A LOCAL-GLOBAL PRINCIPLE FOR LINEAR DEPENDENCE IN ENVELOPING ALGEBRAS OF LIE ALGEBRAS

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Abstract. For an associative algebra A and a class $\mathcal{C}$ of representations of A the following question (related to nullstellensatz) makes sense: Characterize all tuples of elements $a_1, \ldots, a_n \in A$ such that vectors $\pi(a_1)v, \ldots, \pi(a_n)v$ are linearly dependent for every $\pi \in \mathcal{C}$ and every v in the representation space of π. We answer this question in the following cases:

(1) $A = U(L)$ is the enveloping algebra of a finite-dimensional complex Lie algebra L and $\mathcal{C}$ is the class of all finite-dimensional representations of A.
(2) $A = U(\mathfrak{sl}_2(\mathbb{C}))$ and $\mathcal{C}$ is the class of all finite-dimensional irreducible representations of A.
(3) $A = U(\mathfrak{sl}_3(\mathbb{C}))$ and $\mathcal{C}$ is the class of all finite-dimensional irreducible representations of A with sufficiently high weights.

In case (1) the answer is: tuples that are linearly dependent over $\mathbb{C}$ while in cases (2) and (3) the answer is: tuples that are linearly dependent over the center of A. Similar results have been proved before for free algebras and Weyl algebras.

Let A be a complex associative algebra and let $\mathcal{C}$ be a class of representations of A. We say that the elements $p_1, \ldots, p_k \in A$ are *$\mathcal{C}$-locally linearly dependent* (abbreviated as $\mathcal{C}$-LLD) if for every representation $\pi : A \to \mathrm{End}(V_\pi)$ in $\mathcal{C}$ we have that $\pi(p_1), \ldots, \pi(p_k)$ are linearly dependent. We say that elements $p_1, \ldots, p_k \in A$ are *$\mathcal{C}$-locally directionally linearly dependent* (abbreviated as $\mathcal{C}$-LDLD) if for every representation $\pi : A \to \mathrm{End}(V_\pi)$ in $\mathcal{C}$ and every vector $v \in V_\pi$ we have that $\pi(p_1)v, \ldots, \pi(p_k)v$ are linearly dependent. Clearly, linear dependence implies $\mathcal{C}$-LLD which implies $\mathcal{C}$-LDLD. The opposite implications are false in general. The motivation for this terminology comes from [2].

Our first main result is the following theorem, proved in Section 1.

Theorem 1. *Let L be a finite-dimensional complex Lie algebra, $U(L)$ its universal enveloping algebras and $\mathcal{R}$ the class of all finite-dimensional representations of $U(L)$. For any elements $p_1, \ldots, p_k \in U(L)$ the following are equivalent:*

(1) $p_1, \ldots, p_k$ are linearly dependent.
(2) $p_1, \ldots, p_k$ are $\mathcal{R}$-locally linearly dependent.
(3) $p_1, \ldots, p_k$ are $\mathcal{R}$-locally directionally linearly dependent.

The analogue of Theorem 1 for free algebras was proved in [2]. The analogue for the algebra $M_n(\mathbb{C})$ of all complex $n \times n$ matrices is trivial. Namely, let π be the direct sum of n copies of the identity representation of $M_n(\mathbb{C})$ and let v be the direct sum of all elements of

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the standard basis of $\mathbb{C}^n$. Then (3) implies that $\pi(p_1)v, \ldots, \pi(p_n)v$ are linearly dependent which implies (1) since each $\pi(p_i)v$ is just a vectorization of p_i. See also Lemma 1 below.

Our second main result, whose proof is given in Section 2, is:

Theorem 2. *Let $\mathfrak{sl}_2$ be the Lie algebra of trace-zero 2×2 complex matrices and $\mathcal{I}$ the class of all finite-dimensional irreducible representations of its universal enveloping algebra $U(\mathfrak{sl}_2)$. For any elements $p_1, \ldots, p_k \in U(\mathfrak{sl}_2)$ the following are equivalent:*

(1) There exist $z_1, \ldots, z_k$ in the center of $U(\mathfrak{sl}_2)$ which are not all zero such that $z_1p_1 + \ldots + z_kp_k = 0$.
(2) $p_1, \ldots, p_k$ are $\mathcal{I}$-locally linearly dependent.
(3) $p_1, \ldots, p_k$ are $\mathcal{I}$-locally directionally linearly dependent.

To obtain the analogue of Theorem 2 for the enveloping algebra $U(\mathfrak{sl}_3)$, which is our third main result, proved in Section 4, we consider a smaller class of irreducible representations. Namely, for each $d \in \mathbb{N}$ we define $\mathcal{I}_d$ to be the class of all finite-dimensional irreducible representations of $\mathfrak{sl}_3$ with highest weights (m_1, m_2) satisfying $m_1 \geq d$, $m_2 \geq d$.

Theorem 3. *Let $\mathfrak{sl}_3$ be the Lie algebra of trace-zero 3×3 complex matrices. For any elements $p_1, \ldots, p_k \in U(\mathfrak{sl}_3)$ the following are equivalent:*

(1) There exist $z_1, \ldots, z_k$ in the center of $U(\mathfrak{sl}_3)$ which are not all zero such that $z_1p_1 + \ldots + z_kp_k = 0$.
(2) There exists $d \in \mathbb{N}$ such that $p_1, \ldots, p_k$ are $\mathcal{I}_d$-locally linearly dependent.
(3) There exists $d \in \mathbb{N}$ such that $p_1, \ldots, p_k$ are $\mathcal{I}_d$-locally directionally linearly dependent.

Here is a list of a few results related to Theorem 2 and 3 that are either known or trivial:

(1) If $A = M_n(\mathbb{C})$ and $\mathcal{C} = \{\mathrm{id}\}$ then $\mathcal{C}$-LLD is equivalent to linear dependence but $\mathcal{C}$-LDLD is not as it is equivalent to the usual notion of locally linearly dependent matrices; see [3]. For $n \geq 2$ the coordinate matrices $E_{ij} \in M_n(\mathbb{C})$ are $\mathcal{C}$-LDLD although they are linearly independent.

(2) If $A = M_n(\mathbb{C}[X_1, \ldots, X_m])$ and $\mathcal{C} = \{\mathrm{ev}_a \mid a \in \mathbb{C}^m\}$ is the set of all evaluations at n-tuplets of complex numbers, then $\mathcal{C}$-LLD is equivalent to linear dependence over $\mathbb{C}[X_1, \ldots, X_n]$ (see below), but $\mathcal{C}$-LDLD is not (it suffices to consider constant matrices: see (1) above).

Pick any matrices $P_1, \ldots, P_k \in A$ and consider the matrix $P = [\mathbf{p}_1, \ldots, \mathbf{p}_k]$ where $\mathbf{p}_i$ is the vectorization of P_i. Note that $P_1, \ldots, P_k$ are $\mathcal{C}$-LLD iff for every $a \in \mathbb{C}^n$ every maximal subdeterminant of $P(a)$ is zero iff every maximal subdeterminant of P is zero iff $P_1, \ldots, P_k$ are linearly dependent over $\mathbb{C}(X_1, \ldots, X_n)$.

(3) If $A = A_n(\mathbb{C})$ is the n-th Weyl algebra and $\mathcal{C} = \{\pi_0\}$, where π_0 is the Schrödinger representation of A, then $\mathcal{C}$-LLD and $\mathcal{C}$-LDLD are equivalent to linear dependence; see [5]. Recall that $A_n(\mathbb{C})$ has generators $x_1, \ldots, x_n, y_1, \ldots, y_n$ and relations $y_ix_j - x_jy_i = \delta_{ij}$, $x_ix_j = x_jx_i$ and $y_iy_j = y_jy_i$ for $i, j = 1, \ldots, n$ and that π_0 acts on the vector space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing C^∞-functions $f \colon \mathbb{R}^n \to \mathbb{C}$ by $(x_jf)(t) = t_jf(t)$ and $(y_if)(t) = \frac{\partial f}{\partial t_i}(t)$, where $f \in \mathcal{S}(\mathbb{R}^n)$ and $t := (t_1, \ldots, t_n) \in \mathbb{R}^n$. Note that the center of A is $\mathbb{C}$; see [12, Example 2.5.2].

Recall from linear algebra that the span of elements $p_1, \ldots, p_k$ in a complex vector space is the set $\mathrm{span}\{p_1, \ldots, p_k\}$ of all complex linear combinations of $p_1, \ldots, p_k$. For an algebra A and a class $\mathcal{C}$ of representations of A, two more notions of span of elements $p_1, \ldots, p_k \in A$ will be used throughout the paper:

the $\mathcal{C}$-*local linear span* of $p_1, \ldots, p_k$, denoted by $\mathrm{Loc}_{\mathcal{C}}\{p_1, \ldots, p_k\}$, is the set of all $q \in A$ such that

$$\text{(A)} \qquad \pi(q) \in \operatorname{span}\{\pi(p_1), \ldots, \pi(p_k)\} \quad \text{for all } \pi \in \mathcal{C};$$

and the $\mathcal{C}$-*reflexive closure* of $p_1, \ldots, p_k$, denoted by $\mathrm{Ref}_{\mathcal{C}}\{p_1, \ldots, p_k\}$, is the set of all $q \in A$ with

$$\text{(B)} \qquad \pi(q)v \in \operatorname{span}\{\pi(p_1)v, \ldots, \pi(p_k)v\}, \text{ for every } \pi : A \to \mathrm{End}(V_\pi) \text{ in } \mathcal{C} \text{ and } v \in V_\pi.$$

Clearly, $\operatorname{span}\{p_1, \ldots, p_k\} \subseteq \mathrm{Loc}_{\mathcal{C}}\{p_1, \ldots, p_k\} \subseteq \mathrm{Ref}_{\mathcal{C}}\{p_1, \ldots, p_k\}$, and Theorem 1 implies that $\operatorname{span}\{p_1, \ldots, p_k\} = \mathrm{Loc}_{\mathcal{R}}\{p_1, \ldots, p_k\} = \mathrm{Ref}_{\mathcal{R}}\{p_1, \ldots, p_k\}$ for every $p_1, \ldots, p_k \in U(L)$. On the other hand, we do not have a similar result in $U(\mathfrak{sl}_2)$ for $\mathrm{Ref}_{\mathcal{I}}$ and $\mathrm{Loc}_{\mathcal{I}}$. We will provide several counterexamples in Section 3.1 (see Theorem 4). For finite-dimensional complex solvable Lie algebras we will give explicit descriptions of $\mathrm{Ref}_{\mathcal{I}}$ and $\mathrm{Loc}_{\mathcal{I}}$ in Section 3.2.

Our motivation for studying Loc and Ref comes from their relation to nullstellensatz. Namely, assume that the class $\mathcal{C}$ contains only finite-dimensional representations. Then the properties (A) and (B) are respectively equivalent to the properties (A') and (B') below:

(A') For every $\pi : A \to \mathrm{End}(V_\pi)$ in $\mathcal{C}$ and a matrix $B \in \mathrm{End}(V_\pi)$,

$$\operatorname{tr}(\pi(p_1)B) = \ldots = \operatorname{tr}(\pi(p_k)B) = 0 \text{ implies } \operatorname{tr}(\pi(q)B) = 0.$$

(B') For every $\pi \in \mathcal{C}$, $v \in V_\pi$ and $w \in V_\pi^*$,

$$\langle \pi(p_1)v, w\rangle = \ldots = \langle \pi(p_k)v, w\rangle = 0 \text{ implies } \langle \pi(q)v, w\rangle = 0.$$

Here V_π^* stands for the dual of V_π and $\langle u, w\rangle = w(u)$. These equivalences are pretty easy to prove: the proof of the equivalence of (A) and (A') uses the fact that the span of $\pi(p_1), \ldots, \pi(p_k)$ is equal to its second orthogonal complement in $\mathrm{End}(V_\pi)$ with inner product defined by the trace map; the proof of the equivalence of (B) and (B') is based on the span of $\pi(p_1)v, \ldots, \pi(p_k)v$ being equal to its second annihilator in V_π.

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1. Proof of Theorem 1

Let $\mathfrak{sl}_n$ denote the Lie algebra of all complex $n \times n$ matrices with zero trace. A theorem of Ado, see [6, 2.5.6], implies that for every finite-dimensional complex Lie algebra L there exists an embedding $\iota \colon L \to \mathfrak{sl}_n$ for some n.

Let $U(L)$ be the universal enveloping algebra of L. By the PBW theorem [11, §17.3] ι induces an embedding of $U(L)$ into $U(\mathfrak{sl}_n)$. If $f_1, \ldots, f_{n^2-1}$ is a basis of $\mathfrak{sl}_n$, then the monomials $f_1^{m_1} \cdots f_{n^2-1}^{m_{n^2-1}}, m_j \in \mathbb{N}_0$, form a basis of $U(\mathfrak{sl}_n)$.

We write $\mathcal{R}$ for the class of all finite-dimensional representations of L. Proposition 1 below reduces Theorem 1 to a special linearly independent set in $\mathfrak{sl}_n$.

Proposition 1. *The following statements are equivalent:*

1. *For every finite-dimensional Lie algebra L over $\mathbb{C}$ we have that every finite $\mathcal{R}$-locally directionally linearly dependent subset of $U(L)$ is linearly dependent.*
2. *For every finite-dimensional Lie algebra L over $\mathbb{C}$ and every linearly independent set $p_1, \ldots, p_k \in U(L)$ there exists $\pi : U(L) \to \mathrm{End}(V_\pi)$ in $\mathcal{R}$ and a vector $v \in V_\pi$ such that $\pi(p_1)v, \ldots, \pi(p_k)v$ are linearly independent.*
3. *For every $n, d \in \mathbb{N}$ there exists a finite-dimensional representation $\pi_{n,d}\colon U(\mathfrak{sl}_n) \to \mathrm{End}(V_{\pi_{n,d}})$ and a vector $v_{n,d} \in V_{\pi_{n,d}}$ such that all vectors of the form*
$$\pi_{n,d}(f_1)^{m_1} \cdots \pi_{n,d}(f_{n^2-1})^{m_{n^2-1}} v_{n,d}$$
where $f_1, \ldots, f_{n^2-1}$ is a basis for $\mathfrak{sl}_n$ and $\sum_{i=1}^{n^2-1} m_i \leq d$, are linearly independent.

Proof. Clearly, (1) is equivalent to (2) and (3) is a special case of (2).

It remains to prove the implication (3) $\Rightarrow$ (2). Let L be a finite-dimensional complex Lie algebra and let $p_1, \ldots, p_k \in U(L)$ be linearly independent. We first identify $U(L)$ with a subalgebra of $U(\mathfrak{sl}_n)$ for some n. Then using the basis $f_1, \ldots, f_{n^2-1}$ of $U(\mathfrak{sl}_n)$, for each $\ell \in \mathbb{N}$, let W_ℓ denote the subspace of $U(\mathfrak{sl}_n)$ spanned by all elements $\prod_{i=1}^{n^2-1} f_i^{m_i}$ where $\sum_{i=1}^{n^2-1} m_i \leq \ell$.

Let $d \in \mathbb{N}$ be such that $p_1, \ldots, p_k \in W_d$. Choose $p_{k+1}, \ldots, p_N \in W_d$, so that $p_1, \ldots, p_N$ is a basis of W_d. Let $q_1, \ldots, q_N$ be another basis of W_d consisting of all monomials $\prod_{i=1}^{n^2-1} f_i^{m_i}$ with $\sum_{i=1}^{n^2-1} m_i \leq d$. By assumption (3) there exists a representation $\pi_{n,d}\colon U(\mathfrak{sl}_n) \to \mathrm{End}(V_{\pi_{n,d}})$ and a vector $v_{n,d} \in V_{\pi_{n,d}}$ such that the vectors $\pi_{n,d}(q_1)v_{n,d}, \ldots, \pi_{n,d}(q_N)v_{n,d}$ are linearly independent. Claim (2) will follow from the linear independence of
$$\pi_{n,d}(p_1)v_{n,d}, \ldots, \pi_{n,d}(p_N)v_{n,d}$$
which we now show. There are $\gamma_{ij} \in \mathbb{C}$ such that $p_i = \sum_{j=1}^N \gamma_{ij} q_j$ for $i = 1, \ldots, N$. Assume that $\sum_{i=1}^N \alpha_i \pi_{n,d}(p_i)v_{n,d} = 0$ for some $\alpha_i \in \mathbb{C}$. Then $\sum_{j=1}^N \beta_j \pi_{n,d}(q_j)v_{n,d} = 0$ where $\beta_j = \sum_{i=1}^N \alpha_i \gamma_{ij}$ for every $j = 1, \ldots, N$. Since $\pi_{n,d}(q_j)v_{n,d}$ are linearly independent it follows that $\beta_j = 0$ for $j = 1, \ldots, N$. Since the matrix $[\gamma_{ij}]_{i,j}$ is invertible, it follows that $\alpha_i = 0$ for $i = 1, \ldots, N$. ∎

Let $\rho_n\colon \mathfrak{sl}_n \to \mathrm{End}(\mathbb{C}^n)$ be the standard representation of $\mathfrak{sl}_n$ defined by $\rho_n(X)u := Xu$ for every $X \in \mathfrak{sl}_n$ and $u \in \mathbb{C}^n$. Its unique extension to $U(\mathfrak{sl}_n)$ will be denoted by the same symbol. Let $\pi_n = \bigoplus_{i=1}^n \rho_n$ be the direct sum of n copies of ρ_n and let $v = \bigoplus_{i=1}^n e_i$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$. Note that v belongs to $V := \oplus_{i=1}^n \mathbb{C}^n = \mathbb{C}^{n^2}$ and that π_n maps into $\mathrm{End}(V)$. Let $f_1, \ldots, f_{n^2-1}$ be a basis of $\mathfrak{sl}_n$. The following is clear:

Lemma 1. *With the above notation, the vectors v, $\pi_n(f_1)v, \ldots, \pi_n(f_{n^2-1})v$ are linearly independent.*

For every $k \in \mathbb{N}$ let $V^{\otimes k}$ be the k-th tensor power of V and let $\mathrm{Sym}^k(V)$ be the k-th symmetric power of V. Recall that $\mathrm{Sym}^k(V)$ is the subset of $V^{\otimes k}$ consisting of all elements that are invariant under the natural action of the symmetric group S_k on $V^{\otimes k}$. We define a representation $\mathrm{Sym}^k(\pi_n)\colon \mathfrak{sl}_n \to \mathrm{End}(\mathrm{Sym}^k(V))$ by
$$\mathrm{Sym}^k(\pi_n)(x) := \sum_{i=0}^{k-1} I^{\otimes i} \otimes \pi_n(x) \otimes I^{\otimes(k-i-1)}$$

where $I \in \mathrm{End}(V)$ is the identity. Its extension to $U(\mathfrak{sl}_n)$ is unique and it will be denoted by the same symbol.

Lemma 2. *Let $\pi_n\colon \mathfrak{sl}_n \to \mathrm{End}(V)$, $v \in V$ and $f_1, \ldots, f_{n^2-1} \in \mathfrak{sl}_n$ be as in Lemma 1. Let F_k be the subspace of $Sym^k(V)$ generated by all elements of the form*

$$\sum_{\sigma \in S_k} u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \cdots \otimes u_{\sigma(k)}, \text{ where } u_1, \ldots, u_k \in V \text{ and } u_1 = v.$$

Then the vectors

$$\mathrm{Sym}^k(\pi_n)(f_{i_1} \ldots f_{i_k})v^{\otimes k}, \text{ where } 1 \le i_1 \le i_2 \le \ldots \le i_k \le n^2 - 1,$$

are linearly independent in $\mathrm{Sym}^k(V)/F_k$.

Proof. By Lemma 1, the vectors $v_i := \pi_n(f_i)v$, $1 \le i \le n^2 - 1$, and $v_0 = v$ are linearly independent. We have that

$$\mathrm{Sym}^k(\pi_n)(f_{i_1} \cdots f_{i_k})v^{\otimes k} - \sum_{\sigma \in S_k} v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \cdots \otimes v_{i_{\sigma(k)}} \in F_k. \tag{1}$$

Note that the projection of the set

$$\Big\{ \sum_{\sigma \in S_k} v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \cdots \otimes v_{i_{\sigma(k)}} : 1 \le i_1 \le i_2 \le \ldots \le i_k \le n^2 - 1 \Big\} \tag{2}$$

into the vector space $\mathrm{Sym}^k(V)/F_k$ is linearly independent. By (1) and (2) the conclusion of the lemma follows. ∎

Proof of Theorem 1. The implications (1) ⇒ (2) ⇒ (3) are trivial.

It remains to prove the implication (3) ⇒ (1). It suffices to prove statement (3) of Proposition 1. Fix $n, d \in \mathbb{N}$. With the notation from Lemma 2 we define a representation $\pi_{n,d} := \bigoplus_{k=1}^{d} \mathrm{Sym}^k(\pi_n)$ and a vector $v_{n,d} := \bigoplus_{k=1}^{d} v^{\otimes k}$.

To prove that the vectors

$$\pi_{n,d}(f_1)^{m_1} \cdots \pi_{n,d}(f_{n^2-1})^{m_{n^2-1}} v_{n,d} \quad \text{where} \quad m_1 + \ldots + m_{n^2-1} \le d \tag{3}$$

are linearly independent we assume that

$$\sum_{\sum_{i=1}^{n^2-1} m_i \le d} \lambda_{m_1,\ldots,m_{n^2-1}} \pi_{n,d}(f_1)^{m_1} \cdots \pi_{n,d}(f_{n^2-1})^{m_{n^2-1}} v_{n,d} = 0. \tag{4}$$

Project this onto

$$\bigoplus_{k=1}^{d} \mathrm{Sym}^k(V) / \Big(\bigoplus_{k=1}^{d-1} \mathrm{Sym}^k(V) \oplus F_d \Big) \cong \mathrm{Sym}^d(V)/F_d$$

to conclude, by Lemma 2, that $\lambda_{m_1,\ldots,m_{n^2-1}} = 0$ whenever $\sum_{i=1}^{n^2-1} m_i = d$. Repeating this argument for $d-1, d-2, \ldots$ in place of d, we prove that $\lambda_{m_1,\ldots,m_{n^2-1}} = 0$ for all $\sum_{i=1}^{n^2-1} m_i \le d$. ∎

2. Proof of Theorem 2

2.1. Irreducible representations of $\mathfrak{sl}_2$. The main result of this subsection, Proposition 2, describes an irreducible representation of the Lie algebra $\mathfrak{sl}_2$ of 2×2 complex traceless matrices and a vector v making monomials of the form (5) linearly independent. This result will be needed in the proof of Theorem 2, given in Subsection 2.2 below.

Let $e_1, \ldots, e_k$ be the standard basis of $\mathbb{C}^k$, let E_{ij}, $1 \leq i, j \leq 2$, be the standard basis of $M_2(\mathbb{C})$ and let $X := E_{12}$, $Y := E_{21}$ and $H := E_{11} - E_{22}$ be the standard basis of $\mathfrak{sl}_2$. Recall [6, §1.8] that for every $k \in \mathbb{N}$ there is a unique (up to equivalence) irreducible representation $\rho_k : \mathfrak{sl}_2 \to \operatorname{End}(\mathbb{C}^k)$ defined by

$$\rho_k(X)e_i = x_{k,i-1}e_{i-1}, \quad \rho_k(Y)e_i = y_{k,i}e_{i+1}, \quad \rho_k(H)e_i = h_{k,i}e_i,$$

where

$$\begin{aligned} x_{k,i} &:= \begin{cases} k-i, \ \text{if} \ \ i = 1, \ldots, k-1 \\ 0, \qquad\qquad \text{otherwise} \end{cases}, \\ y_{k,i} &:= \begin{cases} i, \ \text{if} \ \ i = 1, \ldots, k-1 \\ 0, \qquad\qquad \text{otherwise} \end{cases}, \\ h_{k,i} &:= (k+1-2i), \quad \text{for} \quad i = 1, \ldots, k. \end{aligned}$$

We denote by 0_ℓ a sequence of ℓ zeroes.

Proposition 2. *Assume the notation as above. For every $t \in \mathbb{N} \cup \{0\}$ and a vector $v \in \mathbb{C}^{(d+1)^2+t}$, $d \in \mathbb{N}$, of the form*

$$v = [0_d, 1, 0_d, 0_{d-1}, 1, 0_{d-1}, \ldots, 0_2, 1, 0_2, 0_1, 1, 0_1, 1, 0_t]^T$$

all vectors of the form

$$\rho_{(d+1)^2+t}(X)^{m_1}\rho_{(d+1)^2+t}(Y)^{m_2}\rho_{(d+1)^2+t}(H)^{m_3}v \tag{5}$$

with $m_1, m_2, m_3 \in \mathbb{N}_0$, $0 \leq m_1 + m_2 + m_3 \leq d$ and $m_1 m_2 = 0$ are linearly independent.

Proof. Let $e_1, \ldots, e_{(d+1)^2+t}$ be the standard basis of $\mathbb{C}^{(d+1)^2+t}$. Then

$$v = e_{i_1} + \ldots + e_{i_{d+1}} \tag{6}$$

where $i_1 = d+1$ and $i_{k+1} = i_k + 2(d-k+1)$ for $k = 1, \ldots, d$. Note that $i_{d+1} = (d+1)^2$.

For every $k = -d, \ldots, d$ and $\ell = 0, \ldots, d$ we write

$$Z_k = \begin{cases} \rho_{(d+1)^2+t}(X)^k & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ \rho_{(d+1)^2+t}(Y)^{-k} & \text{if } k < 0 \end{cases} \quad \text{and} \quad H_\ell = \rho_{(d+1)^2+t}(H)^\ell. \tag{7}$$

To prove that all $Z_k H_\ell v$ with $|k| + \ell \leq d$ are linearly independent we assume that

$$\sum_{|k|+\ell \leq d} \alpha_{k,\ell} Z_k H_\ell v = 0. \tag{8}$$

Since $(d+1)^2 + t$ is fixed in the proof, we abbreviate $x_j := x_{(d+1)^2+t,j}$, $y_j := y_{(d+1)^2+t,j}$ and $h_j := h_{(d+1)^2+t,j}$. Thus

$$Z_k H_\ell e_j = z_{j,k}(h_j)^\ell e_{j-k} \tag{9}$$

where $z_{j,k} = 0$ if $j - k \notin \{1, \ldots, (d+1)^2 + t\}$ while in other cases

$$z_{j,k} = \begin{cases} x_{j-k} \cdots x_{j-1} & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ y_j \cdots y_{j-k-1} & \text{if } k < 0 \end{cases}$$

Since x_j and y_j are nonzero for $j = 1, \ldots, (d+1)^2 + t - 1$, it follows that $z_{j,k}$ are also nonzero when $1 \le j - k \le (d+1)^2 + t$. If we substitute (6) and (9) into (8), we get

$$\sum_{k=-d}^{d} \sum_{r=1}^{d+1} (\sum_{\ell=0}^{d-|k|} \alpha_{k,\ell} (h_{i_r})^\ell) z_{i_r,k} e_{i_r-k} = 0. \tag{10}$$

We prove by backward induction on $|k|$ that the equation (10) implies $\alpha_{k,\ell} = 0$ for all k and ℓ such that $|k| + \ell \le d$. This means we prove:

- **Induction base:** $\alpha_{d,0} = \alpha_{-d,0} = 0$.
- **Induction step:** Fix $m \in \{0, \ldots, d-1\}$. Suppose $\alpha_{k,\ell} = 0$ for $|k| \ge m+1$ and prove that $\alpha_{k,\ell} = 0$ for $|k| = m$.

To establish the base of induction we first compute the coefficient of e_1 in (10). Note that $e_{i_r-k} = e_1$ iff $r = 1$ and $k = d$, so that $|k| + \ell \le d$ forces $\ell = 0$. Since $z_{i_1,d} \ne 0$ and $h_{i_1} \ne 0$, it follows that $\alpha_{d,0} = 0$. Next we compute the coefficient of e_{2d+1} in (10). Note that $e_{i_r-k} = e_{2d+1}$ iff $r = 1, k = -d$ or $r = 2, k = d$. In both cases, it follows that $\ell = 0$. Since $\alpha_{d,0} = 0$ and $z_{i_1,-d} \ne 0$ and $h_{i_1} \ne 0$, it follows that $\alpha_{-d,0} = 0$.

To prove induction step we assume that $\alpha_{k,\ell} = 0$ for every k with $|k| \ge m+1$. Then equation (10) implies that

$$\sum_{k=-m}^{m} \sum_{r=1}^{d+1} (\sum_{\ell=0}^{d-|k|} \alpha_{k,\ell} (h_{i_r})^\ell) z_{i_r,k} e_{i_r-k} = 0. \tag{11}$$

Suppose that $s \in \{1, \ldots, d-m+1\}$. We claim that equation (11) contains only one term with e_{i_s-m} and only one term with e_{i_s+m}. Namely, if $i_r - k = i_s - m$ for some $r = 1, \ldots, d+1$ and $k = -m, \ldots, m$ then $m - k = i_s - i_r$. Clearly, $0 \le m - k \le 2m$, so $s \ge r$. If $r = s$ then $k = m$ and we are done. Otherwise, $2m \ge i_s - i_r \ge i_s - i_{s-1} = 2(d - s + 2)$ which implies that $s \ge d - m + 2$. The other case is similar. It follows that for every $s = 1, \ldots, d-m+1$

$$(\sum_{\ell=0}^{d-m} \alpha_{m,\ell} (h_{i_s})^\ell) z_{i_s,m} = 0 \quad \text{and} \quad (\sum_{\ell=0}^{d-m} \alpha_{-m,\ell} (h_{i_s})^\ell) z_{i_s,-m} = 0 \tag{12}$$

We divide out by $z_{i_s,m}$ and $z_{i_s,-m}$ to obtain two Vandermonde systems

$$\sum_{\ell=0}^{d-m} \alpha_{m,\ell} (h_{i_s})^\ell = 0 \quad \text{and} \quad \sum_{\ell=0}^{d-m} \alpha_{-m,\ell} (h_{i_s})^\ell = 0, \quad \text{for } s = 1, \ldots, d-m+1. \tag{13}$$

Since the h_{i_s} are distinct for different s, the Vandermonde coefficient matrices in both are invertible. It follows that

$$\alpha_{m,0} = \ldots = \alpha_{m,d-m} = 0 \quad \text{and} \quad \alpha_{-m,0} = \ldots = \alpha_{-m,d-m} = 0 \tag{14}$$

which completes the proof of the induction step. ∎

2.2. **$\mathcal{I}$-local directional linear dependence in $\mathfrak{sl}_2$.** In this subsection we prove Theorem 2, which is a characterization of the situation when finitely many elements of $U(\mathfrak{sl}_2)$ are $\mathcal{I}$-locally directionally linearly independent, where $\mathcal{I}$ stands for the class of all finite-dimensional irreducible representations of $U(\mathfrak{sl}_2)$.

Recall from the previous subsection that E_{ij}, $1 \le i,j \le 2$, is the standard basis of $M_2(\mathbb{C})$, and $X := E_{12}$, $Y := E_{21}$, $H := E_{11} - E_{22}$ is the standard basis of $\mathfrak{sl}_2$. Let ρ_k, for $k \in \mathbb{N}$, be the unique (up to equivalence) irreducible representation of $\mathfrak{sl}_2$ of dimension k. The element

$$C := XY + \frac{1}{2}H^2 + YX = 2XY + \frac{1}{2}H^2 - H$$

of the enveloping algebra $U(\mathfrak{sl}_2)$ is called the *Casimir element*. It is well-known that C generates the center Z of $U(\mathfrak{sl}_2)$, i.e., $Z = \mathbb{C}[C]$, and that $\rho_k(C) = \frac{1}{2}(k^2-1)I_k$ where I_k is the identity matrix of size k (see [11]). We write $c_k := \frac{1}{2}(k^2-1)$ for all $k \in \mathbb{N}$. Moreover, every element $p \in U(\mathfrak{sl}_2)$ can be written in the form $p = \sum_{i=1}^m f_i s_i$ where f_i are monomials of the form $X^{i_1}Y^{i_2}H^{i_3}$ with $i_1 i_2 = 0$ and $s_i \in \mathbb{C}[C]$ are central elements.

Before we proceed with the proof of Theorem 2, we need the following lemma:

Lemma 3. *Suppose $u_1, \ldots, u_k \in \mathbb{C}(z)^\ell$, for $k, \ell \in \mathbb{N}$, are linearly dependent for infinitely many complex values of z. Then they are linearly dependent over $\mathbb{C}(z)$.*

Proof. Assume to the contrary that $u_1, \ldots, u_k$ are linearly independent over $\mathbb{C}(z)$. Then we can add vectors $u_{k+1}, \ldots, u_\ell \in \mathbb{C}(z)^\ell$ such that $u_1, \ldots, u_\ell$ form a basis for $\mathbb{C}(z)^\ell$ over $\mathbb{C}(z)$. The determinant of the matrix with columns $u_1, \ldots, u_\ell$ is a non-zero rational function $\frac{p(z)}{r(z)} \in \mathbb{C}(z)$ which has only finitely many zeros, a contradiction with the hypothesis that infinitely many evaluations of $u_1, \ldots, u_k$ are $\mathbb{C}$-linearly dependent. ■

Proof of Theorem 2. To prove the implication (1) ⇒ (2), we first divide out the greatest common divisor of $z_1, \ldots, z_k \in \mathbb{C}[C]$ from the equation $\sum_{i=1}^k z_i p_i = 0$. Hence, we can assume WLOG that $z_1, \ldots, z_k$ do not have a common zero. Applying each $\rho_n \in \mathcal{I}$, for $n \in \mathbb{N}$, to $\sum_{i=1}^k z_i p_i = 0$ one gets $0 = \sum_{i=1}^k z_i(c_n)\rho_n(p_i)$. Since $z_1, \ldots, z_k$ are without common zeroes, this linear combination is non-trivial and hence $p_1, \ldots, p_k$ are $\mathcal{I}$-locally linearly dependent.

The implication (2) ⇒ (3) is trivial.

It remains to prove the implication (3) ⇒ (1). We write $p_j = \sum_{i=1}^m f_i t_{ij}$, $m \in \mathbb{N}$, where $t_{ij} \in \mathbb{C}[C]$ are central elements and f_i are different monomials of the form $X^{i_1}Y^{i_2}H^{i_3}$ with $i_1, i_2, i_3 \in \mathbb{N}_0$ and $i_1 i_2 = 0$. By Proposition 2 for all $n \in \mathbb{N}_0$ sufficiently large there exist vectors $v_n \in V_{\rho_n}$ such that vectors $\rho_n(f_i)v_n$, $i = 1, \ldots, m$, are linearly independent. Therefore, for those n, the vectors $\rho_n(p_1)v_n, \ldots, \rho_n(p_k)v_n$, are linearly dependent if and only if the vectors $[t_{1j}(c_n), \ldots, t_{mj}(c_n)]^T, j = 1, \ldots, k$, are linearly dependent. Since this is true for infinitely many n-s, this implies by Lemma 3 that the vectors $[t_{1j}(C), \ldots, t_{mj}(C)]^T, j = 1, \ldots, k$, are $\mathbb{C}(C)$-linearly dependent and hence there exist $v_j(C) \in \mathbb{C}(C)$, $j = 1, \ldots, k$, not all zero such that $0 = \sum_{j=1}^k v_j(C)[t_{1j}(C), \ldots, t_{mj}(C)]^T$. Multiplying by the least common denominator $z_0 \in \mathbb{C}[C]$ of nonzero $v_1, \ldots, v_k$ we obtain $0 = \sum_{j=1}^k z_j[t_{1j}(C), \ldots, t_{mj}(C)]^T$ for some $z_1, \ldots, z_k \in \mathbb{C}[C]$, not all zero and hence $0 = z_1 p_1 + \ldots + z_k p_k$. ■

3. Reflexive closures

3.1. **Reflexive closures in $\mathfrak{sl}_2$.** Assume the notation from the previous section. Let $q, p_1, \ldots, p_k$ be elements of $U(\mathfrak{sl}_2)$. Theorem 4 gives a closely related sufficient condition (1) and a necessary condition (4) for q to belong to the $\mathcal{I}$-local span, resp. the $\mathcal{I}$-reflexive closure, of $p_1, \ldots, p_k$. The conditions differ only in the assumptions on the zero set of the central element z_0.

Theorem 4. *Let $q, p_1, \ldots, p_k$ be elements of $U(\mathfrak{sl}_2)$ and consider the following statements:*

(1) There exist central elements $z_0, z_1, \ldots, z_k \in \mathbb{C}[C]$ such that z_0 is nonzero, $z_0(c_n) \neq 0$ if $\rho_n(q) \neq 0$ and $z_0 q = z_1 p_1 + \ldots + z_k p_k$.

(2) $q \in \mathrm{Loc}_{\mathcal{I}}\{p_1, \ldots, p_k\}$.

(3) $q \in \mathrm{Ref}_{\mathcal{I}}\{p_1, \ldots, p_k\}$.

(4) There exist central elements $z_0, z_1, \ldots, z_k \in \mathbb{C}[C]$ such that z_0 is nonzero and $z_0 q = z_1 p_1 + \ldots + z_k p_k$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and the reverse implications do not hold.

The proof of Theorem 4 uses the following trivial consequence of Lemma 3.

Lemma 4. *Suppose $s, u_1, \ldots, u_k \in \mathbb{C}(z)^\ell$ for $k, \ell \in \mathbb{N}$, have the property that $s(t) \in \mathrm{span}_{\mathbb{C}}\{u_1(t), \ldots, u_k(t)\}$ for infinitely many $t \in \mathbb{C}$. Then*

$$s(z) \in \mathrm{span}_{\mathbb{C}(z)}\{u_1(z), \ldots, u_k(z)\}.$$

Proof of Theorem 4. To prove (1) $\Rightarrow$ (2) note that $z_0 q = z_1 p_1 + \ldots + z_k p_k$ implies that $z_0(c_n)\rho_n(q) = z_1(c_n)\rho_n(p_1) + \ldots + z_k(c_n)\rho_n(p_k)$. If $\rho_n(q) = 0$, then clearly $\rho_n(q) \in \mathrm{span}\{\rho_n(p_1), \ldots, \rho_n(p_k)\}$. Otherwise $\rho_n(q) \neq 0$ which implies by assumption that $z_0(c_n) \neq 0$ and hence again $\rho_n(q) \in \mathrm{span}\{\rho_n(p_1), \ldots, \rho_n(p_k)\}$.

The implication (2) $\Rightarrow$ (3) is trivial.

The proof of (3) $\Rightarrow$ (4) is analogous to the proof of the implication (3) $\Rightarrow$ (1) in Theorem 2 only that we use Lemma 4 instead of Lemma 3.

It remains to construct counterexamples for the reverse implications. To prove (1) $\not\Leftarrow$ (2) take $q = H$, $p_1 = CX^2 + c_2 H$ and $p_2 = c_2 X^2 + CH$. First, we prove that $q \in \mathrm{Loc}_{\mathcal{I}}\{p_1, p_2\}$. Since $\rho_2(X^2) = 0$ we have $\rho_2(p_1) = \rho_2(p_2) = c_2\rho_2(H) = c_2\rho_2(q)$, so $\rho_2(q) \in \mathrm{span}\{\rho_2(p_1), \rho_2(p_2)\}$. For $n > 2$ we have $(c_2^2 - c_n^2)\rho_n(q) = c_2\rho_n(p_1) - c_n\rho_n(p_2)$, which also implies that $\rho_n(q) \in \mathrm{span}\{\rho_n(p_1), \rho_n(p_2)\}$. Second, we show that each triplet of central elements z_0, z_1, z_2 that satisfy $z_0 q = z_1 p_1 + z_2 p_2$ must have that $z_0(c_2) = 0$. By comparing the coefficients at X^2 and H we get the system $0 = z_1 C + z_2 c_2$ and $z_0 = z_1 c_2 + z_2 C$. Hence $c_2 z_0 = z_1(c_2^2 - C^2)$ and $z_0(c_2) = 0$.

To prove (2) $\not\Leftarrow$ (3) take $q = X$, $p_1 = I + H$, $p_2 = X + Y$ and $p_3 = (C - c_2)X$. Clearly

$$\rho_2(q) = E_{12} \notin \mathrm{span}\{2E_{11}, E_{12} + E_{21}, 0\} = \mathrm{span}\{\rho_2(p_1), \rho_2(p_2), \rho_2(p_3)\},$$

which implies that $q \notin \mathrm{Loc}_{\mathcal{I}}\{p_1, p_2, p_3\}$. Since

$$[y, 0]^T \in \mathrm{span}\{2[x, 0]^T, [y, x]^T\}$$

for every x and y, we have that $\rho_2(q)v \in \mathrm{span}\{\rho_2(p_1)v, \rho_2(p_2)v, \rho_2(p_3)v\}$. Clearly, we also have that $\rho_n(q)v = \frac{1}{c_n - c_2}\rho_n(p_3)v$ for all $n \geq 3$ and $v \in \mathbb{C}^n$, which implies that $q \in \mathrm{Ref}_I\{p_1, p_2, p_3\}$.

To prove (3) $\not\Leftarrow$ (4) take $q = I$ and $p = (C - c_2)I$ and notice that $(C - c_2)q = p$ but $q \notin \mathrm{Ref}_{\mathcal{I}}\{p\}$ since $\rho_2(q)e_1 = e_1 \notin \{0\} = \mathrm{span}\{\rho_2(p)e_1\}$. ∎

As seen in the proof (4) of Theorem 4 does not suffice to conclude $q \in \mathrm{Ref}_{\mathcal{I}}\{p_1, \ldots, p_k\}$. The failure of the reverse implications in Theorem 4 are caused by representations ρ_n of small dimension n. The following theorem says that, for n big enough, the same reverse implications hold true.

Theorem 5. *Let $q, p_1, \ldots, p_k$ be elements from $U(\mathfrak{sl}_2)$. Then the following statements are equivalent:*

(1) There exist central elements $z_0, z_1, \ldots, z_k \in \mathbb{C}[C]$ such that $z_0 \neq 0$ and $z_0 q = z_1 p_1 + \ldots + z_k p_k$.
(2) $\rho_n(q) \in \mathrm{span}\{\rho_n(p_1), \ldots, \rho_n(p_k)\}$ for every $n \in \mathbb{N}$ big enough.
(3) $\rho_n(q)v \in \mathrm{span}\{\rho_n(p_1)v, \ldots, \rho_n(p_k)v\}$ for every $n \in \mathbb{N}$ big enough and every vector v.

Proof. To prove (1) $\Rightarrow$ (2) one takes n big enough such that $z_0(c_r) \neq 0$ for every $r \geq n$. Notice that for all such r we have that $\rho_r(z_0) = z_0(c_r) \neq 0$ and hence $\rho_r(q) = \frac{1}{\rho_r(z_0)} \sum_{i=1}^k \rho_r(z_i)\rho_r(p_i)$. The implication (2) $\Rightarrow$ (3) is clear. The implication (3) $\Rightarrow$ (1) follows easily from the proof of the implication (3) $\Rightarrow$ (4) in Theorem 4, since for n big enough, there exist vectors $v_n \in V_{\rho_n}$ such that $\rho_n(q)v_n \in \mathrm{span}\{\rho_n(p_i)v_n \colon i = 1, \ldots, k\}$. ∎

3.2. **Reflexive closures in solvable Lie algebras.** By Lie's theorem [8, Theorem 9.11], every irreducible representation π of a finite-dimensional complex solvable Lie algebra L is one-dimensional. It follows that π annihilates $L_1 := [L, L]$, hence it factors through the abelian Lie algebra L/L_1. Let R be the left (equivalently the right) ideal of $U(L)$ generated by L_1. By [6, Proposition 2.2.14], the canonical homomorphism from $U(L)$ to $U(L/L_1)$ is surjective with kernel R and so $U(L)/R \cong U(L/L_1)$. Clearly, every irreducible representation of $U(L)$ factors through $U(L)/R$.

Theorem 6. *Let L be a finite-dimensional complex solvable Lie algebra and R the two-sided ideal of $U(L)$ generated by $L_1 = [L, L]$. Pick $p_1, \ldots, p_k, q \in U(L)$ and write I for the two-sided ideal of $U(L)$ generated by $p_1, \ldots, p_k$. The following are equivalent:*

(1) For some $n \in \mathbb{N}$ we have that $q^n \in I + R$.
(2) Every irreducible representation of $U(L)$ which anihilates $p_1, \ldots, p_k$ also annihilates q.
(3) $q \in \mathrm{Loc}_{\mathcal{I}}\{p_1, \ldots, p_k\}$.
(4) $q \in \mathrm{Ref}_{\mathcal{I}}\{p_1, \ldots, p_k\}$.

Proof. The equivalence of (1) and (2) follows from Hilbert's Nullsellensatz and $U(L)/R \cong U(L/L_1)$. Namely, since $U(L/L_1)$ is isomorphic to a polynomial algebra, the following are equivalent for any $p'_1, \ldots, p'_k, q' \in U(L/L_1)$:

- q' belongs to the radical of the ideal generated by $p'_1, \ldots, p'_k$.
- Every character ϕ of $U(L/L_1)$ which anihilates $p'_1, \ldots, p'_k$ also anihilates q'.

The equivalence of (2) and (3) follows from the trivial observation that for complex numbers $\alpha_1, \ldots, \alpha_k, \beta$ we have that $\beta \in \mathrm{span}\{\alpha_1, \ldots, \alpha_k\}$ iff $\alpha_1 = \ldots = \alpha_k = 0$ implies $\beta = 0$.

Since all irreducible representations are one-dimensional, (3) is equivalent to (4). ∎

4. Proof of Theorem 3

4.1. **$\mathcal{I}$-local directional linear dependence in $\mathfrak{sl}_3$.** The Lie algebra of all trace-zero complex 3×3 matrices is denoted by $\mathfrak{sl}_3$. We refer the reader to [10, Chapter 6] for the theory of representations of $\mathfrak{sl}_3$; here we write the basics. The standard basis of $\mathfrak{sl}_3$ is

$$
\begin{aligned}
&X_1 := E_{12},\ X_2 := E_{23},\ X_3 := E_{13},\ Y_1 := E_{21},\ Y_2 := E_{32},\\
&Y_3 := E_{31},\ H_1 := E_{11}-E_{22},\ H_2 := E_{22}-E_{33}.
\end{aligned}
\tag{15}
$$

We write $V_1 = V_2 = \mathbb{C}^3$. Let e_1, e_2, e_3 be the standard basis of V_1 and let $f_1 = e_3, f_2 = -e_2, f_3 = e_1$ be a basis of V_2. The action of $\mathfrak{sl}_3$ on V_1 is defined by $\pi_1(Z)v := Zv$ and its action on V_2 is defined by $\pi_2(Z)v := -Z^T v$. (Note that π_1 is the standard representation and π_2 is its adjoint.) For every $m_1, m_2 \in \mathbb{N}$, we identify the m_1-th symmetric power $\mathrm{Sym}^{m_1}(V_1)$ of V_1 with the vector space of all homogeneous polynomials of degree m_1 in e_1, e_2, e_3. Similarly, we identify $\mathrm{Sym}^{m_2}(V_2)$ with the vector space of all homogeneous polynomials of degree m_2 in f_1, f_2, f_3. Let ψ_1 be the representation of $\mathfrak{sl}_3$ on $\mathrm{Sym}^{m_1}(V_1)$ defined by

$$\psi_1(e_{i_1}e_{i_2}\cdots e_{i_m}) := \sum_{j=1}^{m} e_{i_1}\cdots e_{i_{j-1}}\pi_1(e_{i_j})e_{i_{j+1}}\cdots e_{i_m}.$$

ψ_2 is defined analogously. The representations ψ_1 and ψ_2 are irreducible but their tensor product $\psi := \psi_1 \otimes \psi_2$, defined by

$$\psi(v_1\otimes v_2) := \psi_1(v_1)\otimes v_2 + v_1 \otimes \psi_2(v_2)$$

is not irreducible. Let W be the subspace of $\mathrm{Sym}^{m_1}(V_1)\otimes \mathrm{Sym}^{m_2}(V_2)$ generated by all elements of the form

$$v_{i,j,k} := \psi(Y_1^i Y_2^j Y_3^k)(e_1^{m_1}\otimes f_1^{m_2}), \quad i,j,k\in\mathbb{N}_0. \tag{16}$$

It turns out that W is an invariant subspace for $\psi(\mathfrak{sl}_3)$ and the subrepresentation $\pi_{m_1,m_2} := \psi|_W$ is irreducible. Recall that a *weight* of a representation π is a pair of integers z_1, z_2 such that $\pi(H_i)v = z_i v$ for $i = 1, 2$ where v is some nonzero vector, called a *weight vector*. The weight (m_1, m_2) is the *highest weight* if for every weight (m_1', m_2') we have

$$(m_1, m_2) - (m_1', m_2') = a(2,-1) + b(-1,2)$$

for some $a, b \geq 0$. The highest weight of the representation with the irreducible subspace generated by (16) is (m_1, m_2) and its highest weight vector is $v := e_1^{m_1}\otimes f_1^{m_2}$.

In order to prove an analogue of Proposition 2, we start with the following proposition.

Proposition 3. *For every $d, m_1, m_2 \in \mathbb{N}_0$ with $m_1 \geq d$ and $m_2 \geq d$, the vectors $v_{k,\ell,m}$ with $k,\ell,m\in\mathbb{N}_0$ such that $k+\ell+m\leq d$ are linearly independent.*

Proof. Denote $S_d := \{(k,\ell,m)\in\mathbb{N}_0^3\colon k+\ell+m\leq d\}$ and assume that

$$\sum_{(k,\ell,m)\in S_d} \alpha_{k,\ell,m} v_{k,\ell,m} = 0 \tag{17}$$

for some $\alpha_{k,\ell,m}\in\mathbb{R}$. We have to prove that each $\alpha_{k,\ell,m}$ is zero. After a short computation which depends on the formula

$$\pi_{m_1,m_2}(Y_i^j)(u_1\otimes u_2) = \sum_{q=0}^{j}\binom{j}{q}\psi_1(Y_i^q)u_1\otimes\psi_2(Y_i^{j-q})u_2$$

for each i and j we get that

$$v_{k,\ell,m} = \sum_{t=0}^{m}\sum_{s=0}^{k} \beta_{s,t}^{k,\ell,m} e_1^{m_1-s-t} e_2^s e_3^t \otimes f_1^{m_2+t-\ell-k} f_2^{\ell-k+s} f_3^{m+k-s-t} \tag{18}$$

where $\beta_{s,t}^{k,\ell,m} \in \mathbb{R}$ and in particular $\beta_{k,m}^{k,\ell,m} = \binom{m_1}{m}\binom{m_2}{\ell}\binom{m_1-m}{k} \neq 0$. For $a,b,c \in \mathbb{N}_0$ we denote by $P_{a,b,c}$ the projection to the linear subspace $\mathrm{Lin}\{e_1^{i_1} e_2^b e_3^a \otimes f_1^{i_2} f_2^c f_3^{i_3} : i_j \in \mathbb{N}_0\}$. Applying projections $P_{a,b,c}$ repeatedly in the lexicographic ordering of indices (a,b,c) where $(b,c,a) \in S_d$ on (17) and using (18) we deduce inductively that each $\alpha_{k,\ell,m}$ in (17) is zero. Namely, first

$$0 = P_{d,0,0}\Big(\sum_{(k,\ell,m)\in S_d} \alpha_{k,\ell,m} v_{k,\ell,m}\Big) = \alpha_{0,0,d} \cdot \beta_{0,d}^{0,0,d} e_1^{m_1-d} e_3^d \otimes f_1^{m_2}$$

implies that $\alpha_{0,0,d} = 0$ (since $\beta_{0,d}^{0,0,d} \neq 0$). Now fix (a_0,b_0,c_0) and assume that $\alpha_{b,c,a} = 0$ for all $(a,b,c) \succ_{\text{lex}} (a_0,b_0,c_0)$. Then

$$\begin{aligned} 0 &= P_{a_0,b_0,c_0}\Big(\sum_{(k,\ell,m)\in S_d} \alpha_{k,\ell,m} v_{k,\ell,m}\Big) \\ &= \alpha_{b_0,c_0,a_0} \cdot \beta_{b_0,a_0}^{b_0,c_0,a_0} e_1^{m_1-b_0-a_0} e_2^{b_0} e_3^{a_0} \otimes f_1^{m_2-c_0} f_2^{c_0} \end{aligned}$$

implies that $\alpha_{b_0,c_0,a_0} = 0$ (since $\beta_{b_0,a_0}^{b_0,c_0,a_0} \neq 0$). ■

Lemma 5. *For every $d, m_1, m_2 \in \mathbb{N}_0$ with $m_1 \geq d$ and $m_2 \geq d$, generators of $\mathfrak{sl}_3$ map vectors $v_{k,\ell,m}$, $k,\ell,m \in \mathbb{N}_0$, by the following rules:*

$$\pi_{m_1,m_2}(H_1)v_{k,\ell,m} = \alpha v_{k,\ell,m}, \tag{19}$$
$$\pi_{m_1,m_2}(H_2)v_{k,\ell,m} = \beta v_{k,\ell,m}, \tag{20}$$
$$\pi_{m_1,m_2}(Y_1)v_{k,\ell,m} = v_{k+1,\ell,m}, \tag{21}$$
$$\pi_{m_1,m_2}(Y_2)v_{k,\ell,m} = v_{k,\ell+1,m} + k v_{k-1,\ell,m+1}, \tag{22}$$
$$\pi_{m_1,m_2}(Y_3)v_{k,\ell,m} = v_{k,\ell,m+1}, \tag{23}$$
$$\pi_{m_1,m_2}(X_1)v_{k,\ell,m} = \gamma v_{k-1,\ell,m} - m v_{k,\ell+1,m-1} \tag{24}$$
$$\pi_{m_1,m_2}(X_2)v_{k,\ell,m} = \delta v_{k,\ell-1,m} + m v_{k+1,\ell,m-1}. \tag{25}$$
$$\pi_{m_1,m_2}(X_3)v_{k,\ell,m} = \xi v_{k-1,\ell-1,m} + \zeta v_{k,\ell,m-1}, \tag{26}$$

where

$$\begin{aligned} \alpha &= (m_1 - 2k + \ell - m), & \beta &= (m_2 + k - 2\ell - m), \\ \gamma &= k(m_1 - k + 1 + \ell - m), & \delta &= \ell(m_2 - \ell + 1), \\ \xi &= -k\ell(m_2 - \ell + 1), & \zeta &= m(m_1 + m_2 + 1 - \ell - k - m). \end{aligned}$$

Proof. We write $\pi := \pi_{m_1,m_2}$. Equalities (19) and (20) follow by the following facts:

- $v_{0,0,0}$ is a weight vector corresponding to the weight (m_1, m_2).
- $\pi(Y_1)$, $\pi(Y_2)$, $\pi(Y_3)$ are root vectors corresponding to the roots $(-2,1)$, $(1,-2)$, $(-1,-1)$.
- A vector $v_{k,\ell,m}$ is nonzero by Proposition 3.

The equality (21) is clear while (23) follows by the fact that Y_3 commutes with Y_1 and Y_2 in $U(\mathfrak{sl}_3)$.

The remaining equalities can be proved by induction on lexicographically increasing triples (k, ℓ, m). For examples we will prove (22) and (24).

The base of induction $(k, \ell, m) = (0, 0, 0)$ for (22) is established by calculating $\pi(Y_2)v_{0,0,0} = v_{0,1,0}$. Now fix a triple (k_0, ℓ_0, m_0) and assume that (22) is true for every triple (k, ℓ, m) such that $(k_0, \ell_0, m_0) \succ_{\text{lex}} (k, \ell, m)$. We separate two cases:

Case 1: $k_0 > 0$. By the relation $Y_2Y_1 = Y_1Y_2 + Y_3$ from $U(\mathfrak{sl}_3)$ and the fact that Y_3 commutes with Y_1 and Y_2 we have that

$$\pi(Y_2)v_{k_0,\ell_0,m_0} = \pi(Y_1)\pi(Y_2)v_{k_0-1,\ell_0,m_0} + v_{k_0-1,\ell_0,m_0+1},$$

Now we use the induction hypothesis for $(k_0 - 1, \ell_0, m_0)$ and get

$$\pi(Y_2)v_{k_0,\ell_0,m_0} = v_{k_0,\ell_0+1,m_0} + k_0v_{k_0-1,\ell_0,m_0+1}.$$

Case 2: $k_0 = 0$. We have $\pi(Y_2)v_{0,\ell_0,m_0} = v_{0,\ell_0+1,m_0}$ which is (22).

Now we prove (24). The base of induction $(k, \ell, m) = (0, 0, 0)$ is established by calculating

$$\pi(X_1)v_{0,0,0} = \psi_1(X_1)e_1^{m_1} \otimes f_1^{m_2} + e_1^{m_1} \otimes \psi_1(X_1)f_1^{m_2} = 0.$$

Now fix a triple (k_0, ℓ_0, m_0) and assume that (24) is true for every triple (k, ℓ, m) such that $(k_0, \ell_0, m_0) \succ_{\text{lex}} (k, \ell, m)$. We separate three cases:

Case 1: $k_0 > 0$. By the relation $X_1Y_1 = Y_1X_1 + H_1$ from $U(\mathfrak{sl}_3)$ we have that

$$\pi(X_1)v_{k_0,\ell_0,m_0} = \pi(Y_1)\pi(X_1)v_{k_0-1,\ell_0,m_0} + \pi(H_1)v_{k_0-1,\ell_0,m_0},$$

Now we use the induction hypothesis for $(k_0 - 1, \ell_0, m_0)$ for the first term, the equality (19) for the second term and after a short calculation we get (24).

Case 2: $k_0 = 0$, $\ell_0 > 0$. By the relation $X_1Y_2 = Y_2X_1$ from $U(\mathfrak{sl}_3)$ we have that

$$\pi(X_1)v_{0,\ell_0,m_0} = \pi(Y_2)\pi(X_1)v_{0,\ell_0-1,m_0},$$

and by the induction hypothesis for $(0, \ell_0 - 1, m_0)$ we get (24).

Case 3: $k_0 = 0$, $\ell_0 = 0$, $m_0 > 0$. By the relation $X_1Y_3 = Y_3X_1 - Y_2$ from $U(\mathfrak{sl}_3)$ we have that

$$\pi(X_1)v_{0,0,m_0} = \pi(Y_3)\pi(X_1)v_{0,0,m_0-1} - v_{0,1,m_0-1},$$

and by the induction hypothesis for $(0, 0, m_0 - 1)$ we get (24). ∎

Proposition 4. *For every $d, m_1, m_2 \in \mathbb{N}_0$ with m_1, m_2 big enough, the vectors*

$$\pi_{m_1,m_2}(Y_1^{j_1}Y_2^{j_2}Y_3^{j_3}X_1^{\ell_1}X_2^{\ell_2}X_3^{\ell_3}H_1^{r_1}H_2^{r_2})\Big(\sum_{t=1}^{L} v_{k(t),\ell(t),m(t)}\Big), \tag{27}$$

are linearly independent, where the powers $j_1, j_2, j_3, \ell_1, \ell_2, \ell_3, r_1, r_2 \in \mathbb{N}_0$ are such that $\sum_{i=1}^{3} j_i + \sum_{i=1}^{3} \ell_i + \sum_{i=1}^{2} r_i \leq d$, $j_2\ell_2 = 0$, $r_2 \leq 2$ and the indices $k(t), \ell(t), m(t)$ for $t = 1, \ldots, L$, *with* $L := 4d^3 + 4d^2 + 2d + 1$, *are defined by*

$$k(t) = (3d+1)t, \quad \ell(t) = t^{2d+1}, \quad m(t) = t^{4d^2+2d+1}.$$

Proof. We write $\vec{Y} := (Y_1, Y_2, Y_3)$, $\vec{X} := (X_1, X_2, X_3)$, $\vec{H} := (H_1, H_2)$, $\vec{j} := (j_1, j_2, j_3)$, $\vec{\ell} := (\ell_1, \ell_2, \ell_3)$, $\vec{r} := (r_1, r_2)$ and

$$\vec{Y}^{\vec{j}}\vec{X}^{\vec{\ell}}\vec{H}^{\vec{r}} := Y_1^{j_1}Y_2^{j_2}Y_3^{j_3}X_1^{\ell_1}X_2^{\ell_2}X_3^{\ell_3}H_1^{r_1}H_2^{r_2}.$$

Lemma 5 implies that

$$\pi_{m_1,m_2}(\vec{Y}^{\vec{j}}\vec{X}^{\vec{\ell}}\vec{H}^{\vec{r}})v_{k,\ell,m} = \sum_{s=0}^{j_2+\ell_1+\ell_2+\ell_3} c_{\vec{j},\vec{\ell},\vec{r},s}(k,\ell,m)\cdot v_{k-\ell_3-\ell_1-j_2+j_1+s,\ell-\ell_3-\ell_2+s,m+j_2+j_3-s} \tag{28}$$

where $c_{\vec{j},\vec{\ell},\vec{r},s}(k,\ell,m)$ are polynomials in k, ℓ, m. Let S be the endomorphism of $V_{\pi_{m_1,m_2}}$ defined by

$$S(v_{k,\ell,m}) = \begin{cases} v_{k+1,\ell+1,m-1} & \text{if } m \geq 1, \\ 0 & \text{if } m = 0. \end{cases} \tag{29}$$

Consider the operator

$$C_{\vec{j},\vec{\ell},\vec{r}}(k,\ell,m,S) := \sum_{s=0}^{j_2+\ell_1+\ell_2+\ell_3} c_{\vec{j},\vec{\ell},\vec{r},s}(k,\ell,m)S^s$$

The equation (28) can now be rewritten as

$$\pi_{m_1,m_2}(\vec{Y}^{\vec{j}}\vec{X}^{\vec{\ell}}\vec{H}^{\vec{r}})v_{k,\ell,m} = C_{\vec{j},\vec{\ell},\vec{r}}(k,\ell,m,S)v_{k-\ell_3-\ell_1-j_2+j_1,\ell-\ell_3-\ell_2,m+j_2+j_3} \tag{30}$$

To compute the leading term of $C_{\vec{j},\vec{\ell},\vec{r}}(k,\ell,m,S)$ with respect to a monomial ordering $\succ$ defined below, we first introduce new variables

$$x := k - 2\ell - m, \quad y := -k + \ell - m, \quad z := -2k + \ell - m. \tag{31}$$

Note that we have that

$$k = y - z, \quad \ell = \frac{1}{3}(-x + 3y - 2z), \quad m = \frac{1}{3}(-x - 3y + z). \tag{32}$$

Now consider the lexicographic ordering induced by

$$x \succ y \succ z \succ S. \tag{33}$$

Using Lemma 5 we see that the leading term of $C_{\vec{j},\vec{\ell},\vec{r}}(k,\ell,m,S)$ is the same as the leading term of

$$(k+S)^{j_2}(k(\ell-k-m)-mS)^{\ell_1}(-\ell^2+mS)^{\ell_2}(k\ell^2-m(k+\ell+m)S)^{\ell_3}\cdot$$
$$\cdot(-2k+\ell-m)^{r_1}(-2\ell+k-m)^{r_2}$$

which is equal to

$$y^{j_2}\Big(\frac{Sx}{3}\Big)^{\ell_1}\Big(-\frac{x^2}{9}\Big)^{\ell_2}\Big(\frac{x^2y}{9}\Big)^{\ell_3}z^{r_1}x^{r_2} = \frac{(-1)^{\ell_2}}{3^{\ell_1+2(\ell_2+\ell_3)}}x^{\ell_1+2(\ell_2+\ell_3)+r_2}y^{j_2+\ell_3}z^{r_1}S^{\ell_1}.$$

We denote by Γ_d the set of all tuples $(\vec{j},\vec{\ell},\vec{r})$ satisfying

$$\sum_{i=1}^{3} j_i + \sum_{i=1}^{3}\ell_i + \sum_{i=1}^{2} r_i \leq d, \quad j_2\ell_2 = 0 \quad \text{and} \quad r_2 \leq 2.$$

Assume that

$$\sum_{(\vec{j},\vec{\ell},\vec{r})\in\Gamma_d} \lambda_{\vec{j},\vec{\ell},\vec{r}} \cdot \pi_{m_1,m_2}(\vec{Y}^{\vec{j}}\vec{X}^{\vec{\ell}}\vec{H}^{\vec{r}})\Big(\sum_{t=1}^{L} v_{k(t),\ell(t),m(t)}\Big) = 0 \tag{34}$$

By the choice of $k(t), \ell(t), m(t)$ we have that triples of v-indices appearing in

$$\pi_{m_1,m_2}(\vec{Y}^{\vec{j}}\vec{X}^{\vec{\ell}}\vec{H}^{\vec{r}})v_{k(t),\ell(t),m(t)}$$

are always different from triples of v-indices appearing in

$$\pi_{m_1,m_2}(\vec{Y}^{\vec{j}}\vec{X}^{\vec{\ell}}\vec{H}^{\vec{r}})v_{k(t'),\ell(t'),m(t')}$$

if $t \neq t'$. Therefore, the equation (34) implies that for every $t = 1, \ldots, L$, we have that

$$\sum_{(\vec{j},\vec{\ell},\vec{r})\in\Gamma_d} \lambda_{\vec{j},\vec{\ell},\vec{r}} \cdot \pi_{m_1,m_2}(\vec{Y}^{\vec{j}}\vec{X}^{\vec{\ell}}\vec{H}^{\vec{r}})v_{k(t),\ell(t),m(t)} = 0.$$

The equation (30) implies that

$$0 = \sum_{(\vec{j},\vec{\ell},\vec{r})\in\Gamma_d} \lambda_{\vec{j},\vec{\ell},\vec{r}} \cdot C_{\vec{j},\vec{\ell},\vec{r}}(k(t), \ell(t), m(t), S)v_{k(t)-\ell_3-\ell_1-j_2+j_1,\ell(t)-\ell_3-\ell_2,m(t)+j_2+j_3} \tag{35}$$

For every $(\vec{j},\vec{\ell},\vec{r}) \in \Gamma_d$ let us define the set

$$\begin{aligned}\Delta_{(\vec{j},\vec{\ell},\vec{r})} \quad := \quad & \{(d_1, d_2, d_3) \in \mathbb{Z}^3 \colon d_1 = -\ell_3 - \ell_1 - j_2 + j_1 + s, d_2 = -\ell_3 - \ell_2 + s, \\ & d_3 = j_2 + j_3 - s \text{ for some } 0 \le s \le j_2 + \sum_{i=1}^{3} \ell_i\}.\end{aligned}$$

Fix a vector $\vec{e} := (e_1, e_2) \in \mathbb{Z}^2$ and define a set

$$\begin{aligned}\Lambda_{\vec{e}} \quad & := \quad \{(\vec{j},\vec{\ell},\vec{r}) \in \Gamma_d \colon (e_1 + d_2, d_2, e_2 - d_2) \in \Delta_{(\vec{j},\vec{\ell},\vec{r})} \text{ for some } d_2 \in \mathbb{Z}\} \\ & = \quad \{(\vec{j},\vec{\ell},\vec{r}) \in \Gamma_d \colon j_1 = j_2 + \ell_1 - \ell_2 + e_1, j_3 = -j_2 + \ell_2 + \ell_3 + e_2\}\end{aligned}$$

Note that sets $\Lambda_{\vec{e}}$ are pairwise disjoint and that they cover Γ_d. Let us define a vector function $\vec{f}$ of $j_2, \vec{\ell}, \vec{e}$ by

$$\vec{f}(j_2, \vec{\ell}, \vec{e}) = (j_2 + \ell_1 - \ell_2 + e_1, j_2, -j_2 + \ell_2 + \ell_3 + e_2).$$

Clearly, $\Lambda_{\vec{e}} = \{(\vec{j},\vec{\ell},\vec{r}) \in \Gamma_d \colon (j_1, j_2, j_3) = \vec{f}(j_2, \vec{\ell}, \vec{e})\}$. Let $\Lambda'_{\vec{e}}$ be the projection of $\Lambda_{\vec{e}}$ along j_1 and j_3. The equation (35) implies that

$$\begin{aligned} 0 \quad & = \sum_{(j_2,\vec{\ell},\vec{r})\in\Lambda'_{\vec{e}}} \lambda_{\vec{f}(j_2,\vec{\ell},\vec{e}),\vec{\ell},\vec{r}} \cdot C_{\vec{f}(j_2,\vec{\ell},\vec{e}),\vec{\ell},\vec{r}}(k(t), \ell(t), m(t), S) \\ & \qquad v_{k(t)+e_1-\ell_2-\ell_3,\ell(t)-\ell_2-\ell_3,m(t)+e_2+\ell_2+\ell_3}, \\ & = \sum_{(j_2,\vec{\ell},\vec{r})\in\Lambda'_{\vec{e}}} \Big(\lambda_{\vec{f}(j_2,\vec{\ell},\vec{e}),\vec{\ell},\vec{r}} \cdot C_{\vec{f}(j_2,\vec{\ell},\vec{e}),\vec{\ell},\vec{r}}(k(t), \ell(t), m(t), S) \\ & \qquad S^{d-\ell_2-\ell_3}\Big)\, v_{k(t)+e_1-d,\ell(t)-d,m(t)+e_2+d}.\end{aligned}$$

Defining operators

$$P_{j_2,\vec{\ell},\vec{r}} := C_{\vec{f}(j_2,\vec{\ell},\vec{e}),\vec{\ell},\vec{r}}(k(t), \ell(t), m(t), S)S^{d-\ell_2-\ell_3},$$

we get that

$$0 = \sum_{(j_2,\vec{\ell},\vec{r})\in\Lambda'_{\vec{e}}} \left(\lambda_{\vec{f}(j_2,\vec{\ell},\vec{e}),\vec{\ell},\vec{r}} P_{j_2,\vec{\ell},\vec{r}}\right) v_{k(t)+e_1-d,\ell(t)-d,m(t)+e_2+d}. \tag{36}$$

We will prove by contradiction that $\lambda_{\vec{f}(j_2,\vec{\ell},\vec{e}),\vec{\ell},\vec{r}} = 0$ for all $j_2,\vec{\ell},\vec{r}$ and hence $\lambda_{\vec{j},\vec{\ell},\vec{r}} = 0$ for all $\vec{j},\vec{\ell},\vec{r}\in\Gamma_d$ in (34). Among tuples $(j_2,\vec{\ell},\vec{r})$ with $\lambda_{\vec{f}(j_2,\vec{\ell},\vec{e}),\vec{\ell},\vec{r}}\neq 0$ choose a tuple $(j_2',\vec{\ell}',\vec{r}')$ such that the operator $P_{j_2,\vec{\ell},\vec{r}}$ has the highest leading term with respect to the monomial ordering (33). By the following claim such tuple is unique.

Claim 1: Different operators $P_{j_2,\vec{\ell},\vec{r}}$ have different leading terms.

From the discussion above, it follows that the leading term of the operator $P_{j_2,\vec{\ell},\vec{r}}$ is

$$\frac{(-1)^{\ell_2}}{3^{\ell_1+2(\ell_2+\ell_3)}} x(t)^{\ell_1+2(\ell_2+\ell_3)+r_2} y(t)^{j_2+\ell_3} z(t)^{r_1} S^{\ell_1+d-\ell_2-\ell_3}.$$

Pick any $\alpha,\beta,\gamma,\delta\in\mathbb{N}_0$. We will show that there exists at most one tuple $(j_2,\vec{\ell},\vec{r})\in\mathbb{N}_0^6$ such that

$$\ell_1+2(\ell_2+\ell_3)+r_2=\alpha \tag{37}$$
$$j_2+\ell_3=\beta \tag{38}$$
$$r_1=\gamma \tag{39}$$
$$\ell_1+d-\ell_2-\ell_3=\delta \tag{40}$$
$$j_2\ell_2=0 \tag{41}$$
$$r_2\leq 2 \tag{42}$$

Subtracting (40) from (37) we obtain

$$3(\ell_2+\ell_3)+r_2=\alpha-\delta+d \tag{43}$$

which together with (42) implies that

$$\ell_2+\ell_3=(\alpha-\delta+d)\,\mathrm{div}\,3 =: \varepsilon \tag{44}$$
$$r_2=(\alpha-\delta+d)\bmod 3 \tag{45}$$

Equations (40) and (44) imply that

$$\ell_1=\delta+\varepsilon-d. \tag{46}$$

Subtracting (44) from (38) we obtain

$$j_2-\ell_2=\beta-\varepsilon \tag{47}$$

which together with (41) implies that

$$j_2=(\beta-\varepsilon)^+ := \max\{\beta-\varepsilon,0\} \tag{48}$$
$$\ell_2=(\beta-\varepsilon)^- := \max\{\varepsilon-\beta,0\} \tag{49}$$

From (44) and (49) we obtain

$$\ell_3=\varepsilon-(\beta-\varepsilon)^- \tag{50}$$

We already know that $r_1=\gamma$ from (39). This proves Claim 1.

For the tuple $(j_2', \vec{\ell}', \vec{r}')$ let $\alpha', \beta', \gamma', \delta'$ be defined as in (37)-(40). Now we observe the coefficients at the vector

$$v_{k(t)+e_1-d+\delta', \ell(t)-d+\delta', m(t)+e_2+d-\delta'}$$

on both sides of (36) and get

$$0 = \frac{(-1)^{\ell_2'}}{3^{\ell_1'+2(\ell_2'+\ell_3')}} x(t)^{\alpha'} y(t)^{\beta'} z(t)^{\gamma'} + \sum_{\substack{\alpha,\beta,\gamma\in\mathbb{N}_0,\\ 0\le\alpha+\beta+\gamma\le 2d,\\ (\alpha',\beta',\gamma')\succ(\alpha,\beta,\gamma)}} c_{\alpha,\beta,\gamma} x(t)^{\alpha} y(t)^{\beta} z(t)^{\gamma},$$

for some $c_{\alpha,\beta,\gamma} \in \mathbb{C}$. Since this must hold for all $t = 1, \ldots, L$, this is a contradiction by the following claim.

Claim 2: All vectors

$$\big(x(t)^{\alpha_1} y(t)^{\alpha_2} z(t)^{\alpha_3}\big)_{t=1,\ldots,L}$$

where $0 \le \sum_{i=1}^3 \alpha_i \le 2d$ are linearly independent.

By Vandermonde determinant one can show that all vectors

$$\big(k(t)^{\alpha_1} \ell(t)^{\alpha_2} m(t)^{\alpha_3}\big)_{t=1,\ldots,L} =$$

$$\big((3d+1)^{\alpha_1} \cdot t^{\alpha_1+\alpha_2\cdot(2d+1)+\alpha_3\cdot(4d^2+4d+1)}\big)_{t=1,\ldots,L}$$

where $0 \le \sum_{i=1}^3 \alpha_i \le 2d$ are linearly independent. Indeed, for different triples $(\alpha_1, \alpha_2, \alpha_3)$ satisfying $0 \le \sum_{i=1}^3 \alpha_i \le 2d$, the exponents

$$\alpha_1 + \alpha_2 \cdot (2d+1) + \alpha_3 \cdot (4d^2 + 4d + 1)$$

are different, with the highest exponent $L-1$ reached at $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 2d$. By using (31) and (32) we see that

$$\operatorname{span}\{\big(x(t)^{\alpha_i} y(t)^{\alpha_i} z(t)^{\alpha_i}\big)_{t=1,\ldots,L} : 0 \le \alpha_1 + \alpha_2 + \alpha_3 \le 2d\}$$

is equal to

$$\operatorname{span}\{\big(k(t)^{\alpha_i} \ell(t)^{\alpha_i} m(t)^{\alpha_i}\big)_{t=1,\ldots,L} : 0 \le \alpha_1 + \alpha_2 + \alpha_3 \le 2d\}.$$

Therefore also all vectors

$$\big(x(t)^{\alpha_1} y(t)^{\alpha_2} z(t)^{\alpha_3}\big)_{t=1,\ldots,L}$$

where $0 \le \sum_{i=1}^3 \alpha_i \le 2d$ are linearly independent. This proves Claim 2. ■

4.2. **An explicit basis over the center of $U(\mathfrak{sl}_3)$.** It is well-known that the center of $U(\mathfrak{sl}_3)$ is generated by two algebraically independent elements Z_2 and Z_3 which are also called Casimir operators. The algorithm for computing Z_2 and Z_3 can be found in [9], while explicit expressions are in [4, p. 984]. We have

$$Z_2 \;=\; H_1^2 + H_1H_2 + H_2^2 + 3Y_1X_1 + 3Y_2X_2 + 3Y_3X_3 + 3H_1 + 3H_2$$

and

$$\begin{aligned} Z_3 = 3Y_1Y_2X_3 + 3Y_3X_1X_2 + \frac{1}{9}(H_1+2H_2)(6+2H_1+H_2)(-3+H_1-H_2) \\ + Y_1X_1(H_1+2H_2) - Y_2X_2(6+2H_1+H_2) + Y_3X_3(-3+H_1-H_2) \end{aligned}$$

(Our choice of Z_2 and Z_3 is equal to h and $-\frac{1}{9}k - h$ in the notation of [4, p. 984].)

By [11, Schur's Lemma] an irreducible representation maps a central element into a scalar multiple of identity. Therefore it is enough to calculate $\pi_{m_1,m_2}(Z_i)v_{0,0,0}$ to determine this scalar. From Lemma 5, we get that

$$\begin{aligned} d_2(m_1,m_2) &:= \pi_{m_1,m_2}(Z_2) = m_1^2 + m_1m_2 + m_2^2 + 3m_1 + 3m_2, \\ d_3(m_1,m_2) &:= \pi_{m_1,m_2}(Z_3) = \frac{1}{9}(m_1+2m_2)(6+2m_1+m_2)(-3+m_1-m_2). \end{aligned} \tag{51}$$

Proposition 5. *Monomials*

$$Y_1^{j_1}Y_2^{j_2}Y_3^{j_3}X_1^{\ell_1}X_2^{\ell_2}X_3^{\ell_3}H_1^{r_1}H_2^{r_2} \tag{52}$$

where the powers $j_1,j_2,j_3,\ell_1,\ell_2,\ell_3,r_1,r_2 \in \mathbb{N}_0$ are such that $j_2\ell_2 = 0$ and $r_2 \le 2$ form a basis of $U(\mathfrak{sl}_3)$ over its center.

Proof. Linear independence of monomials (52) follows from Proposition 4. It remains to prove that they span $U(\mathfrak{sl}_3)$ over its center.

Let $U(\mathfrak{sl}_3)_k$ denote the $\mathbb{C}$-linear span of monomials of the form

$$Y_2^{\ell_2}X_2^{j_2}H_2^{r_2}Y_3^{\ell_3}X_3^{j_3}Y_1^{\ell_1}X_1^{j_1}H_1^{r_1} \tag{53}$$

of degree at most k where the degree equals to the sum of the exponents. We write $\deg(m)$ for the degree of the monomial of the form (53). We define the set

$$M_k := \{m \text{ of the form (53)} \colon \deg(m) \le k,\ j_2\ell_2 = 0 \text{ and } r_2 \le 2\}.$$

We will prove that

$$U(\mathfrak{sl}_3)_k = \operatorname{span}_Z(M_k) \tag{54}$$

where Z stands for the center of $U(\mathfrak{sl}_3)$. It suffices to prove that every monomial of the form (53) belongs to $\operatorname{span}_Z(M_k)$. Let us order the monomials (53) with respect to the degree reverse lexicographic ordering. Note that the largest monomial in the definition of Z_2 is $3Y_2X_2$ and that the largest monomial in the definition of $Z_3 + \frac{1}{3}Z_2(6 + 2H_1 + H_2)$ is $\frac{1}{9}H_2^3$. If we express Y_2X_2 by Z_2 and other monomials and similarly H_2^3 by $Z_3 + \frac{1}{3}Z_2(6 + 2H_1 + H_2)$ and other monomials we get two substitution rules. (Note that the first substitution rule decreases $\min\{j_2, \ell_2\}$ but it can increase r_2 and that the second substitution rule decreases r_2 but can increase $\min\{j_2, \ell_2\}$.) If we start with a monomial with either $j_2\ell_2 > 0$ or $r_2 \ge 3$ and keep applying these substitution rules whenever possible we get a decreasing sequence of expressions with respect to the degree reverse lexicographic ordering. Since this ordering is known to be a well-ordering, this sequence must stop at some point. This means that we finish with an expression whose monomials all satisfy $j_2\ell_2 = 0$ and $r_2 \le 2$. This proves (54).

By the PBW theorem we know that every element of $U(\mathfrak{sl}_3)$ belongs to $U(\mathfrak{sl}_3)_k$ for some $k \in \mathbb{N}_0$. We define the set

$$\widetilde{M}_k := \{m \text{ of the form (52)} \colon \deg(m) \le k,\ j_2\ell_2 = 0 \text{ and } r_2 \le 2\},$$

where $\deg(m)$ is a sum of exponents in m. To finish the proof of the proposition it remains to prove that

$$\operatorname{span}_Z M_k = \operatorname{span}_Z \widetilde{M}_k. \tag{55}$$

We prove (55) by induction on k. The base of induction $k = 1$ is clear. We assume that (55) for all $k \leq n$ for some $n \in \mathbb{N}$. By the relations in $U(\mathfrak{sl}_3)$ we have that

$$Y_2^{\ell_2} X_2^{j_2} H_2^{r_2} Y_3^{\ell_3} X_3^{j_3} Y_1^{\ell_1} X_1^{j_1} H_1^{r_1} = Y_1^{\ell_1} Y_2^{\ell_2} Y_3^{\ell_3} X_1^{\ell_1} X_2^{\ell_2} X_3^{\ell_3} H_1^{r_1} H_2^{r_2} + m'$$

where m' is a $\mathbb{Z}$-linear combination of monomials of the form (53) of degree at most $n - 1 := \sum_{i=1}^{3}(\ell_i + j_i) + r_1 + r_2 - 1$. By (54) we have that $m' \in \operatorname{span}_Z M_{n-1}$ and by the induction hypothesis we have that $m' \in \operatorname{span}_Z \widetilde{M}_{n-1}$. This proves (55). ∎

Lemma 6. *(1) Every polynomial $g \in \mathbb{C}[x, y]$ which satisfies*

$$g(m_1, m_2) = 0$$

for all sufficiently large integers m_1, m_2 is equal to zero.

(2) Every polynomial $f \in \mathbb{C}[x, y]$ which satisfies

$$f(d_2(m_1, m_2), d_3(m_1, m_2)) = 0 \tag{56}$$

for all sufficiently large integers m_1, m_2 is equal to zero.

(3) Every vectors $u_1, \dots, u_k \in \mathbb{C}[x, y]^n$, such that the vectors

$$u_i(d_2(m_1, m_2), d_3(m_1, m_2)), i = 1, \dots, k \tag{57}$$

are linearly dependent over $\mathbb{C}$ for all sufficiently large integers m_1, m_2, are linearly dependent over $\mathbb{C}[x, y]$.

Proof. Part (1) is well-known and easy to prove.

To prove part (2), assume that (56) is true for all sufficiently large integers m_1, m_2. By part (1), it follows that (56) is true for all $m_1, m_2 \in \mathbb{C}$. Let us compute the partial derivatives of (56) with respect to m_1 and m_2 by using the chain rule. We get that

$$\left[\frac{\partial f}{\partial x}(d_2, d_3), \frac{\partial f}{\partial y}(d_2, d_3)\right] \begin{bmatrix} \frac{\partial d_2}{\partial m_1} & \frac{\partial d_2}{\partial m_2} \\ \frac{\partial d_3}{\partial m_1} & \frac{\partial d_3}{\partial m_2} \end{bmatrix} = [0, 0] \tag{58}$$

Since

$$\det \begin{bmatrix} \frac{\partial d_2}{\partial m_1} & \frac{\partial d_2}{\partial m_2} \\ \frac{\partial d_3}{\partial m_1} & \frac{\partial d_3}{\partial m_2} \end{bmatrix} = -3(1 + m_1)(1 + m_2)(2 + m_1 + m_2) \tag{59}$$

is nonzero for all positive reals m_1 and m_2 it follows that

$$\frac{\partial f}{\partial x}(d_2(m_1, m_2), d_3(m_1, m_2)) = 0 \tag{60}$$

$$\frac{\partial f}{\partial y}(d_2(m_1, m_2), d_3(m_1, m_2)) = 0 \tag{61}$$

for all positive reals m_1 and m_2, thus for all complex m_1, m_2 by part (1). By induction, we show that

$$\frac{\partial^{i+j} f}{\partial^i x\, \partial^j y}(d_2(m_1, m_2), d_3(m_1, m_2)) = 0 \tag{62}$$

for all $i, j \in \mathbb{N}_0$ and all $m_1, m_2 \in \mathbb{C}$. It follows that $f \equiv 0$.

To prove part (3), consider the matrix $U = [u_1, \dots, u_k]$. By assumption, each maximal subdeterminant of $U(d_2(m_1, m_2), d_3(m_1, m_2))$ is zero for all sufficiently large integers m_1 and m_2. By part (2), it follows that each maximal subdeterminant of U is identically zero. Thus $u_1, \dots, u_k$ are linearly dependent over $\mathbb{C}(x, y)$. By clearing denominators, we see that they are also linearly dependent over $\mathbb{C}[x, y]$. ∎

We are now ready to prove Theorem 3.

Proof of Theorem 3. To prove the implication (1) $\Rightarrow$ (2), one has to show that there exists $d \in \mathbb{N}$ such that for every $m_1, m_2 \in \mathbb{N}$, $m_1 \geq d$, $m_2 \geq d$, the elements $\pi_{m_1,m_2}(p_1), \ldots, \pi_{m_1,m_2}(p_k)$ are linearly dependent, where $\pi_{m_1,m_2} \in \mathcal{I}_d$. Dividing the equation $\sum_{i=1}^k z_i p_i = 0$ by the common factor of $z_1, \ldots, z_k$, where $z_i \in \mathbb{C}[Z_2, Z_3]$ for each i, we may assume that $z_1, \ldots, z_k$ do not share a common nontrivial factor. Applying π_{m_1,m_2} to the equation $\sum_{i=1}^k z_i p_i = 0$, one gets

$$0 = \sum_{i=1}^{k} z_i(d_2(m_1, m_2), d_3(m_1, m_2))\pi_{m_1,m_2}(p_i).$$

It suffices to prove that there exists $d \in \mathbb{N}$ such that for every $m_1 \geq d, m_2 \geq d$ at least one of the coefficients $z_i(d_2(m_1, m_2), d_3(m_1, m_2))$ is nonzero. Let us assume on the contrary that such d does not exist. Then there exists a sequence $(m_1^{(n)}, m_2^{(n)}) \in \mathbb{N}^2$, $n \in \mathbb{N}$, satisfying $\max\{m_1^{(n)}, m_2^{(n)}\} < \min\{m_1^{(n+1)}, m_2^{(n+1)}\}$ for every $n \in \mathbb{N}$ and $z_i(d_2(m_1^{(n)}, m_2^{(n)}), d_3(m_1^{(n)}, m_2^{(n)})) = 0$ for each i and every $n \in \mathbb{N}$. By the form of $d_2(m_1, m_2)$, the sequence $d_2(m_1^{(n)}, m_2^{(n)})$ is strictly increasing and hence the polynomials $z_1, \ldots, z_k$ share infinitely many common zeroes. This implies by Bezout's theorem (see [7]) that they share a nontrivial factor, leading to a contradiciton.

The implication (2) $\Rightarrow$ (3) is trivial.

The proof of the implication (3) $\Rightarrow$ (1) is almost the same as the proof of the same implication of Theorem 2. Namely, the form of monomials f_i is given by Proposition 5, the coefficients t_i belong to $\mathbb{C}[Z_2, Z_3]$, while Proposition 2 and Lemma 3 are replaced by Proposition 4 and part (3) of Lemma 6, respectively. ■

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