(x_0) \neq 0$. Contradiction. Hence Subcase 2.3 never happens.

To avoid repetition in what follows we define $k_0 = l_0$ if we are in Case 1. If we write $T_{k_0l_0} G T^*_{k_0l_0} = \begin{bmatrix} \tilde{g}_{k_0l_0} & \tilde{\beta} \\ \tilde{\beta}^* & \tilde{C} \end{bmatrix}$ with $\tilde{\beta} \in M_{1,n-1}(\mathbb{C}[x])$ and $\tilde{C} \in M_{n-1}(\mathbb{C}[x])$, then $T_{k_0l_0} F T^*_{k_0l_0} = \begin{bmatrix} c^m \tilde{g}_{k_0l_0} & c^m \tilde{\beta} \\ (c^m \tilde{\beta})^* & c^m \tilde{C} \end{bmatrix} =: \begin{bmatrix} a & \beta \\ \beta^* & C \end{bmatrix}$. Therefore by part (i) of Lemma 2.4 and dividing by c^{4m}, it follows that

$$\tilde{g}^2 F = T^*_{k_0l_0} \begin{bmatrix} \tilde{g}^*_{k_0l_0} & 0 \\ \beta^* & \tilde{g}^*_{k_0l_0} I_{n-1} \end{bmatrix} \begin{bmatrix} d & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \tilde{g}_{k_0l_0} & \tilde{\beta} \\ 0 & \tilde{g}_{k_0l_0} I_{n-1} \end{bmatrix} T_{k_0l_0},$$

where

$$\begin{aligned} \tilde{g} &= \tilde{g}^2_{k_0l_0} \in \mathbb{H}_1(\mathbb{C}[x]) = \mathbb{R}[x] \\ d &= c^m \tilde{g}^3_{k_0l_0} \in \mathbb{H}_1(\mathbb{C}[x]) = \mathbb{R}[x], \\ D &= c^m \tilde{g}_{k_0l_0} \left(\tilde{g}_{k_0l_0} \tilde{C} - \tilde{\beta}^* \tilde{\beta} \right) \in \mathbb{H}_{n-1}(\mathbb{C}[x]). \end{aligned}$$

By part (ii) of Lemma 2.4 and dividing by c^{2m}, we have also

$$\begin{bmatrix} d & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} \tilde{g}^*_{k_0l_0} & 0 \\ -\beta^* & \tilde{g}^*_{k_0l_0} I_{n-1} \end{bmatrix} T_{k_0l_0} F T^*_{k_0l_0} \begin{bmatrix} \tilde{g}_{k_0l_0} & -\tilde{\beta} \\ 0 & \tilde{g}_{k_0l_0} I_{n-1} \end{bmatrix}$$

It follows that $d \geq 0$, $D \succeq 0$ on K. By the induction hypothesis used for the polynomial $D \in \mathbb{H}_{n-1}(\mathbb{C}[x])$, there exists $h_1 \in \mathbb{R}[x]$ such that $h_1(x_0) \neq 0$ and $h_1^2 D \in M_S^{n-1}$. By the scalar case [21, Theorem 5.17] and [22, Corollary 4.4], $h_1^2 d \in M_S^1$. Hence $h^2 F \in M_S^n$ where $h = h_1 \tilde{g} \in \mathbb{R}[x]$ and $h(x_0) \neq 0$. This concludes the proof. □

Remark 2.5. By keeping track on the degree of h and using [12, Theorem 4.1], we can prove more in Proposion 2.2 above. Namely, h can be chosen of degree at most $\deg(F)(3^n - 1)$ and if $S = \{g_1, \ldots, g_s\}$ is the natural description of K, then $F = \sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e \in T_S^n$ for some $\sigma_e \in M_n(\mathbb{C}[x])^2$ with $\deg(\sigma_e \underline{g}^e) \leq \deg(h^2 F)$.

2.2. Getting rid of h^2 in "h^2F-proposition"

To get rid of h^2 in "h^2F-proposition", which proves Theorem 2.1, we will use [20, Proposition 2.7]:

Proposition 2.6. *Suppose R is a commutative ring with 1 and $\mathbb{Q} \subseteq R$. Let $\Phi : R \to C(K, \mathbb{R})$ be a ring homomorphism, where K is a topological space which is compact and Hausdorff. Suppose $\Phi(R)$ separates points in K. Suppose $f_1, \ldots, f_k \in R$ are such that $\Phi(f_j) \geq 0$, $j = 1, \ldots, k$ and $(f_1, \ldots, f_k) = (1)$. Then there exist $s_1, \ldots, s_k \in R$ such that $s_1 f_1 + \ldots + s_k f_k = 1$ and such that each $\Phi(s_j)$ is strictly positive.*

Proof of Theorem 2.1. By [22, Corollary 4.4] and [21, Theorem 5.17], M_S^1 is saturated if and only if S is a saturated description of K. Therefore we have to prove only the if part. Let S be a saturated description of K. We will prove that M_S^n is saturated for every $n \in \mathbb{N}$. Let $R := \mathbb{R}[x]$ and $\Phi : R \to C(K, \mathbb{R})$ be the natural map, i.e., $\Phi(f) = f|_K$. Take $F \in \mathrm{Pos}^n_{\succeq 0}(K)$. We will prove that $F \in M_S^n$. Let $I := \left\langle h^2 \in \mathbb{R}[x] \colon h^2 F \in M_S^n \right\rangle$ be the ideal in $\mathbb{R}[x]$ generated by all h^2 where $h \in \mathbb{R}[x]$ is such that $h^2 F \in M_S^n$. Since $\mathbb{R}[x]$ is a principal ideal domain, there exists a polynomial $p \in \mathbb{R}[x]$ such that $I = \langle p \rangle$. If I was a proper ideal, all its elements would have a common zero $x_0 \in \mathbb{C}$. By Proposition 2.2, there exists $h \in \mathbb{R}[x]$ such that $h(x_0) \neq 0$ and $h^2 F \in M_S^n$. Since h belongs to I, it follows that I is not a proper ideal and hence $I = \mathbb{R}[x]$. By Proposition 2.6, there exist $s_1, \ldots, s_k \in \mathrm{Pos}^1_{\succ 0}(K)$ and $h_1, \ldots, h_k \in I$ such that $\sum_{j=1}^k s_j h_j^2 = 1$. Hence $\sum_{j=1}^k s_j h_j^2 F = F \in M_S^n$, which concludes the proof. □

Remark 2.7.
1. There is another proof of Theorem 2.1 which uses Proposition 2.2 just for the boundary points of K. We outline the main idea. There exists $h \in \mathbb{R}[x]$ such that $h \in \mathrm{Pos}^1_{\succeq 0}(\mathbb{R})$, $h(x_0) > 0$ for every boundary point of K and $hF \in M_S^n$ (Take $h = \sum_{x_0 \in \partial K} h_{x_0}^2$ where ∂K is the boundary of K and h_{x_0} is the polynomial from Proposition 2.2 for the point x_0.). Now multiply every member of the set S by h to obtain the set S_1 which satisfies conditions of [21, Corollary 5.17]. Thus $M_S^1 = M_{S_1}^1$ and $hF \in M_{S_1}^n$. This means there exist $\sigma_j \in \sum M_n(\mathbb{C}[x])^2$ such that $hF = \sigma_0 + \sigma_1 h g_1 + \ldots + \sigma_s h g_s$. From here it is easy to see that $F = \tau_0 + \sigma_1 g_1 + \ldots + \sigma_s g_s$ for some $\tau_0 \in \sum M_n(\mathbb{C}[x])^2$ and hence $F \in M_S^n$.
2. By Remark 2.5, the degree of h in Proposition 2.2 and the degrees of summands in the expression of $h^2 F$ as the element of the preordering T_S^n generated by the natural description S of K can be bounded by the degree of F and n. It would be interesting to know if the same holds for F and an arbitrary compact set K. It can be shown this is true for a finite set K. The degrees can be bounded by $\max(\deg(F), |K| - 1)$.

3. Unbounded sets K without saturated T_S^2 for any finite sets S with $K_S = K$

The answer to the question of Problem' for unbounded sets K is positive for an unbounded interval by Theorem 1.1' (if $K = \mathbb{R}$) and [24, Theorem 8] (if $K = [a, \infty)$). It is also easy to derive a positive answer for a union of two unbounded intervals from the case $K = [a, b]$:

Proposition 3.1. *Let $K = (-\infty, a] \cup [b, \infty)$ be a union of two unbounded intervals where $a, b \in \mathbb{R}$ and $a < b$. Then the quadratic module $M^n_{\{(x-a)(x-b)\}}$ is saturated for every $n \in \mathbb{N}$.*

Proof. By a linear change of variables, we may assume that $K = (-\infty, -1] \cup [1, \infty)$. Note that $F \in \mathrm{Pos}^n_{\succeq 0}(K)$ is of even degree. We define

$$F_1(x) = x^{\deg(F)} F\left(\frac{1}{x}\right)$$

and observe that $F_1 \succeq 0$ on $[-1, 1]$. By [5, Theorem 2.5] and by the identity

$$1 \pm x = \frac{(1 \pm x)^2 + (x+1)(1-x)}{2},$$

there exist matrix polynomials G_1, H_1 such that

$$F_1(x) = G_1(x)^* G_1(x) + H_1(x)^* H_1(x)(x+1)(1-x),$$

$$\deg(G_1) \leq \left\lfloor \frac{\deg(F_1)}{2} \right\rfloor \leq \frac{\deg(F)}{2},$$

$$\deg(H_1) \leq \left\lfloor \frac{\deg(F_1) - 1}{2} \right\rfloor \leq \left\lfloor \frac{\deg(F) - 1}{2} \right\rfloor = \frac{\deg(F)}{2} - 1.$$

Therefore

$$\begin{aligned} F(x) &= x^{\deg(F)} F_1(\frac{1}{x}) \\ &= x^{\deg(F)} (G_1(\frac{1}{x})^* G_1(\frac{1}{x}) + H_1(\frac{1}{x})^* H_1(\frac{1}{x})(\frac{1}{x} + 1)(1 - \frac{1}{x})) \\ &=: G(x)^* G(x) + H(x)^* H(x)(1+x)(x-1), \end{aligned}$$

where

$$G(x) := x^{\frac{\deg(F)}{2}} G_1\left(\frac{1}{x}\right), \quad H := x^{\frac{\deg(F)}{2} - 1} H_1\left(\frac{1}{x}\right)$$

are matrix polynomials. □

The negative answer to the question of Problem' for almost all remaining unbounded sets K (except for a union of an unbounded interval and a point or a union of two unbounded intervals and a point) and all $n \geq 2$ is the main result of this section.

Theorem 3.2. *Let an unbounded closed semialgebraic set $K \subseteq \mathbb{R}$ satisfy either of the following:*

1. *K contains at least two intervals with at least one of them bounded.*
2. *K is a union of an unbounded interval and m isolated points with $m \geq 2$.*

3. *K is a union of two unbounded intervals and m isolated points with $m \geq 2$.*

If $S \subset \mathbb{R}[x]$ is a finite set with $K_S = K$, then the 2-nd matrix preordering T_S^2 is not saturated.

It is sufficient to prove Theorem 3.2 for the natural description S of K by the following lemma.

Lemma 3.3. *Let $K \subseteq \mathbb{R}$ be an unbounded closed semialgebraic set with the natural description S. Let $S_1 \subset \mathbb{R}[x]$ be a finite set such that $K_{S_1} = K$. For every $n \in \mathbb{N}$ such that the n-th matrix preordering T_S^n is not saturated, also the n-th matrix preordering $T_{S_1}^n$ is not saturated.*

Proof. Let us write $S := \{g_1, \ldots, g_s\}$ and $S_1 := \{f_1, \ldots, f_t\}$. We have to show that every matrix polynomial F from $T_{S_1}^n$ also belongs to T_S^n. A matrix polynomial F from $T_{S_1}^n$ is of the form

$$F = \sum_{e' \in \{0,1\}^t} \tau_{e'} f_1^{e'_1} \ldots f_t^{e'_t}, \tag{2}$$

where $e' := (e'_1, \ldots, e'_t)$ and $\tau_{e'} \in \sum M_n(\mathbb{C}[x])^2$. By [11, Theorem 2.2], the preordering T_S^1 is saturated and thus for each j there exist $\sigma_{e,j} \in \sum \mathbb{R}[x]^2$ such that

$$f_j = \sum_{e \in \{0,1\}^s} \sigma_{e,j}\, g_1^{e_1} \cdots g_s^{e_s}, \tag{3}$$

where $e := (e_1, \ldots, e_s)$. Plugging (3) into (2) and rearranging terms we obtain $F \in T_S^n$. This concludes the proof. □

In the remaining part of this section we will prove Theorem 3.2. The major step will be Proposition 3.4.

Let K be a closed semialgebraic set with a natural description $S = \{g_1, \ldots, g_s\}$. For $n \in \mathbb{N}$ and $d \in \mathbb{N} \cup \{0\}$ we define the set

$$T_{S,d}^n := \left\{ \sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e : \sigma_e \in \sum M_n(\mathbb{C}[x])^2 \text{ and } \deg(\sigma_e \underline{g}^e) \leq d \;\forall e \in \{0,1\}^s \right\}.$$

Proposition 3.4. *Let $K = [x_1, x_2] \cup [x_3, \infty)$ be a union of a bounded and an unbounded interval where $x_1 < x_2 < x_3$. Let us define the polynomial*

$$F_k(x) := \begin{bmatrix} x + A(k) & D(k) \\ D(k) & x^2 + B(k)x + C(k) \end{bmatrix},$$

where

$$\begin{aligned} A(k) &:= k - x_1, \\ B(k) &:= -k - x_2 - x_3, \\ C(k) &:= k^2 + k(-x_1 + x_2 + x_3) + x_2 x_3, \\ D(k) &:= \sqrt{A(k)C(k) + x_1 x_2 x_3} = \\ &= \sqrt{k^3 + k^2(-2x_1 + x_2 + x_3) + k(x_2 x_3 + x_1^2 - x_1 x_2 - x_1 x_3)}. \end{aligned}$$

We define $p_k(x) := x^2 + B(k)x + C(k)$. *For every* $k \in \mathbb{R}$ *which satisfies*

$$k > 0, \tag{4}$$

$$D(k)^2 = k^3 + k^2(-2x_1 + x_2 + x_3) + k(x_2x_3 + x_1^2 - x_1x_2 - x_1x_3) > 0, \tag{5}$$

$$p_k\left(-\frac{B(k)}{2}\right) = \frac{3}{4}k^2 + k\left(-x_1 + \frac{x_2+x_3}{2}\right) - \left(\frac{x_2-x_3}{2}\right)^2 > 0, \tag{6}$$

the matrix polynomials $F_k(x)$ *belongs to* $Pos^2_{\succeq 0}(K)$, *but:*

Claim 1. $F_k \notin T^2_{S_1}$ *where* S_1 *is the natural description of any set* K_1 *of the form*

$$[x_1, x_2] \cup \cup_{j=1}^m [x_{2j+1}, x_{2j+2}] \cup [x_{2m+3}, \infty) \subseteq K$$

with $m \in \mathbb{N} \cup \{0\}$ *and* $x_j \leq x_{j+1}$ *for each* j *(and* $x_1 < x_2 < x_3$*). In particular,*

$$F_k(x) \notin T^2_S,$$

where S *is the natural description of* K.

Claim 2. $F_k \notin T^2_{S_2,2}$ *where* S_2 *is the natural description of any set* K_2 *of the form*

$$[x_1, x_2] \cup \cup_{j=3}^m \{x_j\} \subset K$$

with $m \in \mathbb{N}$, $m \geq 4$ *and* $x_j < x_{j+1}$ *for each* j.

Proof. First we will prove that $F_k(x)$ belongs to $\mathrm{Pos}^2_{\succeq 0}(K)$ for every $k \in \mathbb{R}$ satisfying the conditions (4)-(6). Note that every sufficiently large k satisfies the conditions (4)-(6). Condition (5) ensures that $D(k) \in \mathbb{R}$ and hence $F \in \mathbb{H}_n(\mathbb{R}[x])$. The determinant of $F_k(x)$ is $(x - x_1)(x - x_2)(x - x_3) \in \mathrm{Pos}^1_{\succeq 0}(K)$. The upper left corner of F is non-negative for $x \geq x_1 - k$ and hence it belongs to $\mathrm{Pos}^1_{\succeq 0}(K)$ by (4). The lower right corner is a quadratic polynomial $p_k(x)$ with a vertex in $x = \frac{-B(k)}{2}$. Since k satisfies (6), $p_k\left(\frac{-B(k)}{2}\right) > 0$. So $p_k(x)$ is positive on $\mathbb{R}$ and hence $p_k \in \mathrm{Pos}^1_{\succeq 0}(K)$. Since all principal minors of $F_k(x)$ are non-negative on K, the conclusion $F_k(x) \in \mathrm{Pos}^2_{\succeq 0}(K)$ follows.

We will separately prove both claims of the theorem.

Proof of Claim 1. The set

$$\{\underbrace{x - x_1}_{g_1(x)}, \underbrace{(x - x_2)(x - x_3)}_{g_2(x)}, \ldots, \underbrace{(x - x_{2m+2})(x - x_{2m+3})}_{g_{m+2}(x)}\}$$

is the natural description S_1 of K_1. We will prove that $F_k(x) \notin T^2_{S_1}$ by contradiction. Let us assume $F_k \in T^2_{S_1}$. Then for every $e := (e_1, \ldots, e_{m+2}) \in \{0,1\}^{m+2}$ there exists $\sigma_e \in \sum M_n(\mathbb{C}[x])^2$, such that

$$F_k = \sum_{e \in \{0,1\}^{m+2}} \sigma_e g_1^{e_1} \cdots g_{m+2}^{e_{m+2}}. \tag{7}$$

By the degree comparison of both sides of (7), there exist $\sigma_j \in \sum M_n(\mathbb{C}[x])^2$, such that

$$F_k(x) = \sigma_0 + \sigma_1(x - x_1) + \sum_{j=1}^{m+1} \sigma_{j+1}(x - x_{2j})(x - x_{2j+1}), \tag{8}$$

$$\deg(\sigma_0) \le 2, \quad \deg(\sigma_j) = 0 \text{ for } j = 1, \dots, m+2.$$

By observing the monomial x^2 on both sides of (8), it follows that $\sigma_2 = \begin{bmatrix} 0 & 0 \\ 0 & k_0 \end{bmatrix}$ for some $k_0 \in [0, 1]$. Equivalently, (8) can be written as

$$F_k(x) - \sigma_2(x - x_2)(x - x_3) = \sigma_0 + \sigma_1(x - x_1) + \sum_{j=2}^{m+1} \sigma_{j+1}(x - x_{2j})(x - x_{2j+1}).$$

The right-hand side belongs to $\mathrm{Pos}^2_{\succeq 0}(\hat{K}_1)$ where $\hat{K}_1 = K_1 \cup [x_2, x_3]$. We will prove that the left-hand side does not belong to $\mathrm{Pos}^2_{\succeq 0}(\hat{K}_1)$, which is a contradiction. The determinant of the left-hand side is

$$q(x) := (x - x_2)(x - x_3)(x(1 - k_0) - (x_1 - x_1 k_0 + k k_0)).$$

There are two cases two consider: $k_0 = 0$ and $k_0 > 0$. In the first case, $q(x) = (x - x_1)(x - x_2)(x - x_3)$ which is negative on (x_2, x_3), a contradiction with $q|_{\hat{K}_1} \ge 0$. In the second case, $q(x_1) = (x_1 - x_2)(x_1 - x_3)(-k k_0) < 0$, which is also a contradiction with $q|_{\hat{K}_1} \ge 0$. Thus

$$F_k(x) - \sigma_2(x - x_2)(x - x_3) \notin \mathrm{Pos}^2_{\succeq 0}(\hat{K}_1),$$

which is a contradiction. Therefore F_k cannot be expressed in the form (7) and so $F_k \notin T^2_{S_1}$.

Proof of Claim 2. The set

$$\{\underbrace{x - x_1}_{g_1(x)}, \underbrace{(x - x_2)(x - x_3)}_{g_2(x)}, \dots, \underbrace{(x - x_{m-1})(x - x_m)}_{g_{m-1}(x)}, \underbrace{x_m - x}_{g_m(x)}\}$$

is the natural description S_2 of K_2. If $F_k \in T^2_{S_2,2}$, then there exist $\tau_j \in \sum M_n(\mathbb{C}[x])^2$ such that

$$F_k(x) = \tau_0 + \tau_1(x - x_1) + \sum_{j=2}^{m-1} \tau_j(x - x_j)(x - x_{j+1}) + \tau_m(x_m - x) + \tau_{m+1}(x - x_1)(x_m - x),$$

$$\deg(\tau_0) \le 2, \ \deg(\tau_j) = 0 \text{ for } j = 1, \dots, m+1. \tag{9}$$

From (9) it follows that

$$(F_k(x) - \tau_j(x - x_j)(x - x_{j+1}))|_{K_2} \succeq 0 \quad \text{for } j = 2, \dots, m-1. \tag{10}$$

From (10) it follows that

$$\ker F_k(x_1) \subseteq \ker \tau_j, \ \ker F_k(x_2) \subseteq \ker \tau_j \quad \text{for } j = 3, \ldots, m-1.$$

Since $\ker F_k(x_1) \oplus \ker F_k(x_2) = \mathbb{C}^2$, we conclude that $\tau_j = 0$ for $j = 3, \ldots, m-1$.

Hence (9) becomes

$$F_k(x) = \tau_0 + \tau_1(x-x_1) + \tau_2(x-x_2)(x-x_3) + \tau_m(x_m-x) + \tau_{m+1}(x-x_1)(x_m-x),$$

or equivalently,

$$F_k(x) - \tau_2(x-x_2)(x-x_3) = \tau_0 + \tau_1(x-x_1) + \tau_m(x_m-x) + \tau_{m+1}(x-x_1)(x_m-x). \tag{11}$$

Since the determinant of the left hand side is of degree 4 and is divisible by $(x-x_1)(x-x_2)(x-x_3)$ (divisibility by $x-x_1$ is due to $\ker F_k(x_1) \neq \{0\}$ and (10) for $j=2$), it cannot be non-negative on $[x_1, x_m]$ (This follows by a simple geometric argument.). Hence the left-hand side of (11) does not belong to $\mathrm{Pos}^2_{\succeq 0}([x_1, x_m])$, while the right-hand side does. This is a contradiction and thus $F_k \notin T^2_{S_2,2}$. □

Proof of Theorem 3.2.1. By Lemma 3.3, we may assume that S is the natural description of K. Let us write K in the form $K_0 \cup K_1$ where K_0 is the set of isolated points of K and K_1 is the regular part of K (i.e., does not have isolated points). We separate three cases depending on the form of K_1.

Case 1: *K_1 is bounded from below and unbounded from above.* Let us divide the isolated part K_0 into disjoint sets K_{01}, K_{02} where in K_{01} are all those points which are smaller than the minimum of K_1 and in K_{02} all the others. The set $K_2 := K_1 \cup K_{02}$ is of the form

$$[x_1, x_2] \cup \cup_{j=1}^{p} [x_{2j+1}, x_{2j+2}] \cup [x_{2p+3}, \infty),$$

where $p \in \mathbb{N} \cup \{0\}$, $x_1 < x_2 < x_3$ and $x_j \leq x_{j+1}$ for each $j \geq 3$. Let us take a polynomial $F_1 \in \mathrm{Pos}^2_{\succeq 0}(K_2)$ and define the polynomial

$$F(x) := \prod_{y \in K_{01}} (x-y) \cdot F_1(x) \in \mathrm{Pos}^2_{\succeq 0}(K). \tag{12}$$

Let $S := \{g_1, \ldots, g_s\}$ be the natural description of K. If F belongs to T^2_S, then for every $e \in \{0,1\}^s$ there exists $\sigma_e \in \sum M_n(\mathbb{C}[x])^2$ such that

$$F = \sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e. \tag{13}$$

Since for every $y \in K_{01}$ and every $e \in \{0,1\}^s$ we have $F(y) = 0$ and $\sigma_e \underline{g}^e(y) \succeq 0$, it follows from (13) that $\sigma_e \underline{g}^e(y) = 0$. Therefore $\prod_{y \in K_{01}}(x-y)$ divides each $\sigma_e \underline{g}^e$.

Claim. There exist $\tau_e \in \sum M_n(\mathbb{C}[x])^2$ and $h_e \in \mathrm{Pos}^1_{\succeq 0}(K_2)$ such that

$$\frac{\sigma_e \underline{g}^e}{\prod_{y \in K_{01}}(x-y)} = \tau_e h_e.$$

Proof of Claim. Let us take $y \in K_{01}$. We separate two possibilities.

1. *$x-y$ divides σ_e:* Then $\sigma_e \underline{g}^e = \hat{\sigma}_e \cdot (x-y)^2 \underline{g}^e$ where $\hat{\sigma}_e \in \sum M_n(\mathbb{C}[x])^2$ and $\frac{(x-y)^2 \underline{g}^e}{x-y} = (x-y)\underline{g}^e \in \mathrm{Pos}^1_{\succeq 0}(K_2)$.
2. *$x-y$ does not divide σ_e:* Then $x-y$ divides $\underline{g}^e$ and hence $\sigma_e \underline{g}^e = \sigma_e \cdot (x-y)\hat{g}_e$ where $\hat{g}_e := \frac{\underline{g}^e}{x-y} \in \mathrm{Pos}^1_{\succeq 0}(K_2)$.

Repeating the above procedure for every $y \in K_{01}$ we obtain τ_e and h_e proving Claim.

Let S_2 be the natural description of K_2. By [11, Theorem 2.2], $h_e \in T^1_{S_2}$. It follows that $F_1 = \sum_e \tau_e h_e \in T^2_{S_2}$.

We have proved that for $F_1 \in \mathrm{Pos}^2_{\succeq 0}(K_2)$ and $F \in \mathrm{Pos}^2_{\succeq 0}(K)$ defined by (12), from $F \in T^2_S$ it follows that $F_1 \in T^2_{S_2}$. Therefore, to find $F \in \mathrm{Pos}^2_{\succeq 0}(K)$ and $F \notin T^2_S$, it is sufficient to find $F_1 \in \mathrm{Pos}^2_{\succeq 0}(K_2)$ and $F_1 \notin T^2_{S_2}$. Let us define the set $K_3 := [x_1, x_2] \cup [x_3, \infty)$. By Claim 1 of Proposition 3.4, there exists a polynomial $F_1 \in \mathrm{Pos}^2_{\succeq 0}(K_3) \subseteq \mathrm{Pos}^2_{\succeq 0}(K_2)$ such that $F_1 \notin T^2_{S_2}$. This proves Case 1.

Case 2: *K_1 is unbounded from below and bounded from above.* Make a substitution $x \mapsto -x$ and observe that the set $-K_1$ is of the form in Case 1 and that the natural description of K maps into the natural description of $-K$.

Case 3: *K_1 is unbounded from below and above.* Let $d \in \mathbb{R}$ be the smallest endpoint of K_1. Define the map $\lambda_d : \mathbb{R} \setminus \{d\} \to \mathbb{R}$ with $\lambda_d(x) := \frac{1}{d-x}$. Observe that $\lambda_d(K_1) =: K_2$ is the set of the form $[x_1, x_2] \cup [x_3, x_4] \cup \ldots \cup [\hat{x}_{2m+1}, \infty)$ where $m \in \mathbb{N}$ and $x_j < x_{j+1}$ for every j. Let S_3 be the natural description of $\lambda_d(K)$. As in Case 1, construct the polynomial $F \in \mathrm{Pos}^2_{\succeq 0}(\lambda_d(K))$ such that $F \notin T^2_{S_3}$. Now $G(x) = x^{\left(2\lceil \frac{\deg(F)}{2} \rceil\right)} \cdot F\left(d - \frac{1}{x}\right) \in \mathrm{Pos}^2_{\succeq 0}(K)$ and $G \notin T^2_S$. □

Proof of Theorem 3.2.2 and 3.2.3. By Lemma 3.3, we may assume that S is the natural description of K. Let $d \in \mathbb{R}$ be an arbitrary point such that $d \notin K$. Define the map $\lambda_d : \mathbb{R} \setminus \{d\} \to \mathbb{R}$ with $\lambda_d(x) := \frac{1}{d-x}$. Observe that $\lambda_d(K)$ is the set of the form $[x_1, x_2] \cup \cup_{j=3}^m \{x_j\}$ where $m \geq 4$ and the points x_j are pairwise different. Further on, we may choose $d \in \mathbb{R}$ such that $x_1 < x_2 < x_3 < \ldots < x_m$ or $x_m < x_{m-1} < \ldots < x_3 < x_1 < x_2$. By substitution $x \mapsto -x$, we may assume that $x_1 < x_2 < x_3 < \ldots < x_m$. Let $S_1 = \{g_1, \ldots, g_s\}$ be the natural description of $\lambda_d(K)$. Notice that to prove the statement of the theorem, it is sufficient to find $F \in \mathrm{Pos}^2_{\succeq 0}(\lambda_d(K))$ of degree $2k$ such that $F \notin T^2_{S_1, 2k}$. By Claim 2 of Proposition 3.4, there is $F \in \mathrm{Pos}^2_{\succeq 0}(\lambda_d(K))$ of degree 2 such that $F \notin T^2_{S_1,2}$. This concludes the proof. □

Theorem 3.5 gives a characterization of the set $\mathrm{Pos}^n_{\succeq 0}(K)$ for unbounded sets K.

Theorem 3.5. *Suppose K is an unbounded closed semialgebraic set in $\mathbb{R}$ and S the natural description of K. Then, for any $F \in \mathbb{H}_n(\mathbb{C}[x])$, the following are equivalent:*

1. *$F \in Pos^n_{\succeq 0}(K)$.*
2. *There exists a polynomial $h \in \mathbb{R}[x]$ such that for every isolated point $w \in K$, $h(w) \neq 0$ and $h^2 F \in T^n_S$.*
3. *For every point $w \in \mathbb{C}$ there exists a polynomial $h \in \mathbb{R}[x]$ such that $h(w) \neq 0$ and $h^2 F \in T^n_S$.*

Proof. For the implication (3) ⇒ (2) construct h in the same way as in Remark 2.7 (replace the boundary of K with the set of its isolated points). The implication (2) ⇒ (1) is trivial. The proof of direction (1) ⇒ (3) is the same as the proof of Proposition 2.2, just that we use [11, Theorem 2.2] for the $n = 1$ case instead of [22, Theorem 5.17]. □

4. Generalizations of the results to curves

In this section Theorem 2.1 is generalized to curves in $\mathbb{R}^n$. A characterization of sets S satisfying Theorem 4.1.1 was proved by Scheiderer in [21, Theorem 5.17] and [22, Corollary 4.4]. Using the same method as in the proof of Theorem 2.1 we obtain the implication 1. ⇒ 2. of the following theorem.

Theorem 4.1. *Suppose I is a prime ideal of $\mathbb{R}[\underline{x}]$ with $\dim(\frac{\mathbb{R}[\underline{x}]}{I}) = 1$ and let $\mathcal{Z}(I) := \{\underline{x} \in \mathbb{R}^d \colon f(\underline{x}) = 0 \text{ for every } f \in I\}$ be its vanishing set. Let $S := \{g_1, \ldots, g_s\}$ be a finite subset of $\mathbb{R}[\underline{x}]$ and $K_S = \{\underline{x} \in \mathbb{R}^d \colon g_1(\underline{x}) \geq 0, \ldots, g_s(\underline{x}) \geq 0\}$ the associated semialgebraic set. Suppose the set $K_S \cap \mathcal{Z}(I)$ is compact. Then the following are equivalent:*

1. *The quadratic module $M^1_S + I$ is saturated.*
2. *The n-th quadratic module $M^n_S + M_n(I)$ is saturated for every $n \in \mathbb{N}$.*

An example of a non-singular curve is the unit circle. Theorem 1.1 has an equivalent version for the unit complex circle $\mathbb{T}$ (see [19] or [16]). By passing from complex numbers to pairs of real numbers and by Theorem 4.1, we obtain a generalization of this equivalent version to an arbitrary semialgebraic set in the unit circle. To explain this generalization we need some notation. Let us equip the set of $n \times n$ matrix Laurent polynomials $M_n(\mathbb{C}\left[z, \frac{1}{z}\right])$ with an involution $A(z)^* := \overline{A(\frac{1}{\bar{z}})}^T$. We denote by $\mathbb{H}_n(\mathbb{C}[z, \frac{1}{z}])$ the set of all $B \in M_n(\mathbb{C}\left[z, \frac{1}{z}\right])$ such that $B^* = B$, and by $\sum M_n(\mathbb{C}\,[z])^2$ the set of all finite sums of elements of the form B^*B where $B \in M_n(\mathbb{C}\,[z])$. Let $\mathscr{S} = \{b_1, \ldots, b_s\}$ be a finite set from $\mathbb{H}_1(\mathbb{C}\left[z, \frac{1}{z}\right])$ and $\mathscr{K}_{\mathscr{S}} = \{z \in \mathbb{T} \colon b_j(z) \geq 0,\ j = 1, \ldots, s\}$ the associated semialgebraic set. Let the *n-th matrix quadratic module generated by $\mathscr{S}$* in $\mathbb{H}_n(\mathbb{C}[z, \frac{1}{z}]$ be

$$\mathcal{M}^n_{\mathscr{S}} := \{\tau_0 + \tau_1 b_1 + \ldots + \tau_s b_s \colon \tau_j \in \sum M_n\left(\mathbb{C}\,[z]\right)^2 \text{ for } j = 0, \ldots, s\}.$$

We write $\mathrm{Pos}^n_{\succeq 0}(\mathscr{K}_{\mathscr{S}})$ for the set of elements from $\mathbb{H}_n(\mathbb{C}[z,\frac{1}{z}])$ which are positive semidefinite on $\mathscr{K}_{\mathscr{S}}$.

Corollary 4.2. *$\mathcal{M}^n_{\mathscr{S}} = Pos^n_{\succeq 0}(\mathscr{K}_{\mathscr{S}})$ iff $\mathscr{S}$ satisfies the following conditions:*

(a) For every boundary point $a \in \mathscr{K}_{\mathscr{S}}$ which is not isolated there exists $k \in \{1,\dots,s\}$ such that $b_k(a) = 0$ and $\frac{db_k}{dz}(a) \neq 0$.

(b) For every isolated point $a \in \mathscr{K}_{\mathscr{S}}$ there exist $k, l \in \{1,\dots,s\}$ such that $b_k(a) = b_l(a) = 0$, $\frac{db_k}{dz}(a) \neq 0$, $\frac{db_l}{dz}(a) \neq 0$ and $b_k b_l \leq 0$ on some neighborhood of a.

As an application of Corollary 4.2 we obtain the following improvement of Theorem 3.5:

Corollary 4.3. *Suppose K is an unbounded closed semialgebraic set in $\mathbb{R}$ and S the natural description of K. Then, for $F \in \mathbb{H}_n(\mathbb{C}[x])$, the following are equivalent:*

1. *$F \in Pos^n_{\succeq 0}(K)$.*
2. *For every $w \in \mathbb{C}\setminus\mathbb{R}$ there exists $k_w \in \mathbb{N}\cup\{0\}$ such that*

$$((x-\overline{w})(x-w))^{k_w} F \in M^n_S.$$

To prove Corollary 4.3 we need some preliminaries. Möbius transformations that map $\mathbb{R}\cup\{\infty\}$ bijectively into $\mathbb{T}$ are exactly the maps of the form

$$\lambda_{z_0,w_0} : \mathbb{R}\cup\{\infty\} \to \mathbb{T}, \quad \lambda_{z_0,w_0}(x) := z_0 \frac{x-w_0}{x-\overline{w_0}},$$

where $z_0 \in \mathbb{T}$ and $w_0 \in \mathbb{C}\setminus\mathbb{R}$. Notice that $\lambda^{-1}_{z_0,w_0}(x) = \frac{z\overline{w_0}-z_0 w_0}{z-z_0}$. If $F(x)$ is a matrix polynomial from $M_n(\mathbb{C}[x])$, then

$$\Lambda_{z_0,w_0,F}(z) := ((z-z_0)^*(z-z_0))^{\left\lceil \frac{\deg(F)}{2}\right\rceil} \cdot F\left(\lambda^{-1}_{z_0,w_0}(z)\right)$$

is a matrix polynomial from $M_n(\mathbb{C}[z,\frac{1}{z}])$. Observe that

$$F(x) = \left(\frac{(x-\overline{w_0})(x-w_0)}{4\cdot\Im(w_0)^2}\right)^{\left\lceil \frac{\deg(F)}{2}\right\rceil} \cdot \Lambda_{z_0,w_0,F}(\lambda_{z_0,w_0}(x)),$$

where $\Im(w_0)$ is the imaginary part of w_0.

Proof of Corollary 4.3. The non-trivial direction is 1. $\Rightarrow$ 2. Choose $w_0 \in \mathbb{C}\setminus\mathbb{R}$. Observe that $\Lambda_{1,w_0,F}(z)$ belongs to the set $\mathrm{Pos}^n_{\succeq 0}(\mathscr{K}_{w_0})$ where $\mathscr{K}_{w_0} := \mathrm{Cl}(\lambda_{1,w_0}(K))$ and $\mathrm{Cl}(\cdot)$ is the closure operator. Let $S = \{g_1,\dots,g_s\}$ be the natural description of K. Then $\mathscr{S} := \{\Lambda_{1,w_0,g_1}(z),\dots,\Lambda_{1,w_0,g_s}(z)\}$ satisfies the conditions of Corollary 4.2 and hence $\Lambda_{1,w_0,F} \in \mathcal{M}^n_{\mathscr{S}}$. Therefore

$$\left(\frac{(x-\overline{w_0})(x-w_0)}{4\cdot\mathrm{Im}(w_0)^2}\right)^{k_{w_0}} \cdot F(x) \in M^n_S,$$

where $k_{w_0} \in \mathbb{N}\cup\{0\}$ equals $k - \left\lceil \frac{\deg(F)}{2}\right\rceil$ with k being the degree of the summand of the highest degree in the expression of $\Lambda_{1,w_0,F}(z)$ as the element of $\mathcal{M}^n_{\mathscr{S}}$. $\square$

Remark 4.4. By a similar but more technical proof we can show, that Corollary 4.3.2 is true for all $w \in \mathbb{C} \setminus K$, i.e., it is true also for $w \in \mathbb{R} \setminus K$.

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