THE MATRICIAL UNIVARIATE RATIONAL TRUNCATED MOMENT PROBLEM

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ABSTRACT. We solve the matricial univariate rational truncated moment problem on the real line and on a half-line. We give explicit necessary and sufficient conditions for the existence of a positive matrix-valued representing measure and show that every solvable problem admits a finitely atomic representing measure whose total atomic multiplicity equals the rank of the associated moment matrix. The rational problem is reduced to an ordinary matricial moment problem with the additional requirement that the representing measure avoid the real poles of the rational data. The main new ingredient is a simultaneous prescribed-node result: the smallest attainable multiplicities at finitely many prescribed points can be realized by a single minimal representing measure. Our proofs are constructive and use flat extensions, block-column relations, and localizing conditions.

1. INTRODUCTION

Let $i, j \in \mathbb{Z}$ with $i \leq j$ and $\ell \in \mathbb{N}$, where $\mathbb{N} := \{1, 2, \ldots\}$. We use the notation

$$[i; j] := \{i, i+1, \ldots, j\} \qquad \text{and} \qquad [\ell] := [1; \ell] = \{1, 2, \ldots, \ell\}, \quad [0] := \emptyset.$$

For $k \in \mathbb{N} \cup \{0\}$ and $p \in \mathbb{N}$, let $\mathbb{R}[x]_{\leq k}$ denote the vector space of real univariate polynomials of degree at most k, and let $\mathbb{S}_p(\mathbb{R})$ denote the space of real symmetric $p \times p$ matrices.

Let $K \subseteq \mathbb{R}$ be closed. Fix pairwise distinct real numbers

$$t_1, \ldots, t_{k_1} \in \mathbb{R}, \qquad k_1 \in \mathbb{N} \cup \{0\},$$

and pairwise distinct real numbers

$$u_1, \ldots, u_{k_2} \in \mathbb{R}, \quad v_1, \ldots, v_{k_2} \in (0, \infty), \qquad k_2 \in \mathbb{N} \cup \{0\}.$$

Let

$$e_0 \in \mathbb{N} \cup \{0\},\ e_1, \ldots, e_{k_1} \in \mathbb{N} \qquad \text{and} \qquad f_1, \ldots, f_{k_2} \in \mathbb{N},$$

and define

$$q(x) := \prod_{j=1}^{k_1} (x - t_j)^{2e_j} \prod_{j=1}^{k_2} ((x - u_j)^2 + v_j)^{f_j}. \tag{1.1}$$

Set

$$2n := \sum_{j=0}^{k_1} 2e_j + \sum_{j=1}^{k_2} 2f_j,$$

and consider the finite-dimensional vector space of rational functions

$$\mathcal{R}^{(2n)} := \left\{ \frac{g}{q} : g \in \mathbb{R}[x]_{\leq 2n} \right\}. \tag{1.2}$$

The **matricial univariate rational K–truncated moment problem** asks for a characterization of the linear mappings

$$\mathcal{L} \colon \mathcal{R}^{(2n)} \longrightarrow \mathbb{S}_p(\mathbb{R})$$

Date: August 28, 2026.

2020 *Mathematics Subject Classification.* Primary 44A60; Secondary 65D32, 47A57, 47A20, 15A04, 47N40.

Key words and phrases. Rational moment problem, truncated matricial moment problem, matrix-valued representing measure, prescribed atoms, Gaussian quadrature, moment matrix.

[1]Supported by the ARIS (Slovenian Research and Innovation Agency) research core funding No. P1-0288 and grants No. J1-50002, J1-60011, 70017.

[2]Supported by the ARIS (Slovenian Research and Innovation Agency) research core funding No. P1-0288.

for which there exists a positive $\mathbb{S}_p(\mathbb{R})$-valued measure μ, supported on K, such that

$$\mathcal{L}(R) = \int_K R\, d\mu \qquad \text{for every } R \in \mathcal{R}^{(2n)}.$$

The notion of a positive matrix-valued measure is recalled in Subsection 2.2. Any measure μ satisfying the preceding conditions is called a **K–representing measure for $\mathcal{L}$.**

Equivalently, the matricial rational moment problem can be formulated by prescribing the integrals of monomials and rational functions of the form x^i, $\frac{1}{(x-t_j)^\alpha}$, $\frac{1}{((x-u_j)^2+v_j)^\beta}$, and $\frac{x}{((x-u_j)^2+v_j)^\beta}$, where the ranges of the indices are determined by the multiplicities appearing in (1.1). This explicit formulation is the same in the scalar and matrix-valued settings, since it follows solely from the partial-fraction decomposition of the rational functions involved; see [NZ25] for the full formulation.

The equivalent formulation in terms of prescribed integrals connects the matricial rational moment problem directly with the classical theory of univariate rational moment problems. In the scalar case, rational moment problems on bounded and unbounded intervals have a long history. Their full versions were studied extensively by Jones, Njåstad and Thron [JTW80, JT81, JNT84, Nja85, NT86, Nja87, Nja88], primarily by means of orthogonal and quasi-orthogonal rational functions. Rational truncated moment problems were subsequently considered from the viewpoint of positivity and duality; see [Cha94] and the subsequent work [NZ25], which also corrects some results from [Cha94]. The approach in [NZ25] relies, in the case of the real line, on the classical solution of the truncated Hamburger moment problem due to Curto and Fialkow [CF91]. For general closed semialgebraic subsets of the real line, the nonsingular case in [NZ25] essentially uses a result of di Dio and Schmüdgen [dDS18, Proposition 2 and Corollary 6] (or [Sch17, Theorem 1.30]). In the singular case, the approach relies on the truncated analogue of the Riesz–Haviland theorem [CF08], together with the descriptions of univariate polynomials that are nonnegative on such sets due to Kuhlmann, Marshall, and Schwartz [KM02, KMS05].

The present paper makes two main contributions. First, using the characterization in [ZZ+] of the smallest attainable multiplicity at a prescribed point, we show that the smallest attainable multiplicities at any finite collection of prescribed points can be realized simultaneously by a single minimal matrix-valued representing measure. Second, by applying this result to the real poles of the common denominator, we obtain explicit necessary and sufficient conditions for the rational truncated moment problem on $\mathbb{R}$ and $[0, \infty)$. We also aim to apply the resulting theory to bivariate matricial truncated moment problems on suitable algebraic curves, thereby extending results known in the scalar setting [CF02, CF04, CF05]. Following the reduction methods developed for scalar problems in [Zal21, Zal22a, Zal22b, Zal23, YZ24, YZ26], and analogously to those used for tracial moment problems in [BZ18, BZ21], such problems can be reduced to univariate rational moment problems. The theory developed here therefore provides the machinery needed to address their matrix-valued counterparts. The study of bivariate matrix-valued truncated moment problems was initiated by Kimsey and Trachana in [KT22], where the quadratic and cubic cases were investigated.

A basic idea in the study of rational moment problems is to convert them into ordinary polynomial moment problems. In our setting, the rational functions have the common denominator q defined in (1.1), whose real zeros have even multiplicity. Given a linear mapping $\mathcal{L}$ on the corresponding finite-dimensional space of rational functions, we associate with it a linear mapping L on the space of polynomials of the appropriate degree, defined by $L(f) := \mathcal{L}\left(\frac{f}{q}\right)$. A representing matrix measure for the rational problem corresponds to a representing matrix measure for L that assigns zero mass to every real zero of q. Once such a measure has been found, the representing measure for the original rational data is recovered by multiplying it by q. Thus, the rational moment problem reduces to an ordinary truncated matricial moment problem with prescribed support constraints. Related truncated matrix moment problems with a prescribed gap in the support were studied by Zagorodnyuk [Zag15] using generalized-resolvent methods. In the scalar setting, this correspondence was used in [NZ25] to solve the truncated rational moment problem for unbounded closed sets, as well as to complete the solution in the compact case by adding the missing support-avoidance conditions. Questions concerning the existence of quadrature rules with few nodes have also been studied using optimization and moment-theoretic methods; see [RS18, RT26, BKM26]. Of particular relevance to the present setting is the recent work of Bechere, Kuhlmann, and Mourrain [BKM26] on scalar quadrature formulas whose nodes avoid a prescribed finite set.

Viewing the rational moment problem as an ordinary matricial truncated moment problem whose representing measures must avoid the real poles connects it naturally with the theory of matricial Gaussian quadrature rules with prescribed atoms. Indeed, excluding a real pole λ amounts to finding a representing measure for the associated polynomial data whose mass at λ is zero. Results on minimal representing measures with prescribed atoms can then be used to construct such a measure. More generally, one may prescribe both an atom and the multiplicity of its mass. In the nonsingular case, where the truncated moment matrix is positive definite, this problem was solved in [ZZ25]; the corresponding scalar result was established earlier in [BKRSV20]. The singular matricial case was treated in [ZZ+], where the column relations of the moment matrix and their propagation must also be taken into account. The scalar moment-theoretic approach of [NZ+] provides an alternative proof of the one-node result and extends it to several prescribed nodes. Matricial Gaussian quadrature rules have also been studied through orthogonal matrix polynomials and the zeros of the associated matrix polynomial; see [DLR96, DD02, DS03]. In these approaches, the atoms and matricial masses are determined after a suitable odd moment, or equivalently an appropriate extension of the original moment data, has been fixed. The perspective adopted in [ZZ25, ZZ+] is different: rather than fixing the extension in advance, one constructs an extension whose minimal representing measure has the prescribed behavior at a given point. A related problem was studied in [FKM24], where the set of all possible masses at a fixed point, over all representing measures for a given truncated matricial moment sequence, was described and the maximal possible mass was identified. In the multivariate setting, possible atoms of representing measures were characterized in [MS24a], while possible masses at a prescribed point were investigated in [MS24b]. The focus here is instead on minimal representing measures and on simultaneously attaining the smallest possible multiplicities at finitely many prescribed points. The resulting conditions are expressed in terms of ranks, kernels, and column relations of moment and localizing moment matrices. The present rational problem follows the same perspective, with the real poles playing the role of points at which the mass of the representing measure must vanish. The main new ingredient needed for the rational problem is the simultaneous nature of this construction. The main technical result shows that the individual lower bounds for the multiplicities at these points can be attained simultaneously by a single minimal representing matrix measure. This simultaneous attainment is then combined with the Hamburger or Stieltjes support conditions to solve the rational problem.

For a real $m \times n$ matrix A, let $\mathcal{C}(A) := \{Ax : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$ denote its column space. For matrices $B_1, \dots, B_\ell$ with the same number of columns, write

$$\operatorname{col}(B_i)_{i\in[\ell]} := \begin{pmatrix} B_1 \\ \vdots \\ B_\ell \end{pmatrix}.$$

The main result of this paper is the following.

Theorem 1.1. *Let $K = \mathbb{R}$ or $K = [0, \infty)$, let $\Lambda := \{t_1, \dots, t_{k_1}\} \subseteq \mathbb{R}$ be the set of real zeros of the polynomial q defined in* (1.1)*, and let*

$$\mathcal{L} \colon \mathcal{R}^{(2n)} \longrightarrow \mathbb{S}_p(\mathbb{R})$$

be a linear mapping, where $\mathcal{R}^{(2n)}$ is defined in (1.2)*. Define the associated linear mapping*

$$L \colon \mathbb{R}[x]_{\leq 2n} \longrightarrow \mathbb{S}_p(\mathbb{R}), \qquad L(f) := \mathcal{L}(fq^{-1}). \tag{1.3}$$

Let $S_i := L(x^i)$, $i \in [0; 2n]$, and let

$$M(n) := \left(S_{i+j-2}\right)_{i,j=1}^{n+1} = \left(L(x^{i+j-2})\right)_{i,j=1}^{n+1}$$

be the truncated moment matrix of L.

For each $r \in [k_1]$, define the shifted moment matrix

$$\mathcal{M}^{(r)} := \left(L((x - t_r)^{i+j-2})\right)_{i,j=1}^{n+1},$$

and denote its j-th column by $\mathbf{y}_j^{(r)}$, $j \in [(n+1)p]$. *Set*

$$A^{(r)} := \Big\{ i \in [p] \colon \text{ there exists } k \in [n] \text{ such that } \\ \mathbf{y}^{(r)}_{kp+i} \in \operatorname{span}\Big\{\mathbf{y}^{(r)}_{p+1}, \ldots, \mathbf{y}^{(r)}_{kp+i-1}\Big\}, \\ \mathbf{y}^{(r)}_{(k-1)p+i} \notin \operatorname{span}\Big\{\mathbf{y}^{(r)}_{1}, \ldots, \mathbf{y}^{(r)}_{(k-1)p+i-1}\Big\}\Big\},$$

where $\operatorname{span}(\emptyset) = \{0\}$. *Equivalently,* $A^{(r)} = A_{\mathcal{T}_{t_r}}$ *in the notation of Subsection 2.7. Then the following statements are equivalent:*

1. *The mapping* $\mathcal{L}$ *admits a* K*–representing measure.*
2. *The mapping* $\mathcal{L}$ *admits a finitely atomic* K*–representing measure.*
3. *The mapping* $\mathcal{L}$ *admits a* $(\operatorname{rank} M(n))$*–atomic* K*–representing measure.*
4. *The following conditions hold:*
   (a) *The moment matrix* $M(n)$ *is positive semidefinite.*
   (b) *If* $K = \mathbb{R}$, *then* $\mathcal{C}\Big(\operatorname{col}(S_{n+i})_{i\in[n]}\Big) \subseteq \mathcal{C}(M(n-1))$.
   (c) *If* $K = [0,\infty)$, *then* $\mathcal{H}_x := \big(S_{i+j-1}\big)_{i,j=1}^n$ *is positive semidefinite and*
   $$\mathcal{C}\Big(\operatorname{col}(S_{n+i})_{i\in[n]}\Big) \subseteq \mathcal{C}(\mathcal{H}_x).$$
   (d) $A^{(r)} = \emptyset$ *for every* $r \in [k_1]$.

Remark 1.2. The set $A^{(r)}$ identifies the coordinate indices whose first dependence occurs at the same block level with and without the zeroth block column. Its cardinality is the lower bound for the multiplicity at t_r established in Corollary 3.3 (where $t = t_r$).

Reader's guide. Section 2 introduces the notation and preliminary results used throughout the paper. In particular, we recall the basic notions concerning matrix-valued measures, Riesz mappings, moment matrices, block column relations, changes of basis, and the truncated matricial Hamburger and Stieltjes moment problems. Section 3 derives a formula for the multiplicity of a prescribed atom in a given minimal representing measure and establishes the consequences needed in the proofs of the main results. Section 4 treats representing measures on $\mathbb{R}$. There we prove that the minimal possible multiplicities at finitely many prescribed points can be attained simultaneously, and use this result to prove Theorem 1.1 for $K = \mathbb{R}$. Section 5 develops the corresponding results for representing measures supported on $[0,\infty)$ and completes the proof of Theorem 1.1 in the half-line case. Finally, in Section 6 we demonstrate the statement of Theorem 1.1 on a numerical example.

2. Preliminaries

2.1. **Notation.** Let $m, m_1, m_2 \in \mathbb{N}$. We denote by $M_{m_1\times m_2}(\mathbb{R})$ the space of real $m_1 \times m_2$ matrices and write $M_m(\mathbb{R}) := M_{m\times m}(\mathbb{R})$ for short. For $A \in M_{m_1\times m_2}(\mathbb{R})$, we denote the linear span of its columns, called the **column space** of A, by $\mathcal{C}(A)$.

We denote by I_m the $m \times m$ identity matrix and by $\mathbf{0}_{m_1\times m_2}$ the $m_1 \times m_2$ zero matrix. We also write $\mathbf{0}_m := \mathbf{0}_{m\times m}$. Finally, $M_m(\mathbb{R}[x])$ denotes the space of $m \times m$ matrices with entries in $\mathbb{R}[x]$. Its elements are called **matrix polynomials**.

Let $p \in \mathbb{N}$. For $A \in \mathbb{S}_p(\mathbb{R})$, the notation $A \succeq 0$ (respectively, $A \succ 0$) means that A is positive semidefinite (psd) (respectively, positive definite (pd)). We denote the cone of positive semidefinite matrices in $\mathbb{S}_p(\mathbb{R})$ by $\mathbb{S}_p^{\succeq 0}(\mathbb{R})$.

For a polynomial $f \in \mathbb{R}[x]$, we denote its zero set by

$$\mathcal{Z}(f) := \{x \in \mathbb{R} : f(x) = 0\}.$$

If $a \in \mathcal{Z}(f)$ and $f \neq 0$, we denote the multiplicity of a as a zero of f by $\operatorname{mult}_f a$; if $a \notin \mathcal{Z}(f)$, we write $\operatorname{mult}_f a = 0$.

Let A_k, $k \in [i;j]$, be matrices of compatible sizes. We define their block row and block column concatenations, respectively, by

$$\operatorname{row}(A_k)_{k\in[i;j]} \equiv \operatorname{row}(A_i,\dots,A_j) := \begin{pmatrix} A_i & A_{i+1} & \cdots & A_j \end{pmatrix}$$

and

$$\operatorname{col}(A_k)_{k\in[i;j]} \equiv \operatorname{col}(A_i,\dots,A_j) := \begin{pmatrix} A_i \\ A_{i+1} \\ \vdots \\ A_j \end{pmatrix}.$$

Throughout, matrix dimensions are allowed to be zero; a matrix with zero rows or zero columns is regarded as an empty matrix. In particular, a block row or block column concatenation over an empty index set is understood as the empty matrix of the compatible size.

2.2. **Matrix measures.** Let $\operatorname{Bor}(\mathbb{R})$ denote the Borel σ-algebra on $\mathbb{R}$. A mapping

$$\mu = (\mu_{ij})_{i,j=1}^{p} \colon \operatorname{Bor}(\mathbb{R}) \to \mathbb{S}_p(\mathbb{R})$$

is called a $p \times p$ **Borel matrix-valued measure**, or equivalently a **positive** $\mathbb{S}_p(\mathbb{R})$**-valued measure**, if the following conditions hold:

(1) For every $i, j \in [p]$, the set function $\mu_{ij} \colon \operatorname{Bor}(\mathbb{R}) \to \mathbb{R}$ is a signed measure.
(2) For every $\Delta \in \operatorname{Bor}(\mathbb{R})$, one has $\mu(\Delta) \succeq 0$.

A positive $\mathbb{S}_p(\mathbb{R})$-valued measure μ is called **finitely atomic** if there exists a finite set $F \subseteq \mathbb{R}$ such that $\mu(\mathbb{R} \setminus F) = \mathbf{0}_p$. Equivalently, μ can be written as

$$\mu = \sum_{j=1}^{\ell} \delta_{x_j} A_j$$

for some $\ell \in \mathbb{N}$, points $x_j \in \mathbb{R}$, and matrices $A_j \in \mathbb{S}_p^{\succeq 0}(\mathbb{R})$.

A point $x \in \mathbb{R}$ is called an **atom** of μ if $\mu(\{x\}) \neq \mathbf{0}_p$. If $x_1, \dots, x_\ell$ are pairwise distinct and $A_j \neq \mathbf{0}_p$ for every $j \in [\ell]$, then the atoms of μ are precisely $x_1, \dots, x_\ell$, and

$$A_j = \mu(\{x_j\})$$

is called the **mass** of μ at x_j. The rank of A_j is called the **multiplicity** of x_j in μ and is denoted by

$$\operatorname{mult}_\mu x_j := \operatorname{rank} A_j.$$

For a point $x \in \mathbb{R}$ that is not an atom of μ, we set $\operatorname{mult}_\mu x := 0$. We say that μ is r**–atomic** if

$$\sum_{j=1}^{\ell} \operatorname{rank} A_j = r.$$

Thus, throughout this paper, the size of a finitely atomic matrix-valued measure is counted with multiplicity, that is, by the sum of the ranks of its matricial masses, rather than by the number of distinct support points.

Let μ be a positive $\mathbb{S}_p(\mathbb{R})$-valued measure. Its **trace measure** is defined by $\tau := \operatorname{tr}(\mu) := \sum_{i=1}^{p} \mu_{ii}$. A polynomial $f \in \mathbb{R}[x]$ is called μ**-integrable** if $f \in L^1(\tau)$. In this case, its matrix integral with respect to μ is defined entrywise by

$$\int_{\mathbb{R}} f \, d\mu := \left(\int_{\mathbb{R}} f \, d\mu_{ij} \right)_{i,j=1}^{p}.$$

2.3. **Representing matrix measures for linear operators.** Assume the notation introduced in Section 1. Given a linear mapping

$$\mathcal{L}\colon \mathcal{R}^{(2n)} \longrightarrow \mathbb{S}_p(\mathbb{R}),$$

we associate with it the linear operator

$$L\colon \mathbb{R}[x]_{\leq 2n} \longrightarrow \mathbb{S}_p(\mathbb{R}), \qquad L(f) := \mathcal{L}(fq^{-1}).$$

A positive $\mathbb{S}_p(\mathbb{R})$-valued measure μ, supported on K, is called a **K–representing measure for** L if

$$L(f) = \int_K f\, d\mu \qquad \text{for every } f \in \mathbb{R}[x]_{\leq 2n}.$$

Let $\Lambda \subseteq K$ be a Borel set. Define

$$\begin{aligned}
\mathcal{M}_{\mathcal{L},K} &:= \{\mu : \mu \text{ is a } K\text{–representing measure for } \mathcal{L}\},\\
\mathcal{M}_{L,K} &:= \{\mu : \mu \text{ is a } K\text{–representing measure for } L\},\\
\mathcal{M}_{L,K,\Lambda} &:= \{\mu \in \mathcal{M}_{L,K} : \mu(\Lambda) = \mathbf{0}_p\}.
\end{aligned}$$

We denote by $\mathcal{M}^{(\mathrm{fa})}_{\mathcal{L},K}$, $\mathcal{M}^{(\mathrm{fa})}_{L,K}$, $\mathcal{M}^{(\mathrm{fa})}_{L,K,\Lambda}$ the subsets of $\mathcal{M}_{\mathcal{L},K}$, $\mathcal{M}_{L,K}$, and $\mathcal{M}_{L,K,\Lambda}$, respectively, consisting of all finitely atomic measures.

Let μ be a positive $\mathbb{S}_p(\mathbb{R})$-valued measure with $\mathrm{supp}(\mu) \subseteq K$, and let f be μ-integrable. We denote by $f \cdot \mu$ the $\mathbb{S}_p(\mathbb{R})$-valued measure defined by

$$(f \cdot \mu)(E) := \int_E f\, d\mu, \qquad E \in \mathrm{Bor}(\mathbb{R}).$$

If μ is regarded as a measure on K, it is enough to take $E \in \mathrm{Bor}(K)$.

Proposition 2.1. *Let q be as in* (1.1)*, and set $\Lambda := K \cap \{t_1, \ldots, t_{k_1}\}$. Then the following statements hold:*

(1) $\mathcal{M}_{\mathcal{L},K} \neq \emptyset \Longleftrightarrow \mathcal{M}_{L,K,\Lambda} \neq \emptyset$.

(2) $\mathcal{M}^{(\mathrm{fa})}_{\mathcal{L},K} \neq \emptyset \Longleftrightarrow \mathcal{M}^{(\mathrm{fa})}_{L,K,\Lambda} \neq \emptyset$.

(3) The map

$$\Phi\colon \mathcal{M}_{L,K,\Lambda} \Longrightarrow \mathcal{M}_{\mathcal{L},K}, \qquad \Phi(\mu) := q \cdot \mu,$$

is a bijection, with inverse

$$\Phi^{-1}(\nu) = q^{-1} \cdot \nu.$$

Proof. The proof is identical to that of the scalar case; see [NZ25, Proposition 2.1]. □

2.4. **Riesz mapping and the moment matrix.** Let $L\colon \mathbb{R}[x]_{\leq 2n} \to \mathbb{S}_p(\mathbb{R})$ be a linear operator. The matrices $S_i := L(x^i)$, $i \in [0; 2n]$, are called the **matricial moments** of L.

Conversely, given a sequence

$$\mathcal{S} \equiv \mathcal{S}^{(2n)} := (S_0, S_1, \ldots, S_{2n}) \in \big(\mathbb{S}_p(\mathbb{R})\big)^{2n+1}, \tag{2.1}$$

we denote by

$$L_{\mathcal{S}}\colon \mathbb{R}[x]_{\leq 2n} \to \mathbb{S}_p(\mathbb{R})$$

the unique linear operator satisfying

$$L_{\mathcal{S}}(x^i) = S_i, \qquad i \in [0; 2n],$$

and call it the **Riesz mapping associated with** $\mathcal{S}$. A K–representing measure for $L_{\mathcal{S}}$ will also be called a **K–representing measure for** $\mathcal{S}$.

For a sequence $\mathcal{S}$ as in (2.1), we define its **n-th truncated moment matrix** by

$$
(2.2)\qquad M(n) \equiv M_{\mathcal{S}}(n) := \left(S_{i+j-2}\right)_{i,j=1}^{n+1} = \begin{array}{c} \\ \mathit{1} \\ X \\ X^2 \\ \vdots \\ X^n \end{array} \begin{array}{c} \begin{array}{ccccc} \mathit{1} & X & X^2 & \cdots & X^n \end{array} \\ \begin{pmatrix} S_0 & S_1 & S_2 & \cdots & S_n \\ S_1 & S_2 & \iddots & \iddots & S_{n+1} \\ S_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & S_{2n-1} \\ S_n & S_{n+1} & \cdots & S_{2n-1} & S_{2n} \end{pmatrix} \end{array}.
$$

For each $i \in [0; n]$, we write

$$
(2.3)\qquad X^i_{\mathcal{S}} := \operatorname{col}(S_{i+\ell})_{\ell\in[0;n]}
$$

for the $(i+1)$-st block column of $M_{\mathcal{S}}(n)$. Whenever the underlying sequence $\mathcal{S}$ is clear from the context, we write X^i instead of $X^i_{\mathcal{S}}$.

2.5. **Evaluation of a matrix polynomial on $M(n)$.** Let $\mathcal{S}$ be a sequence as in (2.1), and let $M_{\mathcal{S}}(n)$ be the corresponding truncated moment matrix. Recall from (2.3) that $X^i_{\mathcal{S}}$ denotes the $(i+1)$-st block column of $M_{\mathcal{S}}(n)$.

Given a matrix polynomial

$$
(2.4)\qquad P(x) = \sum_{i=0}^{n} x^i P_i, \qquad P_i \in M_p(\mathbb{R}),
$$

we define its **evaluation on** $M_{\mathcal{S}}(n)$ by

$$
(2.5)\qquad P(X_{\mathcal{S}}) := \sum_{i=0}^{n} X^i_{\mathcal{S}} P_i = X^0_{\mathcal{S}} P_0 + X^1_{\mathcal{S}} P_1 + \cdots + X^n_{\mathcal{S}} P_n \in M_{(n+1)p\times p}(\mathbb{R}).
$$

When the underlying sequence $\mathcal{S}$ is clear from the context, we write $P(X)$ instead of $P(X_{\mathcal{S}})$.

If

$$
P(X_{\mathcal{S}}) = \mathbf{0}_{(n+1)p\times p},
$$

then $P(x)$ is called a **block column relation** of $M_{\mathcal{S}}(n)$.

For a matrix polynomial $P(x)$ as in (2.4), we denote the block column vector of its matrix coefficients by

$$
(2.6)\qquad \operatorname{coef}(P) := \operatorname{col}(P_i)_{i\in[0;n]}.
$$

Finally, the moment matrix $M_{\mathcal{S}}(n)$ is called **block–recursively generated** if multiplication by x preserves every block column relation of degree at most $n-1$. More precisely, if

$$
P(x) = \sum_{i=0}^{j} x^i P_i \in M_p(\mathbb{R}[x]), \qquad j \in [0; n-1],
$$

satisfies $P(X_{\mathcal{S}}) = \mathbf{0}_{(n+1)p\times p}$, then the matrix polynomial

$$
(xP)(x) := xP(x) = \sum_{i=0}^{j} x^{i+1} P_i
$$

also satisfies $(xP)(X_{\mathcal{S}}) = \mathbf{0}_{(n+1)p\times p}$.

2.6. **Change of basis in the moment matrix.** Let $\mathcal{S} \equiv \mathcal{S}^{(2n)}$ be as in (2.1), and fix $t \in \mathbb{R}$. Consider the invertible affine transformation

$$\phi_t(x) := x - t.$$

For each $i \in [0; 2n]$, define the shifted moment

$$T_i := L_{\mathcal{S}}\big((x-t)^i\big) = \sum_{\ell=0}^{i} \binom{i}{\ell} (-1)^\ell t^\ell S_{i-\ell}, \tag{2.7}$$

and set

$$\mathcal{T}_t \equiv \mathcal{T}_t^{(2n)} := (T_0, T_1, \ldots, T_{2n}). \tag{2.8}$$

Let $M_{\mathcal{T}_t}(n)$ be the moment matrix associated with $\mathcal{T}_t$, and let $X^i_{\mathcal{T}_t}$ denote its $(i+1)$-st block column for $i \in [0; n]$.

Let

$$J_t \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

be the linear automorphism induced by the substitution $x \mapsto x - t$ with respect to the monomial basis $1, x, \ldots, x^n$. Thus,

$$J_t \operatorname{coef}(f) = \operatorname{coef}(f \circ \phi_t) = \operatorname{coef}\big(f(x-t)\big), \qquad f \in \mathbb{R}[x]_{\leq n}. \tag{2.9}$$

Proposition 2.2 ([ZZ+, Proposition 2.7]). *With the notation above, the following statements hold:*

(1) For every matrix polynomial $P \in M_p(\mathbb{R}[x])$ of degree at most n,

$$(J_t \otimes I_p) \operatorname{coef}(P) = \operatorname{coef}(P \circ \phi_t) = \operatorname{coef}\big(P(x-t)\big).$$

(2) The moment matrices $M_{\mathcal{S}}(n)$ and $M_{\mathcal{T}_t}(n)$ are related by

$$M_{\mathcal{T}_t}(n) = (J_t \otimes I_p)^{\mathsf{T}} M_{\mathcal{S}}(n) (J_t \otimes I_p).$$

(3) The linear maps J_t and $J_t \otimes I_p$ are invertible.

(4) Positive semidefiniteness is preserved under the change of basis:

$$M_{\mathcal{T}_t}(n) \succeq 0 \quad \Longleftrightarrow \quad M_{\mathcal{S}}(n) \succeq 0.$$

(5) Rank is preserved under the change of basis:

$$\operatorname{rank} M_{\mathcal{T}_t}(n) = \operatorname{rank} M_{\mathcal{S}}(n). \tag{2.10}$$

(6) For every matrix polynomial $P \in M_p(\mathbb{R}[x])$ of degree at most n,

$$P(X_{\mathcal{T}_t}) = (J_t \otimes I_p)^{\mathsf{T}} (P \circ \phi_t)(X_{\mathcal{S}}).$$

Proposition 2.3 ([ZZ+, Corollary 2.8]). *A matrix polynomial $P \in M_p(\mathbb{R}[x])$ of degree at most n is a block column relation of $M_{\mathcal{T}_t}(n)$ if and only if $P(x-t)$ is a block column relation of $M_{\mathcal{S}}(n)$.*

Proposition 2.4 ([ZZ+, Proposition 2.9]). *The moment matrix $M_{\mathcal{S}}(n)$ is block–recursively generated if and only if $M_{\mathcal{T}_t}(n)$ is block–recursively generated.*

2.7. **Column relations of the moment matrix.** Fix $t \in \mathbb{R}$. Using the notation introduced in Subsection 2.6, write

$$M_{\mathcal{T}_t}(n) = \operatorname{row}\big(X^i_{\mathcal{T}_t}\big)_{i \in [0;n]}.$$

For each $i \in [0; n]$, denote the columns of the block column $X^i_{\mathcal{T}_t}$ by $(x_{\mathcal{T}_t})^{(i)}_j$, so that

$$X^i_{\mathcal{T}_t} = \begin{pmatrix} (x_{\mathcal{T}_t})^{(i)}_1 & (x_{\mathcal{T}_t})^{(i)}_2 & \cdots & (x_{\mathcal{T}_t})^{(i)}_p \end{pmatrix} = \operatorname{row}\big((x_{\mathcal{T}_t})^{(i)}_j\big)_{j \in [p]}. \tag{2.11}$$

For $i \in [0; n]$ and $j \in [p]$, define

$$C_{\mathcal{T}_t}(i,j) := \operatorname{row}\Big(X^0_{\mathcal{T}_t}, \ldots, X^{i-1}_{\mathcal{T}_t}, (x_{\mathcal{T}_t})^{(i)}_1, \ldots, (x_{\mathcal{T}_t})^{(i)}_j\Big),$$
$$C_{\mathcal{T}_t}(i,0) := \operatorname{row}\Big(X^0_{\mathcal{T}_t}, \ldots, X^{i-1}_{\mathcal{T}_t}\Big).$$

Here, when $i = 0$, the block columns preceding $X^0_{\mathcal{T}_t}$ are absent. Thus, $C_{\mathcal{T}_t}(0,0)$ is the empty matrix.

We define

$$\operatorname{Dep}\Big(X^i_{\mathcal{T}_t}\Big) := \{j \in [p] \colon \operatorname{rank} C_{\mathcal{T}_t}(i,j) = \operatorname{rank} C_{\mathcal{T}_t}(i,j-1)\}\,. \tag{2.12}$$

Equivalently, $j \in \operatorname{Dep}(X^i_{\mathcal{T}_t})$ if and only if the column $(x_{\mathcal{T}_t})^{(i)}_j$ belongs to the column space of $C_{\mathcal{T}_t}(i,j-1)$.

The following result shows that these dependence indices are invariant under translations of the polynomial variable.

Proposition 2.5. *For every $t, u \in \mathbb{R}$ and every $i \in [0;n]$, one has*

$$\operatorname{Dep}\Big(X^i_{\mathcal{T}_t}\Big) = \operatorname{Dep}\Big(X^i_{\mathcal{T}_u}\Big)\,.$$

Proof. Fix $i \in [0;n]$. By symmetry, it suffices to prove that

$$\operatorname{Dep}\Big(X^i_{\mathcal{T}_t}\Big) \subseteq \operatorname{Dep}\Big(X^i_{\mathcal{T}_u}\Big)\,.$$

Let $j \in \operatorname{Dep}(X^i_{\mathcal{T}_t})$. By the definition of $\operatorname{Dep}(X^i_{\mathcal{T}_t})$, the column $(x_{\mathcal{T}_t})^{(i)}_j$ belongs to the column space of $C_{\mathcal{T}_t}(i,j-1)$. Hence, there exist vectors

$$s_0, \ldots, s_{i-1} \in \mathbb{R}^p \quad \text{and} \quad s_{i,(0)} \in \mathbb{R}^{j-1}$$

such that

$$(x_{\mathcal{T}_t})^{(i)}_j = \sum_{k=0}^{i-1} X^k_{\mathcal{T}_t} s_k + X^i_{\mathcal{T}_t} \operatorname{col}\Big(s_{i,(0)}, \mathbf{0}_{(p-j+1)\times 1}\Big)\,. \tag{2.13}$$

Let $e_j \in \mathbb{R}^p$ be the j-th standard basis vector and set

$$v := e_j - \operatorname{col}\Big(s_{i,(0)}, \mathbf{0}_{(p-j+1)\times 1}\Big)\,. \tag{2.14}$$

Since $(x_{\mathcal{T}_t})^{(i)}_j = X^i_{\mathcal{T}_t} e_j$, equation (2.13) is equivalent to

$$X^i_{\mathcal{T}_t} v = \sum_{k=0}^{i-1} X^k_{\mathcal{T}_t} s_k. \tag{2.15}$$

For $k \in [0;i]$, define

$$p_k := \begin{cases} -s_k, & k \in [0;i-1], \\ v, & k = i, \end{cases} \qquad P_k := \begin{pmatrix} p_k & \mathbf{0}_{p\times(p-1)} \end{pmatrix} \in M_p(\mathbb{R}).$$

Then (2.15) is equivalent to

$$\sum_{k=0}^{i} X^k_{\mathcal{T}_t} P_k = \mathbf{0}_{(n+1)p\times p}.$$

Thus, the matrix polynomial

$$P(x) := \sum_{k=0}^{i} x^k P_k$$

is a block column relation of $M_{\mathcal{T}_t}(n)$.

The sequence $\mathcal{T}_t$ is obtained from $\mathcal{T}_u$ by the change of basis corresponding to

$$x \longmapsto x - (t-u).$$

Therefore, by Proposition 2.3, the matrix polynomial

$$Q(x) := P(x-(t-u)) = P(x-t+u)$$

is a block column relation of $M_{\mathcal{T}_u}(n)$. Write

$$Q(x) = \sum_{k=0}^{i} x^k Q_k.$$

Since the last $p-1$ columns of every coefficient P_k are zero, the same is true of every Q_k. Moreover, translation preserves the leading coefficient, so $Q_i = P_i$. Consequently, there exist vectors $q_0, \dots, q_i \in \mathbb{R}^p$ such that

$$Q_k = \begin{pmatrix} q_k & \mathbf{0}_{p\times(p-1)} \end{pmatrix}, \qquad k \in [0;i],$$

and $q_i = v$.

Evaluating Q on $M_{\mathcal{T}_u}(n)$ and taking the first column gives

$$\sum_{k=0}^{i} X^k_{\mathcal{T}_u} q_k = \mathbf{0}_{(n+1)p}.$$

Hence,

$$X^i_{\mathcal{T}_u} v = \sum_{k=0}^{i-1} X^k_{\mathcal{T}_u}(-q_k).$$

Recalling the definition of v in (2.14), we obtain

$$(x_{\mathcal{T}_u})^{(i)}_j = \sum_{k=0}^{i-1} X^k_{\mathcal{T}_u}(-q_k) + X^i_{\mathcal{T}_u} \operatorname{col}\left(s_{i,(0)}, \mathbf{0}_{(p-j+1)\times 1}\right).$$

Therefore, $(x_{\mathcal{T}_u})^{(i)}_j$ belongs to the column space of $C_{\mathcal{T}_u}(i, j-1)$, and hence $j \in \operatorname{Dep}\left(X^i_{\mathcal{T}_u}\right)$. This proves the desired inclusion. The reverse inclusion follows by interchanging t and u. □

Similarly, for $i \in [n]$ and $j \in [p]$, define

$$C^{(1)}_{\mathcal{T}_t}(i,j) := \operatorname{row}\left(X^1_{\mathcal{T}_t}, \dots, X^{i-1}_{\mathcal{T}_t}, (x_{\mathcal{T}_t})^{(i)}_1, \dots, (x_{\mathcal{T}_t})^{(i)}_j\right),$$
$$C^{(1)}_{\mathcal{T}_t}(i,0) := \operatorname{row}\left(X^1_{\mathcal{T}_t}, \dots, X^{i-1}_{\mathcal{T}_t}\right).$$

When $i = 1$, the block columns preceding $X^1_{\mathcal{T}_t}$ are absent, so $C^{(1)}_{\mathcal{T}_t}(1,0)$ is the empty matrix.

Remark 2.6. The difference from $C_{\mathcal{T}_t}(i,j)$ is that $C^{(1)}_{\mathcal{T}_t}(i,j)$ does not contain the zeroth block column $X^0_{\mathcal{T}_t}$. Thus, $C_{\mathcal{T}_t}(i,j)$ records column dependencies relative to all preceding block columns, whereas $C^{(1)}_{\mathcal{T}_t}(i,j)$ records column dependencies relative only to the preceding block columns of positive degree. In particular, $\mathcal{C}\left(C^{(1)}_{\mathcal{T}_t}(i,j)\right) \subseteq \mathcal{C}(C_{\mathcal{T}_t}(i,j))$.

Set $\operatorname{Dep}_1\left(X^0_{\mathcal{T}_t}\right) := \emptyset$, and, for $i \in [n]$, define

$$\operatorname{Dep}_1\left(X^i_{\mathcal{T}_t}\right) := \left\{ j \in [p] \colon \operatorname{rank} C^{(1)}_{\mathcal{T}_t}(i,j) = \operatorname{rank} C^{(1)}_{\mathcal{T}_t}(i,j-1) \right\}.$$

Equivalently, $j \in \operatorname{Dep}_1(X^i_{\mathcal{T}_t})$ if and only if $(x_{\mathcal{T}_t})^{(i)}_j$ belongs to the column space of $C^{(1)}_{\mathcal{T}_t}(i,j-1)$.

For each $j \in [p]$, define the first-dependence indices

$$\zeta_{\mathcal{T}_t}(j) := \min\left\{ i \in [0;n] : j \in \operatorname{Dep}\left(X^i_{\mathcal{T}_t}\right)\right\},$$
$$\zeta^{(1)}_{\mathcal{T}_t}(j) := \min\left\{ i \in [n] : j \in \operatorname{Dep}_1\left(X^i_{\mathcal{T}_t}\right)\right\},$$

where $\min \emptyset := +\infty$. For each $i \in [0;n]$, set

$$\operatorname{DepN}\left(X^i_{\mathcal{T}_t}\right) := \{ j \in [p] \colon \zeta_{\mathcal{T}_t}(j) = i \},$$
$$\operatorname{DepN}_1\left(X^i_{\mathcal{T}_t}\right) := \left\{ j \in [p] \colon \zeta_{\mathcal{T}_t}(j) = \zeta^{(1)}_{\mathcal{T}_t}(j) = i \right\},$$
$$\operatorname{DepN}_0\left(X^i_{\mathcal{T}_t}\right) := \operatorname{DepN}\left(X^i_{\mathcal{T}_t}\right) \setminus \operatorname{DepN}_1\left(X^i_{\mathcal{T}_t}\right).$$

Thus, $\operatorname{DepN}(X^i_{\mathcal{T}_t})$ consists of those indices that first become dependent in the block column $X^i_{\mathcal{T}_t}$, while $\operatorname{DepN}_1(X^i_{\mathcal{T}_t})$ consists of those indices for which the first dependence occurs at the same block level even after the block column $X^0_{\mathcal{T}_t}$ is omitted.

Finally, define

$$A_{\mathcal{T}_t} := \bigcup_{i=1}^{n} \operatorname{DepN}_1\left(X^i_{\mathcal{T}_t}\right). \tag{2.16}$$

For $t=0$, we write $X^i := X^i_{\mathcal{T}_0}$ and omit the subscript $\mathcal{T}_0$ whenever no confusion can arise.

Set

$$p_0 := p - \operatorname{card}\operatorname{Dep}(X^n).$$

Let

$$\mathcal{D} \equiv \{z_1, \ldots, z_s\} := \left\{i \in [0;n] : \operatorname{DepN}\left(X^i_{\mathcal{T}_t}\right) \neq \emptyset\right\}, \tag{2.17}$$

where the elements are listed in increasing order:

$$0 \leq z_1 < \cdots < z_s \leq n.$$

If $\mathcal{D} \neq \emptyset$, it follows by Proposition 2.5 and the definition of z_s that $p_0 = p - \operatorname{card}\operatorname{Dep}\left(X^{z_s}_{\mathcal{T}_t}\right)$; if $\mathcal{D} = \emptyset$, we have $p_0 = p$. For $j \in [s]$, define

$$p_j := \operatorname{card}\operatorname{DepN}\left(X^{z_{s-j+1}}_{\mathcal{T}_t}\right), \qquad r_j := \sum_{\ell=1}^{j} p_\ell. \tag{2.18}$$

If $\mathcal{D} = \emptyset$, we set $s := 0$ and $z_s \equiv z_0 := 0$ so that the formulas in the statements below remain well-defined.

Lemma 2.7 ([ZZ+, Section 3, Claim 2]). *Let $\mathcal{S} \equiv \mathcal{S}^{(2n)}$ be as in* (2.1)*, and assume that the corresponding moment matrix $M(n)$ is block–recursively generated. Fix $t \in \mathbb{R}$, and let $\mathcal{D}$ be as in* (2.17)*, with p_j and r_j as in* (2.18).

Then there exist matrices

$$\widetilde{H}_0, \widetilde{H}_1, \ldots, \widetilde{H}_n \in M_{p\times(p-p_0)}(\mathbb{R})$$

satisfying the following properties:

(1) $\operatorname{rank}\widetilde{H}_n = p - p_0$.

(2)

$$\sum_{i=0}^{n} X^i_{\mathcal{T}_t}\widetilde{H}_i = \mathbf{0}_{(n+1)p\times(p-p_0)}. \tag{2.19}$$

(3) If $\widetilde{s}_i := \dim\left(\bigcap_{j=0}^{i}\operatorname{Ker}\widetilde{H}_j\right)$, $i \in [0;n-1]$, then

$$\sum_{i=0}^{n-1}\widetilde{s}_i \geq (n-z_s)(p-p_0) + \sum_{j=1}^{s-1}(z_{j+1}-z_j)(p-p_0-r_{s-j}) + \operatorname{card}A_{\mathcal{T}_t}. \tag{2.20}$$

Proposition 2.8. *Let $\mathcal{S} \equiv \mathcal{S}^{(2n)}$ be as in* (2.1)*, and assume that the corresponding moment matrix $M(n)$ is block–recursively generated. Fix $t \in \mathbb{R}$, and let $\mathcal{D}$ be as in* (2.17)*, with p_j and r_j as in* (2.18).

Then there exists a block column relation $H(x) = \sum_{i=0}^{n} x^i H_i$ of $M_{\mathcal{T}_t}(n)$ satisfying the following properties:

(1) $\operatorname{rank}H_n = p - p_0$.

(2)

$$\sum_{i=0}^{n}\dim\left(\bigcap_{j=0}^{i}\operatorname{Ker}H_j\right) \geq (n+1)p_0 + (n-z_s)(p-p_0) + \sum_{j=1}^{s-1}(z_{j+1}-z_j)(p-p_0-r_{s-j}) + \operatorname{card}A_{\mathcal{T}_t}, \tag{2.21}$$

where $A_{\mathcal{T}_t}$ is defined in (2.16).

(3) each coefficient H_i is of the form

$$H_i = \begin{pmatrix}\mathbf{0}_{p\times p_0} & \widetilde{H}_i\end{pmatrix}, \qquad i \in [0;n], \qquad \text{where } \widetilde{H}_i \in M_{p\times(p-p_0)}(\mathbb{R}). \tag{2.22}$$

Proof. By Proposition 2.4, the moment matrix $M_{\mathcal{T}_t}(n)$ is block–recursively generated. Hence, Lemma 2.7 yields matrices $\widetilde{H}_0, \widetilde{H}_1, \ldots, \widetilde{H}_n \in M_{p\times(p-p_0)}(\mathbb{R})$ satisfying $\operatorname{rank} \widetilde{H}_n = p - p_0$, together with (2.19) and (2.20).

For each $i \in [0; n]$, define $H_i := \begin{pmatrix} \mathbf{0}_{p\times p_0} & \widetilde{H}_i \end{pmatrix}$, and set $H(x) := \sum_{i=0}^n x^i H_i$. By (2.19), $H(x)$ is a block column relation of $M_{\mathcal{T}_t}(n)$. Moreover,

$$\operatorname{rank} H_n = \operatorname{rank} \widetilde{H}_n = p - p_0.$$

For every $i \in [0; n]$, the block form of the matrices H_j gives

$$\bigcap_{j=0}^{i} \operatorname{Ker} H_j = \mathbb{R}^{p_0} \oplus \bigcap_{j=0}^{i} \operatorname{Ker} \widetilde{H}_j.$$

Consequently,

$$\dim\left(\bigcap_{j=0}^{i} \operatorname{Ker} H_j\right) = p_0 + \dim\left(\bigcap_{j=0}^{i} \operatorname{Ker} \widetilde{H}_j\right).$$

Since $\widetilde{H}_n$ has full column rank, one has $\operatorname{Ker} \widetilde{H}_n = \{\mathbf{0}\}$, and hence $\bigcap_{j=0}^n \operatorname{Ker} \widetilde{H}_j = \{\mathbf{0}\}$. It follows that

$$\sum_{i=0}^{n} \dim\left(\bigcap_{j=0}^{i} \operatorname{Ker} H_j\right) = (n+1)p_0 + \sum_{i=0}^{n-1} \dim\left(\bigcap_{j=0}^{i} \operatorname{Ker} \widetilde{H}_j\right). \tag{2.23}$$

Combining (2.23) with (2.20) yields (2.21). □

2.8. **Support of a representing measure and block column relations.** The following lemma relates the support and atomic multiplicities of a representing measure to the zeros of the determinant of a block column relation.

Lemma 2.9 ([ZZ+, Lemma 2.4])**.** *Let $n, p \in \mathbb{N}$, and let*

$$\mathcal{S} := (S_0, S_1, \ldots, S_{2n}) \in \left(\mathbb{S}_p(\mathbb{R})\right)^{2n+1}$$

be a sequence with moment matrix $M(n)$ and an $\mathbb{R}$–representing measure μ. Suppose that

$$H(x) = \sum_{i=0}^{n} x^i H_i \in M_p(\mathbb{R}[x])$$

is a block column relation of $M(n)$ and that H_n is invertible. Then:

(1) $\operatorname{supp}(\mu) \subseteq \mathcal{Z}(\det H)$.

(2) $\operatorname{mult}_\mu \xi \leq \operatorname{mult}_{\det H} \xi$ *for every $\xi \in \mathbb{R}$.*

2.9. **Determinant of a matrix polynomial.** The following result relates the vanishing order of the determinant of a matrix polynomial at a prescribed point to the common kernels of its coefficients.

Lemma 2.10 ([ZZ+, Lemma 2.10])**.** *Let $n, p \in \mathbb{N}$, let $t \in \mathbb{R}$, and let*

$$H(x) = \sum_{i=0}^{n} (x-t)^i H_i \in M_p(\mathbb{R}[x])$$

be a nonzero matrix polynomial. Let H_n be invertible. For $k \in [0; n-1]$ define

$$W_k := \bigcap_{i=0}^{k} \operatorname{Ker} H_i, \qquad s_k := \dim W_k.$$

Then

$$\det H(x) = (x-t)^{\sum_{k=0}^{n-1} s_k} g(x), \tag{2.24}$$

where $0 \neq g(x) \in \mathbb{R}[x]$.

2.10. **Lemma on the rank of a matrix.** The following technical lemma will be used in the proof of Theorem 4.2.

Lemma 2.11 ([ZZ+, Lemma 2.14]). *Let $m_1, m_2, c, k, \ell \in \mathbb{N}$, and let*

$$A_1 \in M_{m_1\times k}(\mathbb{R}), \qquad A_2 \in M_{m_2\times k}(\mathbb{R}), \qquad C \in M_{c\times k}(\mathbb{R}),$$

and

$$B_1 \in M_{m_1\times \ell}(\mathbb{R}), \qquad B_2 \in M_{m_2\times \ell}(\mathbb{R}), \qquad D \in M_{c\times \ell}(\mathbb{R}).$$

Assume that

(1)
$$\operatorname{rank} A_2 = \operatorname{rank}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix};$$

(2)
$$\operatorname{rank}\begin{pmatrix} A_2 \\ C \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A_2 & B_2 \\ C & D \end{pmatrix}.$$

Then
$$\operatorname{rank}\begin{pmatrix} A_1 \\ A_2 \\ C \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \\ C & D \end{pmatrix}.$$

2.11. **Solutions to the truncated matricial Hamburger and Stieltjes moment problem.** We conclude this section by recalling the characterizations of the solvability of the truncated matricial Hamburger and Stieltjes moment problems due to Bakonyi and Woerdeman and to Kimsey, respectively. These results also guarantee the existence of finitely atomic representing matrix measures of minimal size.

Theorem 2.12 ([BW11, Theorem 2.7.6]). *Let $n, p \in \mathbb{N}$ and let $\mathcal{S} := (S_0, S_1, \ldots, S_{2n}) \in (\mathbb{S}_p(\mathbb{R}))^{2n+1}$ be a sequence with a moment matrix $M(n)$. Then the following statements are equivalent:*

(1) There exists a $\mathbb{R}$–representing matrix measure for $\mathcal{S}$.
(2) There exists a $(\operatorname{rank} M(n))$–atomic $\mathbb{R}$–representing matrix measure for $\mathcal{S}$.
(3) $M(n)$ is positive semidefinite and $\mathcal{C}\big(\operatorname{col}(S_{n+i})_{i\in[n]}\big) \subseteq \mathcal{C}(M(n-1))$.

Theorem 2.13 ([Kim22, Theorem 2.18]). *Let $n, p \in \mathbb{N}$ and let $\mathcal{S} := (S_0, S_1, \ldots, S_{2n}) \in (\mathbb{S}_p(\mathbb{R}))^{2n+1}$ be a sequence with a moment matrix $M(n)$. Then the following statements are equivalent:*

(1) There exists a $[0,\infty)$–representing matrix measure for $\mathcal{S}$.
(2) There exists a $(\operatorname{rank} M(n))$–atomic $[0,\infty)$–representing matrix measure for $\mathcal{S}$.
(3) $M(n)$ and $\mathcal{H}_x := \big(S_{i+j-1}\big)_{i,j=1}^{n}$ are positive semidefinite, and $\mathcal{C}\big(\operatorname{col}(S_{n+i})_{i\in[n]}\big) \subseteq \mathcal{C}(\mathcal{H}_x)$.

3. Multiplicity of an Atom in a given Minimal Measure

The formula (3.1) below is implicit in the proof of [ZZ+, Theorem 1.1], where it is used for a single prescribed point. We isolate it here because its precise equality, rather than only the resulting lower bound, will be applied at several prescribed points in Sections 4 and 5. It expresses the multiplicity of an atom at a prescribed point t in terms of the moments written in the shifted basis associated with $x - t$.

Theorem 3.1. *Let $n, p \in \mathbb{N}$, and let*

$$\mathcal{S} := (S_0, S_1, \ldots, S_{2n}) \in \big(\mathbb{S}_p(\mathbb{R})\big)^{2n+1}$$

be a sequence with moment matrix $M(n)$. Suppose that $\mathcal{S}$ admits a $(\operatorname{rank} M(n))$–atomic $\mathbb{R}$–representing measure μ. Set

$$S_{2n+1} := \int_{\mathbb{R}} x^{2n+1}\, d\mu,$$

and fix $t \in \mathbb{R}$. For $i \in [0; 2n+1]$, define

$$\mathcal{E}_t(S_i) := L_{\mathcal{S}}\big((x-t)^i\big) = \sum_{\ell=0}^{i} \binom{i}{\ell} (-1)^{\ell} t^{\ell} S_{i-\ell},$$

where, for $i = 2n+1$, the operator $L_{\mathcal{S}}$ is extended to $\mathbb{R}[x]_{\leq 2n+1}$ by setting $L_{\mathcal{S}}(x^{2n+1}) := S_{2n+1}$. Set

$$\mathcal{T}_t \equiv \mathcal{T}_t^{(2n)} := (\mathcal{E}_t(S_0), \mathcal{E}_t(S_1), \ldots, \mathcal{E}_t(S_{2n})).$$

Set

$$\mathcal{H}_t := \big(\mathcal{E}_t(S_{i+j-1})\big)_{i,j=1}^n, \qquad K_t := \operatorname{col}\big(\mathcal{E}_t(S_{n+i})\big)_{i\in[n]}.$$

Then

$$\operatorname{mult}_\mu t = p - \operatorname{card}\operatorname{Dep}(X_{\mathcal{T}_t}^n) + \operatorname{card} A_{\mathcal{T}_t} - \left[\operatorname{rank}\begin{pmatrix} \mathcal{H}_t & K_t \\ K_t^{\mathsf{T}} & \mathcal{E}_t(S_{2n+1}) \end{pmatrix} - \operatorname{rank}\begin{pmatrix} \mathcal{H}_t \\ K_t^{\mathsf{T}} \end{pmatrix}\right], \tag{3.1}$$

where $A_{\mathcal{T}_t} := \bigcup_{i=1}^n \operatorname{DepN}_1\Big(X_{\mathcal{T}_t}^i\Big)$ is as in (2.16).

Proof. Let m denote the right-hand side of (3.1). We need to prove that

$$m = \operatorname{mult}_\mu t. \tag{3.2}$$

The proof has two parts. First, we construct a matrix-polynomial column relation whose determinant yields the lower bound $\operatorname{mult}_\mu t \geq m$. Second, we remove the mass at t and compare the ranks of the resulting flat moment matrices to obtain the reverse inequality.

Define

$$\mathcal{H}_{0,t} := \operatorname{col}\big(\mathcal{E}_t(S_{i-1})\big)_{i\in[n]}.$$

Since μ is a $(\operatorname{rank} M(n))$–atomic representing measure, its moment sequence has a positive semidefinite flat extension $M(n+1)$. More precisely, $M(n+1) \succeq 0$ and $\operatorname{rank} M(n+1) = \operatorname{rank} M(n)$. Set

$$S_{2n+2} := \int_{\mathbb{R}} x^{2n+2}\, d\mu \qquad \text{and} \qquad \mathcal{E}_t(S_{2n+2}) := \int_{\mathbb{R}} (x-t)^{2n+2}\, d\mu.$$

Write

$$\mathcal{T}_t^{(2n+2)} := (\mathcal{E}_t(S_0), \mathcal{E}_t(S_1), \ldots, \mathcal{E}_t(S_{2n+2})).$$

The moment matrices

$$M(n+1) := (S_{i+j-2})_{i,j=1}^{n+2} \qquad \text{and} \qquad M_{\mathcal{T}_t^{(2n+2)}}(n+1) := \big(\mathcal{E}_t(S_{i+j-2})\big)_{i,j=1}^{n+2}$$

are positive semidefinite and satisfy

$$\operatorname{rank} M_{\mathcal{T}_t^{(2n+2)}}(n+1) = \operatorname{rank} M_{\mathcal{T}_t}(n) = \operatorname{rank} M(n).$$

Consequently,

$$\mathcal{C}\Big(\operatorname{col}\big(\mathcal{E}_t(S_{n+i})\big)_{i\in[n+1]}\Big) \subseteq \mathcal{C}(M_{\mathcal{T}_t}(n))\,.$$

Equivalently,

$$\operatorname{rank}\begin{pmatrix} \mathcal{H}_{0,t} & \mathcal{H}_t & K_t \\ \mathcal{E}_t(S_n) & K_t^{\mathsf{T}} & \mathcal{E}_t(S_{2n+1}) \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \mathcal{H}_{0,t} & \mathcal{H}_t \\ \mathcal{E}_t(S_n) & K_t^{\mathsf{T}} \end{pmatrix}. \tag{3.3}$$

Let $P \in M_p(\mathbb{R})$ be a permutation matrix that arranges the columns in the order C_1, C_2, C_3, where

$$\begin{aligned} C_1 &:= \operatorname{DepN}_0\Big(X^{n+1}_{\mathcal{T}_t^{(2n+2)}}\Big), \\ C_2 &:= \operatorname{DepN}_1\Big(X^{n+1}_{\mathcal{T}_t^{(2n+2)}}\Big), \\ C_3 &:= \operatorname{Dep}\Big(X^{n}_{\mathcal{T}_t^{(2n+2)}}\Big). \end{aligned}$$

Set

$$p_0 := p - \operatorname{card}\operatorname{Dep}(X^n_{\mathcal{T}_t})\,, \qquad r_t := p_0 - m + \operatorname{card} A_{\mathcal{T}_t}, \tag{3.4}$$

and

$$s_t := m - \operatorname{card} A_{\mathcal{T}_t}, \qquad q := p - p_0.$$

By the definitions of C_1, C_2, and C_3, and by the definitions of m, r_t, s_t, and q, we have

$$\operatorname{card} C_1 = r_t, \qquad \operatorname{card} C_2 = s_t, \qquad \operatorname{card} C_3 = q.$$

Decompose

$$\widehat{K}_t \equiv \begin{pmatrix} \widehat{K}_{t,1} & \widehat{K}_{t,2} & \widehat{K}_{t,3} \end{pmatrix} := K_t P, \tag{3.5}$$

where the three blocks have r_t, s_t, and q columns, respectively. Also write

$$\widehat{Z}_t := P^{\mathsf{T}} \mathcal{E}_t(S_{2n+1}) P \equiv \begin{pmatrix} \widehat{Z}_{t,1} & \widehat{Z}_{t,2} & \widehat{Z}_{t,3} \\ \widehat{Z}_{t,2}^{\mathsf{T}} & \widehat{Z}_{t,4} & \widehat{Z}_{t,5} \\ \widehat{Z}_{t,3}^{\mathsf{T}} & \widehat{Z}_{t,5}^{\mathsf{T}} & \widehat{Z}_{t,6} \end{pmatrix}. \tag{3.6}$$

The block sizes are

$$\begin{aligned} &\widehat{Z}_{t,1} \in M_{r_t}(\mathbb{R}), && \widehat{Z}_{t,2} \in M_{r_t \times s_t}(\mathbb{R}), && \widehat{Z}_{t,3} \in M_{r_t \times q}(\mathbb{R}), \\ &\widehat{Z}_{t,4} \in M_{s_t}(\mathbb{R}), && \widehat{Z}_{t,5} \in M_{s_t \times q}(\mathbb{R}), && \widehat{Z}_{t,6} \in M_{q}(\mathbb{R}). \end{aligned}$$

Since P is a permutation matrix,

$$\begin{pmatrix} \mathcal{H}_t & \widehat{K}_t \\ \widehat{K}_t^{\mathsf{T}} & \widehat{Z}_t \end{pmatrix} = (I_{np} \oplus P^{\mathsf{T}}) \begin{pmatrix} \mathcal{H}_t & K_t \\ K_t^{\mathsf{T}} & \mathcal{E}_t(S_{2n+1}) \end{pmatrix} (I_{np} \oplus P), \tag{3.7}$$

It follows from the definition of r_t and rank invariance under multiplication by invertible matrices that

$$\operatorname{rank} \begin{pmatrix} \mathcal{H}_t & \widehat{K}_{t,1} \\ \widehat{K}_{t,1}^{\mathsf{T}} & \widehat{Z}_{t,1} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \mathcal{H}_t & \widehat{K}_t \\ \widehat{K}_t^{\mathsf{T}} & \widehat{Z}_t \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \mathcal{H}_t \\ \widehat{K}_t^{\mathsf{T}} \end{pmatrix} + r_t. \tag{3.8}$$

Define

$$\widehat{T}_n \equiv \begin{pmatrix} \widehat{T}_{n,1} & \widehat{T}_{n,2} & \widehat{T}_{n,3} \end{pmatrix} := \mathcal{E}_t(S_n) P, \tag{3.9}$$

where the three blocks have r_t, s_t, and q columns, respectively. Then

$$\begin{pmatrix} \mathcal{H}_{0,t} & \mathcal{H}_t & \widehat{K}_t \\ \widehat{T}_n^{\mathsf{T}} & \widehat{K}_t^{\mathsf{T}} & \widehat{Z}_t \end{pmatrix} = (I_{np} \oplus P^{\mathsf{T}}) \begin{pmatrix} \mathcal{H}_{0,t} & \mathcal{H}_t & K_t \\ \mathcal{E}_t(S_n) & K_t^{\mathsf{T}} & \mathcal{E}_t(S_{2n+1}) \end{pmatrix} (I_{(n+1)p} \oplus P). \tag{3.10}$$

Thus, (3.3) implies

$$\operatorname{rank} \begin{pmatrix} \mathcal{H}_{0,t} & \mathcal{H}_t & \widehat{K}_t \\ \widehat{T}_n^{\mathsf{T}} & \widehat{K}_t^{\mathsf{T}} & \widehat{Z}_t \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \mathcal{H}_{0,t} & \mathcal{H}_t \\ \widehat{T}_n^{\mathsf{T}} & \widehat{K}_t^{\mathsf{T}} \end{pmatrix} = \operatorname{rank} M_{\mathcal{T}_t}(n). \tag{3.11}$$

We next construct a block column relation of $M_{\mathcal{T}_t^{(2n+2)}}(n+1)$ whose determinant has a zero at the origin of sufficiently high order.

By Proposition 2.8, there exists a block column relation

$$H(x) = \sum_{i=0}^{n} x^i H_i$$

of $M_{\mathcal{T}_t}(n)$ satisfying the lower bound (2.21), with

$$H_i = \begin{pmatrix} \mathbf{0}_{p \times p_0} & \widehat{H}_i \end{pmatrix}, \qquad i \in [0;n], \tag{3.12}$$

and

$$\operatorname{rank} H_n = p - p_0 = q.$$

Since $M_{\mathcal{T}_t^{(2n+2)}}(n+1)$ is block–recursively generated, the relation H propagates to the extended moment matrix:

$$\begin{pmatrix} \mathcal{H}_t & K_t \\ K_t^{\mathsf{T}} & \mathcal{E}_t(S_{2n+1}) \end{pmatrix} \operatorname{col}(H_i)_{i \in [0;n]} = \mathbf{0}_{(n+1)p \times p}. \tag{3.13}$$

Applying the permutation gives

$$\begin{pmatrix} \mathcal{H}_t & \widehat{K}_t \\ \widehat{K}_t^{\mathsf{T}} & \widehat{Z}_t \end{pmatrix} (I_{np} \oplus P^{\mathsf{T}}) \operatorname{col}(H_i)_{i \in [0;n]} = \mathbf{0}_{(n+1)p \times p}. \tag{3.14}$$

In accordance with the partition induced by P, write

$$P^{\mathsf{T}} H_n = \begin{pmatrix} \mathbf{0}_{r_t\times p_0} & G_1 \\ \mathbf{0}_{s_t\times p_0} & G_2 \\ \mathbf{0}_{q\times p_0} & G_3 \end{pmatrix}. \tag{3.15}$$

By the definition of P, the last q columns correspond precisely to the dependent columns of $X^n_{\mathcal{T}_t}$. Since $\operatorname{rank} H_n = q$, the matrix $G_3 \in M_q(\mathbb{R})$ is invertible.

The first equality in (3.8) shows that the columns of the block containing $\widehat{K}_{t,2}$ and the corresponding blocks of $\widehat{Z}_t$ belong to the column space generated by the preceding blocks. Hence, there exist matrices

$$J_1 \in M_{np\times s_t}(\mathbb{R}), \qquad J_2 \in M_{r_t\times s_t}(\mathbb{R})$$

such that

$$\begin{pmatrix} \mathcal{H}_t & \widehat{K}_{t,1} & \widehat{K}_{t,2} \\ \widehat{K}^{\mathsf{T}}_{t,1} & \widehat{Z}_{t,1} & \widehat{Z}_{t,2} \\ \widehat{K}^{\mathsf{T}}_{t,2} & \widehat{Z}^{\mathsf{T}}_{t,2} & \widehat{Z}_{t,4} \\ \widehat{K}^{\mathsf{T}}_{t,3} & \widehat{Z}^{\mathsf{T}}_{t,3} & \widehat{Z}^{\mathsf{T}}_{t,5} \end{pmatrix} \begin{pmatrix} -J_1 \\ -J_2 \\ I_{s_t} \end{pmatrix} = \mathbf{0}_{(n+1)p\times s_t}. \tag{3.16}$$

Similarly, the first equality in (3.11) implies that the relevant block of r_t columns belongs to the column space generated by the preceding blocks. Therefore, there exist matrices

$$U_1 \in M_{p\times r_t}(\mathbb{R}), \qquad U_2 \in M_{np\times r_t}(\mathbb{R})$$

such that

$$\begin{pmatrix} \mathcal{H}_{0,t} & \mathcal{H}_t & \widehat{K}_{t,1} \\ \widehat{T}^{\mathsf{T}}_{n,1} & \widehat{K}^{\mathsf{T}}_{t,1} & \widehat{Z}_{t,1} \\ \widehat{T}^{\mathsf{T}}_{n,2} & \widehat{K}^{\mathsf{T}}_{t,2} & \widehat{Z}^{\mathsf{T}}_{t,2} \\ \widehat{T}^{\mathsf{T}}_{n,3} & \widehat{K}^{\mathsf{T}}_{t,3} & \widehat{Z}^{\mathsf{T}}_{t,3} \end{pmatrix} \begin{pmatrix} -U_1 \\ -U_2 \\ I_{r_t} \end{pmatrix} = \mathbf{0}_{(n+1)p\times r_t}. \tag{3.17}$$

The second equality in (3.8) implies

$$\operatorname{rank} U_1 = r_t. \tag{3.18}$$

Combining (3.14), (3.12), (3.15), (3.16), and (3.17), we obtain

$$\begin{pmatrix} \mathcal{H}_{0,t} & \mathcal{H}_t & \widehat{K}_t \\ \widehat{T}^{\mathsf{T}}_n & \widehat{K}^{\mathsf{T}}_t & \widehat{Z}_t \end{pmatrix} \begin{pmatrix} -U_1 & \mathbf{0}_{p\times s_t} & \mathbf{0}_{p\times q} \\ -U_2 & -J_1 & \operatorname{col}(\widehat{H}_i)_{i\in[0;n-1]} \\ I_{r_t} & -J_2 & G_1 \\ \mathbf{0}_{s_t\times r_t} & I_{s_t} & G_2 \\ \mathbf{0}_{q\times r_t} & \mathbf{0}_{q\times s_t} & G_3 \end{pmatrix} = \mathbf{0}_{(n+1)p\times p}.$$

Therefore,

$$\begin{pmatrix} \widehat{K}_t \\ \widehat{Z}_t \end{pmatrix} \widehat{Q}_n = \operatorname{row}(\widehat{X}^i_{\mathcal{T}_t})_{i\in[n]} \begin{pmatrix} U_2 & J_1 & \operatorname{col}(-\widehat{H}_i)_{i\in[0;n-1]} \end{pmatrix} + \widehat{X}^0_{\mathcal{T}_t} \begin{pmatrix} U_1 & \mathbf{0}_{p\times(s_t+q)} \end{pmatrix}, \tag{3.19}$$

where

$$\widehat{Q}_n := \begin{pmatrix} I_{r_t} & -J_2 & G_1 \\ \mathbf{0}_{s_t\times r_t} & I_{s_t} & G_2 \\ \mathbf{0}_{q\times r_t} & \mathbf{0}_{q\times s_t} & G_3 \end{pmatrix}, \tag{3.20}$$

and

$$\operatorname{row}(\widehat{X}^i_{\mathcal{T}_t})_{i\in[0;n]} := (I_{np} \oplus P^{\mathsf{T}}) \operatorname{row}\Big(\operatorname{col}(\mathcal{E}_t(S_{i+j}))_{j\in[0;n]}\Big)_{i\in[0;n]}. \tag{3.21}$$

Since G_3 is invertible, so is $\widehat{Q}_n$.

Decompose

$$\begin{pmatrix} U_2 & J_1 \end{pmatrix} = \begin{pmatrix} U_{2,0} & J_{1,0} \\ U_{2,1} & J_{1,1} \\ \vdots & \vdots \\ U_{2,n-1} & J_{1,n-1} \end{pmatrix}, \tag{3.22}$$

where

$$U_{2,i} \in M_{p\times r_t}(\mathbb{R}), \qquad J_{1,i} \in M_{p\times s_t}(\mathbb{R}).$$

For $i \in [0; n-1]$, define

$$Q_i := \begin{pmatrix} U_{2,i} & J_{1,i} & -\widehat{H}_i \end{pmatrix}. \tag{3.23}$$

Then (3.19) becomes

$$\begin{pmatrix} \widehat{K}_t \\ \widehat{Z}_t \end{pmatrix} \widehat{Q}_n = \sum_{i=1}^{n} \widehat{X}^i_{\mathcal{T}_t} Q_{i-1} + \widehat{X}^0_{\mathcal{T}_t} \begin{pmatrix} U_1 & \mathbf{0}_{p\times(s_t+q)} \end{pmatrix}. \tag{3.24}$$

Set

$$Q_n := P\widehat{Q}_n. \tag{3.25}$$

Undoing the permutation in (3.24) yields

$$\operatorname{col}\big(\mathcal{E}_t(S_{n+i})\big)_{i\in[n+1]} Q_n = \sum_{i=1}^{n} X^i_{\mathcal{T}_t} Q_{i-1} + X^0_{\mathcal{T}_t} \begin{pmatrix} U_1 & \mathbf{0}_{p\times(s_t+q)} \end{pmatrix}. \tag{3.26}$$

Define

$$Q(x) := x^{n+1} Q_n - \sum_{i=1}^{n} x^i Q_{i-1} - \begin{pmatrix} U_1 & \mathbf{0}_{p\times(s_t+q)} \end{pmatrix}. \tag{3.27}$$

By (3.26), the polynomial $Q(x)$ is a block column relation of $M_{\mathcal{T}_t^{(2n+2)}}(n+1)$. Hence, Proposition 2.3 shows that $Q(x-t)$ is a block column relation of $M(n+1)$.

By (3.18),

$$\operatorname{rank} \begin{pmatrix} U_1 & \mathbf{0}_{p\times(s_t+q)} \end{pmatrix} = r_t. \tag{3.28}$$

Define

$$\widetilde{W}_0 := \operatorname{Ker}\begin{pmatrix} U_1 & \mathbf{0}_{p\times(s_t+q)} \end{pmatrix}, \qquad \widetilde{W}_i := \widetilde{W}_0 \cap \bigcap_{j=0}^{i-1} \operatorname{Ker} Q_j, \quad i \in [n].$$

Since Q_i ends with the block $-\widehat{H}_i$, for every $i \in [0; n-1]$ one has

$$v \in \bigcap_{j=0}^{i} \operatorname{Ker} \widehat{H}_j \quad \Longrightarrow \quad \begin{pmatrix} \mathbf{0}_{r_t+s_t} \\ v \end{pmatrix} \in \widetilde{W}_{i+1}. \tag{3.29}$$

Let

$$\widetilde{s}_i := \dim \widetilde{W}_i, \qquad i \in [0; n].$$

It follows from (3.29) that

$$\widetilde{s}_i \geq \dim\left(\bigcap_{j=0}^{i-1} \operatorname{Ker} \widehat{H}_j \right), \qquad i \in [n]. \tag{3.30}$$

Defining

$$\Phi(H) := \sum_{i=0}^{n} \dim\left(\bigcap_{j=0}^{i} \operatorname{Ker} H_j \right),$$

we have

$$\Phi(H) \geq (n+1)p_0 + (n - z_s)q + \sum_{j=1}^{s-1} (z_{j+1} - z_j)(q - r_{s-j}) + \operatorname{card} A_{\mathcal{T}_t}. \tag{3.31}$$

Moreover, (3.12) implies that, for every $i \in [0;n]$,

$$\bigcap_{j=0}^{i} \operatorname{Ker} H_j = \mathbb{R}^{p_0} \oplus \bigcap_{j=0}^{i} \operatorname{Ker} \widehat{H}_j.$$

Therefore, for every $w \in [0;n]$,

$$\sum_{i=0}^{w} \dim \left(\bigcap_{j=0}^{i} \operatorname{Ker} H_j \right) = (w+1)p_0 + \sum_{i=0}^{w} \dim \left(\bigcap_{j=0}^{i} \operatorname{Ker} \widehat{H}_j \right). \tag{3.32}$$

We have

$$\begin{aligned}
\sum_{i=0}^{n} \widetilde{s}_i &\geq \widetilde{s}_0 + \sum_{i=1}^{n} \dim \left(\bigcap_{j=0}^{i-1} \operatorname{Ker} \widehat{H}_j \right) && \text{by (3.30)} \\
&= \widetilde{s}_0 + \sum_{i=0}^{n-1} \dim \left(\bigcap_{j=0}^{i} \operatorname{Ker} \widehat{H}_j \right) && \text{by reindexing} \\
&= s_t + q + \sum_{i=0}^{n-1} \dim \left(\bigcap_{j=0}^{i} \operatorname{Ker} \widehat{H}_j \right) && \text{by (3.28)} \\
&= s_t + q + \sum_{i=0}^{n-1} \dim \left(\bigcap_{j=0}^{i} \operatorname{Ker} H_j \right) - np_0 && \text{by (3.32) with } w = n-1 \\
&= s_t + q + \Phi(H) - (n+1)p_0 && \text{since } \operatorname{Ker} \widehat{H}_n = \{\mathbf{0}\} \\
&\geq m + (n - z_s + 1)q + \sum_{j=1}^{s-1} (z_{j+1} - z_j)(q - r_{s-j}) && \text{by (3.31)} \\
&= m + (n+1)p - \left((n+1)p_0 + \sum_{j=1}^{s} z_{s-j+1} p_j \right) && \text{by the definitions of } p_j,\, r_j, \text{ and } q \\
&= m + (n+1)p - \operatorname{rank} M_{\mathcal{T}_t}(n) && \text{by block–recursive generation of } M_{\mathcal{T}_t}(n) \\
&= m + (n+1)p - \operatorname{rank} M(n) && \text{by Proposition 2.2.}
\end{aligned}$$

Since Q_n is invertible, Lemma 2.10, together with the estimate above, yields

$$\det Q(x-t) = (x-t)^{m+(n+1)p-\operatorname{rank} M(n)} g(x), \tag{3.33}$$

where $0 \neq g(x) \in \mathbb{R}[x]$. Moreover,

$$\deg \big(\det Q(x-t) \big) = (n+1)p,$$

and therefore

$$\deg g = \operatorname{rank} M(n) - m. \tag{3.34}$$

Write

$$\mu = \sum_{j=1}^{\ell} \delta_{x_j} A_j,$$

where the points $x_1, \ldots, x_\ell$ are pairwise distinct, $A_j \in \mathbb{S}_p^{\succeq 0}(\mathbb{R})$, and

$$\sum_{j=1}^{\ell} \operatorname{rank} A_j = \operatorname{rank} M(n).$$

By Lemma 2.9 and (3.33), every atom of μ different from t is a zero of g, and its multiplicity as an atom is at most its multiplicity as a zero of g. By (3.34) and since the total atomic multiplicity of μ equals $\operatorname{rank} M(n)$, it follows that $\operatorname{mult}_\mu t \geq m$. If $t \notin \operatorname{supp}(\mu)$, then $\operatorname{mult}_\mu t = 0$, so the preceding inequality implies that $m = 0$. Hence, $\operatorname{mult}_\mu t = m$. We may therefore assume that $t \in \operatorname{supp}(\mu)$. Write $t = x_{j'}$ and set

$$m' := \operatorname{rank} A_{j'} = \operatorname{mult}_\mu t.$$

Suppose, toward a contradiction, that $m' > m$. Define

$$\widetilde{\mu} := \mu - \delta_t A_{j'},$$

and, for $i \in [0; 2n+2]$, set

$$\widetilde{S}_i := \int_{\mathbb{R}} x^i \, d\widetilde{\mu}.$$

Let

$$\widetilde{\mathcal{S}} := (\widetilde{S}_0, \widetilde{S}_1, \ldots, \widetilde{S}_{2n+2}).$$

Since

$$M(n) - M_{\widetilde{\mathcal{S}}}(n)$$

is the moment matrix associated with the measure $\delta_t A_{j'}$, its rank is at most m'. Consequently,

$$\operatorname{rank} M(n) - m' \leq \operatorname{rank} M_{\widetilde{\mathcal{S}}}(n) \leq \operatorname{rank} M_{\widetilde{\mathcal{S}}}(n+1).$$

On the other hand, $\widetilde{\mu}$ is an atomic representing measure for $\widetilde{\mathcal{S}}$, and hence

$$\operatorname{rank} M_{\widetilde{\mathcal{S}}}(n+1) \leq \sum_{\substack{j=1 \\ j \neq j'}}^{\ell} \operatorname{rank} A_j = \operatorname{rank} M(n) - m'.$$

Therefore,

$$\operatorname{rank} M_{\widetilde{\mathcal{S}}}(n) = \operatorname{rank} M_{\widetilde{\mathcal{S}}}(n+1) = \operatorname{rank} M(n) - m'. \tag{3.35}$$

For $i \in [0; 2n+2]$, define the shifted moments

$$\widetilde{T}_i := \int_{\mathbb{R}} (x-t)^i \, d\widetilde{\mu}.$$

Since the removed mass is supported at t, the zeroth shifted moment changes by $A_{j'}$, whereas all shifted moments of positive degree remain unchanged. More precisely,

$$\widetilde{T}_0 = \mathcal{E}_t(S_0) - A_{j'},$$

while

$$\widetilde{T}_i = \mathcal{E}_t(S_i), \qquad i \in [2n+2]. \tag{3.36}$$

Thus, $i = 0$ is the only index for which the shifted moments associated with $\widetilde{\mu}$ and μ differ.

Let

$$\widetilde{X}^i := \operatorname{col}\big(\widetilde{T}_{i+j}\big)_{j \in [0; n+1]}, \qquad i \in [0; n+1].$$

By (3.35) and Proposition 2.2,

$$m' = \operatorname{rank} M(n) - \operatorname{rank} \operatorname{row}\big(\widetilde{X}^i\big)_{i \in [n+1]}. \tag{3.37}$$

Let

$$X^i := \operatorname{col}\big(\mathcal{E}_t(S_{i+j})\big)_{j \in [0; n+1]}, \qquad i \in [0; n+1].$$

By (3.36),

$$\operatorname{row}\big(\widetilde{X}^i\big)_{i \in [n+1]} = \operatorname{row}\big(X^i\big)_{i \in [n+1]}.$$

Moreover, the definitions of the dependence indices give

$$\operatorname{rank} \operatorname{row}\big(X^i\big)_{i \in [n]} = \operatorname{rank} M_{\mathcal{T}_t}(n) - p_0 - \operatorname{card} A_{\mathcal{T}_t}.$$

Passing from the block columns indexed by $[n]$ to those indexed by $[n+1]$ increases the rank by

$$r_t = p_0 - m + \operatorname{card} A_{\mathcal{T}_t}.$$

It follows that

$$\operatorname{rank} \operatorname{row}\big(\widetilde{X}^i\big)_{i \in [n+1]} = \operatorname{rank} \operatorname{row}\big(X^i\big)_{i \in [n+1]}$$

$$= \operatorname{rank} M_{\mathcal{T}_t}(n) - p_0 - \operatorname{card} A_{\mathcal{T}_t} + r_t$$
$$= \operatorname{rank} M(n) - m,$$

where the last equality also uses Proposition 2.2. Combining this identity with (3.37) yields

$$m' = \operatorname{rank} M(n) - (\operatorname{rank} M(n) - m) = m,$$

contradicting $m' > m$. Therefore, $\operatorname{rank} A_{j'} \leq m$. Together with the previously established inequality $\operatorname{rank} A_{j'} \geq m$, this gives $\operatorname{rank} A_{j'} = m$. Consequently,

$$\operatorname{mult}_\mu t = m,$$

which completes the proof. □

The following immediate consequence of Theorem 3.1 will be used in the proof of the main result.

Corollary 3.2. *Let $n, p \in \mathbb{N}$, and let*

$$\mathcal{S} := (S_0, S_1, \ldots, S_{2n}) \in (\mathbb{S}_p(\mathbb{R}))^{2n+1}$$

be a sequence with moment matrix $M(n)$. Suppose that $\mathcal{S}$ admits a $(\operatorname{rank} M(n))$–atomic $\mathbb{R}$–representing measure μ. Set $S_{2n+1} := \int_{\mathbb{R}} x^{2n+1}\, d\mu$, fix $t \in \mathbb{R}$, and adopt the notation $\mathcal{T}_t$, $\mathcal{H}_t$, K_t, and $A_{\mathcal{T}_t}$ from Theorem 3.1.

Then the following statements are equivalent:

(1)
$$\operatorname{mult}_\mu t = \operatorname{card} A_{\mathcal{T}_t}.$$

(2)
$$\operatorname{rank} \begin{pmatrix} \mathcal{H}_t & K_t \\ K_t^{\mathsf{T}} & \mathcal{E}_t(S_{2n+1}) \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \mathcal{H}_t \\ K_t^{\mathsf{T}} \end{pmatrix} + p - \operatorname{card} \operatorname{Dep}(X^n_{\mathcal{T}_t}) .$$

Proof. By Theorem 3.1,

$$\operatorname{mult}_\mu t - \operatorname{card} A_{\mathcal{T}_t} = p - \operatorname{card} \operatorname{Dep}(X^n_{\mathcal{T}_t}) - \left[\operatorname{rank} \begin{pmatrix} \mathcal{H}_t & K_t \\ K_t^{\mathsf{T}} & \mathcal{E}_t(S_{2n+1}) \end{pmatrix} - \operatorname{rank} \begin{pmatrix} \mathcal{H}_t \\ K_t^{\mathsf{T}} \end{pmatrix}\right].$$

Hence, $\operatorname{mult}_\mu t = \operatorname{card} A_{\mathcal{T}_t}$ if and only if the rank identity in (2) holds. □

The following consequence of Theorem 3.1 applies to arbitrary finitely atomic representing measures.

Corollary 3.3. *Let $n, p \in \mathbb{N}$, and let*

$$\mathcal{S} := (S_0, S_1, \ldots, S_{2n}) \in (\mathbb{S}_p(\mathbb{R}))^{2n+1}$$

be a sequence with moment matrix $M(n)$. Suppose that $\mathcal{S}$ admits a finitely atomic $\mathbb{R}$–representing measure μ. Fix $t \in \mathbb{R}$. Then

$$\operatorname{mult}_\mu t \geq \operatorname{card} A_{\mathcal{T}_t},$$

where $A_{\mathcal{T}_t}$ is defined as in Theorem 3.1.

Proof. Extend the moment sequence by setting

$$S_i := \int_{\mathbb{R}} x^i\, d\mu, \qquad i \geq 2n+1.$$

Since μ is finitely atomic, there exists $k \in \mathbb{N}$ such that

$$\operatorname{rank} M(n+k) = \operatorname{rank} M(n+k-1) = \sum_{x \in \operatorname{supp} \mu} \operatorname{mult}_\mu x.$$

Thus, μ is a $(\operatorname{rank} M(n+k))$–atomic representing measure for the extended sequence $(S_0, \ldots, S_{2n+2k})$. Set

$$\mathcal{T}_t := (\mathcal{E}_t(S_0), \mathcal{E}_t(S_1), \ldots, \mathcal{E}_t(S_{2n+2k})), \qquad \text{where } \mathcal{E}_t(S_i) := \int_{\mathbb{R}} (x-t)^i\, d\mu, \quad i \geq 0.$$

For $r \in [n+k]$, define

$$A_{\mathcal{T}_t}(r) := \bigcup_{i=1}^{r} \mathrm{DepN}_1\Big(X_{\mathcal{T}_t}^{i}\Big).$$

The flatness condition gives $p = \operatorname{card}\mathrm{Dep}\Big(X_{\mathcal{T}_t}^{n+k}\Big)$. Hence, that theorem yields

$$\begin{aligned}
\mathrm{mult}_\mu t &= p - \operatorname{card}\mathrm{Dep}\Big(X_{\mathcal{T}_t}^{n+k-1}\Big) + \operatorname{card} A_{\mathcal{T}_t}(n+k-1) \\
&= \operatorname{card}\mathrm{Dep}\Big(X_{\mathcal{T}_t}^{n+k}\Big) - \operatorname{card}\mathrm{Dep}\Big(X_{\mathcal{T}_t}^{n+k-1}\Big) + \operatorname{card} A_{\mathcal{T}_t}(n+k-1) \\
&= \operatorname{card}\mathrm{DepN}\Big(X_{\mathcal{T}_t}^{n+k}\Big) + \operatorname{card} A_{\mathcal{T}_t}(n+k-1) \\
&\geq \operatorname{card}\mathrm{DepN}_1\Big(X_{\mathcal{T}_t}^{n+k}\Big) + \operatorname{card} A_{\mathcal{T}_t}(n+k-1) \\
&= \operatorname{card} A_{\mathcal{T}_t}(n+k).
\end{aligned}$$

Since

$$A_{\mathcal{T}_t} = A_{\mathcal{T}_t}(n) \subseteq A_{\mathcal{T}_t}(n+k),$$

we conclude that

$$\mathrm{mult}_\mu t \geq \operatorname{card} A_{\mathcal{T}_t}(n+k) \geq \operatorname{card} A_{\mathcal{T}_t},$$

as claimed. □

4. Minimal $\mathbb{R}$-representing Measures with Minimal Multiplicities at Prescribed Atoms

Let $L\colon \mathbb{R}[x]_{\leq 2n} \to \mathbb{S}_p(\mathbb{R})$ be a linear operator, and let $t_1, \ldots, t_k \in \mathbb{R}$ be pairwise distinct. In this section, we characterize when L admits a $\mathbb{R}$–representing matrix measure μ satisfying $\mathrm{mult}_\mu t_j = \operatorname{card} A_{\mathcal{T}_{t_j}}$, $j \in [k]$, where $A_{\mathcal{T}_{t_j}}$ is defined in (2.16). By Corollary 3.3, every finitely atomic representing measure ν for L satisfies $\mathrm{mult}_\nu t_j \geq \operatorname{card} A_{\mathcal{T}_{t_j}}$, $j \in [k]$. Thus, the multiplicity of each prescribed point t_j in μ is the smallest possible. We use this result to prove Theorem 1.1 for $K = \mathbb{R}$.

We begin with an auxiliary rank lemma.

Lemma 4.1. *Let $m, n \in \mathbb{N}$, let $A \in \mathbb{S}_m(\mathbb{R})$, $B \in M_{m\times n}(\mathbb{R})$, and $P \in \mathbb{S}_n(\mathbb{R})$. If P is either positive definite or negative definite, then*

$$\operatorname{rank}\begin{pmatrix} A & B \\ B^\mathsf{T} & B^\mathsf{T} A^\dagger B + P \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A & B \end{pmatrix} + n,$$

where $A^\dagger$ denotes the Moore–Penrose pseudoinverse of A.

Proof. Set $r := \operatorname{rank} A$. Choose an orthogonal matrix $Q \in M_m(\mathbb{R})$ such that

$$Q^\mathsf{T} A Q = A_1 \oplus \mathbf{0}_{m-r},$$

where $A_1 \in \mathbb{S}_r(\mathbb{R})$ is invertible, and write

$$Q^\mathsf{T} B = \mathrm{col}(B_1, B_2), \qquad B_1 \in M_{r\times n}(\mathbb{R}), \quad B_2 \in M_{(m-r)\times n}(\mathbb{R}).$$

Since

$$A^\dagger = Q \begin{pmatrix} A_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} Q^\mathsf{T},$$

one has

$$B^\mathsf{T} A^\dagger B = B_1^\mathsf{T} A_1^{-1} B_1.$$

Hence, by a congruence transformation and the Schur complement with respect to A_1,

$$\operatorname{rank}\begin{pmatrix} A & B \\ B^\mathsf{T} & B^\mathsf{T} A^\dagger B + P \end{pmatrix} = r + \operatorname{rank}\begin{pmatrix} \mathbf{0}_{m-r} & B_2 \\ B_2^\mathsf{T} & P \end{pmatrix}. \tag{4.1}$$

Since P is definite, it is invertible. Taking the Schur complement with respect to P gives

$$\operatorname{rank}\begin{pmatrix}\mathbf{0}_{m-r} & B_2\\ B_2^\mathsf{T} & P\end{pmatrix} = n + \operatorname{rank}\Big(B_2 P^{-1} B_2^\mathsf{T}\Big)\,.$$

Because P^{-1} is definite,

$$\operatorname{rank}\Big(B_2 P^{-1} B_2^\mathsf{T}\Big) = \operatorname{rank} B_2.$$

Therefore,

$$\operatorname{rank}\begin{pmatrix}A & B\\ B^\mathsf{T} & B^\mathsf{T} A^\dagger B + P\end{pmatrix} = r + \operatorname{rank} B_2 + n. \tag{4.2}$$

On the other hand, the same orthogonal change of basis yields

$$\operatorname{rank}\begin{pmatrix}A & B\end{pmatrix} = \operatorname{rank}\begin{pmatrix}A_1 & \mathbf{0} & B_1\\ \mathbf{0} & \mathbf{0} & B_2\end{pmatrix} = r + \operatorname{rank} B_2.$$

Combining this identity with (4.2) proves the result. □

The following result shows that the multiplicities identified in Theorem 3.1 can be attained simultaneously at any finite collection of prescribed points. More precisely, among all representing matrix measures, one can choose a minimal measure whose multiplicity at each prescribed point t_j is the smallest possible value $\operatorname{card} A_{\mathcal{T}_{t_j}}$.

Theorem 4.2. *Let $n, p \in \mathbb{N}$, and let $t_1, \ldots, t_{k_1} \in \mathbb{R}$ be pairwise distinct. Let*

$$L\colon \mathbb{R}[x]_{\leq 2n} \to \mathbb{S}_p(\mathbb{R})$$

be a linear operator with a $\mathbb{R}$–representing matrix measure. Then L admits a $(\operatorname{rank} M(n))$–atomic $\mathbb{R}$–representing matrix measure μ such that

$$\operatorname{mult}_\mu t_j = \operatorname{card} A_{\mathcal{T}_{t_j}}, \qquad j \in [k_1],$$

where $A_{\mathcal{T}_{t_j}}$ is defined in (2.16).

Proof. Set $S_i := L(x^i)$, $i \in [0; 2n]$, and write $\mathcal{S} := (S_0, S_1, \ldots, S_{2n})$. For each $t \in \mathbb{R}$, let $\mathcal{T}_t$ be the shifted sequence defined in (2.8), and adopt the notation introduced in Subsections 2.6 and 2.7.

For $t \in \mathbb{R}$, write

$$X^0_{\mathcal{T}_t} =: \begin{pmatrix}\mathcal{H}_{0,t}\\ \mathcal{E}_t(S_n)\end{pmatrix}, \qquad \operatorname{row}(X^i_{\mathcal{T}_t})_{i\in[n]} =: \begin{pmatrix}\mathcal{H}_t\\ K_t^\mathsf{T}\end{pmatrix} \in \begin{pmatrix}\mathbb{S}_{np}(\mathbb{R})\\ M_{p\times np}(\mathbb{R})\end{pmatrix}. \tag{4.3}$$

By Proposition 2.5, the set $\operatorname{Dep}(X^n_{\mathcal{T}_t})$ is independent of t. Set

$$p_0 := p - \operatorname{card}\operatorname{Dep}(X^n_{\mathcal{T}_t})\,.$$

Choose a permutation matrix $P \in M_p(\mathbb{R})$ such that, upon multiplication from the right, the first p_0 columns correspond to $[p] \setminus \operatorname{Dep}(X^n_{\mathcal{T}_t})$ and the remaining $p - p_0$ columns correspond to $\operatorname{Dep}(X^n_{\mathcal{T}_t})$. For each $t \in \mathbb{R}$, define

$$\widehat{K_t} \equiv \begin{pmatrix}\widehat{K}_{t,1} & \widehat{K}_{t,3}\end{pmatrix} := K_t P, \tag{4.4}$$

where $\widehat{K}_{t,1}$ has p_0 columns. By the definition of $\operatorname{Dep}(X^n_{\mathcal{T}_t})$,

$$\operatorname{rank}\begin{pmatrix}\mathcal{H}_t & \widehat{K}_{t,1}\end{pmatrix} = \operatorname{rank}\begin{pmatrix}\mathcal{H}_t & \widehat{K_t}\end{pmatrix}, \qquad t \in \mathbb{R}. \tag{4.5}$$

Fix $t \in \mathbb{R}$. By Proposition 2.8, there exists a block column relation

$$H(x) = \sum_{i=0}^n x^i H_i$$

of $M_{\mathcal{T}_t}(n)$ such that

$$\operatorname{rank} H_n = p - p_0, \qquad H_i = \begin{pmatrix}\mathbf{0}_{p\times p_0} & \widehat{H}_i\end{pmatrix}, \qquad i \in [0; n].$$

In the block decomposition

$$P^{\mathsf{T}} H_n = \begin{pmatrix} \mathbf{0}_{p_0\times p_0} & G_1 \\ \mathbf{0}_{(p-p_0)\times p_0} & G_3 \end{pmatrix}, \tag{4.6}$$

the matrix $G_3 \in M_{p-p_0}(\mathbb{R})$ is invertible. Consequently, there exists a matrix $\widehat{G}$ such that

$$\widehat{K_{t,3}} = \begin{pmatrix} \mathcal{H}_t & \widehat{K_{t,1}} \end{pmatrix} \widehat{G}. \tag{4.7}$$

For each $j \in [k_1]$, set

$$E_j := \sum_{i=1}^{2n+1} \binom{2n+1}{i} (-1)^i (t_j^i - t^i) \widehat{S}_{2n+1-i,1} - \widehat{K_{t_j,1}^{\mathsf{T}}} \mathcal{H}_{t_j}^{\dagger} \widehat{K_{t_j,1}}, \tag{4.8}$$

where

$$\widehat{S}_i := P^{\mathsf{T}} S_i P$$

and $\widehat{S}_{i,1}$ denotes the upper-left $p_0 \times p_0$ principal submatrix of $\widehat{S}_i$.

Choose a symmetric matrix $\widehat{Z}_{t,1} \in \mathbb{S}_{p_0}(\mathbb{R})$ such that

$$\widehat{Z}_{t,1} + E_j \succ 0, \qquad j \in [k_1]. \tag{4.9}$$

Define

$$\widehat{Z}_{t,3} := \begin{pmatrix} \widehat{K_{t,1}^{\mathsf{T}}} & \widehat{Z}_{t,1} \end{pmatrix} \widehat{G}, \qquad \widehat{Z}_{t,6} := \begin{pmatrix} \widehat{K_{t,3}^{\mathsf{T}}} & \widehat{Z}_{t,3}^{\mathsf{T}} \end{pmatrix} \widehat{G}, \tag{4.10}$$

and set

$$\widehat{Z}_t := \begin{pmatrix} \widehat{Z}_{t,1} & \widehat{Z}_{t,3} \\ \widehat{Z}_{t,3}^{\mathsf{T}} & \widehat{Z}_{t,6} \end{pmatrix}, \qquad Z_t := P \widehat{Z}_t P^{\mathsf{T}}. \tag{4.11}$$

The matrix $\widehat{Z}_t$ is symmetric. Indeed, (4.7) and (4.10) give

$$\widehat{Z}_{t,6} = \widehat{G}^{\mathsf{T}} \begin{pmatrix} \mathcal{H}_t & \widehat{K_{t,1}} \\ \widehat{K_{t,1}^{\mathsf{T}}} & \widehat{Z}_{t,1} \end{pmatrix} \widehat{G}.$$

We next verify that the choice of $\widehat{Z}_{t,1}$ does not obstruct the construction of a flat extension. More precisely, we claim that

$$\operatorname{rank} \begin{pmatrix} \mathcal{H}_{0,t} & \mathcal{H}_t & \widehat{K_t} \\ \mathcal{E}_t(S_n) & \widehat{K_t^{\mathsf{T}}} & \widehat{Z}_t \end{pmatrix} = \operatorname{rank} M_{\mathcal{T}_t}(n). \tag{4.12}$$

In particular,

$$\mathcal{C}\big(\operatorname{col}(K_t, Z_t)\big) \subseteq \mathcal{C}\big(M_{\mathcal{T}_t}(n)\big). \tag{4.13}$$

Indeed, by (4.7), (4.10), and (4.11), we have

$$\begin{pmatrix} \widehat{K_{t,3}} \\ \widehat{Z}_{t,3} \end{pmatrix} = \begin{pmatrix} \mathcal{H}_t & \widehat{K_{t,1}} \\ \widehat{K_{t,1}^{\mathsf{T}}} & \widehat{Z}_{t,1} \end{pmatrix} \widehat{G}. \qquad \text{and} \qquad \widehat{Z}_{t,6} = \begin{pmatrix} \widehat{K_{t,3}^{\mathsf{T}}} & \widehat{Z}_{t,3}^{\mathsf{T}} \end{pmatrix} \widehat{G}.$$

Consequently, the blocks $\widehat{K_{t,3}}$, $\widehat{Z}_{t,3}$, and $\widehat{Z}_{t,6}$ do not increase the rank, and hence

$$\operatorname{rank} \begin{pmatrix} \mathcal{H}_{0,t} & \mathcal{H}_t & \widehat{K_t} \\ \mathcal{E}_t(S_n) & \widehat{K_t^{\mathsf{T}}} & \widehat{Z}_t \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \mathcal{H}_{0,t} & \mathcal{H}_t & \widehat{K_{t,1}} \\ \mathcal{E}_t(S_n) & \widehat{K_{t,1}^{\mathsf{T}}} & \widehat{Z}_{t,1} \end{pmatrix}. \tag{4.14}$$

We prove that the rank on the right-hand side of (4.14) equals $\operatorname{rank} M_{\mathcal{T}_t}(n)$. Set

$$\widehat{T}_n \equiv \begin{pmatrix} \widehat{T}_{n,1} & \widehat{T}_{n,3} \end{pmatrix} := \mathcal{E}_t(S_n) P,$$

where $\widehat{T}_{n,1}$ has p_0 columns. By the same column relations that give

$$\widehat{K_{t,3}} = \begin{pmatrix} \mathcal{H}_t & \widehat{K_{t,1}} \end{pmatrix} \widehat{G},$$

we also have

$$\widehat{T}_{n,3} = \begin{pmatrix} \mathcal{H}_{0,t}^{\mathsf{T}} & \widehat{T}_{n,1} \end{pmatrix} \widehat{G}.$$

Consequently,

$$\operatorname{rank} M_{\mathcal{T}_t}(n) = \operatorname{rank} \begin{pmatrix} \mathcal{H}_{0,t} & \mathcal{H}_t \\ \widehat{T}_{n,1}^{\mathsf{T}} & \widehat{K}_{t,1}^{\mathsf{T}} \end{pmatrix}. \tag{4.15}$$

Moreover, by the definition of p_0,

$$\operatorname{rank} M_{\mathcal{T}_t}(n) = \operatorname{rank} \begin{pmatrix} \mathcal{H}_{0,t} & \mathcal{H}_t \end{pmatrix} + p_0. \tag{4.16}$$

Permute the rows of $\begin{pmatrix} \mathcal{H}_{0,t} & \mathcal{H}_t \end{pmatrix}$ using a permutation matrix $P_2 \in M_{np}(\mathbb{R})$ such that

$$P_2^T \begin{pmatrix} \mathcal{H}_{0,t} & \mathcal{H}_t \end{pmatrix} = \begin{pmatrix} \mathcal{H}_{0,t,1} & \mathcal{H}_{t,1} \\ \mathcal{H}_{0,t,2} & \mathcal{H}_{t,2} \end{pmatrix},$$

where

$$\operatorname{rank} \begin{pmatrix} \mathcal{H}_{0,t,1} & \mathcal{H}_{t,1} \\ \mathcal{H}_{0,t,2} & \mathcal{H}_{t,2} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \mathcal{H}_{0,t,2} & \mathcal{H}_{t,2} \end{pmatrix}. \tag{4.17}$$

Apply the same permutation to $\widehat{K}_{t,1}^{\mathsf{T}}$ and write

$$\widehat{K}_{t,1}^{\mathsf{T}} P_2 =: \begin{pmatrix} (\widehat{K}_{t,1}^{(1)})^{\mathsf{T}} & (\widehat{K}_{t,1}^{(2)})^{\mathsf{T}} \end{pmatrix}.$$

By (4.15) and (4.16), the permutation P_2 can be chosen so that

$$\operatorname{Ker} \begin{pmatrix} \mathcal{H}_{0,t,2}^{\mathsf{T}} & \widehat{T}_{n,1} \\ \mathcal{H}_{t,2}^{\mathsf{T}} & \widehat{K}_{t,1} \end{pmatrix} = \{0\}. \tag{4.18}$$

Equivalently, the matrix

$$\begin{pmatrix} \mathcal{H}_{0,t,2} & \mathcal{H}_{t,2} \\ \widehat{T}_{n,1}^{\mathsf{T}} & \widehat{K}_{t,1}^{\mathsf{T}} \end{pmatrix}$$

is surjective. Therefore, for every $\widehat{Z}_{t,1} \in \mathbb{S}_{p_0}(\mathbb{R})$,

$$\operatorname{rank} \begin{pmatrix} \mathcal{H}_{0,t,2} & \mathcal{H}_{t,2} & \widehat{K}_{t,1}^{(2)} \\ \widehat{T}_{n,1}^{\mathsf{T}} & \widehat{K}_{t,1}^{\mathsf{T}} & \widehat{Z}_{t,1} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \mathcal{H}_{0,t,2} & \mathcal{H}_{t,2} \\ \widehat{T}_{n,1}^{\mathsf{T}} & \widehat{K}_{t,1}^{\mathsf{T}} \end{pmatrix}. \tag{4.19}$$

Since the original sequence admits an $\mathbb{R}$–representing measure, it admits a positive semidefinite extension. Thus, there exist matrices $Z_t^{(e)}$ and $W_t^{(e)}$ such that

$$\begin{pmatrix} M_{\mathcal{T}_t}(n) & \operatorname{col}(K_t, Z_t^{(e)}) \\ \operatorname{row}(K_t^{\mathsf{T}}, (Z_t^{(e)})^{\mathsf{T}}) & W_t^{(e)} \end{pmatrix} \succeq 0.$$

The generalized Schur complement criterion [Alb69] implies

$$\mathcal{C}(\operatorname{col}(K_t, Z_t^{(e)})) \subseteq \mathcal{C}(M_{\mathcal{T}_t}(n)).$$

In particular,

$$\operatorname{rank} \begin{pmatrix} \mathcal{H}_{0,t,1} & \mathcal{H}_{t,1} & \widehat{K}_{t,1}^{(1)} \\ \mathcal{H}_{0,t,2} & \mathcal{H}_{t,2} & \widehat{K}_{t,1}^{(2)} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \mathcal{H}_{0,t,1} & \mathcal{H}_{t,1} \\ \mathcal{H}_{0,t,2} & \mathcal{H}_{t,2} \end{pmatrix}. \tag{4.20}$$

By Lemma 2.11, (4.17), (4.19), and (4.20), we obtain

$$\operatorname{rank} \begin{pmatrix} \mathcal{H}_{0,t} & \mathcal{H}_t & \widehat{K}_{t,1} \\ \widehat{T}_{n,1}^{\mathsf{T}} & \widehat{K}_{t,1}^{\mathsf{T}} & \widehat{Z}_{t,1} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \mathcal{H}_{0,t} & \mathcal{H}_t \\ \widehat{T}_{n,1}^{\mathsf{T}} & \widehat{K}_{t,1}^{\mathsf{T}} \end{pmatrix} = \operatorname{rank} M_{\mathcal{T}_t}(n),$$

where the last equality follows from (4.15). Together with (4.14), this proves (4.12).

For each $j \in [k_1]$, define

$$\widehat{Z}_{t_j} := \widehat{Z}_t + \sum_{i=1}^{2n+1} \binom{2n+1}{i} (-1)^i (t_j^i - t^i) \widehat{S}_{2n+1-i}. \tag{4.21}$$

Let $\widehat{Z}_{t_j,1}$ denote its upper-left $p_0 \times p_0$ principal submatrix. By (4.8) and (4.21),

$$\widehat{Z}_{t_j,1} = \widehat{K}_{t_j,1}^{\mathsf{T}} \mathcal{H}_{t_j}^{\dagger} \widehat{K}_{t_j,1} + \widehat{Z}_{t,1} + E_j.$$

Hence, by (4.9) and Lemma 4.1 (applied with $A = \mathcal{H}_{t_j}$, $B = \widehat{K}_{t_j,1}$, and $P = \widehat{Z}_{t,1} + E_j$),

$$\operatorname{rank} \begin{pmatrix} \mathcal{H}_{t_j} & \widehat{K}_{t_j,1} \\ \widehat{K}_{t_j,1}^{\mathsf{T}} & \widehat{Z}_{t_j,1} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \mathcal{H}_{t_j} \\ \widehat{K}_{t_j,1}^{\mathsf{T}} \end{pmatrix} + p_0, \qquad j \in [k_1]. \tag{4.22}$$

We now extend the shifted moment sequence by setting

$$T_{2n+1} \coloneqq Z_t, \qquad T_{2n+2} \coloneqq \operatorname{row}(K_t^{\mathsf{T}}, Z_t) M_{\mathcal{T}_t}(n)^{\dagger} \operatorname{col}(K_t, Z_t). \tag{4.23}$$

Let

$$\mathcal{T}_t^{(2n+2)} \coloneqq (T_0, T_1, \ldots, T_{2n+2}).$$

By [Alb69], (4.13) and (4.23), the corresponding moment matrix $M_{\mathcal{T}_t^{(2n+2)}}(n+1)$ is positive semidefinite and

$$\operatorname{rank} M_{\mathcal{T}_t^{(2n+2)}}(n+1) = \operatorname{rank} M_{\mathcal{T}_t}(n).$$

Define the unshifted moments S_{2n+1} and S_{2n+2}

$$\begin{aligned} S_{2n+1} &\coloneqq \mathcal{E}_t^{-1}(T_{2n+1}) = T_{2n+1} - \sum_{i=1}^{2n+1} \binom{2n+1}{i} (-1)^i t^i S_{2n+1-i}, \\ S_{2n+2} &\coloneqq \mathcal{E}_t^{-1}(T_{2n+2}) = T_{2n+2} - \sum_{i=1}^{2n+2} \binom{2n+2}{i} (-1)^i t^i S_{2n+2-i}, \end{aligned} \tag{4.24}$$

and set

$$\mathcal{S}^{(2n+2)} \coloneqq (S_0, S_1, \ldots, S_{2n+2}). \tag{4.25}$$

Proposition 2.2 yields

$$M_{\mathcal{S}^{(2n+2)}}(n+1) \succeq 0, \qquad \operatorname{rank} M_{\mathcal{S}^{(2n+2)}}(n+1) = \operatorname{rank} M(n).$$

Hence $\mathcal{S}^{(2n+2)}$ admits a $(\operatorname{rank} M(n))$–atomic $\mathbb{R}$–representing measure

$$\mu = \sum_{\ell=1}^{N} \delta_{x_\ell} A_\ell.$$

Its restriction to moments of degree at most $2n$ is an $\mathbb{R}$–representing measure for L.

It remains to determine the multiplicities of the prescribed points. By the inverse change-of-basis formula,

$$P \widehat{Z}_{t_j} P^{\mathsf{T}} = \mathcal{E}_{t_j}(S_{2n+1}), \qquad j \in [k_1]. \tag{4.26}$$

Indeed, fix $j \in [k_1]$. Using (4.11) and (4.21), we have

$$\widehat{Z}_{t_j} = \widehat{Z}_t + P^T \sum_{i=1}^{2n+1} \binom{2n+1}{i} (-1)^i (t_j^i - t^i) S_{2n+1-i} P.$$

By (4.11), (4.23), and (4.24), we have

$$\widehat{Z}_t = P^T T_{2n+1} P = P^T \mathcal{E}_t(S_{2n+1}) P = P^T \left(S_{2n+1} + \sum_{i=1}^{2n+1} \binom{2n+1}{i} (-1)^i t^i S_{2n+1-i} \right) P.$$

Substituting this into the previous expression yields

$$\begin{aligned} \widehat{Z}_{t_j} = {} & P^T \left(S_{2n+1} + \sum_{i=1}^{2n+1} \binom{2n+1}{i} (-1)^i t^i S_{2n+1-i} \right) P \\ & + P^T \sum_{i=1}^{2n+1} \binom{2n+1}{i} (-1)^i (t_j^i - t^i) S_{2n+1-i} P \end{aligned}$$

$$= P^T \left(S_{2n+1} + \sum_{i=1}^{2n+1} \binom{2n+1}{i} (-1)^i t_j^i S_{2n+1-i} \right) P$$
$$= P^T \, \mathcal{E}_{t_j}(S_{2n+1}) \, P.$$

Multiplying by P on the left and by P^T on the right give (4.26).

Fix $j \in [k_1]$ and define the extended shifted sequence

$$\mathcal{T}_{t_j}^{(2n+2)} := (\mathcal{E}_{t_j}(S_0), \mathcal{E}_{t_j}(S_1), \ldots, \mathcal{E}_{t_j}(S_{2n+2})).$$

By Proposition 2.5,

$$\operatorname{Dep}\Big(X^n_{\mathcal{T}_t^{(2n+2)}} \Big) = \operatorname{Dep}\Big(X^n_{\mathcal{T}_{t_j}^{(2n+2)}} \Big). \tag{4.27}$$

Moreover,

$$X^n_{\mathcal{T}_{t_j}^{(2n+2)}} = \begin{pmatrix} \mathcal{E}_{t_j}(S_n) \\ K_{t_j} \\ Z_{t_j} \end{pmatrix}.$$

Since P places the dependent columns in the last $p - p_0$ positions, (4.27) implies that the last $p - p_0$ columns of $X^n_{\mathcal{T}_{t_j}^{(2n+2)}} P$ are linear combinations of the preceding columns. Consequently, the blocks $\widehat{K}_{t_j,3}$, $\widehat{Z}_{t_j,3}$, and $\widehat{Z}_{t_j,6}$ do not increase the column rank. Since the full matrix is symmetric, the corresponding block rows do not increase its rank either. Therefore,

$$\operatorname{rank} \begin{pmatrix} \mathcal{H}_{t_j} & \widehat{K}_{t_j} \\ \widehat{K}_{t_j}^{\mathsf{T}} & \widehat{Z}_{t_j} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \mathcal{H}_{t_j} & \widehat{K}_{t_j,1} \\ \widehat{K}_{t_j,1}^{\mathsf{T}} & \widehat{Z}_{t_j,1} \end{pmatrix}. \tag{4.28}$$

Moreover, the columns corresponding to $\widehat{K}_{t_j,3}$ and the associated blocks of $\widehat{Z}_{t_j}$ are linear combinations of the columns corresponding to $\mathcal{H}_{t_j}$ and $\widehat{K}_{t_j,1}$. Therefore,

$$\begin{aligned}
&\operatorname{rank} \begin{pmatrix} \mathcal{H}_{t_j} & K_{t_j} \\ K_{t_j}^{\mathsf{T}} & \mathcal{E}_{t_j}(S_{2n+1}) \end{pmatrix} \\
&= \operatorname{rank} \begin{pmatrix} \mathcal{H}_{t_j} & \widehat{K}_{t_j} \\ \widehat{K}_{t_j}^{\mathsf{T}} & \widehat{Z}_{t_j} \end{pmatrix} && \text{(by the permutation congruence induced by } P\text{)} \\
&= \operatorname{rank} \begin{pmatrix} \mathcal{H}_{t_j} & \widehat{K}_{t_j,1} \\ \widehat{K}_{t_j,1}^{\mathsf{T}} & \widehat{Z}_{t_j,1} \end{pmatrix} && \text{by (4.28)} \\
&= \operatorname{rank} \begin{pmatrix} \mathcal{H}_{t_j} \\ \widehat{K}_{t_j,1}^{\mathsf{T}} \end{pmatrix} + p_0 && \text{by (4.22)} \\
&= \operatorname{rank} \begin{pmatrix} \mathcal{H}_{t_j} \\ \widehat{K}_{t_j}^{\mathsf{T}} \end{pmatrix} + p_0 && \text{by (4.5)} \\
&= \operatorname{rank} \begin{pmatrix} \mathcal{H}_{t_j} \\ K_{t_j}^{\mathsf{T}} \end{pmatrix} + p - \operatorname{card} \operatorname{Dep}\Big(X^n_{\mathcal{T}_{t_j}} \Big) && \text{by } \widehat{K}_{t_j} = K_{t_j} P \text{ and the definition of } p_0.
\end{aligned}$$

Corollary 3.2 now implies that

$$\operatorname{mult}_\mu t_j = \operatorname{card} A_{\mathcal{T}_{t_j}}, \qquad j \in [k_1].$$

This completes the proof. □

The following corollary characterizes when the representing measure in Theorem 4.2 can be chosen so that none of the prescribed points is an atom.

Corollary 4.3. *Let $n, p \in \mathbb{N}$, and let $t_1, \ldots, t_{k_1} \in \mathbb{R}$ be pairwise distinct. Let*

$$L \colon \mathbb{R}[x]_{\leq 2n} \to \mathbb{S}_p(\mathbb{R})$$

be a linear operator which admits an $\mathbb{R}$–representing matrix measure. Then the following statements are equivalent:

(1) There exists a finitely atomic $\mathbb{R}$–representing matrix measure μ for L such that

$$t_j \notin \operatorname{supp}(\mu), \qquad j \in [k_1].$$

(2) There exists a $(\operatorname{rank} M(n))$–atomic $\mathbb{R}$–representing matrix measure μ for L such that

$$t_j \notin \operatorname{supp}(\mu), \qquad j \in [k_1].$$

(3) For every $j \in [k_1]$, we have

$$A_{\mathcal{T}_{t_j}} = \emptyset,$$

where $A_{\mathcal{T}_{t_j}}$ is defined in (2.16)*.*

Proof. Assume first that (1) holds. Then $\operatorname{mult}_\mu t_j = 0$, $j \in [k_1]$. It follows from Corollary 3.3 that $A_{\mathcal{T}_{t_j}} = \emptyset$ for every $j \in [k_1]$. Hence, (3) holds.

Assume that (3) holds. By Theorem 4.2, there exists a $(\operatorname{rank} M(n))$–atomic $\mathbb{R}$–representing matrix measure μ for L such that $\operatorname{mult}_\mu t_j = \operatorname{card} A_{\mathcal{T}_{t_j}} = 0$, $j \in [k_1]$. Therefore, $t_j \notin \operatorname{supp}(\mu)$ for every $j \in [k_1]$, proving (2).

The implication (2) $\Rightarrow$ (1) is clear. □

We are now ready to prove Theorem 1.1 in the case $K = \mathbb{R}$.

Proof of Theorem 1.1 for $K = \mathbb{R}$. The implication (1) $\Rightarrow$ (2) follows from [MS24a, Corollary 5.3].

We next prove (2) $\Rightarrow$ (4). Let ν be a finitely atomic $\mathbb{R}$-representing measure for $\mathcal{L}$. By Proposition 2.1, the measure $\mu := q^{-1} \cdot \nu$ is a finitely atomic $\mathbb{R}$-representing measure for L and satisfies $\mu(\Lambda) = \mathbf{0}_p$. By Theorem 2.12, (4a) and (4b) hold. It remains to prove (4d). Consider the extended moment sequence

$$\widetilde{\mathcal{S}} := (S_0, \ldots, S_{2n+2k}), \qquad S_i := \int_{\mathbb{R}} x^i \, d\mu,$$

satisfying

$$\operatorname{rank} M_{\widetilde{\mathcal{S}}}(n+k) = \operatorname{rank} M_{\widetilde{\mathcal{S}}}(n+k-1).$$

Fix $t_j \in \Lambda$. In what follows, let $\mathcal{H}_{t_j}$ and K_{t_j} denote the matrices defined as in Theorem 3.1, with $\widetilde{\mathcal{S}}$ in place of $\mathcal{S}$ and $n+k-1$ in place of n. The flatness condition implies that

$$p = \operatorname{card} \operatorname{Dep}\Big(X^{n+k}_{\mathcal{T}_{t_j}}\Big)$$

and

$$\operatorname{rank} \begin{pmatrix} \mathcal{H}_{t_j} & K_{t_j} \\ K_{t_j}^{\dagger} & \mathcal{E}_{t_j}(S_{2n+2k-1}) \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \mathcal{H}_{t_j} \\ K_{t_j}^{\dagger} \end{pmatrix}.$$

Thus, condition (2) of Corollary 3.2 is satisfied. Consequently,

$$\operatorname{mult}_\mu t_j = \operatorname{card} A_{\mathcal{T}_{t_j}}(n+k),$$

where

$$A_{\mathcal{T}_{t_j}}(r) := \bigcup_{i=1}^{r} \operatorname{DepN}_1\Big(X^i_{\mathcal{T}_{t_j}}\Big), \qquad r \in [n+k].$$

Since $t_j \in \Lambda$, we have $\operatorname{mult}_\mu t_j = 0$, and hence

$$A_{\mathcal{T}_{t_j}}(n+k) = \emptyset.$$

The sets $A_{\mathcal{T}_{t_j}}(r)$ are increasing in r, so

$$A_{\mathcal{T}_{t_j}}(r) = \emptyset, \qquad r \in [n+k].$$

Since $t_j \in \Lambda$ was arbitrary, this proves (4).

The implication (4) $\Rightarrow$ (3) follows from Corollary 4.3. Finally, the implication (3) $\Rightarrow$ (2) is immediate. □

5. MINIMAL $[0,\infty)$-REPRESENTING MEASURES WITH MINIMAL MULTIPLICITIES AT PRESCRIBED ATOMS

In this section, we establish analogues of the results from Section 4 for the case where the support of representing measures is contained in $[0,\infty)$. We then use these results to complete the proof of Theorem 1.1 for $K=[0,\infty)$.

The following result is an analogue of Theorem 4.2, where $\mathbb{R}$ is replaced by $[0,\infty)$.

Theorem 5.1. *Let $n,p\in\mathbb{N}$, and let $t_1,\ldots,t_{k_1}\in[0,\infty)$ be pairwise distinct. Let*

$$L\colon \mathbb{R}[x]_{\le 2n}\to \mathbb{S}_p(\mathbb{R})$$

be a linear operator with an $[0,\infty)$–representing matrix measure. Then L admits a $(\operatorname{rank} M(n))$–atomic $[0,\infty)$–representing matrix measure μ such that

$$\operatorname{mult}_\mu t_j=\operatorname{card} A_{\mathcal{T}_{t_j}},\qquad j\in[k_1],$$

where $A_{\mathcal{T}_{t_j}}$ is defined in (2.16).

Proof. The construction of the flat extension of $M(n)$ must meet two requirements simultaneously. First, we choose $L(x^{2n+1})$ so that the rank identities at all prescribed points hold. Second, we choose the same matrix sufficiently positive to ensure the Stieltjes localizing condition. The remainder of the argument then completes the flat extension as in the proof of Theorem 4.2.

Set $S_i:=L(x^i)$, $i\in[0;2n]$, and write $\mathcal{S}:=(S_0,S_1,\ldots,S_{2n})$. For each $t\in\mathbb{R}$, let $\mathcal{T}_t$ be the shifted sequence defined in (2.8), and adopt the notation introduced in Subsections 2.6 and 2.7. We follow the construction in the proof of Theorem 4.2, taking the reference point $t=0$, up to the choice of the block $\widehat{Z}_{0,1}$. We retain all notation introduced there. In particular,

$$\widehat{K}_0=\begin{pmatrix}\widehat{K}_{0,1} & \widehat{K}_{0,3}\end{pmatrix},$$

where $\widehat{K}_{0,1}$ has p_0 columns, and

$$\widehat{K}_{0,3}=\begin{pmatrix}\mathcal{H}_0 & \widehat{K}_{0,1}\end{pmatrix}\widehat{G}. \tag{5.1}$$

Since the reference point is $t=0$, we have

$$\mathcal{H}_0=\mathcal{H}_x(n-1).$$

In addition to the rank conditions required in the proof of Theorem 4.2, we must choose the subsequent moment S_{2n+1} so that

$$S_{2n+1}-\operatorname{row}(S_{n+i})_{i\in[n]}\mathcal{H}_x(n-1)^\dagger\operatorname{col}(S_{n+i})_{i\in[n]}\succeq 0. \tag{5.2}$$

For each $j\in[k_1]$, define

$$F_j:=\widehat{K}_{0,1}^{\mathsf T}\mathcal{H}_0^\dagger\widehat{K}_{0,1}+\sum_{i=1}^{2n+1}\binom{2n+1}{i}(-1)^i t_j^i\widehat{S}_{2n+1-i,1}-\widehat{K}_{t_j,1}^{\mathsf T}\mathcal{H}_{t_j}^\dagger\widehat{K}_{t_j,1}, \tag{5.3}$$

where $\widehat{S}_{i,1}$ denotes the upper-left $p_0\times p_0$ principal submatrix of

$$\widehat{S}_i:=P^{\mathsf T}S_iP.$$

The sum in (5.3) is the change-of-basis correction from the basis centered at 0 to the basis centered at t_j.

Since the family $(F_j)_{j\in[k_1]}$ is finite, we may choose $\alpha>0$ sufficiently large that

$$\alpha I_{p_0}+F_j\succ 0,\qquad j\in[k_1].$$

Set

$$P_+:=\alpha I_{p_0}. \tag{5.4}$$

In particular, $P_+\succ 0$. Define

$$\widehat{Z}_{0,1}:=\widehat{K}_{0,1}^{\mathsf T}\mathcal{H}_0^\dagger\widehat{K}_{0,1}+P_+. \tag{5.5}$$

As in the proof of Theorem 4.2, define

$$\widehat{Z}_{0,3} \coloneqq \begin{pmatrix} \widehat{K}_{0,1}^{\mathsf{T}} & \widehat{Z}_{0,1} \end{pmatrix} \widehat{G}, \qquad \widehat{Z}_{0,6} \coloneqq \begin{pmatrix} \widehat{K}_{0,3}^{\mathsf{T}} & \widehat{Z}_{0,3}^{\mathsf{T}} \end{pmatrix} \widehat{G}, \tag{5.6}$$

and set

$$\widehat{Z}_0 \coloneqq \begin{pmatrix} \widehat{Z}_{0,1} & \widehat{Z}_{0,3} \\ \widehat{Z}_{0,3}^{\mathsf{T}} & \widehat{Z}_{0,6} \end{pmatrix}, \qquad Z_0 \coloneqq P\widehat{Z}_0 P^{\mathsf{T}}. \tag{5.7}$$

Since the reference point is 0, we set

$$S_{2n+1} \coloneqq Z_0.$$

We claim that

$$\widehat{Z}_0 - \widehat{K}_0^{\mathsf{T}} \mathcal{H}_0^{\dagger} \widehat{K}_0 \succeq 0. \tag{5.8}$$

Write

$$\widehat{G} = \begin{pmatrix} \widehat{G}_0 \\ \widehat{G}_1 \end{pmatrix}, \qquad \widehat{G}_0 \in M_{np\times(p-p_0)}(\mathbb{R}), \quad \widehat{G}_1 \in M_{p_0\times(p-p_0)}(\mathbb{R}).$$

Then

$$\widehat{K}_{0,3} = \mathcal{H}_0 \widehat{G}_0 + \widehat{K}_{0,1}\widehat{G}_1. \tag{5.9}$$

Moreover, by the definitions of $\widehat{Z}_{0,3}$ and $\widehat{Z}_{0,6}$,

$$\widehat{Z}_{0,3} = \widehat{K}_{0,1}^{\mathsf{T}} \widehat{G}_0 + \widehat{Z}_{0,1}\widehat{G}_1, \tag{5.10}$$

$$\widehat{Z}_{0,6} = \widehat{G}_0^{\mathsf{T}} \mathcal{H}_0 \widehat{G}_0 + \widehat{G}_1^{\mathsf{T}} \widehat{K}_{0,1}^{\mathsf{T}} \widehat{G}_0 + \widehat{G}_0^{\mathsf{T}} \widehat{K}_{0,1} \widehat{G}_1 + \widehat{G}_1^{\mathsf{T}} \widehat{Z}_{0,1} \widehat{G}_1. \tag{5.11}$$

Recall the definition of $\widehat{Z}_{0,1}$ from (5.5). Since

$$\mathcal{C}(\widehat{K}_{0,1}) \subseteq \mathcal{C}(\mathcal{H}_0),$$

and $\mathcal{H}_0$ is symmetric, we have

$$\mathcal{H}_0 \mathcal{H}_0^{\dagger} \widehat{K}_{0,1} = \widehat{K}_{0,1}, \qquad \widehat{K}_{0,1}^{\mathsf{T}} \mathcal{H}_0^{\dagger} \mathcal{H}_0 = \widehat{K}_{0,1}^{\mathsf{T}}. \tag{5.12}$$

We now compute the four blocks of

$$\widehat{Z}_0 - \widehat{K}_0^{\mathsf{T}} \mathcal{H}_0^{\dagger} \widehat{K}_0, \qquad \widehat{K}_0 = \begin{pmatrix} \widehat{K}_{0,1} & \widehat{K}_{0,3} \end{pmatrix}.$$

For the upper-left block, (5.5) gives

$$\widehat{Z}_{0,1} - \widehat{K}_{0,1}^{\mathsf{T}} \mathcal{H}_0^{\dagger} \widehat{K}_{0,1} = P_+. \tag{5.13}$$

Using (5.9), (5.10), and (5.12), we obtain

$$\begin{aligned} \widehat{Z}_{0,3} - \widehat{K}_{0,1}^{\mathsf{T}} \mathcal{H}_0^{\dagger} \widehat{K}_{0,3} &= \widehat{K}_{0,1}^{\mathsf{T}} \widehat{G}_0 + \widehat{Z}_{0,1} \widehat{G}_1 \\ &\quad - \widehat{K}_{0,1}^{\mathsf{T}} \mathcal{H}_0^{\dagger} (\mathcal{H}_0 \widehat{G}_0 + \widehat{K}_{0,1} \widehat{G}_1) \\ &= \widehat{K}_{0,1}^{\mathsf{T}} \widehat{G}_0 + \widehat{Z}_{0,1} \widehat{G}_1 - \widehat{K}_{0,1}^{\mathsf{T}} \widehat{G}_0 - \widehat{K}_{0,1}^{\mathsf{T}} \mathcal{H}_0^{\dagger} \widehat{K}_{0,1} \widehat{G}_1 \\ &= \left(\widehat{Z}_{0,1} - \widehat{K}_{0,1}^{\mathsf{T}} \mathcal{H}_0^{\dagger} \widehat{K}_{0,1} \right) \widehat{G}_1 \\ &= P_+ \widehat{G}_1. \end{aligned} \tag{5.14}$$

Taking transposes yields

$$\widehat{Z}_{0,3}^{\mathsf{T}} - \widehat{K}_{0,3}^{\mathsf{T}} \mathcal{H}_0^{\dagger} \widehat{K}_{0,1} = \widehat{G}_1^{\mathsf{T}} P_+. \tag{5.15}$$

Finally, by (5.9) and (5.11),

$$\begin{aligned} &\widehat{Z}_{0,6} - \widehat{K}_{0,3}^{\mathsf{T}} \mathcal{H}_0^{\dagger} \widehat{K}_{0,3} \\ &= \widehat{G}_0^{\mathsf{T}} \mathcal{H}_0 \widehat{G}_0 + \widehat{G}_1^{\mathsf{T}} \widehat{K}_{0,1}^{\mathsf{T}} \widehat{G}_0 + \widehat{G}_0^{\mathsf{T}} \widehat{K}_{0,1} \widehat{G}_1 + \widehat{G}_1^{\mathsf{T}} \widehat{Z}_{0,1} \widehat{G}_1 \\ &\quad - (\widehat{G}_0^{\mathsf{T}} \mathcal{H}_0 + \widehat{G}_1^{\mathsf{T}} \widehat{K}_{0,1}^{\mathsf{T}}) \mathcal{H}_0^{\dagger} (\mathcal{H}_0 \widehat{G}_0 + \widehat{K}_{0,1} \widehat{G}_1) \\ &= \widehat{G}_1^{\mathsf{T}} \left(\widehat{Z}_{0,1} - \widehat{K}_{0,1}^{\mathsf{T}} \mathcal{H}_0^{\dagger} \widehat{K}_{0,1} \right) \widehat{G}_1 \end{aligned}$$

$$= \widehat{G}_1^{\mathsf{T}} P_+ \widehat{G}_1. \tag{5.16}$$

Here we used $\mathcal{H}_0\mathcal{H}_0^\dagger\mathcal{H}_0 = \mathcal{H}_0$ together with (5.12).

Combining (5.13), (5.14), (5.15), and (5.16), we obtain

$$\widehat{Z}_0 - \widehat{K_0}^{\mathsf{T}}\mathcal{H}_0^\dagger\widehat{K_0} = \begin{pmatrix} P_+ & P_+\widehat{G}_1 \\ \widehat{G}_1^{\mathsf{T}} P_+ & \widehat{G}_1^{\mathsf{T}} P_+ \widehat{G}_1 \end{pmatrix}$$
$$= \begin{pmatrix} I_{p_0} \\ \widehat{G}_1^{\mathsf{T}} \end{pmatrix} P_+ \begin{pmatrix} I_{p_0} & \widehat{G}_1 \end{pmatrix} \succeq 0. \tag{5.17}$$

This proves (5.8).

Since

$$\widehat{K_0} = K_0 P \quad \text{and} \quad S_{2n+1} = P\widehat{Z}_0 P^{\mathsf{T}},$$

congruence by P in (5.8) yields

$$S_{2n+1} - K_0^{\mathsf{T}}\mathcal{H}_0^\dagger K_0 \succeq 0.$$

Since

$$K_0^{\mathsf{T}} = \operatorname{row}(S_{n+i})_{i\in[n]}, \qquad \mathcal{H}_0 = \mathcal{H}_x(n-1),$$

condition (5.2) follows.

We next verify the rank conditions at the prescribed points. For each $j \in [k_1]$, define

$$\widehat{Z}_{t_j} := \widehat{Z}_0 + \sum_{i=1}^{2n+1} \binom{2n+1}{i} (-1)^i t_j^i \widehat{S}_{2n+1-i}. \tag{5.18}$$

Let $\widehat{Z}_{t_j,1}$ denote its upper-left $p_0 \times p_0$ principal submatrix. By (5.3), (5.5), and (5.18), one has

$$\widehat{Z}_{t_j,1} = \widehat{K}_{t_j,1}^{\mathsf{T}}\mathcal{H}_{t_j}^\dagger\widehat{K}_{t_j,1} + P_+ + F_j.$$

Since $P_+ + F_j \succ 0$, Lemma 4.1 gives

$$\operatorname{rank}\begin{pmatrix} \mathcal{H}_{t_j} & \widehat{K}_{t_j,1} \\ \widehat{K}_{t_j,1}^{\mathsf{T}} & \widehat{Z}_{t_j,1} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \mathcal{H}_{t_j} \\ \widehat{K}_{t_j,1}^{\mathsf{T}} \end{pmatrix} + p_0, \qquad j \in [k_1]. \tag{5.19}$$

We now complete the extension exactly as in the proof of Theorem 4.2. Choose

$$S_{2n+2} := \operatorname{row}(S_{n+i})_{i\in[n+1]} M(n)^\dagger \operatorname{col}(S_{n+i})_{i\in[n+1]}.$$

Define

$$\mathcal{S}^{(2n+2)} := (S_0, S_1, \ldots, S_{2n+2}).$$

The corresponding moment matrix $M(n+1)$ satisfies

$$M(n+1) \succeq 0, \qquad \operatorname{rank} M(n+1) = \operatorname{rank} M(n). \tag{5.20}$$

Since by assumption L admits an $[0,\infty)$–representing measure, Theorem 2.13 implies that

$$\mathcal{H}_x(n-1) \succeq 0 \qquad \text{and} \qquad \mathcal{C}\Big(\operatorname{col}(S_{n+i})_{i\in[n]}\Big) \subseteq \mathcal{C}(\mathcal{H}_x(n-1)).$$

Together with (5.2), the generalized Schur complement criterion gives

$$\mathcal{H}_x(n) = \begin{pmatrix} \mathcal{H}_x(n-1) & \operatorname{col}(S_{n+i})_{i\in[n]} \\ \operatorname{row}(S_{n+i})_{i\in[n]} & S_{2n+1} \end{pmatrix} \succeq 0.$$

By the rank equality in (5.20), the last block column of $M(n+1)$ belongs to the column space of its first $n+1$ block columns. Restricting the corresponding column relation to the last $(n+1)p$ rows of $M(n+1)$, whose first $n+1$ block columns form $\mathcal{H}_x(n)$, gives

$$\mathcal{C}\Big(\operatorname{col}(S_{n+1+i})_{i\in[n+1]}\Big) \subseteq \mathcal{C}(\mathcal{H}_x(n)).$$

Therefore, Theorem 2.13, applied with n replaced by $n+1$, shows that the Riesz mapping $L_{\mathcal{S}^{(2n+2)}}$ admits a $(\operatorname{rank} M(n+1))$–atomic $[0,\infty)$–representing matrix measure μ. By (5.20),

$$\operatorname{rank} M(n+1) = \operatorname{rank} M(n),$$

and hence μ is $(\operatorname{rank} M(n))$–atomic. Its restriction to $\mathbb{R}[x]_{\leq 2n}$ represents the original operator L.

Finally, the change-of-basis and rank arguments from the proof of Theorem 4.2 apply without modification. In particular, for every $j \in [k_1]$,

$$P\widehat{Z}_{t_j}P^{\mathsf{T}} = \mathcal{E}_{t_j}(S_{2n+1}),$$

and

$$\begin{aligned}\operatorname{rank}\begin{pmatrix}\mathcal{H}_{t_j} & K_{t_j}\\ K_{t_j}^{\mathsf{T}} & \mathcal{E}_{t_j}(S_{2n+1})\end{pmatrix} &= \operatorname{rank}\begin{pmatrix}\mathcal{H}_{t_j} & \widehat{K}_{t_j,1}\\ \widehat{K}_{t_j,1}^{\mathsf{T}} & \widehat{Z}_{t_j,1}\end{pmatrix}\\ &= \operatorname{rank}\begin{pmatrix}\mathcal{H}_{t_j}\\ K_{t_j}^{\mathsf{T}}\end{pmatrix} + p - \operatorname{card}\operatorname{Dep}\Big(X^n_{\mathcal{T}_{t_j}}\Big),\end{aligned}$$

where the second equality follows from (5.19) and the definition of p_0.

Corollary 3.2 now yields

$$\operatorname{mult}_\mu t_j = \operatorname{card} A_{\mathcal{T}_{t_j}}, \qquad j \in [k_1].$$

This completes the proof. □

The following corollary characterizes when the representing measure in Theorem 5.1 can be chosen so that none of the prescribed points is an atom.

Corollary 5.2. *Let $n, p \in \mathbb{N}$, and let $t_1, \ldots, t_{k_1} \in [0,\infty)$ be pairwise distinct. Let*

$$L\colon \mathbb{R}[x]_{\leq 2n} \to \mathbb{S}_p(\mathbb{R})$$

be a linear operator with an $[0,\infty)$–representing matrix measure. Then the following statements are equivalent:

(1) There exists a finitely atomic $[0,\infty)$–representing matrix measure μ for L such that

$$t_j \notin \operatorname{supp}(\mu), \qquad j \in [k_1].$$

(2) There exists a $(\operatorname{rank} M(n))$–atomic $[0,\infty)$–representing matrix measure μ for L such that

$$t_j \notin \operatorname{supp}(\mu), \qquad j \in [k_1].$$

(3) For every $j \in [k_1]$, we have

$$A_{\mathcal{T}_{t_j}} = \emptyset,$$

where $A_{\mathcal{T}_{t_j}}$ is defined in (2.16)*.*

Proof. Assume first that (1) holds. Then $\operatorname{mult}_\mu t_j = 0$, $j \in [k_1]$. It follows from Corollary 3.3 that $A_{\mathcal{T}_{t_j}} = \emptyset$ for every $j \in [k_1]$. Hence, (3) holds.

Assume that (3) holds. By Theorem 5.1, there exists a $(\operatorname{rank} M(n))$–atomic $[0,\infty)$–representing matrix measure μ for L such that $\operatorname{mult}_\mu t_j = \operatorname{card} A_{\mathcal{T}_{t_j}} = 0$, $j \in [k_1]$. Therefore, $t_j \notin \operatorname{supp}(\mu)$, $j \in [k_1]$, which proves (2).

The implication (2) ⇒ (1) is clear. □

We are now ready to prove Theorem 1.1 in the case $K = [0,\infty)$.

Proof of Theorem 1.1 for $K = [0,\infty)$. The argument is identical to that for $K = \mathbb{R}$ in Section 4, except that Theorem 2.12 and Corollary 4.3 are replaced by Theorem 2.13 and Corollary 5.2, respectively. □

6. Example for Theorem 1.1

We illustrate the finite linear-algebraic test in Theorem 1.1. We construct a flat extension and recover the representing measure from it. The example also shows that the pole-avoidance condition contains information not captured by the ordinary matricial Hamburger conditions.

Let $p = 2$, $K = \mathbb{R}$, and $q(x) = x^2(x-1)^2$. Thus $e_0 = 0$, $e_1 = e_2 = 1$, $2n = 4$ (see Section 1), and the real poles of q are 0 and 1. Define $\mathcal{L}\colon \mathcal{R}^{(4)} \to \mathbb{S}_2(\mathbb{R})$ by prescribing the moments of its associated polynomial mapping $L(g) := \mathcal{L}(g/q)$:

$$S_0 = \begin{pmatrix}4 & 1\\ 1 & 1\end{pmatrix}, \quad S_1 = \begin{pmatrix}2 & 3\\ 3 & 3\end{pmatrix}, \quad S_2 = \begin{pmatrix}18 & 9\\ 9 & 9\end{pmatrix}, \quad S_3 = \begin{pmatrix}26 & 27\\ 27 & 27\end{pmatrix}, \quad S_4 = \begin{pmatrix}114 & 81\\ 81 & 81\end{pmatrix}.$$

The corresponding moment matrix is

$$M(2) = \begin{pmatrix} 4 & 1 & 2 & 3 & 18 & 9 \\ 1 & 1 & 3 & 3 & 9 & 9 \\ 2 & 3 & 18 & 9 & 26 & 27 \\ 3 & 3 & 9 & 9 & 27 & 27 \\ 18 & 9 & 26 & 27 & 114 & 81 \\ 9 & 9 & 27 & 27 & 81 & 81 \end{pmatrix}.$$

It is easy to check that

$$M(2) \succeq 0, \qquad \operatorname{rank} M(1) = 3 < 4 = \operatorname{rank} M(2), \qquad \text{and} \qquad \mathcal{C}(\operatorname{col}(S_3, S_4)) \subseteq \mathcal{C}(M(1)).$$

Thus, (4a) and (4b) of Theorem 1.1 are satisfied. It remains to check (4d). Let $M_{\mathcal{T}_t}(2)$ be the moment matrix written in the shifted basis $1, x-t, (x-t)^2$, and let $y_1^{(t)}, \ldots, y_6^{(t)}$ denote its columns in their natural block order. Set

$$r_j(t) := \operatorname{rank}(y_1^{(t)}, \ldots, y_j^{(t)}) \quad (1 \le j \le 6)$$

and

$$s_j(t) := \operatorname{rank}(y_3^{(t)}, \ldots, y_j^{(t)}) \quad (3 \le j \le 6).$$

The shifted moments at $t = 1$ are

$$T_0 = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} -2 & 2 \\ 2 & 2 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 18 & 4 \\ 4 & 4 \end{pmatrix}, \quad T_3 = \begin{pmatrix} -26 & 8 \\ 8 & 8 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 114 & 16 \\ 16 & 16 \end{pmatrix}.$$

Exact column elimination gives

t	$(r_1, \ldots, r_6)$	$(s_3, \ldots, s_6)$	$(\zeta_{\mathcal{T}_t}(1), \zeta_{\mathcal{T}_t}(2))$	$(\zeta^{(1)}_{\mathcal{T}_t}(1), \zeta^{(1)}_{\mathcal{T}_t}(2))$	$A_{\mathcal{T}_t}$
0	$(1,2,3,3,4,4)$	$(1,2,3,3)$	$(\infty, 1)$	$(\infty, 2)$	$\varnothing$
1	$(1,2,3,3,4,4)$	$(1,2,3,3)$	$(\infty, 1)$	$(\infty, 2)$	$\varnothing$

in the notation of Subsection 2.7. Thus condition (4d) of Theorem 1.1 holds at both poles. The rational moment problem is solvable.

A flat extension. Set $S_5 = \begin{pmatrix} 242 & 243 \\ 243 & 243 \end{pmatrix}$, $S_6 = \begin{pmatrix} 858 & 729 \\ 729 & 729 \end{pmatrix}$, such that $\operatorname{rank} M(3) = 4 = \operatorname{rank} M(2)$. Thus $M(3)$ is a flat extension of $M(2)$. We now recover the measure. Write $y_j := y_j^{(0)}$, now regarding the columns as columns of $M(3)$, and group them into block columns

$$X^0 = (y_1\; y_2), \qquad X^1 = (y_3\; y_4), \qquad X^2 = (y_5\; y_6), \qquad X^3 = (y_7\; y_8).$$

We have that

$$y_7 = 4y_1 + 20y_2 + 4y_3 - y_5, \qquad y_8 = 27y_2.$$

Equivalently,

$$X^3 = X^0 H_0 + X^1 H_1 + X^2 H_2,$$

where

$$H_0 = \begin{pmatrix} 4 & 0 \\ 20 & 27 \end{pmatrix}, \qquad H_1 = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Define

$$H(x) = -H_0 - xH_1 - x^2 H_2 + x^3 I_2 = \begin{pmatrix} x^3 + x^2 - 4x - 4 & 0 \\ -20 & x^3 - 27 \end{pmatrix}.$$

By the above, H is a block column relation of $M(3)$. Moreover,

$$\begin{aligned} \det H(x) &= (x^3 + x^2 - 4x - 4)(x^3 - 27) \\ &= (x+2)(x+1)(x-2)(x-3)(x^2 + 3x + 9). \end{aligned}$$

Since $x^2 + 3x + 9 > 0$ for every $x \in \mathbb{R}$,

$$\mathcal{Z}(\det H) = \{-2, -1, 2, 3\}.$$

By Lemma 2.9 and $\operatorname{rank} M(3) = 4$, it follows that $x_1 = -2$, $x_2 = -1$, $x_3 = 2$, $x_4 = 3$ are the atoms of the measure. Solving the matrix Vandermonde system

$$S_k = \sum_{j=1}^{4} x_j^k A_j, \qquad k = 0, 1, 2, 3,$$

gives $A_1 = A_2 = A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Hence a representing measure is

$$\mu = \delta_{-2} A_1 + \delta_{-1} A_2 + \delta_2 A_3 + \delta_3 A_4.$$

The support avoids both zeros of q, as predicted by the pole test. The representing measure for the original rational data is

$$\nu = q \cdot \mu = 36\delta_{-2} A_1 + 4\delta_{-1} A_2 + 4\delta_2 A_3 + 36\delta_3 A_4.$$

Indeed, for every $g \in \mathbb{R}[x]_{\leq 4}$,

$$\int_{\mathbb{R}} \frac{g}{q} \, d\nu = \sum_{j=1}^{4} g(x_j) A_j = L(g) = \mathcal{L}\left(\frac{g}{q}\right).$$

Each mass has rank one, so the total atomic multiplicity is $4 = \operatorname{rank} M(2)$.

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