MATRIX FEJÉR-RIESZ TYPE THEOREM FOR A UNION OF AN INTERVAL AND A POINT

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Abstract. The matrix Fejér-Riesz theorem characterizes positive semidefinite matrix polynomials on the real line. In [30] this was extended to the characterization on arbitrary closed semialgebraic sets $K \subseteq \mathbb{R}$ by using matrix quadratic modules from real algebraic geometry. In the compact case there is a denominator-free characterization, while in the non-compact case denominators are needed except when K is the whole line, an unbounded interval, a union of two unbounded intervals, and according to a conjecture of [30] also when K is a union of an unbounded interval and a point or a union of two unbounded intervals and a point. In this paper, we confirm this conjecture by solving the truncated matrix-valued moment problem (TMMP) on a union of a bounded interval and a point. The presented technique for solving the corresponding TMMP can potentially be used to determine degree bounds in the positivity certificates for matrix polynomials on compact sets K [30, Theorem C].

1. Introduction

The matrix Fejér-Riesz theorem characterizes positive semidefinite (psd) matrix polynomials on the real line (see [11, 12, 23, 24, 25] for some proofs). The statement in [23, Appendix B] provides the most information, since it is shown that for every factorization of the determinant of the given psd matrix polynomial there is a corresponding factorization of the matrix polynomial. In [30] the second named author has extended a characterization to arbitrary closed semialgebraic sets $K \subseteq \mathbb{R}$ by using matrix quadratic modules from real algebraic geometry. Except for two special cases, the following characterizations were obtained:

Theorem 1.1 (Theorem C and D in [30]). *Let p be the size of matrix and $K \subseteq \mathbb{R}$ be a closed semialgebraic set.*

(1) If K is compact, then the matrix quadratic module is saturated for every $p \geq 1$ and saturated *description of K.*
(2) If $K = \mathbb{R}$ or K is a union of two unbounded intervals, then the matrix quadratic module is saturated for every $p \geq 1$ and the natural *description of K.*
(3) If K is not compact, and is not
   (i) $\mathbb{R}$ or a union of two unbounded intervals,
   (ii) a union of an unbounded interval and a point, or
   (iii) a union of two unbounded intervals and a point,

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Then the matrix preordering is NOT saturated for every $p \geq 2$ and any valid description of K.

The main technique used in [30] to establish the characterizations is induction on the size of the matrices, where the base case is the scalar result of Kuhlmann and Marshall [17, Theorem 2.2], the diagonalization method for matrix polynomials of Schmüdgen [28, §4.3] and in the compact case the elimination of the denominator by a result of Scheiderer [26, Proposition 2.7]. In the diagonalization method, the degrees of the polynomials can grow exponentially with the size of the matrices, while [26, Proposition 2.7] uses the Stone-Weierstraß theorem, in which the trace of the degrees is lost.

The *natural* description is a special case of saturated descriptions. Thus Theorem 1.1 states that except for the two unresolved cases (ii) and (iii), the natural description is the best possible in terms of saturation of matrix quadratic module and preordering. It was conjectured in [30] that for cases (ii) and (iii), the matrix preordering is saturated for the natural description and every $p \geq 1$.

When the preordering is not saturated for any valid description, there is a weaker characterization with denominators [30, Theorem D]. It turns out that by an appropriate substitution of variables, both unresolved cases (ii) and (iii) above are equivalent. Moreover, they are equivalent to the case of a union of a compact interval and a point, where the degrees in the algebraic certificate of positivity are bounded by the degree of the given psd matrix polynomial. Let us explain. After applying a suitable substitution of the variables ϕ, a union of an unbounded interval and a point K becomes a union of a compact interval and a point $\phi(K)$, while the given matrix polynomial F, psd on K, becomes a matrix polynomial $G = \phi(F)$, psd on $\phi(K)$. By the form of the polynomials in the natural description of K, it is clear the bounds on the degrees of the summands in the representation of F as an element of the matrix preordering are bounded by the degree of F and hence the same is true for the corresponding representation of G on $\phi(K)$. Conversely, any representation of G as an element of the matrix preordering corresponding to the natural description of $\phi(K)$, yields a representation of F as an element of the matrix preordering corresponding to the natural description of K only if the degrees of summands are bounded by the degree of G. Otherwise non-trivial denominators would be present in the representation. See also Remark 1.5 for more details.

In this article we answer the conjecture above affirmatively and conclude all univariate cases of matrix Fejér-Riesz theorem by studying the dual side of solving the corresponding univariate truncated matrix-valued moment problem on K (K–TMMP). The univariate K–TMMP where K is an interval (bounded or unbounded) is well understood and solved using tools from various fields, such as operator theory, complex analysis and linear algebra [2, 8, 7, 9, 3]. Very recently, the study of operator-valued and multivariate TMMPs has attracted interest among several authors [16, 13, 22, 14, 15, 20, 21, 5]. In particular, by [21, Corollary 5.2], the K–TMMP on a compact set K admits a solution if only if the corresponding linear functional is positive on every matrix polynomial of bounded degree, psd on K. Usually, this duality is exploited in the scalar case in the direction ($\Leftarrow$) using certificates of positivity for polynomials. However, our motivation is to use the implication ($\Rightarrow$) and obtain matricial sum-of-squares certificates by solving the K–TMMP for a given K.

1.1. **Main results and reader's guide.** Given a $p \times p$ univariate matrix polynomial $H(\mathtt{x}) = \sum_{i=0}^n H_i \mathtt{x}^i$, where H_i are real matrices, we call an expression of the form $H(\mathtt{x})^T H(\mathtt{x})$ a **symmetric square**, where $H(\mathtt{x})^T = \sum_{i=0}^n H_i^T \mathtt{x}^i$. An algebraic certificate of positivity for matrix polynomials on the union of a bounded interval and a point is indeed the best possible in terms of degree bounds, which solves the conjecture from [30]:

Theorem 1.2. *Let $K = \{a\} \cup [b, c]$, $a, b, c \in \mathbb{R}$, $a < b < c$, and $F(\mathtt{x}) = \sum_{i=0}^n F_i \mathtt{x}^i$ be a $p \times p$ matrix polynomial with coefficients F_i being real symmetric matrices and F_n a nonzero matrix. Assume that $F(x)$ is positive semidefinite for every $x \in K$. Then there are $p \times p$ real matrix polynomials $G_0(\mathtt{x}), G_1(\mathtt{x}), G_2(\mathtt{x}), G_3(\mathtt{x})$, each being a sum of at most two symmetric squares, such that:*

(1) If n is even, then

$$F(\mathtt{x}) = G_0(\mathtt{x}) + (\mathtt{x} - a)(\mathtt{x} - b)G_1(\mathtt{x}) + (\mathtt{x} - a)(c - \mathtt{x})G_2(\mathtt{x}),$$

where the degree of each summand is bounded above by the degree of F.

(2) If n is odd, then

$$\begin{aligned} F(\mathtt{x}) = (\mathtt{x} - a)G_0(\mathtt{x}) + (c - \mathtt{x})G_1(\mathtt{x}) + (\mathtt{x} - a)^2(\mathtt{x} - b)G_2(\mathtt{x}) \\ + (\mathtt{x} - a)(\mathtt{x} - b)(c - \mathtt{x})G_3(\mathtt{x}), \end{aligned}$$

where the degree of each summand is bounded above by the degree of F.

Theorem 1.2 implies the following positivity certificate for K being a union of a point and an unbounded interval, which in particular proves that the matrix preordering generated by the natural description of K is saturated.

Theorem 1.3. *Let $K = \{a\} \cup [b, \infty)$, $a, b \in \mathbb{R}$, $a < b$, and $F(\mathtt{x}) = \sum_{i=0}^n F_i \mathtt{x}^i$ be a $p \times p$ matrix polynomial with coefficients F_i being real symmetric matrices and F_n a nonzero matrix. Assume that $F(x)$ is positive semidefinite for every $x \in K$. Then there exist $p \times p$ real matrix polynomials $G_0(\mathtt{x}), G_1(\mathtt{x}), G_2(\mathtt{x}), G_3(\mathtt{x})$, each being a sum of at most two symmetric squares, such that*

$$F(\mathtt{x}) = G_0(\mathtt{x}) + (\mathtt{x} - a)G_1(\mathtt{x}) + (\mathtt{x} - a)(\mathtt{x} - b)G_2(\mathtt{x}) + (\mathtt{x} - a)^2(\mathtt{x} - b)G_3(\mathtt{x}),$$

where the degree of each summand is bounded above by the degree of F.

When K is a union of a point and two unbounded intervals the positivity certificate is the following, implying that the matrix preordering generated by the natural description of K is saturated.

Theorem 1.4. *Let $K = (-\infty, a] \cup \{b\} \cup [c, \infty)$, $a, b, c \in \mathbb{R}$, $a < b < c$, and $F(\mathtt{x}) = \sum_{i=0}^n F_i \mathtt{x}^i$ be a $p \times p$ matrix polynomial with coefficients F_i being real symmetric matrices and F_n a nonzero matrix. Assume that $F(x)$ is positive semidefinite for every $x \in K$. Then n is even and there exist $p \times p$ real matrix polynomials $G_0(\mathtt{x}), G_1(\mathtt{x}), G_2(\mathtt{x}), G_3(\mathtt{x})$, each being a sum of at most two symmetric squares, such that*

$$\begin{aligned} F(\mathtt{x}) = G_0(\mathtt{x}) + (\mathtt{x} - a)(\mathtt{x} - b)G_1(\mathtt{x}) + (\mathtt{x} - b)(\mathtt{x} - c)G_2(\mathtt{x}) + \\ + (\mathtt{x} - a)(\mathtt{x} - b)^2(\mathtt{x} - c)G_3(\mathtt{x}), \end{aligned}$$

where the degree of each summand is bounded above by the degree of F.

Remark 1.5. Theorems 1.2–1.4 are equivalent to each other if a suitable substitution of the variables is applied. Let us assume, for example, that F is a matrix polynomial, psd on $K_1 = \{a\} \cup [b, \infty)$, $a < b$. Define $\phi(x) = -\frac{1}{x-a+1}$ and note that $\phi(K) = \{-1\} \cup [-\frac{1}{b-a+1}, 0)$. Then the polynomial $G(x) := (-x)^{\deg F} F(-\frac{1}{x}+a-1)$ is psd on $\overline{\phi(K)} = \{-1\} \cup [-\frac{1}{b-a+1}, 0]$. Using the positivity certificate for G given by Theorem 1.2 and the equality

$$F(x) = (x-a+1)^{\deg G} G\big(-\frac{1}{x-a+1}\big),$$

we obtain a positivity certificate for F given by Theorem 1.3. Theorem 1.2 thus implies Theorem 1.3. The other implications follow by similar reasoning.

The paper is organized as follows. In Section 2 we introduce the notation and some preliminary results. We state the truncated matrix-valued moment problem and establish the connection with positive matrix polynomials (see Proposition 2.2). In Section 3 we extend the flat extension theorem for a closed semialgebraic set K in $\mathbb{R}$ from the scalar to the matrix case (see Theorem 3.1). In Section 4 we prove the main technical result, Proposition 4.1, which allows us to manipulate the value of the matrix moment of degree 0, and is used in Section 5 to solve the truncated matrix-valued moment problem on the union of a compact interval and a point (see Theorem 5.1). This result together with Proposition 2.2 then implies Theorem 1.2.

We mention at the end that the approach presented in this paper could be used to provide degree bounds in the positivity certificates for matrix polynomials on a compact set K (see [30, Theorem C]), if Theorem 5.1 can be extended to a given K.

2. Notation and preliminaries

Let $d \in \mathbb{N} \cup \{0\}$, $p \in \mathbb{N}$. We denote by $M_p(\mathbb{R})$ the set of $p \times p$ real matrices and by $\mathbb{S}_p$ its subset of symmetric matrices. For $A \in \mathbb{S}_p$ the notation $A \succeq 0$ (resp. $A \succ 0$) means A is positive semidefinite (psd) (resp. positive definite (pd)). We use $\mathbb{S}_p^{\succeq 0}$ for the subset of all psd matrices in $\mathbb{S}_p$. We write $0_{p\times p}$ for the $p \times p$ zero matrix. Let tr denote the trace and $\langle \cdot, \cdot \rangle$ the usual Frobenius inner product on $M_p(\mathbb{R})$, i.e., $\langle A, B\rangle = \mathrm{tr}(A^T B)$.

Let $\mathbb{R}[\mathtt{x}]_{\leq d}$ stand for the vector space of univariate polynomials of degree at most d. Let $M_p(\mathbb{R}[\mathtt{x}])$ be a set of all $p \times p$ matrix polynomials over $\mathbb{R}[\mathtt{x}]$. We say $F(\mathtt{x}) \in M_p(\mathbb{R}[\mathtt{x}])$ is **symmetric** if $F(\mathtt{x}) = F(\mathtt{x})^T$. We write $\mathbb{S}_p(\mathbb{R}[\mathtt{x}])$ for the set of all symmetric matrix polynomials from $M_p(\mathbb{R}[\mathtt{x}])$.

2.1. **Positive matrix polynomials and matrix quadratic module.**

A matrix polynomial $F(\mathtt{x}) \in \mathbb{S}_p(\mathbb{R}[\mathtt{x}])$ is positive semidefinite in $x_0 \in \mathbb{R}$ if $v^T F(x_0) v \geq 0$ for every $v \in \mathbb{R}^p$. We denote by $\sum M_p(\mathbb{R}[\mathtt{x}])^2$ the set of sums of symmetric squares $H(\mathtt{x})^T H(\mathtt{x})$, where $H(\mathtt{x}) \in M_p(\mathbb{R}[\mathtt{x}])$. Note that by the matrix Fejér-Riesz theorem, $\sum M_p(\mathbb{R}[\mathtt{x}])^2$ is equal to the set of all sums of at most two symmetric squares.

Let $K \subseteq \mathbb{R}$ be a closed nonempty set. We denote by $\mathrm{Pos}_d^{(p)}(K)$ the set of $p \times p$ matrix polynomials over $\mathbb{R}[\mathtt{x}]_{\leq d}$, psd on K, i.e.,

$$\mathrm{Pos}_d^{(p)}(K) := \{F \in \mathbb{S}_p(\mathbb{R}[\mathtt{x}]_{\leq d})\colon F(x) \succeq 0 \text{ for every } x \text{ in } K\}.$$

We call every $F \in \mathrm{Pos}_d^{(p)}(K)$ a K–**positive matrix polynomial**.

Let $S \subset \mathbb{R}[\mathtt{x}]$ be a finite set. We denote by

$$K_S := \{x \in \mathbb{R}\colon f(x) \geq 0 \text{ for each } f \in S\}$$

the semialgebraic set generated by S. The **matrix quadratic module generated by** S in $M_p(\mathbb{R}[\mathtt{x}])$ is defined by

$$\mathrm{QM}_S^{(p)} \;:=\; \Big\{\sum_{s\in S}\sigma_s s \colon \sigma_s \in \sum M_p(\mathbb{R}[\mathtt{x}])^2 \text{ for each } s\Big\}.$$

For $d \in \mathbb{N}\cup\{0\}$ we define the set

$$\mathrm{QM}_{S,d}^{(p)} := \Big\{\sum_{s\in S}\sigma_s s \colon \sigma_s \in \sum M_p(\mathbb{R}[\mathtt{x}])^2 \text{ and } \deg(\sigma_s s) \le d \text{ for each } s\Big\}.$$

We call $\mathrm{QM}_{S,d}^{(p)}$ the **d-th truncated matrix quadratic module generated by** S.

Proposition 2.1. *If K_S has a nonempty interior, then $\mathrm{QM}_{S,d}^{(p)}$ is closed in $\mathbb{S}_p(\mathbb{R}[x]_{\le d})$.*

Proof. The proof is analogous to the proof of the scalar result [19, Theorem 3.49], i.e., $p = 1$. □

2.2. **Matrix measures.** Let $K \subseteq \mathbb{R}$ be a closed set and $\mathrm{Bor}(K)$ the Borel σ-algebra of K. We call

$$\mu = (\mu_{ij})_{i,j=1}^p : \mathrm{Bor}(K) \to \mathbb{S}_p$$

a $p\times p$ **Borel matrix-valued measure** supported on K (or **positive $\mathbb{S}_p$-valued measure** for short) if:

(1) $\mu_{ij} : \mathrm{Bor}(K) \to \mathbb{R}$ is a real measure for every $i, j = 1, \ldots, p$ and
(2) $\mu(\Delta) \succeq 0_{p\times p}$ for every $\Delta \in \mathrm{Bor}(K)$.

A point $x \in K$ is called an atom of μ if $\mu(\{x\}) \ne 0_{p\times p}$. A $\mathbb{S}_p$-valued measure μ is **finitely atomic**, if there exists a finite set $M \in \mathrm{Bor}(K)$ such that $\mu(K \setminus M) = 0_{p\times p}$ or equivalently, $\mu = \sum_{j=1}^k A_j\delta_{x_j}$ for some $k \in \mathbb{N}$, $x_j \in K$, $A_j \in \mathbb{S}_p^{\succeq 0}$, where δ_{x_j} stands for the Dirac measure.

We denote by $\mathcal{M}_p(K, \mathrm{Bor}(K))$ the set of all $\mathbb{S}_p$-valued measures and by $\mathcal{M}_p^{(\mathrm{fa})}(K, \mathrm{Bor}(K))$ the set of all finitely atomic $\mathbb{S}_p$-valued measures.

Let $\mu \in \mathcal{M}_p(K, \mathrm{Bor}(K))$ and $\tau := \mathrm{tr}(\mu) = \sum_{i=1}^p \mu_{ii}$ denote its trace measure. A polynomial $f \in \mathbb{R}[x]_{\le n}$ is μ-integrable if $f \in L^1(\tau)$. We define its integral by

$$\int_K f\, d\mu = \Big(\int_K f\, d\mu_{ij}\Big)_{i,j=1}^p.$$

2.3. **Truncated matrix-valued moment problem.** Let $n, p \in \mathbb{N}$. Given a linear mapping

$$L : \mathbb{R}[x]_{\le n} \to \mathbb{S}_p, \tag{2.1}$$

the **truncated matrix-valued moment problem** supported on K **(K–TMMP)** asks to characterize the existence of a $\mathbb{S}_p$-valued measure $\mu \in \mathcal{M}(K, \mathrm{Bor}(K))$ such that

$$L(f) = \int_K f\, d\mu \quad \text{for every} \quad f \in \mathbb{R}[x]_{\le n}. \tag{2.2}$$

If such a measure exists, we say that L is **a K–matrix moment functional on** $\mathbb{R}[x]_{\le n}$ **(K–mmf)** and μ is its **K–representing matrix–valued measure (K–rmm).** We denote by $\mathcal{M}_L$ the set of all K–rmm$s$ for L.

Equivalently, one can define L as in (2.1) by a sequence of its values on monomials x^i, $i = 0, \ldots, n$. Throughout the paper we will denote these values by $\Gamma_i := L(x^i)$. If

$$\Gamma := (\Gamma_0, \Gamma_1, \ldots, \Gamma_n) \in (\mathbb{S}_p)^{n+1} \tag{2.3}$$

is given, then we denote the corresponding linear mapping on $\mathbb{R}[\mathrm{x}]_{\leq n}$ by L_Γ and call it a **Riesz mapping of** Γ. If L_Γ is a K–mmf, we call Γ a K**–matrix moment sequence** (K**–mms**).

The connection between the K–TMMP and K–positive matrix polynomials is the following.

Proposition 2.2. *Let $n, p \in \mathbb{N}$, Γ as in* (2.3) *and K a compact set. The following statements are equivalent:*

(1) Γ is a K–matrix moment sequence.

(2) $\sum_{i=0}^{n} A_i x^i \in \mathrm{Pos}_n^{(p)}(K)$ implies that $\sum_{i=0}^{n} \mathrm{tr}(\Gamma_i A_i) \geq 0$.

Proof. The proof is verbatim the same to the proof of [8, Lemma 2.3(a)] which deals with the case $K = [0, 1]$. Here we emphasize K needs to be *compact* for the proof to work, since the set

$$\mathcal{M}(K, n, p) := \{\Gamma = (\Gamma_0, \ldots, \Gamma_n) \in (\mathbb{S}_p)^{n+1} \colon \Gamma \text{ is a } K\text{–matrix moment sequence}\}$$

needs to be closed. $\square$

Remark 2.3. In [8, Section 5, sentence 2] the authors claim results of [8, Section 2] remain true if $K = [0, 1]$ is replaced by $K = [0, \infty)$. In particular, also Proposition 2.2 should hold. However, this is not true even in the scalar case ($p = 1$). This is due to the fact that $\mathcal{M}([0, \infty), n, 1)$ is not closed.

2.4. **Moment matrix and localizing moment matrices.** For $m, n \in \mathbb{N}$, $m \leq \frac{n}{2}$ and Γ as in (2.3) we denote by

$$\mathcal{M}_m \equiv \mathcal{M}_m(\Gamma) = \left(\Gamma_{i+j-2}\right)_{i,j=1}^{m+1} = \begin{pmatrix} \Gamma_0 & \Gamma_1 & \Gamma_2 & \cdots & \Gamma_m \\ \Gamma_1 & \Gamma_2 & \iddots & \iddots & \Gamma_{m+1} \\ \Gamma_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \Gamma_{2m-1} \\ \Gamma_m & \Gamma_{m+1} & \cdots & \Gamma_{2m-1} & \Gamma_{2m} \end{pmatrix}$$

the m**–th truncated moment matrix.** For $0 \leq i, j \leq \frac{n}{2}$ we also write

$$\mathbf{v}_i^{(j)} = \left(\Gamma_{i+r-1}\right)_{r=1}^{j+1} = \begin{pmatrix} \Gamma_i \\ \Gamma_{i+1} \\ \vdots \\ \Gamma_{i+j} \end{pmatrix} \tag{2.4}$$

For $f \in \mathbb{R}[\mathrm{x}]_{\leq n}$ an f**–localizing moment matrix** $\mathcal{H}_f$ **of** $L : \mathbb{R}[\mathrm{x}]_{\leq n} \to \mathbb{S}_p$ is a block square matrix of size $s(n, f) \times s(n, f)$, where $s(n, f) = \lfloor \frac{n - \deg f}{2} \rfloor + 1$, with the (i, j)–th entry equal to $L(fx^{i+j-2})$. Writing $\Gamma_i^{(f)} := L(fx^i)$, the **$\ell$-th truncated f–localizing matrix** is

$$\mathcal{H}_f(\ell) := \left(\Gamma^{(f)}_{i+j-2}\right)_{i,j=1}^{\ell+1} = \begin{array}{c} \\ \mathit{1} \\ X \\ X^2 \\ \vdots \\ X^\ell \end{array} \begin{array}{c} \begin{array}{ccccc} \mathit{1} & X & X^2 & \cdots & X^\ell \end{array} \\ \begin{pmatrix} \Gamma^{(f)}_0 & \Gamma^{(f)}_1 & \Gamma^{(f)}_2 & \cdots & \Gamma^{(f)}_\ell \\ \Gamma^{(f)}_1 & \Gamma^{(f)}_2 & \iddots & \iddots & \Gamma^{(f)}_{\ell+1} \\ \Gamma^{(f)}_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \Gamma^{(f)}_{2\ell-1} \\ \Gamma^{(f)}_\ell & \Gamma^{(f)}_{\ell+1} & \cdots & \Gamma^{(f)}_{2\ell-1} & \Gamma^{(f)}_{2\ell} \end{pmatrix} \end{array}$$

For $0 \leq i, j \leq s(n,f)$ we also write

$$(f\cdot \mathbf{v})_i^{(j)} := \left(\Gamma^{(f)}_{i+r-1}\right)_{r=1}^{j+1} = \begin{pmatrix} \Gamma^{(f)}_i \\ \Gamma^{(f)}_{i+1} \\ \vdots \\ \Gamma^{(f)}_{i+j} \end{pmatrix}. \tag{2.5}$$

2.5. **Generalized Schur complements.** Let $m, n \in \mathbb{N}$ and

$$\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{n+m}(\mathbb{R}), \tag{2.6}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$. The **generalized Schur complement** of A (resp. D) in $\mathcal{M}$ is defined by

$$\mathcal{M}/A = D - CA^\dagger B \quad (\text{resp. } \mathcal{M}/D = A - BD^\dagger C),$$

where $A^\dagger$ (resp. $D^\dagger$) stands for the Moore-Penrose inverse of A (resp. D) [32].

For a matrix M we denote by $\mathcal{C}(M)$ its column space, i.e., the linear span of the columns of M.

A characterization of psd 2×2 block matrices in terms of generalized Schur complements is the following.

Theorem 2.4 ([1])**.** *Let $n, m \in \mathbb{N}$ and*

$$\mathcal{M} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \in M_{n+m}(\mathbb{R}),$$

where $A \in \mathbb{S}_n$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{S}_m$. Then:

(1) The following conditions are equivalent:
  - *(a) $\mathcal{M} \succeq 0$.*
  - *(b) $C \succeq 0$, $\mathcal{C}(B^T) \subseteq \mathcal{C}(C)$ and $\mathcal{M}/C \succeq 0$.*
  - *(c) $A \succeq 0$, $\mathcal{C}(B) \subseteq \mathcal{C}(A)$ and $\mathcal{M}/A \succeq 0$.*

(2) If $\mathcal{M} \succeq 0$, then $\operatorname{rank}\mathcal{M} = \operatorname{rank} A$ *if and only if $\mathcal{M}/A = 0$.*

(3) If $\mathcal{M} \succeq 0$, then $\operatorname{rank}\mathcal{M} = \operatorname{rank} C$ *if and only if $\mathcal{M}/C = 0$.*

3. The K–flat extension theorem for a truncated univariate sequence

In this section we extend the abstract solution to the K–TMMP for a semialgebraic set K from the scalar [6, Theorem 5.2] (see also [18, Theorem 1.6]) to the matrix case.

Theorem 3.1. *Let $n, p \in \mathbb{N}$, K be a closed nonempty semialgebraic set such that $K = K_S$, where $S = \{g_1, \dots, g_k\} \subset \mathbb{R}[\mathbf{x}]$, and*

$$\Gamma \equiv \Gamma^{(2n)} = (\Gamma_0, \Gamma_1, \dots, \Gamma_{2n}) \in (\mathbb{S}_p)^{2n+1}$$

be a given sequence. Let $v_j := \lceil \frac{\deg g_j}{2} \rceil$ and $v := \max(\max_j v_j, 1)$. Then the following statements are equivalent:

1. *There exists a $(\operatorname{rank} \mathcal{M}_{n-v})$–atomic K–representing matrix measure μ for Γ.*
2. *The following statemets hold:*
   (a) *$\mathcal{M}_n \succeq 0$.*
   (b) *$\mathcal{H}_{g_j}(n - v_j) \succeq 0$ for $j = 1, \dots, k$.*
   (c) *$\operatorname{rank} \mathcal{M}_{n-v} = \operatorname{rank} \mathcal{M}_n$.*

Moreover, μ has $\operatorname{rank} \mathcal{M}_{n-v} - \operatorname{rank} \mathcal{H}_{g_j}(n - v)$ atoms $x \in \mathbb{R}$ that satisfy $g_j(x) = 0$.

Proof. The nontrivial implication of the theorem is (2) ⇒ (1). Let $r := \operatorname{rank} \mathcal{M}_{n-v}$. By [3, Theorem 2.7.9], there is $C \in \mathbb{R}^{p\times r}$ and a $r \times r$ diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_r)$ such that $\Gamma_i = CD^iC^T$ for $i = 0, \dots, 2n$. It remains to show that $d_1, \dots, d_r \in K$. Let

$$V_l = \begin{pmatrix} C^T & DC^T & \cdots & D^lC^T \end{pmatrix} \quad \text{for} \quad l = 0, \dots, n + 1.$$

Note that

$$\mathcal{M}_l = V_l^T V_l \quad \text{for} \quad l = 0, \dots, n. \tag{3.1}$$

By assumption (2c), it follows that $\operatorname{rank} V_l = r$ for $l = n - v, \dots, n$. Further, for each g_j, $j = 1, \dots, k$, we have that

$$0 \preceq \mathcal{H}_{g_j}(n - v_j) = V_{n-v_j}^T g_j(D) V_{n-v_j} = V_{n-v_j}^T \operatorname{diag}(g_j(d_i))_{i=1}^r V_{n-v_j}. \tag{3.2}$$

Since $\operatorname{rank} V_{n-v_j} = r$, it follows from (3.2) that for each j and each i, $g_j(d_i) \geq 0$. This proves that $d_j \in K$ for each j. Denoting the i–th column of C by $\mathbf{c}_i$, we have that the K–rmm for Γ is equal to $\sum_{i=1}^r \mathbf{c}_i \mathbf{c}_i^T \delta_{d_i}$. This concludes the proof of the implication (2) ⇒ (1).

Let us prove the moreover part. Replacing v_j with v in (3.2) we see that

$$\begin{aligned} \operatorname{rank} \mathcal{H}_{g_j}(n - v) &= r - |\{d_i \colon g_j(d_i) = 0 \text{ and } i \in \{1, \dots, r\}\}| \\ &= \operatorname{rank} \mathcal{M}_{n-v} - |\{d_i \colon g_j(d_i) = 0 \text{ and } i \in \{1, \dots, r\}\}|, \end{aligned}$$

which implies the moreover part. □

Remark 3.2.
1. Note that in Theorem 3.1, $v > \max_j v_j$ iff $v_j = 0$ for every j. But then each g_j is a nonnegative constant (since $K \neq \emptyset$) and $K = \mathbb{R}$. Then Theorem 3.1 is [3, Theorem 2.7.9].
2. A d–variate version of Theorem 3.1 for $K = \mathbb{R}^d$ and $S = \{1\}$ is [22, Proposition 4.3] (see also [15, Theorem 6.2] and [21, Theorem 4.3]).

4. Coflatness implies flatness

Let $n \in \mathbb{N}$, $L : \mathbb{R}[\mathtt{x}]_{\leq n} \to \mathbb{S}_p$ be a linear mapping and $f \in \mathbb{R}[\mathtt{x}]_{\leq n}$. We call:

(1) A moment matrix $\mathcal{M}_{\lfloor \frac{n}{2} \rfloor}$ of L **coflat**, if $\operatorname{rank} \mathcal{M}_{\lfloor \frac{n}{2} \rfloor} = \operatorname{rank} \mathcal{H}_{\mathtt{x}^2}(\lfloor \frac{n}{2} \rfloor - 1)$.

(2) An f–localizing moment matrix $\mathcal{H}_f$ **coflat**, if $\operatorname{rank} \mathcal{H}_f = \operatorname{rank} \mathcal{H}_{\mathtt{x}^2 f}$.

The main result of this section, Proposition 4.1, states that for a compact set $K \subseteq \mathbb{R}$ with $0 = \min K$, under the positivity assumptions, the coflatness of the moment matrix (resp. the $(\max K - \mathtt{x})$–localizing matrix) implies its flatness. This will be essentially used in the solution to the TMMP for a union of an interval and a point, since the coflatness can be easily achieved by manipulating $L(1)$.

Proposition 4.1. *Let K be a bounded closed semialgebraic set with $\min K = 0$ and $\max K = c$, $c > 0$. Let $n \in \mathbb{N}$, $p \in \mathbb{N}$ and $\Gamma = (\Gamma_0, \ldots, \Gamma_n) \in (\mathbb{S}_p)^{n+1}$ be a sequence with the Riesz mapping L_Γ. Assume that $\mathcal{M}_m$ is positive semidefinite. If:*

(1) $n = 2m$, $\mathcal{H}_{\mathtt{x}(c-\mathtt{x})}(m-1)$ is positive semidefinite and $\mathcal{M}_m$ is coflat, then

$$\operatorname{rank} \mathcal{M}_m = \operatorname{rank} \mathcal{M}_{m-1}. \tag{4.1}$$

(2) $n = 2m+1$, $\mathcal{H}_{\mathtt{x}}(m)$ and $\mathcal{H}_{c-\mathtt{x}}(m)$ are positive semidefinite, and $\mathcal{H}_{c-\mathtt{x}}$ is coflat, then

$$\operatorname{rank} \mathcal{H}_{c-\mathtt{x}}(m) = \operatorname{rank} \mathcal{H}_{c-\mathtt{x}}(m-1). \tag{4.2}$$

In the proof we will need a few lemmas.

Lemma 4.2. *Let $\Gamma \equiv (\Gamma_0, \Gamma_1, \ldots, \Gamma_n) \in \big(\mathbb{S}_p\big)^{n+1}$ be a sequence, $f \in \mathbb{R}[\mathtt{x}]_{\leq n}$ a polynomial and $m \in \mathbb{Z}_+$, $m \leq \frac{1}{2}(n - \deg f)$. Then*

$$B_m(t)\mathcal{H}_f(m)\big(B_m(t)\big)^T = \mathcal{H}_{(\mathtt{x}-t)^2 f(\mathtt{x})}(m-1), \tag{4.3}$$

where

$$B_m(t) := \begin{pmatrix} -tI_p & I_p & 0 & \cdots & 0 \\ 0 & -tI_p & I_p & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -tI_p & I_p \end{pmatrix} \in M_{m,m+1}(M_p)$$

is a $m \times (m+1)$ block matrix with blocks of size $p \times p$.

Proof. First we will show that

$$(B_m(t)\mathcal{H}_f(m))_{ij} = \Gamma_{i+j-2}^{((\mathtt{x}-t)f(\mathtt{x}))} \quad \text{for} \quad 1 \leq i \leq m,\ 1 \leq j \leq m+1, \tag{4.4}$$

where $(B_m(t)\mathcal{H}_f(m))_{ij}$ stands for the matrix block in the i–th row and j–th column of $B_m(t)\mathcal{H}_f(m)$. By definition of $\Gamma_i^{((\mathtt{x}-t)f)}$, to establish (4.4) we need to prove that

$$(B_m(t)\mathcal{H}_f(m))_{ij} = L\big((\mathtt{x}-t)f(\mathtt{x})\mathtt{x}^{i+j-2}\big) \quad \text{for each} \quad 1 \leq i \leq m,\ 1 \leq j \leq m+1, \tag{4.5}$$

where L is the Riesz functional of Γ. Let $(f \cdot \mathbf{v})_i^{(j)}$ be as in (2.5). We have that:

$$\begin{aligned} (B_m(t)\mathcal{H}_f(m))_{ij} &= \Big(B_m(t) \Big((f \cdot \mathbf{v})_0^{(m)} \quad \cdots (f \cdot \mathbf{v})_{j-1}^{(m)} \quad \cdots \quad (f \cdot \mathbf{v})_m^{(m)} \quad \Big)\Big)_{ij} \\ &= \sum_{\ell=1}^{m+1} \big(B_m(t)\big)_{i\ell} \Gamma_{\ell+j-2}^{(f)} \end{aligned}$$

$$
\begin{aligned}
&= -t\Gamma^{(f)}_{i+j-2} + \Gamma^{(f)}_{i+j-1} \\
&= L\big(-tf(\mathtt{x})\mathtt{x}^{i+j-2} + f(\mathtt{x})\mathtt{x}^{i+j-1}\big) \\
&= L\Big((\mathtt{x}-t)f(\mathtt{x})\mathtt{x}^{i+j-2}\Big),
\end{aligned}
$$

which is (4.5).

Finally, (4.3) follows by the following computation:

$$
\begin{aligned}
B_m(t)\mathcal{H}_f(m)\big(B_m(t)\big)^T \underbrace{=}_{(4.5)} & (\Gamma^{((\mathtt{x}-t)f(\mathtt{x}))}_{i+j-2})_{\substack{1\le i\le m,\\ 1\le j\le m+1}} \big(B_m(t)\big)^T \\
&= (-t\Gamma^{((\mathtt{x}-t)f(\mathtt{x}))}_{i+j-2} + \Gamma^{((\mathtt{x}-t)f(\mathtt{x}))}_{i+j-1})_{\substack{1\le i\le m,\\ 1\le j\le m}} \\
&= (\Gamma^{(-t(\mathtt{x}-t)f(\mathtt{x}))}_{i+j-2} + \Gamma^{(\mathtt{x}(\mathtt{x}-t)f(\mathtt{x}))}_{i+j-2})_{\substack{1\le i\le m,\\ 1\le j\le m}} \\
&= (\Gamma^{((\mathtt{x}-t)^2f(\mathtt{x}))}_{i+j-2})_{\substack{1\le i\le m,\\ 1\le j\le m}} \\
&= \mathcal{H}_{(\mathtt{x}-t)^2f(\mathtt{x})}(m-1),
\end{aligned}
$$

which concludes the proof of the lemma. □

The following lemma states that the rank of a matrix is a monotone function on the set of psd matrices with respect to the usual Loewner order, i.e., $A \succeq B$ if and only if $A - B \succeq 0$.

Lemma 4.3. *Let $n \in \mathbb{N}$ and $A, B \in \mathbb{S}_n(\mathbb{R})$ such that $A \succeq B \succeq 0$. Then $\mathcal{C}(B) \subseteq \mathcal{C}(A)$ and* $\operatorname{rank} A \geq \operatorname{rank} B$.

Proof. Since for every $X \in \mathbb{S}_n(\mathbb{R})$ it holds that $\mathcal{C}(X)$ is an orthogonal complement of $\ker X$ with respect to the usual Euclidean inner product, it suffices to prove that $\ker A \subseteq \ker B$. Let us take $v \in \ker A$. From $0 = v^TAv \geq v^TBv \geq 0$, it follows that $v^TBv = 0$. By $0 = v^TBv = v^TB^{\frac12}B^{\frac12}v = \|B^{\frac12}v\|^2$, it follows that $v \in \ker B^{\frac12}$ and thus $v \in \ker B$. □

Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. First we prove (1). The rank equality (4.1) will follow once we establish the following two equalities:

$$
\mathcal{C}(\mathcal{M}_{m-1}) = \mathcal{C}(\mathcal{H}_{\mathtt{x}}(m-1)) = \mathcal{C}(\mathcal{H}_{\mathtt{x}^2}(m-1)). \tag{4.6}
$$

From

$$
0 \preceq \mathcal{M}_m = \begin{pmatrix} \mathcal{M}_{m-1} & \mathbf{v}_m^{(m-1)} \\ (\mathbf{v}_m^{(m-1)})^T & \Gamma_{2m} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_0^{(m-1)} & \mathcal{H}_{\mathtt{x}}(m-1) \\ \Gamma_m & (\mathbf{v}_{m+1}^{(m-1)})^T \end{pmatrix},
$$

it follows by Theorem 2.4.(1c) used for the pair $(\mathcal{M}, A) = (\mathcal{M}_m, \mathcal{M}_{m-1})$, that

$$
\mathcal{C}(\mathcal{M}_{m-1}) \supseteq \mathcal{C}(\mathcal{H}_{\mathtt{x}}(m-1)). \tag{4.7}
$$

Similary, from

$$
0 \preceq \mathcal{M}_m = \begin{pmatrix} \Gamma_0 & (\mathbf{v}_1^{(m-1)})^T \\ \mathbf{v}_1^{(m-1)} & \mathcal{H}_{\mathtt{x}^2}(m-1) \end{pmatrix},
$$

it follows by Theorem 2.4.(1b) used for the pair $(\mathcal{M}, C) = (\mathcal{M}_m, \mathcal{H}_{\mathtt{x}^2}(m-1))$, that

$$\mathcal{C}(\mathbf{v}_1^{(m-1)}) \subseteq \mathcal{C}(\mathcal{H}_{\mathtt{x}^2}(m-1)). \tag{4.8}$$

The assumption $\operatorname{rank} \mathcal{M}_m = \operatorname{rank} \mathcal{H}_{\mathtt{x}^2}(m-1)$ implies that

$$\mathcal{C}(\mathcal{M}_{m-1}) \subseteq \mathcal{C}(\mathcal{H}_{\mathtt{x}^2}(m-1)). \tag{4.9}$$

Hence, by (4.7)–(4.9) we have that

$$\mathcal{C}(\mathcal{H}_{\mathtt{x}}(m-1)) \subseteq \mathcal{C}(\mathcal{M}_{m-1}) \subseteq \mathcal{C}(\mathcal{H}_{\mathtt{x}^2}(m-1)). \tag{4.10}$$

Note that

$$\frac{1}{c}\big(\mathcal{H}_{\mathtt{x}(c-\mathtt{x})}(m-1) + \mathcal{H}_{\mathtt{x}^2}(m-1)\big) = \mathcal{H}_{\mathtt{x}}(m-1). \tag{4.11}$$

Since $\mathcal{H}_{\mathtt{x}(c-\mathtt{x})}(m-1)$ and $\mathcal{H}_{\mathtt{x}^2}(m-1)$ are both psd, it follows from (4.11) that

$$0 \preceq \mathcal{H}_{\mathtt{x}^2}(m-1) \preceq \mathcal{H}_{\mathtt{x}(c-\mathtt{x})}(m-1) + \mathcal{H}_{\mathtt{x}^2}(m-1).$$

Therefore

$$\mathcal{C}\big(\mathcal{H}_{\mathtt{x}^2}(m-1)\big) \subseteq \mathcal{C}\big(\mathcal{H}_{\mathtt{x}(c-\mathtt{x})}(m-1) + \mathcal{H}_{\mathtt{x}^2}(m-1)\big) = \mathcal{C}(\mathcal{H}_{\mathtt{x}}(m-1)), \tag{4.12}$$

where the inclusion follows by Lemma 4.3 used for $A = \mathcal{H}_{\mathtt{x}(c-\mathtt{x})}(m-1) + \mathcal{H}_{\mathtt{x}^2}(m-1)$ and $B = \mathcal{H}_{\mathtt{x}^2}(m-1)$, while the equality follows by (4.11). Now (4.12) implies that all inclusions in (4.10) are equalities, which proves (4.6) and concludes the proof of the part (1).

It remains to prove (2). We have that

$$\operatorname{rank} \mathcal{H}_{c-\mathtt{x}}(m) = \operatorname{rank} \mathcal{H}_{\mathtt{x}^2(c-\mathtt{x})}(m-1) \leq \operatorname{rank} \mathcal{H}_{\mathtt{x}(c-\mathtt{x})}(m-1), \tag{4.13}$$

where the equality follows by the assumption, while the inequality follows by the equality

$$\begin{aligned}\mathcal{H}_{\mathtt{x}(c-\mathtt{x})}(m-1) &= \frac{1}{c}\big(\mathcal{H}_{(c-\mathtt{x})^2\mathtt{x}}(m-1) + \mathcal{H}_{\mathtt{x}^2(c-\mathtt{x})}(m-1)\big)\\ &\underbrace{=}_{(4.3)} \frac{1}{c}\Big(\underbrace{B_m(c)\mathcal{H}_{\mathtt{x}}(m)\big(B_m(c)\big)^T}_{\succeq 0} + \underbrace{B_m(0)\mathcal{H}_{c-\mathtt{x}}(m)\big(B_m(0)\big)^T}_{\succeq 0}\Big)\end{aligned}$$

and Lemma 4.3. Similarly, the inequality

$$\begin{aligned}\mathcal{H}_{c-\mathtt{x}}(m-1) &= \frac{1}{c}\big(\mathcal{H}_{(c-\mathtt{x})^2}(m-1) + \mathcal{H}_{\mathtt{x}(c-\mathtt{x})}(m-1)\big)\\ &\underbrace{=}_{(4.3)} \frac{1}{c}\Big(\underbrace{B_m(c)\mathcal{M}(m)\big(B_m(c)\big)^T}_{\succeq 0} + \underbrace{\mathcal{H}_{\mathtt{x}(c-\mathtt{x})}(m-1)}_{\succeq 0}\Big)\end{aligned}$$

and Lemma 4.3 imply that

$$\operatorname{rank} \mathcal{H}_{\mathtt{x}(c-\mathtt{x})}(m-1) \leq \operatorname{rank} \mathcal{H}_{c-\mathtt{x}}(m-1). \tag{4.14}$$

The inequalities (4.13) and (4.14) imply that

$$\operatorname{rank} \mathcal{H}_{c-\mathtt{x}}(m) \leq \operatorname{rank} \mathcal{H}_{c-\mathtt{x}}(m-1),$$

which is only possible in the case of the equality, whence (4.2) holds and concludes the proof of the part (2). $\square$

5. K–TMMP for K being a union of an interval and a point

In this section we solve the K–TMMP for K as in the title of the section. Then we use this solution to prove Theorem 1.2.

Theorem 5.1. *Let $n, p \in \mathbb{N}$, $a, b, c \in \mathbb{R}$, $a < b < c$,*

$$K = K_S = \{a\} \cup [b, c],$$

where $S := \{f_1, f_2, f_3\}$ with $f_1(\mathtt{x}) = \mathtt{x} - a$, $f_2(\mathtt{x}) = (\mathtt{x} - a)(\mathtt{x} - b)$, $f_3(\mathtt{x}) = c - \mathtt{x}$, and

$$\Gamma \equiv \Gamma^{(n)} = (\Gamma_0, \Gamma_1, \ldots, \Gamma_n) \in (\mathbb{S}_p)^{n+1}$$

be a given sequence. Then the following statements are equivalent:

(1) There exists a K–representing matrix measure for Γ.

(2) There exists a finitely–atomic K–representing matrix measure for Γ.

(3) One of the following statements holds:

(a) $n = 2m$ for some $m \in \mathbb{N}$ and

$$\mathcal{M}_m \succeq 0, \quad \mathcal{H}_{f_2}(m-1) \succeq 0, \quad \textit{and} \quad \mathcal{H}_{f_1 f_3}(m-1) \succeq 0. \tag{5.1}$$

(b) $n = 2m + 1$ for some $m \in \mathbb{N}$ and

$$\mathcal{H}_{f_1}(m) \succeq 0, \quad \mathcal{H}_{f_3}(m) \succeq 0, \quad \mathcal{H}_{f_1 f_2}(m-1) \succeq 0 \quad \textit{and} \quad \mathcal{H}_{f_2 f_3}(m-1) \succeq 0. \tag{5.2}$$

Moreover, if $n = 2m$, then there is a $(\operatorname{rank} \mathcal{M}_n)$–atomic K–representing measure for Γ, while if $n = 2m + 1$, there exists at most $(\operatorname{rank} \mathcal{M}_n + p)$–atomic one.

Proof. The nontrivial implication is (3) $\Rightarrow$ (2). By applying an an affine linear transformation of variables we may assume that $a = 0$, $b = 1$ and $c > 1$. Hence, $f_1(\mathtt{x}) = \mathtt{x}$, $f_2(\mathtt{x}) = \mathtt{x}(\mathtt{x} - 1)$, $f_3(\mathtt{x}) = c - \mathtt{x}$.

First assume that $n = 2m$, $m \in \mathbb{N}$. Note that Γ_0 only appears in $\mathcal{M}_m$, but not in any of $\mathcal{H}_{\mathtt{x}(\mathtt{x}-1)}(m-1)$, $\mathcal{H}_{\mathtt{x}(c-\mathtt{x})}(m-1)$. Let us replace Γ_0 by the smallest $\widetilde{\Gamma}_0$ such that $\widetilde{\mathcal{M}}_m \succeq 0$, where $\widetilde{\mathcal{M}}_\ell$ is the moment matrix corresponding to $\widetilde{\Gamma} \equiv (\widetilde{\Gamma}_0, \Gamma_1, \ldots, \Gamma_{2\ell})$, $1 \leq \ell \leq m$. Namely, by Theorem 2.4, used for the pair $(\mathcal{M}, C) = (\widetilde{\mathcal{M}}_m, \mathcal{H}_{\mathtt{x}^2}(m-1))$, we have that

$$\widetilde{\Gamma}_0 = \begin{pmatrix} \Gamma_1 & \cdots & \Gamma_m \end{pmatrix} \left(\mathcal{H}_{\mathtt{x}^2}(m-1)\right)^{\dagger} \begin{pmatrix} \Gamma_1 \\ \cdots \\ \Gamma_m \end{pmatrix}$$

and

$$\operatorname{rank} \widetilde{\mathcal{M}}_m = \operatorname{rank} \mathcal{H}_{\mathtt{x}^2}(m-1).$$

By Proposition 4.1, we have that $\operatorname{rank} \widetilde{\mathcal{M}}_m = \operatorname{rank} \widetilde{\mathcal{M}}_{m-1}$. By Theorem 3.1, it follows that $\widetilde{\Gamma}$ has a K–rmm of the form $\sum_{i=1}^r \mathbf{c}_i \mathbf{c}_i^T \delta_{d_i}$, where $r = \operatorname{rank} \widetilde{\mathcal{M}}_m$ and $c_i \in \mathbb{R}^p$. Then $\sum_{i=1}^r \mathbf{c}_i \mathbf{c}_i^T \delta_{d_i} + (\Gamma_0 - \widetilde{\Gamma}_0)\delta_0$ is a $(\operatorname{rank} \mathcal{M}_m)$–atomic K–rmm for Γ. This proves (3a) $\Rightarrow$ (2).

Now assume that $n = 2m + 1$, $m \in \mathbb{N}$. Note that Γ_0 only appears in $\mathcal{H}_{c-\mathtt{x}}(m)$, but not in any of $\mathcal{H}_{\mathtt{x}}(m)$, $\mathcal{H}_{\mathtt{x}^2(\mathtt{x}-1)}(m-1)$, $\mathcal{H}_{\mathtt{x}(\mathtt{x}-1)(c-\mathtt{x})}(m-1)$. Let us replace Γ_0 by the smallest $\widetilde{\Gamma}_0$ such that $\widetilde{\mathcal{H}}_{c-\mathtt{x}}(m) \succeq 0$, where $\widetilde{\mathcal{H}}_{c-\mathtt{x}}(m)$ is the moment matrix corresponding to $\widetilde{\Gamma} \equiv (\widetilde{\Gamma}_0, \Gamma_1, \ldots, \Gamma_{2m+1})$. Below $\widetilde{\mathcal{M}}_\ell$ and $\widetilde{\mathcal{H}}_f(\ell)$ will refer to the moment matrix and the f–localizing moment matrix of $\widetilde{\Gamma}$, respectively. By Theorem 2.4, used for the pair

$$(\mathcal{M}, C) = (\widetilde{\mathcal{H}}_{c-\mathtt{x}}(m), \mathcal{H}_{\mathtt{x}^2(c-\mathtt{x})}(m-1)),$$

we have that

$$\widetilde{\Gamma}_0 = \frac{1}{c}\Gamma_1 + \frac{1}{c}\begin{pmatrix} c\Gamma_1 - \Gamma_2 & \cdots & c\Gamma_m - \Gamma_{m+1}\end{pmatrix}\left(\mathcal{H}_{\mathtt{x}^2(c-\mathtt{x})}(m-1)\right)^{\dagger}\begin{pmatrix} c\Gamma_1 - \Gamma_2 \\ \cdots \\ c\Gamma_m - \Gamma_{m+1}\end{pmatrix}$$

and

$$\operatorname{rank}\widetilde{\mathcal{H}}_{c-\mathtt{x}}(m) = \operatorname{rank}\mathcal{H}_{\mathtt{x}^2(c-\mathtt{x})}(m-1).$$

By Proposition 4.1, we have that $\operatorname{rank}\widetilde{\mathcal{H}}_{c-\mathtt{x}}(m) = \operatorname{rank}\widetilde{\mathcal{H}}_{c-\mathtt{x}}(m-1)$. Let $Q_0,\ldots,Q_{m-1}\in M_p(\mathbb{R})$ be such that

$$c\Gamma_m - \Gamma_{m+1} = (c\widetilde{\Gamma}_0 - \Gamma_1)Q_0 + \sum_{i=1}^{m-1}(c\Gamma_i - \Gamma_{i+1})Q_i,$$

$$c\Gamma_{m+j} - \Gamma_{m+j+1} = \sum_{i=0}^{m-1}(c\Gamma_{i+j} - \Gamma_{i+1+j})Q_i, \quad j = 1,\ldots,m. \tag{5.3}$$

Equivalently, defining

$$\begin{aligned}\widetilde{Q}_0 &:= -cQ_0,\\ \widetilde{Q}_i &:= Q_{i-1} - cQ_i \quad \text{for} \quad i = 1,\ldots,m-1,\\ \widetilde{Q}_m &:= cI + Q_{m-1},\end{aligned}$$

where I is the identity matrix, we have that

$$\Gamma_{m+1} = \widetilde{\Gamma}_0\widetilde{Q}_0 + \sum_{i=1}^{m}\Gamma_i\widetilde{Q}_i,$$

$$\Gamma_{m+j+1} = \Gamma_j\widetilde{Q}_0 + \sum_{i=1}^{m}\Gamma_{i+j}\widetilde{Q}_i, \quad j = 1,\ldots,m. \tag{5.4}$$

Let $\widetilde{\mathcal{M}}_\ell$ stand for the ℓ–th truncated moment matrix of $\widetilde{\Gamma}$. Observe that

$$\widetilde{\mathcal{M}}_m = \frac{1}{c}\left(\widetilde{\mathcal{H}}_{f_1}(m) + \widetilde{\mathcal{H}}_{f_3}(m)\right),$$

whence $\widetilde{\mathcal{M}}_m \succeq 0$. Let us define $\Gamma_{2m+2},\Gamma_{2m+3},\Gamma_{2m+4}$ by (5.4) used for $j = m+1, m+2, m+3$. Note that

$$\Gamma_{m+3+j} - \Gamma_{m+2+j} = \sum_{i=0}^{m}(\Gamma_{i+2+j} - \Gamma_{i+1+j})\widetilde{Q}_i \quad \text{for} \quad j = 0,\ldots,m+1, \tag{5.5}$$

$$c\Gamma_{m+2+j} - \Gamma_{m+3+j} = \sum_{i=0}^{m}(c\Gamma_{i+1+j} - \Gamma_{i+2+j})\widetilde{Q}_i \quad \text{for} \quad j = 0,\ldots,m+1. \tag{5.6}$$

By definition of Γ_{2m+2} we have that $\operatorname{rank}\widetilde{\mathcal{M}}_{m+1} = \operatorname{rank}\widetilde{\mathcal{M}}_m$. If $\Gamma_{2m+2} \neq \widetilde{\mathcal{M}}_{m+1}/\widetilde{\mathcal{M}}_m$, then the equality of ranks cannot hold, whence $\Gamma_{2m+2} = \widetilde{\mathcal{M}}_{m+1}/\widetilde{\mathcal{M}}_m$, Γ_{2m+2} is symmetric and $\widetilde{\mathcal{M}}_{m+1} \succeq 0$. Similarly, by (5.3) and (5.6) we have that $\operatorname{rank}\widetilde{\mathcal{H}}_{c-\mathtt{x}}(m+1) = \operatorname{rank}\widetilde{\mathcal{H}}_{c-\mathtt{x}}(m)$, whence Γ_{2m+3} is symmetric, $\widetilde{\mathcal{H}}_{c-\mathtt{x}}(m+1) \succeq 0$ and $\widetilde{\mathcal{H}}_{\mathtt{x}^2(c-\mathtt{x})}(m+1) \succeq 0$. Further, by definition of Γ_{2m+4} we have that $\operatorname{rank}\widetilde{\mathcal{M}}_{m+2} = \operatorname{rank}\widetilde{\mathcal{M}}_{m+1}$, whence Γ_{2m+4} is symmetric and $\widetilde{\mathcal{M}}_{m+2} \succeq 0$. By (5.4), it follows that $\operatorname{rank}\widetilde{\mathcal{H}}_{\mathtt{x}}(m+1) = \operatorname{rank}\widetilde{\mathcal{H}}_{\mathtt{x}}(m)$, whence $\widetilde{\mathcal{H}}_{\mathtt{x}}(m+1) \succeq 0$. Finally, by (5.5), we have that $\operatorname{rank}\widetilde{\mathcal{H}}_{\mathtt{x}^2(\mathtt{x}-1)}(m) = \operatorname{rank}\widetilde{\mathcal{H}}_{\mathtt{x}^2(\mathtt{x}-1)}(m-1)$, whence $\widetilde{\mathcal{H}}_{\mathtt{x}^2(\mathtt{x}-1)}(m) \succeq 0$. By Theorem 3.1, it follows that $\widetilde{\Gamma}$ has a K–rmm of the form $\sum_{i=1}^r \mathbf{c}_i\mathbf{c}_i^T\delta_{d_i}$, where $r = \operatorname{rank}\widetilde{\mathcal{M}}_m$ and $c_i \in \mathbb{R}^p$. Then $\sum_{i=1}^r \mathbf{c}_i\mathbf{c}_i^T\delta_{d_i} + (\Gamma_0 - \widetilde{\Gamma}_0)\delta_0$ is at most a $(\operatorname{rank}\widetilde{\mathcal{M}}_m + \operatorname{rank}\Gamma_0 - \widetilde{\Gamma}_0)$–atomic K–rmm for Γ. This proves (3b) $\Rightarrow$ (2). $\square$

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Assume the notation from Theorem 5.1 and let

$$\widetilde{S} = \begin{cases} \{1, f_2, f_1f_3\}, & \text{if } n \text{ is even},\\ \{f_1, f_3, f_1f_2, f_2f_3\}, & \text{if } n \text{ is odd}.\end{cases}$$

We have to prove that

$$\operatorname{Pos}_n^{(p)}(\{a\}\cup[b,c]) = \operatorname{QM}^{(p)}_{\widetilde{S},n}. \tag{5.7}$$

This follows by using Theorem 5.1 and Proposition 2.2 for $K = \{a\} \cup [b, c]$.

Namely, assume that $n = 2m$. Note that

$$\begin{aligned}
&\mathcal{M}_m \succeq 0\\
\Leftrightarrow\ & \langle \mathcal{M}_m, B\rangle \geq 0 \ \text{ for every } B \in \mathbb{S}^{\succeq 0}_{(m+1)p}\\
\Leftrightarrow\ & \langle \mathcal{M}_m, \widetilde{B}\widetilde{B}^T\rangle \geq 0 \ \text{ for every } \widetilde{B} = (\widetilde{B}_i)_{i=0}^m \in (M_p(\mathbb{R}))^{m+1}\\
\Leftrightarrow\ & \sum_{i,j=0}^m \operatorname{tr}(\widetilde{B}_i^T\Gamma_{i+j}\widetilde{B}_j) \geq 0 \ \text{ for every } \widetilde{B} = (\widetilde{B}_i)_{i=0}^m \in (M_p(\mathbb{R}))^{m+1}\\
\Leftrightarrow\ & \sum_{i,j=0}^m \operatorname{tr}(\Gamma_{i+j}\widetilde{B}_j\widetilde{B}_i^T) \geq 0 \ \text{ for every } \widetilde{B} = (\widetilde{B}_i)_{i=0}^m \in (M_p(\mathbb{R}))^{m+1}\\
\Leftrightarrow\ & \sum_{k=0}^n \operatorname{tr}(\Gamma_k A_k) \geq 0 \ \text{ for every } \sum_{i=0}^n A_i\mathtt{x}^i = \Big(\sum_{j=0}^m \widetilde{B}_j\mathtt{x}^j\Big)\Big(\sum_{j=0}^m \widetilde{B}_j\mathtt{x}^j\Big)^T \in M_p(\mathbb{R}[\mathtt{x}]_{\leq n})\\
\Leftrightarrow\ & \sum_{k=0}^n \operatorname{tr}(\Gamma_k A_k) \geq 0 \ \text{ for every } \sum_{i=0}^n A_i\mathtt{x}^i \in \sum M_p(\mathbb{R}[\mathtt{x}])^2.
\end{aligned} \tag{5.8}$$

where the second equivalence follows by noting that every $B \in \mathbb{S}^{\succeq 0}_{(m+1)p}$ is a sum of the form $\widetilde{B}^T\widetilde{B}$ with $\widetilde{B} \in (M_p(\mathbb{R}))^{m+1}$, while the third equivalence follows by definition of the inner product and $\mathcal{M}_m$. Similarly, for

$$f := c_2\mathtt{x}^2 + c_1\mathtt{x} + c_0 \in \{f_2, f_1f_3\},$$

we have that

$$
\begin{aligned}
&\mathcal{H}_f(m-1) \succeq 0\\
\Leftrightarrow\ &\langle \mathcal{H}_f(m-1, C\rangle \geq 0 \ \text{ for every } C \in \mathbb{S}_{mp}^{\succeq 0}\\
\Leftrightarrow\ &\langle \mathcal{H}_f(m-1), \widetilde{C}^T\widetilde{C}\rangle \geq 0 \ \text{ for every } \widetilde{C} = (\widetilde{C}_i)_{i=0}^{m-1} \in (M_p(\mathbb{R}))^m\\
\Leftrightarrow\ &\sum_{k=0}^{n-2} \operatorname{tr}(\Gamma_k^{(f)} A_k) \geq 0 \ \text{ for every } \sum_{i=0}^{n-2} A_i \mathtt{x}^i \in \sum M_p(\mathbb{R}[\mathtt{x}])^2\\
\Leftrightarrow\ &\sum_{k=0}^{n-2} \operatorname{tr}\big((\Gamma_{k+2}c_2 + \Gamma_{k+1}c_1 + \Gamma_k c_0)A_k\big) \geq 0 \ \text{ for every } \sum_{i=0}^{n-2} A_i \mathtt{x}^i \in \sum M_p(\mathbb{R}[\mathtt{x}])^2 \qquad (5.9)\\
\Leftrightarrow\ &\sum_{k=0}^{n} \operatorname{tr}(\Gamma_k \widetilde{A}_k) \geq 0 \ \text{ for every } \sum_{i=0}^{n} \widetilde{A}_i \mathtt{x}^i = f\big(\sum_{i=0}^{n-2} A_i \mathtt{x}^i\big) \ \text{ with}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \big(\sum_{i=0}^{n-2} A_i \mathtt{x}^i\big) \in \sum M_p(\mathbb{R}[\mathtt{x}])^2\\
\Leftrightarrow\ &\sum_{k=0}^{n} \operatorname{tr}(\Gamma_k \widetilde{A}_k) \geq 0 \ \text{ for every } \sum_{i=0}^{n} \widetilde{A}_i \mathtt{x}^i \in \mathrm{QM}_{\{f\},n}^{(p)},
\end{aligned}
$$

By Theorem 5.1, Proposition 2.2 and (5.8), (5.9), we have that

$$
\begin{aligned}
&\sum_{k=0}^{n} \operatorname{tr}(\Gamma_k A_k) \geq 0 \quad \text{for every } \sum_{i=0}^{n} A_i x^i \in \mathrm{Pos}_n^{(p)}(\{a\} \cup [b,c])\\
\Leftrightarrow\ &\sum_{k=0}^{n} \operatorname{tr}(\Gamma_k A_k) \geq 0 \quad \text{for every } \sum_{i=0}^{n} A_i x^i \in \mathrm{QM}_{\widetilde{S},n}^{(p)}.
\end{aligned} \tag{5.10}
$$

Since $\mathrm{QM}_{\widetilde{S},n}^{(p)}$ is closed, (5.10) implies (5.7). Indeed, if $\mathrm{QM}_{\widetilde{S},n}^{(p)} \not\subseteq \mathrm{Pos}_n^{(p)}(\{a\} \cup [b,c])$, then there is $\sum_{k=0}^{n} \widetilde{A}_k \mathtt{x}^k \in \mathrm{Pos}_n^{(p)}\big(\{a\} \cup [b,c]\big) \setminus \mathrm{QM}_{\widetilde{S},n}^{(p)}$. By the Hahn-Banach theorem there is $\widetilde{\Gamma} := (\widetilde{\Gamma}_0, \ldots, \widetilde{\Gamma}_n)$ such that $\sum_{k=0}^{n} \operatorname{tr}(\widetilde{\Gamma}_k \widetilde{A}_k) < 0$ and $\sum_{k=0}^{n} \operatorname{tr}(\widetilde{\Gamma}_k A_k) \geq 0$ for every $\sum_{i=0}^{n} A_i \mathtt{x}^i \in \mathrm{QM}_{\widetilde{S},n}^{(p)}$. But this is a contradiction with (5.10).

The proof for n of odd parity is analogous. □

Remark 5.2. (1) By analogous reasoning as in this section one can obtain an alternative proof of the $[a,b]$–TMMP and the corresponding matrix Positivstellensatz. This proof seems to be more elementary than all of the existing proofs of this case (e.g. [2, 8, 7, 9]).

(2) For the case of a compact set K which is not a union of at most three isolated points or a union of an interval and a point, it is known that the positivity assumptions on localizing matrices at polynomials from the natural description of K, introduced in [17], are only necessary, but not sufficient conditions for the existence of a K–rmm. This is due to the fact that there are positive matrix polynomials for which the degrees in the positivity certificate are not best possible (see [30, Theorem D]). However, the approach presented in this paper could provide degree bounds in the

positivity certificate on a compact set K (see [30, Theorem C]), after Theorem 3.1 is extended to these cases.

(3) The main reason why the approach in this section works is the fact that by subtracting the largest possible matrix mass in the isolated point, we obtain a coflat moment matrix in the case of even degree data and a coflat $(c - \mathtt{x})$–localizing moment matrix in the case of odd degree data. Then Proposition 4.1 implies that the manipulated sequence is K–flat and the conclusion follows by Theorem 3.1. In particular, this implies that if a K–rmm exists, there always exists a K–rmm with the largest matrix mass at the isolated point. If K has more components, more localizing matrices bound from above the mass of the atom in some chosen point, and in general we cannot achieve coflatness (for reasons of (2) above). It would be very interesting to extend the approach to such K, by possibly achieving coflatness in a more involved way and answering the question of degree bounds from (2) above.

(4) It would be interesting to extend the results of this paper from matrix polynomials to operator polynomials where the coefficients are bounded operators on a Hilbert space. The operator Fejér-Riesz theorem for $\mathbb{R}$ is true in this setting [25] and also its version for a bounded or unbounded interval [4, Proposition 3], while it fails for K, which is a single point [4, §5.1].

References

[1] A. Albert, *Conditions for positive and nonnegative definiteness in terms of pseudoinverses.* SIAM J. Appl. Math. **17** (1969), 434–440.

[2] T. Ando, *Truncated moment problems for operator*, Acta Sci. Math. (Szeged) **31** (1970) 319–334.

[3] M. Bakonyi, H.J. Woerdeman, *Matrix Completions, Moments, and Sums of Hermitian Squares*, Princeton University Press, Princeton, 2011.

[4] J. Cimprič, A. Zalar, *Moment problems for operator polynomials*, J. Math. Anal. Appl. **401** (2013), 307–316.

[5] R.E. Curto, A. Ech-charyfy, H. El Azhar, E.H. Zerouali, *The Local Operator Moment Problem on* $\mathbb{R}$, Complex Analysis and Operator Theory **19**, 25 (2025).

[6] R. Curto, L. Fialkow, *The truncated complex K-moment problem*, Trans. Amer. Math. Soc. **352** (2000), 2825–2855.

[7] A.E. Choque Rivero, Y.M. Dyukarev, B. Fritzsche, B. Kirstein, *A truncated matricial moment problem on a finite interval*, In: Interpolation, Schur functions and moment problems. Oper. Theory Adv. Appl. 165, 121–173.

[8] H. Dette, W.J. Studden, *Matrix measures, moment spaces and Favard's theorem for the interval* $[0, 1]$ *and* $[0, \infty)$. Linear Algebra Appl. **345** (2002), 169–193.

[9] Y.M. Dyukarev, B. Fritzsche, B. Kirstein, B., C. Mädler, *On truncated matricial Stieltjes type moment problems*, Complex Anal. Oper. Theory **4** (2010), 905–951.

[10] L. Fialkow, *Positivity, extensions and the truncated complex moment problem*, Contemporary Math. **185** (1995), 133–150.

[11] I.T. Gohberg, M.G. Krein, *A system of integral equation on a semiaxis with kernels depending on different arguments*, Uspekhi matemat. nauk **13** (1958), 3–72.

[12] V.A. Jakubovič, *Factorization of symmetric matrix polynomials*, Dokl. Akad. Nauk **194** (1970), 532–535.

[13] D. Kimsey, *The operator-valued generalization of Tchakaloff's theorem*, J. Funct. Anal. **266(3)** (2014), 1170–1784.

[14] D. Kimsey, *The matricial subnormal completion problem.* Linear Algebra Appl. **645** (2022), 170–193.

[15] D. Kimsey, M. Trachana, *On a solution of the multidimensional truncated matrix-valued moment problem*, Milan J. Math. **90** (2022), 17–101.

[16] D.P. Kimsey, H.J. Woerdeman, *The truncated matrix-valued K-moment problem on $\mathbb{R}^d$, $\mathbb{C}^d$, $\mathbb{T}^d$*, Trans. Amer. Math. Soc. **365(10)** (2013), 5393–5430.

[17] S. Kuhlmann, M. Marshall, *Positivity, sums of squares and the multidimensional moment problem*, Trans. Amer. Math. Soc. **354** (2002), 4285–4301.

[18] M. Laurent, *Revising two theorems of Curto and Fialkow on moment matrices*, Proc. Amer. Math. Soc. **133** (2005), 2965–2976.

[19] M. Laurent, *Sums of squares, moment matrices and optimization over polynomials*, in: Emerging Applications of Algebraic Geometry, IMA Vol. Math. Appl., 149, Springer-Verlag.

[20] C. Mädler, K. Schmüdgen, *On the Truncated Matricial Moment Problem. I*, J. Math. Anal. Appl. **540** (2024), 36 pp.

[21] C. Mädler, K. Schmüdgen, *On the Truncated Matricial Moment Problem. II*, Linear Algebra Appl. **649** (2024), 63–97.

[22] B. Mourrain, K. Schmüdgen, *Flat extensions in $*$-algebras*, Proc. Am. Math. Soc. **144(11)** (2016), 4873–4885.

[23] V.M. Popov, *Hyperstability of control systems*, Springer-Verlag, Berlin, 1973.

[24] M. Rosenblatt, *A multidimensional prediction problem*, Ark. Mat. vol. **3** (1958), 407–424.

[25] M. Rosenblum, *Vectorial Toeplitz operators and the Fejér-Riesz theorem*, J. Math. Anal. Appl. **23** (1968), 139-147.

[26] C. Scheiderer, *Sums of squares on real algebraic surfaces*, Manuscr. Math. **119** (2006), 395–410.

[27] K. Schmüdgen, *On a generalization of the classical moment problem*, J. Math. Anal. Appl. **125** (1987), 461–470.

[28] K. Schmüdgen, *Noncommutative real algebraic geometry - some concepts and first ideas.* in: Emerging applications of algebraic geometry, IMA Vol. Math. Appl., 149, Springer, New York, 2009, pp. 325–350.

[29] K. Schmüdgen, *The moment problem.* Graduate Texts in Mathematics 277, Springer, Cham, 2017.

[30] A. Zalar, *Matrix Fejér-Riesz theorem with gaps*, J. Pure Appl. Algebra **220** (2016), 2533-2548.

[31] A. Zalar. *The truncated moment problem on the union of parallel lines.* Linear Algebra Appl. **649** (2022), 186–239.

[32] F. Zhang, *The Schur Complement and Its Applications.* Springer-Verlag, New York, 2005.

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