IMPRECISE COPULAS

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We dedicate this paper to the memory of an excellent mathematician and friend, Ahmed Sourour. He was born not far from Alexandria, Egypt, and lived most of his life in Victoria, Canada – places connected by names of two historic emperors and two oceans, Mediterranean and Pacific.

Abstract. There are two main goals of this paper: First, we propose a well justified definition of an imprecise copula; and second, we present some deep novel techniques on copulas, inspired by the bivariate "signed patch" construction of the first author and Stopar, *Freedom in constructing quasi-copulas vs. copulas*, *Fuzzy Sets and Systems* **514** (2025) 109399. An imprecise copula according to our definition is determined by two quasi-copulas (the definition is given below) $A \leq B$ satisfying certain conditions; it is based on an interval (A, B) that contains all copulas C such that $A \leq C \leq B$. Explicit construction methods for quasi-copulas are essential in the applications, especially in engineering: they provide concrete and verifiable dependence models and help elucidate the structure of the class.

1. Introduction

Copulas are a central tool for dependence modeling: they are distribution functions with uniform margins, and Sklar's theorem ensures that every multivariate distribution can be decomposed as a copula with given marginals; see, e.g., [6]. In this paper we propose an upgrade of copulas associated with imprecise probability, i.e., to copulas in the presence of probabilistic uncertainty. Let us point out that books have been written in engineering applications of either of the two fields, copulas [9, 36], or imprecise probability [3].

There are various reasons for imprecision, such as scarcity of available information, costs connected to acquiring precise inputs or even inherent uncertainty related to phenomena under consideration. Ignoring imprecision may lead to deceptive conclusions and consequentially to harmful decisions, especially if the conclusions are backed by seemingly precise outputs. The theories of imprecise probabilities that have been developed in recent decades aim at providing methods whose results would faithfully reflect the imprecision of input information. The probabilistic imprecision is quite often described with sets of possible probability distributions, consistent with the available information, instead of a single precise distribution. The sets are represented by various types of constraints, ranging from the most general lower and upper previsions to more specific lower and upper probabilities, p-boxes, belief and possibility functions, and other models. In recent years, methods of imprecise probabilities [2] have been applied to various areas of probabilistic modeling, such as stochastic processes [5, 28], game theory [11, 13], reliability theory [4, 14, 31, 35], decision theory [8, 12, 29], financial risk theory [21, 32] and others; although, only recently imprecise models involving copulas have been proposed. A historic first try was made by Montes et al. [20] introducing a possible approach to imprecise copulas and connecting them to the theory of bivariate p-boxes introduced by Pelessoni et al. [22] (as an extension of univariate p-boxes [7, 30]). More on this subject is presented in Subsection 3.1.

The main purpose of this work is twofold: Firstly, we want to propose a well justified definition of an imprecise copula. And secondly, we want to present some deep novel techniques on copulas and *quasi-copulas*, to which we are naturally led, in order to help practitioners, especially

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engineers, handle the related challenges. (The full definition of a quasi-copula will be given in Subsection 2.1, but for a quicker one see [1].) An imprecise copula according to our definition is determined by two quasi-copulas $A \leq B$ satisfying certain conditions; it is based on an interval (A, B) that contains all copulas C such that $A \leq C \leq B$. From an applied perspective, explicit *construction methods* for quasi-copulas are essential: they provide concrete and verifiable dependence models and help elucidate the structure of the class. The rest of this introduction is split in two parts each aimed at one of the two objectives just explained.

1.1. **Imprecise probability.** We first introduce briefly the basic concepts and ideas of imprecise probability models. For a detailed treatment, the reader is referred to [33, 34] and also to [2]. Let Ω be a possibility space, and $\mathcal{A}$ a collection of its subsets, called *events*. Usually we assume $\mathcal{A}$ to be an algebra, but not necessarily a σ-algebra.

The concept of precise probability on the measurable space $(\Omega, \mathcal{A})$ can be generalised by allowing probabilities of events in $\mathcal{A}$ to be given in terms of intervals $[\underline{P}(A), \overline{P}(A)]$ rather than precise values. The functions $\underline{P} \leqslant \overline{P}$ are mapping events to their lower and upper probability bounds and are respectively called *lower* and *upper probabilities*. Lower and upper probabilities are a special case of even more general lower and upper previsions [34]. If $\mathcal{A}$ is an algebra (or at least closed for complements), then the following conjugacy relation between lower and upper probabilities is usually required:

$$\overline{P}(A) = 1 - \underline{P}(A^c) \text{ for every } A \in \mathcal{A}. \tag{1.1}$$

To every pair of lower and upper probabilities $\underline{P}$ and $\overline{P}$ we can also associate the set

$$\mathcal{M} = \{P\colon P \text{ is a finitely additive probability on } \mathcal{A}, \underline{P} \leqslant P \leqslant \overline{P}\}.$$

It is clear from the above definition that the set $\mathcal{M}$ is non-empty only if $\underline{P} \leq \overline{P}$.

Another central question regarding a pair of lower and upper probabilities is whether the bounds are pointwise limits of the elements in $\mathcal{M}$:

$$\underline{P}(A) = \inf_{P \in \mathcal{M}} P(A), \qquad \overline{P}(A) = \sup_{P \in \mathcal{M}} P(A) \qquad \text{for every} A \in \mathcal{A}.$$

If the above conditions are satisfied, $\underline{P}$ and $\overline{P}$ are said to be *coherent* lower and upper probabilities respectively. In the case of coherence, the conjugacy condition (1.1) is automatically fulfilled, which among others means that if a lower probability $\underline{P}$ is coherent, then it uniquely determines the corresponding upper probability.

A simple characterization of coherence in terms of the properties of $\underline{P}$ and $\overline{P}$ does not seem to be known in literature. Below we give some necessary conditions. A pair $\underline{P}$ and $\overline{P}$ of coherent lower and upper probabilities satisfies the following properties:

(1) $0 \leqslant \underline{P}(A) \leqslant \overline{P}(A) \leqslant 1$ for every $A \in \mathcal{A}$.
(2) If $A \subseteq B$, then $\underline{P}(A) \leqslant \underline{P}(B)$ and $\overline{P}(A) \leqslant \overline{P}(B)$. (Monotonicity)
(3) $\underline{P}(A \cup B) \geqslant \underline{P}(A) + \underline{P}(B)$. (Superadditivity)

Instead of the full structure of probability spaces, we are often concerned only with the distribution functions of specific random variables. The set of relevant events where the probabilities have to be given then shrinks considerably. In the precise case, a single distribution function F describes the distribution of a random variable X, which gives the probabilities of the events of the form $\{X \leqslant x\}$. Thus $F(x) = P(X \leqslant x)$. Sometimes we will also consider the corresponding *survival function*, which we will denote by $\hat{F}(x) = 1 - F(x) = P(X > x)$, which is decreasing and positive.

In the imprecise case, the probabilities of the above form are replaced by the corresponding lower (and upper) probabilities, resulting in sets of distribution functions called p-boxes [7, 30]. A *p-box* is a pair $(\underline{F}, \overline{F})$ of distribution functions with $\underline{F} \leqslant \overline{F}$, where $\underline{F}(x) = \underline{P}(X \leqslant x)$ and $\overline{F}(x) = \overline{P}(X \leqslant x)$. To every p-box we associate the set of all distribution functions with the values between the bounds:

$$\mathcal{F} = \{F\colon F \text{ is a distribution function}, \underline{F} \leqslant F \leqslant \overline{F}\}.$$

Clearly, $\mathcal{F}$ is a convex set of distribution functions. Conversely, since supremum and infimum of any set of distribution functions are themselves distribution functions, every set of distribution functions generates a p-box containing the original set.

In the theory of imprecise probabilities, precise probability denotes a probability measure that is finitely additive, and not necessarily σ-additive, as is the case in most models using classical probabilities. As far as distribution functions and p-boxes are concerned, this implies that a distribution function is any increasing, or more precisely non-decreasing function, mapping $\mathbb{R}$ to $[0, 1]$. This is in contrast with the σ-additive case, where distribution functions are cadlag, i.e., continuous from the right; and their corresponding survival functions are then caglad, i.e., continuous from the left. No such continuity assumptions can therefore be made neither for distribution functions nor for the bounds of p-boxes.

1.2. **Construction techniques.** Our results are inspired by the bivariate "signed patch" construction of the first author and Stopar [18]. Several construction principles are investigated there. In particular, [18, Section 5] develops a procedure for constructing a bivariate quasi-copula whose induced signed mass distribution matches a prescribed *signed mass pattern on a patch*. This approach was motivated by an explicit counterexample related to the definition of imprecise copulas in the sense of [20]: an imprecise copula interval (A, B) for which no copula C satisfies the condition; see [15, 18, 20].

The guiding question in [18, Section 5] can be phrased as follows. Given a rectangular patch with prescribed (possibly negative) masses on its grid cells, can it be extended to a larger patch by adding only nonnegative masses– without changing the original pattern except for an overall rescaling– so that, after normalization, the resulting mass table is the mass distribution of a (discrete) quasi-copula [18]? The main difficulty arises from the geometric characterization of bivariate quasi-copulas: any admissible extension must satisfy a boundary-volume condition, namely that *every rectangle touching at least one boundary has nonnegative volume*; see [18, 23]. This condition is equivalent to monotonicity together with the 1-Lipschitz property in each coordinate, and it is the principal constraint driving the construction in [18].

Accordingly, the procedure in [18, Section 5] begins by adding the minimal amount of positive mass along all four sides of the original patch needed to eliminate negative boundary-touching volumes. Once mass is added outside the original patch, new outer strips appear, and one must also satisfy the discrete boundary constraints: before normalization, every row and every column must have the same total mass. This requires an *equalize-and-spread* phase, in which deficits are distributed across outer strips and any surplus is propagated outward by introducing additional strips while preserving the totals of already balanced rows and columns. A technically delicate step is to show that this procedure terminates, that is, after adding only finitely many strips in each direction, one simultaneously achieves equal row and column sums. The resulting construction is optimal in the sense that the balancing level of mass in each strip is chosen minimally; however, it is not unique, since the allocation of the additional mass across directions depends on the chosen forking strategy.

This paper develops a multidimensional analogue of the patch-based paradigm. We represent a k-dimensional "patch" as a k-way array whose entries specify signed volumes of grid boxes in $[0, 1]^k$ via an associated integral transform. Our objective is to extend a given array to a larger array by adding only nonnegative masses so that, after normalization, the resulting function is a k-dimensional quasi-copula. To achieve this, we use a geometric characterization of multivariate quasi-copulas in terms of volume bounds on a distinguished family of boundary boxes (equivalently, the nonnegativity of volumes of primal sets); see [16, 23]. In the discrete setting, these requirements become explicit mode-wise marginal constraints and bounded multi-index partial-sum inequalities, which together characterize *quasi-copula type arrays*.

Extending the bivariate patch construction to higher dimensions introduces significant additional challenges, as modifying entries within a slice of one mode typically affects partial sums in the other modes. Therefore, a slice-by-slice application of the bivariate procedure is not feasible: (i) the resulting slice-wise extensions may not have a common extension size, and (ii)

the necessary boundary-volume inequalities must be enforced simultaneously across all modes. Designing terminating algorithms that satisfy both constraints is thus nontrivial.

Our main contributions are two explicit extension algorithms in dimension three (Theorems 9 and 12), along with their generalizations to arbitrary dimensions (Theorem 14). The first construction (Theorem 9) is simpler but yields a more restrictive subclass: it produces *strong quasi-copula type arrays* under a sufficient two-axis positivity condition concentrated along a distinguished mode. In particular, we show that any prescribed $n \times n \times n$ array can be embedded (up to affine rescaling) into a strong quasi-copula type array of side length $2n + 1$. The second construction (Theorem 12) is more involved and removes the strong slice-wise requirements, thereby yielding a strictly richer family: we embed an arbitrary $n \times n \times n$ array into a quasi-copula type array of side length $3n + 2$. In both cases, the extension size is explicit– in contrast to the bivariate patch algorithm of [18], where the final size is not prescribed a priori– and the added mass is kept small by construction, although global optimality (in the sense of minimizing the total added mass) remains open. In addition, we provide an application to the theory of imprecise copulas: using Theorem 6, we lift the bivariate discrete counterexample from [15, Section 5] to arbitrary dimension by constructing pairs of quasi-copulas $\mathcal{Q}_{\mathcal{T}_A} \leq \mathcal{Q}_{\mathcal{T}_B}$ for which no copula C satisfies $\mathcal{Q}_{\mathcal{T}_A} \leq C \leq \mathcal{Q}_{\mathcal{T}_B}$ (see Subsection 3.1).

Finally (see Theorem 14), we extend both three-dimensional constructions to tensors of arbitrary order $k \geq 4$ by exploiting a *stacking principle* for quasi-copula type arrays. This yields explicit embeddings into arrays with side length $2n + 1$ (strong/strictly positive variant) and $3n + 2$ (general variant) in every dimension, providing a constructive mechanism for producing multidimensional quasi-copulas from prescribed signed data.

1.3. **Organization of the paper.** Section 2 introduces notation and background, including index set conventions, the notion of A-volume, a geometric characterization of quasi-copulas, and tensor notation for multidimensional arrays. In Section 3, we develop the concept of quasi-copula type arrays, translate quasi-copula constraints into finite tensor conditions, and provide both a characterization and tractable sufficient criteria, along with a construction principle for higher-dimensional arrays. In Subsection 3.1, we apply this construction principle to imprecise copulas, showing how bivariate discrete counterexamples can be lifted to arbitrary dimension via stacking. In the center of this work is Section 4 devoted to the justified definition of an imprecise copula. Section 5 addresses a three-dimensional extension problem under stronger positivity assumptions and gives an explicit embedding into strong quasi-copula type arrays. Finally, Section 6 generalizes both extension constructions from dimension 3 to arbitrary order $k \geq 4$.

2. Preliminaries

This section presents the notation and background required throughout the paper. We first establish basic index-set conventions for $[n]$ and intervals $[i; j]$, along with a compact summation shorthand for multi-index partial sums. Next, we recall the concept of the *A-volume* of an axis-aligned box, which provides a convenient characterization of quasi-copulas in terms of volume bounds on a distinguished family of boxes $\wp^n$ (see Theorem 1). Finally, we introduce our tensor notation for multidimensional arrays, including sections, fibers, and slices, as well as the canonical basis arrays $\mathbb{E}^{(\cdot)}$ and the all-ones array $\mathbb{L}^{(\cdot)}$. These conventions will be used to translate quasi-copula volume constraints into discrete conditions on arrays in later sections.

Let $n, i, j \in \mathbb{N}$ with $i \leq j$. Define

$$[n] = \{1, \ldots, n\}, \qquad [i; j] := \{i, i + 1, \ldots, j\}.$$

For $\ell_1, \ldots, \ell_k \in \mathbb{N}$ and a subset $S = \{s_1, \ldots, s_t\} \subseteq [k]$, we use the shorthand

$$\sum_{\substack{i_r=1 \\ r\in S}}^{\ell_r} := \sum_{i_{s_1}=1}^{\ell_{s_1}} \cdots \sum_{i_{s_t}=1}^{\ell_{s_t}}.$$

2.1. **Quasi-copulas.** Let A be a real-valued function whose domain contains the set of vertices of a rectangle $R := \prod_{i=1}^{n}[x_i, y_i]$, where $x_i \leq y_i$. Denote the set of vertices of R by $\mathrm{ver}(R) = \prod_{i=1}^{n}\{x_i, y_i\}$. For a vertex $v := (v_1, \ldots, v_n) \in \mathrm{ver}(R)$ define

$$\mathrm{sign}_R(v) := \begin{cases} 1, & \text{if } v_j = x_j \text{ for an even number of coordinates } j, \\ -1, & \text{otherwise.} \end{cases}$$

The A**-volume** of the rectangle R is defined by

$$V_A(R) := \sum_{v \in \mathrm{ver}(R)} \mathrm{sign}_R(v)\, A(v).$$

An n-dimensional **quasi-copula** is a function

$$Q : [0,1]^n \to [0,1]$$

satisfying the following conditions:

(1) **Boundary conditions.** For every $j \in [n]$,

$$Q(\underbrace{1, \ldots, 1}_{j-1}, u_j, \underbrace{1, \ldots, 1}_{n-j}) = u_j.$$

(2) **Coordinatewise monotonicity.** For every $j \in [n]$ and every $(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n) \in [0,1]^{n-1}$, the function

$$t \longmapsto Q(u_1, \ldots, u_{j-1}, t, u_{j+1}, \ldots, u_n)$$

is non-decreasing on $[0,1]$.

(3) **Lipschitz continuity.** For all $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $[0,1]^n$,

$$|Q(u) - Q(v)| \leq \sum_{j=1}^{n} |u_j - v_j|.$$

Let $\wp^n$ be the set of n-boxes in $[0,1]^n$ of the form $\prod_{i=1}^{n}[u_i, v_i]$ such that there is $j \in [n]$ satisfying $u_i = 0$ for all $i \neq j$.

The following theorem characterizes quasi-copulas in terms of volumes of sets from $\wp^n$.

Theorem 1 ([23, Theorem 7])**.** *A function $Q : [0,1]^n \to \mathbb{R}$ is a quasi-copula if and only if it satisfies:*

(1) $Q(1, \ldots, 1) = 1$.

(2) For every $j \in [n]$ and every point $(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n) \in [0,1]^{n-1}$ we have

$$Q(u_1, \ldots, u_{j-1}, 0, u_{j+1}, \ldots, u_n) = 0.$$

(3) $0 \leq V_Q(R) \leq v_j - u_j$ for every $R := \prod_{i=1}^{n}[u_i, v_i] \in \wp^n$, and every $j \in [n]$.

Remark 2. In [16, Theorem 2], an equivalent version of Theorem 1 is established using the language of *primal sets*. Specifically, a set $P \subseteq [0,1]^n$ is called a **primal set** if $P \in \wp^n$ or $P \in \wp_c^n$, where $\wp_c^n$ is the set of all subsets of $[0,1]^n$ of the form $\left([0,1]^{j-1} \times [u_j, v_j] \times [0,1]^{n-j-1}\right) \setminus \left(\prod_{i=1}^{j-1}[0, v_i] \times [u_j, v_j] \times \prod_{i=j+1}^{n}[0, v_i]\right)$ where $j \in [n]$. In [16, Theorem 2], the condition (3) in Theorem 1 is replaced by: *(3') The Q-volume of every primal set is nonnegative.*

2.2. **Multidimensional arrays.** Let $k \in \mathbb{N}$ and let $n_1, \ldots, n_k \in \mathbb{N}$. Consider a k-dimensional array

$$\mathcal{T} := \left[\mathcal{T}_{i_1,\ldots,i_k}\right]_{i_1,\ldots,i_k} \in \mathbb{R}^{n_1 \times \cdots \times n_k}$$

(see Figure 1 for the three-dimensional case).

Fix $\ell \in [k]$. Let $(j_1, \ldots, j_\ell)$ be a strictly increasing ℓ-tuple with

$$1 \leq j_1 < j_2 < \cdots < j_\ell \leq k,$$

and fix indices $(i_{j_1}, \ldots, i_{j_\ell}) \in [n_{j_1}] \times \cdots \times [n_{j_\ell}]$. We denote by $\mathcal{T}_{(j_1,\ldots,j_\ell)}^{(i_{j_1},\ldots,i_{j_\ell})}$ the $(k-\ell)$-dimensional subarray obtained from $\mathcal{T}$ by fixing the coordinates $j_1, \ldots, j_\ell$ to $i_{j_1}, \ldots, i_{j_\ell}$ and allowing all remaining coordinates to vary, i.e.,

$$\mathcal{T}_{(j_1,\ldots,j_\ell)}^{(i_{j_1},\ldots,i_{j_\ell})} := \left[\mathcal{T}_{i_1,\ldots,i_k}\right]_{\substack{i_j=1,\ldots,n_j \\ j \notin \{j_1,\ldots,j_\ell\}}}.$$

In particular, when $\ell = 1$, for $j \in [k]$ and $i_j \in [n_j]$ the subarray $\mathcal{T}_{(j)}^{(i_j)}$ is obtained by fixing the jth index and letting all other indices vary. We refer to $\mathcal{T}_{(j)}^{(i_j)}$ as a **mode-j section** of $\mathcal{T}$. More generally, fixing ℓ indices produces a $(k-\ell)$-dimensional section of $\mathcal{T}$. One-dimensional sections,

$$\mathcal{T}_{(j_1,\ldots,j_{k-1})}^{(i_{j_1},\ldots,i_{j_{k-1}})},$$

are called **fibers** of $\mathcal{T}$, while two-dimensional sections,

$$\mathcal{T}_{(j_1,\ldots,j_{k-2})}^{(i_{j_1},\ldots,i_{j_{k-2}})},$$

are called **slices** of $\mathcal{T}$ (see Figure 2).

We write

$$\sum \mathcal{T} := \sum_{i_1=1}^{n_1} \cdots \sum_{i_k=1}^{n_k} \mathcal{T}_{i_1,\ldots,i_k}$$

for the sum of all entries of $\mathcal{T}$.

Let $\mathbb{E}_{i_1,\ldots,i_k}^{(n_1,\ldots,n_k)}$ denote the $n_1 \times \cdots \times n_k$ array with a single nonzero entry equal to 1 at position $(i_1, \ldots, i_k)$ and zeros elsewhere. When $n_1 = \cdots = n_k = n$, we abbreviate

$$\mathbb{E}_{i_1,\ldots,i_k}^{(n,\ldots,n)} =: \mathbb{E}_{i_1,\ldots,i_k}^{(n)}.$$

Finally, let $\mathbb{L}^{(n_1,\ldots,n_k)}$ denote the $n_1 \times \cdots \times n_k$ array whose entries are all equal to 1. When $n_1 = \cdots = n_k = n$, we write

$$\mathbb{L}^{(n,\ldots,n)} =: \mathbb{L}^{(n)}.$$

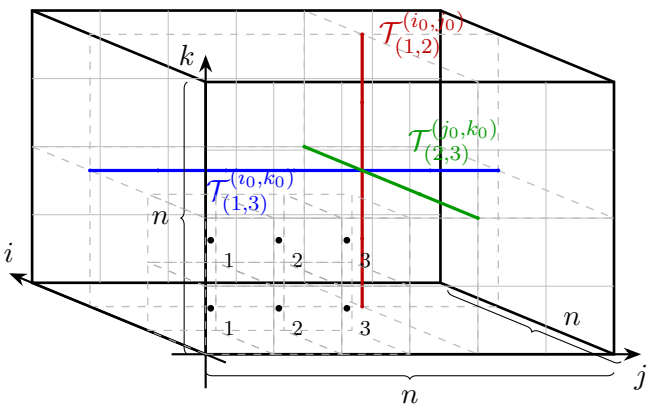
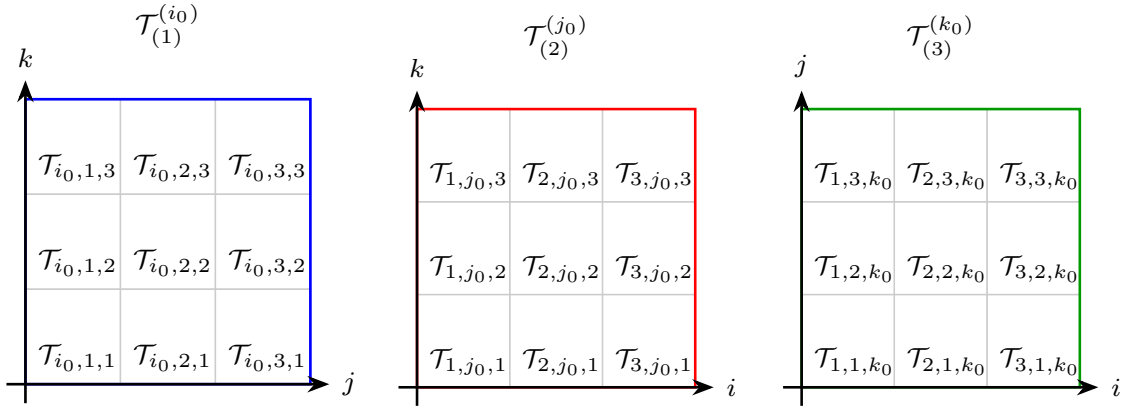


FIGURE 1. 3-dimensional array and its fibers



(a) $(\mathcal{T}_{i,j,k})_{\substack{j=1,\dots,n_2\\k=1,\dots,n_3}}$ (b) $(\mathcal{T}_{i,j,k})_{\substack{i=1,\dots,n_1\\k=1,\dots,n_3}}$ (c) $(\mathcal{T}_{i,j,k})_{\substack{i=1,\dots,n_1\\j=1,\dots,n_2}}$

FIGURE 2. Three slice types of a 3D array

3. Quasi-copula type arrays

In this section we study when a given k-dimensional array $\mathcal{T}$, viewed as prescribing signed volumes of axis-aligned boxes in the unit cube $[0,1]^k$ via the associated integral transform $Q_{\mathcal{T}}$, determines a quasi-copula. Specifically, we associate to $\mathcal{T} \in \mathbb{R}^{n_1\times\cdots\times n_k}$ a piecewise-constant density $q_{\mathcal{T}}$ on the grid partition of $[0,1]^k$ and define $Q_{\mathcal{T}}$ as its iterated integral (see (3.1)). This converts quasi-copula constraints into finite conditions on $\mathcal{T}$: Theorem 3 characterizes quasi-copula type arrays via mode-wise marginal sums and bounded multi-index partial sums, while Theorem 4 provides a simpler sufficient criterion based on stronger two-axis positivity along a single distinguished mode. Finally, Theorem 6 uses this structure to construct higher-dimensional quasi-copula type arrays by stacking lower-dimensional ones.

Let $\mathcal{T} \in \mathbb{R}^{n_1\times\cdots\times n_k}$ be a k-dimensional array. For indices $i_r \in [n_r]$ $(r = 1,\dots,k)$, define the k-box

$$B_{i_1,\dots,i_k} := \left[\frac{i_1-1}{n_1}, \frac{i_1}{n_1}\right] \times \cdots \times \left[\frac{i_k-1}{n_k}, \frac{i_k}{n_k}\right] \subseteq [0,1]^k.$$

For a set $B \subseteq [0,1]^k$, let δ_B denote its indicator (characteristic) function, i.e.,

$$\delta_B(x) = \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{if } x \notin B. \end{cases}$$

Define the piecewise-constant function $q_{\mathcal{T}} : [0,1]^k \to \mathbb{R}$ by

$$q_{\mathcal{T}}(x) := \sum_{i_1=1}^{n_1} \cdots \sum_{i_k=1}^{n_k} (n_1 \cdots n_k)\, \mathcal{T}_{i_1,\dots,i_k}\, \delta_{B_{i_1,\dots,i_k}}(x),$$

and define $Q_{\mathcal{T}} : [0,1]^k \to \mathbb{R}$ by

$$Q_{\mathcal{T}}(x_1,\dots,x_k) := \int_0^{x_1} \cdots \int_0^{x_k} q_{\mathcal{T}}(t_1,\dots,t_k)\, dt_k \cdots dt_1. \tag{3.1}$$

We say that $\mathcal{T}$ is a **quasi-copula type array** if $Q_{\mathcal{T}}$ is a quasi-copula.

The following theorem characterizes quasi-copula–type arrays in terms of the $Q_{\mathcal{T}}$-volumes of sets in $\wp^k$ (see Subsection 2.1).

Theorem 3. *An array $\mathcal{T} \in \mathbb{R}^{n_1\times\cdots\times n_k}$ is of quasi-copula type if and only if it satisfies the following conditions:*

(1) For every $j \in [k]$ and every $i_j \in [n_j]$,

$$\sum \mathcal{T}^{(i_j)}_{(j)} = \frac{1}{n_j}. \tag{3.2}$$

(2) For every $j \in [k]$, every $i_j \in [n_j]$, and every choice of $\ell_r \in [n_r]$ for $r \in [k] \setminus \{j\}$, we have

$$0 \leq \sum_{\substack{i_r=1 \\ r\in[k]\setminus\{j\}}}^{\ell_r} \big(\mathcal{T}_{(j)}^{(i_j)}\big)_{i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_k} \leq \frac{1}{n_j}. \tag{3.3}$$

Proof. We apply Theorem 1 with dimension $n = k$ to the function $Q_{\mathcal{T}}$.

Step 1: Explicit form of $Q_{\mathcal{T}}$ on the grid. By definition, $q_{\mathcal{T}}$ is constant on each box $B_{i_1,\ldots,i_k}$ and equals

$$q_{\mathcal{T}}(x) = (n_1 \cdots n_k)\, \mathcal{T}_{i_1,\ldots,i_k} \qquad \text{for } x \in B_{i_1,\ldots,i_k}.$$

Let

$$\mathcal{G} := \prod_{r=1}^{k} \Big\{0, \frac{1}{n_r}, \ldots, \frac{n_r-1}{n_r}, 1\Big\}$$

be the grid in $[0,1]^k$. For any grid point $x = (x_1, \ldots, x_k) \in \mathcal{G}$ with $x_r = \ell_r/n_r$ (where $\ell_r \in \{0, 1, \ldots, n_r\}$), the iterated integral yields

$$Q_{\mathcal{T}}(x) = \sum_{i_1=1}^{\ell_1} \cdots \sum_{i_k=1}^{\ell_k} \mathcal{T}_{i_1,\ldots,i_k}. \tag{3.4}$$

Indeed, $[0, x_1] \times \cdots \times [0, x_k]$ is a disjoint union of the boxes $B_{i_1,\ldots,i_k}$ with $1 \leq i_r \leq \ell_r$, each of volume $(n_1 \cdots n_k)^{-1}$.

Step 2: Volumes of primal boxes are partial sums of mode-sections. Let $R = \prod_{r=1}^{k} [u_r, v_r] \in \wp^k$, so that there exists $j \in [k]$ with $u_r = 0$ for all $r \neq j$. First assume $u, v \in \mathcal{G}$, and write $v_r = \ell_r/n_r$ for $r \neq j$. For the j-coordinate write

$$u_j = \frac{a-1}{n_j}, \qquad v_j = \frac{b}{n_j} \qquad \text{for some } 1 \leq a \leq b \leq n_j.$$

Using additivity of $V_{Q_{\mathcal{T}}}$ with respect to splitting the interval $[u_j, v_j]$ into grid cells, we obtain

$$V_{Q_{\mathcal{T}}}(R) = \sum_{i_j=a}^{b} V_{Q_{\mathcal{T}}}\Big([0, v_1] \times \cdots \times [0, v_{j-1}] \times \Big[\frac{i_j-1}{n_j}, \frac{i_j}{n_j}\Big] \times [0, v_{j+1}] \times \cdots \times [0, v_k]\Big). \tag{3.5}$$

Now fix $i_j \in [n_j]$ and consider the *cell-primal box*

$$R(i_j) := [0, v_1] \times \cdots \times [0, v_{j-1}] \times \Big[\frac{i_j-1}{n_j}, \frac{i_j}{n_j}\Big] \times [0, v_{j+1}] \times \cdots \times [0, v_k].$$

By the definition of A-volume and (3.4),

$$V_{Q_{\mathcal{T}}}\big(R(i_j)\big) = \sum_{\substack{i_r=1 \\ r\neq j}}^{\ell_r} \big(\mathcal{T}_{(j)}^{(i_j)}\big)_{i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_k}, \tag{3.6}$$

which is exactly the partial sum appearing in (3.3). (Note that $\mathcal{T}_{(j)}^{(i_j)}$ is a $(k-1)$-dimensional *mode-j section* of $\mathcal{T}$.)

Consequently, if (3.3) holds, then (3.6) gives

$$0 \leq V_{Q_{\mathcal{T}}}\big(R(i_j)\big) \leq \frac{1}{n_j} \qquad \text{for all } i_j \in [n_j].$$

Summing these bounds over $i_j = a, \ldots, b$ in (3.5) yields

$$0 \leq V_{Q_{\mathcal{T}}}(R) \leq \frac{b-a+1}{n_j} = v_j - u_j,$$

so the volume condition of Theorem 1 holds for all $R \in \wp^k$ whose endpoints lie on $\mathcal{G}$.

Finally, for general $R \in \wp^k$ (with arbitrary real endpoints), note that on each grid cell $Q_{\mathcal{T}}$ is multilinear, hence $V_{Q_{\mathcal{T}}}(R)$ is affine in each endpoint. Therefore the bounds $0 \leq V_{Q_{\mathcal{T}}}(R) \leq v_j - u_j$,

valid on the grid, extend to all endpoints by linear interpolation. Thus the volume condition in Theorem 1 holds for every $R \in \wp^k$.

Step 3: Boundary conditions. Assume (3.2). Taking $\ell_r = n_r$ for all r in (3.4) gives $Q_{\mathcal{T}}(1,\dots,1) = \sum \mathcal{T}$. Moreover,

$$\sum \mathcal{T} = \sum_{i_j=1}^{n_j} \sum \mathcal{T}_{(j)}^{(i_j)} = \sum_{i_j=1}^{n_j} \frac{1}{n_j} = 1,$$

so $Q_{\mathcal{T}}(1,\dots,1) = 1$. Also, if $x_j = 0$ then (3.4) gives $Q_{\mathcal{T}}(\dots,0,\dots) = 0$.

Conclusion ("⇒"). By Steps 2–3, $Q_{\mathcal{T}}$ satisfies items (1)–(3) of Theorem 1; therefore $Q_{\mathcal{T}}$ is a quasi-copula, i.e., $\mathcal{T}$ is a quasi-copula type array.

Conversely ("⇐"). If $Q_{\mathcal{T}}$ is a quasi-copula, then Theorem 1 gives $0 \le V_{Q_{\mathcal{T}}}(R) \le v_j - u_j$ for all $R \in \wp^k$. Applying this to the cell-primal boxes $R(i_j)$ above and using (3.6) yields (3.3). Moreover, for $x_j = \ell_j/n_j$ and $x_r = 1$ for $r \neq j$, we have

$$Q_{\mathcal{T}}(1,\dots,1,x_j,1,\dots,1) = \sum_{i_j=1}^{\ell_j} \sum \mathcal{T}_{(j)}^{(i_j)}.$$

Since $Q_{\mathcal{T}}(1,\dots,1,x_j,1,\dots,1) = x_j$ by the boundary conditions of a quasi-copula, taking differences in ℓ_j gives $\sum \mathcal{T}_{(j)}^{(i_j)} = 1/n_j$ for all i_j, i.e., (3.2). This completes the proof. □

In Theorem 3, we need to check the positivity conditions in (3.3) for each $(k-1)$-dimensional subarray $\mathcal{T}_{(j)}^{(i_j)}$ of $\mathcal{T}$ in every dimension $j \in [k]$. However, the following result provides sufficient conditions for quasi-copula type arrays under stronger positivity conditions on the $(k-1)$-dimensional subarray $\mathcal{T}_{(j)}^{(i_j)}$ of $\mathcal{T}$ in only *one* dimension $j \in [k]$.

Theorem 4. *$\mathcal{T} \in \mathbb{R}^{n_1 \times \dots \times n_k}$ is a quasi-copula type array if it satisfies:*

(1) For every $j \in [k]$ and every $i_j \in [n_j]$ we have

$$\sum \mathcal{T}_{(j)}^{(i_j)} = \frac{1}{n_j}.$$

(2) There exists $j \in [k]$ such that for every $i_j \in [n_j]$, every $i_\ell \in [n_\ell]$, $\ell \neq j$, and every $\ell_r \in [n_r]$ for $r \notin \{j,\ell\}$, we have

$$0 \le \sum_{\substack{i_r=1,\\ r\notin\{j,\ell\}}}^{\ell_r} \left(\mathcal{T}_{(m,M)}^{(i_m,i_M)}\right)_{i_1,\dots,i_{m-1},i_{m+1},\dots,i_{M-1},i_{M+1},\dots,i_k} \le \frac{1}{n_j n_\ell}, \tag{3.7}$$

where $(m,M) := (\min(j,\ell), \max(j,\ell))$.

Proof. Assumption (1) coincides with condition (3.2) in Theorem 3. Hence, by Theorem 3, it remains to verify (3.3).

By relabeling the coordinate axes (i.e., permuting dimensions), we may assume that the index j from Assumption (2) equals 1. Fix $j' \in [k]$, $i_{j'} \in [n_{j'}]$, and cutoffs $\ell_r \in [n_r]$ for all $r \neq j'$. We consider two cases.

Case 1: $j' = 1$. Choose $\ell = 2$ in (3.7). Then for each $i_2 \in [n_2]$,

$$0 \le \sum_{\substack{i_r=1\\ r\in[3;k]}}^{\ell_r} \left(\mathcal{T}_{(1,2)}^{(i_1,i_2)}\right)_{i_3,\dots,i_k} \le \frac{1}{n_1 n_2}.$$

Therefore,

$$0 \le \sum_{\substack{i_r=1\\ r\in[2;k]}}^{\ell_r} \left(\mathcal{T}_{(1)}^{(i_1)}\right)_{i_2,\dots,i_k} = \sum_{i_2=1}^{\ell_2} \left(\sum_{\substack{i_r=1\\ r\in[3;k]}}^{\ell_r} \left(\mathcal{T}_{(1,2)}^{(i_1,i_2)}\right)_{i_3,\dots,i_k} \right)$$

$$\leq \sum_{i_2=1}^{\ell_2} \frac{1}{n_1 n_2} = \frac{\ell_2}{n_1 n_2} \leq \frac{1}{n_1}.$$

This proves (3.3) for $j' = 1$.

Case 2: $j' \neq 1$**.** Set $\ell = j'$ in (3.7). Then for each $i_1 \in [n_1]$,

$$0 \leq \sum_{\substack{i_r=1 \\ r \notin \{1,j'\}}}^{\ell_r} \left(\mathcal{T}_{(1,j')}^{(i_1,i_{j'})}\right)_{\text{remaining indices}} \leq \frac{1}{n_1 n_{j'}}.$$

Hence,

$$0 \leq \sum_{\substack{i_r=1 \\ r \neq j'}}^{\ell_r} \left(\mathcal{T}_{(j')}^{(i_{j'})}\right)_{\text{remaining indices}} = \sum_{i_1=1}^{\ell_1} \left(\sum_{\substack{i_r=1 \\ r \notin \{1,j'\}}}^{\ell_r} \left(\mathcal{T}_{(1,j')}^{(i_1,i_{j'})}\right)_{\text{remaining indices}} \right)$$
$$\leq \sum_{i_1=1}^{\ell_1} \frac{1}{n_1 n_{j'}} = \frac{\ell_1}{n_1 n_{j'}} \leq \frac{1}{n_{j'}}.$$

This proves (3.3) for all $j' \neq 1$. □

Remark 5. (1) *Interpretation of the sufficient positivity condition.* Condition (3.7) is stronger than the $(k-1)$-dimensional partial-sum bound (3.3) from Theorem 3. Indeed, (3.7) controls partial sums on every $(k-2)$-dimensional section $\mathcal{T}_{(m,M)}^{(i_m,i_M)}$ obtained by fixing the distinguished index j together with any other index $\ell \neq j$. Summing these bounds over the remaining free index yields the corresponding partial-sum bounds for each $(k-1)$-dimensional section $\mathcal{T}_{(j')}^{(i_{j'})}$ appearing in (3.3). Thus, (3.7) can be viewed as a *two-axis* positivity requirement, which implies the *one-axis* positivity needed in Theorem 3.

(2) *Why checking one dimension can suffice.* Fix j as in Theorem 4. The inequalities (3.7) impose uniform bounds $0 \leq \cdot \leq 1/(n_j n_\ell)$ on partial sums of each two-index section $\mathcal{T}_{(j,\ell)}^{(i_j,i_\ell)}$. Consequently, summing over the remaining free index forces the cumulative sum along the j-axis to be at most $1/n_j$, which is exactly the upper bound required for the j-section in (3.3). By exchanging the role of the summed index, the same conclusion holds for every other $(k-1)$–dimensional section. In this way, the theorem provides a convenient sufficient criterion when the array is naturally constructed so that all two-index sections involving a fixed axis already satisfy strong positivity constraints.

We call arrays $\mathcal{T} \in \mathbb{R}^{n_1 \times \cdots \times n_k}$ that satisfy the hypotheses of Theorem 4 **strong quasi-copula type arrays**. Their importance lies in the fact that they provide a simple way to construct higher-dimensional quasi-copula type arrays from lower-dimensional ones.

Theorem 6. *Let $k \geq 3$ and fix $j \in [k]$. Let*

$$\mathcal{S}_1, \ldots, \mathcal{S}_{n_j} \in \mathbb{R}^{n_1 \times \cdots \times n_{j-1} \times n_{j+1} \times \cdots \times n_k}$$

be arbitrary quasi-copula type arrays. Define $\mathcal{T} \in \mathbb{R}^{n_1 \times \cdots \times n_k}$ by

$$\mathcal{T}_{(j)}^{(i_j)} = \frac{1}{n_j} \mathcal{S}_{i_j} \qquad \text{for all } i_j \in [n_j].$$

Then $\mathcal{T}$ is a quasi-copula type array.

Proof. It suffices to verify conditions (1) and (2) of Theorem 4.

Step 1: Verification of Theorem 4.(1)**.** Fix $j' \in [k]$ and $i_{j'} \in [n_{j'}]$.

If $j' = j$, then

$$\sum \mathcal{T}_{(j)}^{(i_j)} = \frac{1}{n_j} \sum \mathcal{S}_{i_j} = \frac{1}{n_j},$$

since each $\mathcal{S}_{i_j}$ is quasi-copula type and hence satisfies the marginal-sum condition in Theorem 3.

If $j' \neq j$, note that $\mathcal{S}_{i_j}$ has dimension $k-1$, so fixing index j' in $\mathcal{T}$ corresponds to fixing index

$$\phi(j') := \begin{cases} j', & j' < j, \\ j'-1, & j' > j, \end{cases}$$

in $\mathcal{S}_{i_j}$. Therefore,

$$\sum \mathcal{T}_{(j')}^{(i_{j'})} = \frac{1}{n_j} \sum_{i_j=1}^{n_j} \sum (\mathcal{S}_{i_j})_{(\phi(j'))}^{(i_{j'})} = \frac{1}{n_j} \sum_{i_j=1}^{n_j} \frac{1}{n_{j'}} = \frac{1}{n_{j'}}.$$

This proves (1).

Step 2: Verification of Theorem 4.(2)**.** We show that the same index j used in the definition of $\mathcal{T}$ satisfies the strong positivity condition. Fix $\ell \neq j$, choose $i_j \in [n_j]$, $i_\ell \in [n_\ell]$, and cutoffs $\ell_r \in [n_r]$ for $r \notin \{j, \ell\}$. Let $(m, M) := (\min\{j, \ell\}, \max\{j, \ell\})$.

Using $\mathcal{T}_{(j)}^{(i_j)} = \frac{1}{n_j}\mathcal{S}_{i_j}$ and the same index shift $\phi(\ell)$ as above, we obtain

$$\sum_{\substack{i_r=1 \\ r\notin\{j,\ell\}}}^{\ell_r} \left(\mathcal{T}_{(m,M)}^{(i_m,i_M)}\right)_{\text{remaining indices}} = \frac{1}{n_j} \sum_{\substack{i_r=1 \\ r\notin\{j,\ell\}}}^{\ell_r} \left((\mathcal{S}_{i_j})_{(\phi(\ell))}^{(i_\ell)}\right)_{\text{remaining indices}}.$$

Since $\mathcal{S}_{i_j}$ is a quasi-copula type array, Theorem 3 implies that the partial sum on the right-hand side lies in $[0, 1/n_\ell]$. Hence

$$0 \leq \sum_{\substack{i_r=1 \\ r\notin\{j,\ell\}}}^{\ell_r} \left(\mathcal{T}_{(m,M)}^{(i_m,i_M)}\right)_{\text{remaining indices}} \leq \frac{1}{n_j} \cdot \frac{1}{n_\ell} = \frac{1}{n_j n_\ell},$$

which is exactly (2). Therefore, $\mathcal{T}$ is a quasi-copula type array. □

3.1. **Application of Theorem 6 to imprecise copulas.** It is time to give some historic remarks on the main subject of this paper together with some hints of how our techniques help understanding the problem. The historic first definition of an **imprecise copula** was given in [20, Definition 6]; they call so a pair (A, B) of functions $A, B : [0,1]^2 \to [0,1]$ if:

- (**A**) $A(0,u) = A(u,0) = 0$ and $A(1,u) = A(u,1) = u$ for all $u \in [0,1]$;
- (**B**) $B(0,u) = B(u,0) = 0$ and $B(1,u) = B(u,1) = u$ for all $u \in [0,1]$;
- for any $u_1 \leq u_2$ and $v_1 \leq v_2$ in $[0,1]$, the following inequalities hold:

(**IC1**) $$A(u_2,v_2) + A(u_1,v_1) - A(u_1,v_2) - B(u_2,v_1) \geq 0,$$

(**IC2**) $$B(u_2,v_2) + A(u_1,v_1) - A(u_1,v_2) - A(u_2,v_1) \geq 0,$$

(**IC3**) $$B(u_2,v_2) + B(u_1,v_1) - B(u_1,v_2) - A(u_2,v_1) \geq 0,$$

(**IC4**) $$B(u_2,v_2) + B(u_1,v_1) - A(u_1,v_2) - B(u_2,v_1) \geq 0.$$

By [20, Proposition 2], we have $A \leq B$ pointwise, and both A and B are quasi-copulas. Observe that this definition is clearly bivariate, We will call it **imprecise copula in the sense of** [20]. However, for the sake of simplicity we will omit the attribute "in the sense of [20]" in this subsection,

Given a mesh $\Delta := \sigma_x \times \sigma_y$, where

$$\sigma_x = \{0 = x_0 < x_1 < \cdots < x_p = 1\} \quad \text{and} \quad \sigma_y = \{0 = y_0 < y_1 < \cdots < y_q = 1\},$$

a pair (A, B) defined on Δ is called a **discrete imprecise copula** [15] if it satisfies **(A)**, **(B)**, and **(IC1)**–**(IC4)**. Discrete imprecise copulas and related construction techniques have been studied very recently also by Košir and Perone in [10]. The first author and Škulj [19] introduced an imprecise version of Marshall-Olkin copulas.

Given a function $A : \Delta \to [0,1]$, we denote by $A^{\mathrm{BL}} : [0,1]^2 \to [0,1]$ the function obtained from A by bilinear interpolation on each mesh cell (equivalently, on the two triangles induced by a fixed diagonalization). By [15, Proposition 4], A is a discrete quasi-copula if and only if A^{BL} is a quasi-copula.

Moreover, [15, Proposition 5] shows that every discrete imprecise copula (A, B) induces an imprecise copula $(A^{\mathrm{BL}}, B^{\mathrm{BL}})$, and conversely, every imprecise copula (A, B) induces a discrete imprecise copula $(A|_\Delta, B|_\Delta)$. In particular, if (A, B) is a discrete imprecise copula, then both A and B are discrete quasi-copulas.

In [15, Section 5], the first author and Stopar established the existence of a discrete imprecise copula (A, B) (and hence also an imprecise copula) for which there exists no copula C satisfying $A \leq C \leq B$. Furthermore, [15, Proposition 16 and Theorem 17] characterize when, for two discrete quasi-copulas (respectively, quasi-copulas) A, B with $A \leq B$, there exists a copula C such that $A \leq C \leq B$. This result was extended to arbitrary dimensions in [16].

Theorem 6 can be used to produce multidimensional examples of pairs (A, B) such that $A \leq B$ and yet no copula C exists with $A \leq C \leq B$. Indeed, on a fixed grid, a discrete (imprecise) copula is equivalently encoded by its array of box masses. Let $\mathcal{S}_A$ and $\mathcal{S}_B$ denote the mass arrays associated with A and B, respectively, and choose arbitrary quasi-copula type arrays $\mathcal{S}_2, \ldots, \mathcal{S}_n$ to serve as the remaining slices in the stacking construction. Applying Theorem 6 to the slice families $(\mathcal{S}_A, \mathcal{S}_2, \ldots, \mathcal{S}_n)$ and $(\mathcal{S}_B, \mathcal{S}_2, \ldots, \mathcal{S}_n)$ yields quasi-copula type arrays $\mathcal{T}_A$ and $\mathcal{T}_B$. The corresponding quasi-copulas $\mathcal{Q}_{\mathcal{T}_A}$ and $\mathcal{Q}_{\mathcal{T}_B}$, obtained via the integral transform (3.1), satisfy $\mathcal{Q}_{\mathcal{T}_A} \leq \mathcal{Q}_{\mathcal{T}_B}$, and there is no copula C such that $\mathcal{Q}_{\mathcal{T}_A} \leq C \leq \mathcal{Q}_{\mathcal{T}_B}$. Consequently, starting from the bivariate example in [15, Section 5], one can iteratively construct such examples in arbitrary dimension.

In Section 4 we present our definition of an imprecise copula and its justification, but before that we exhibit briefly an independent approach to modeling with copulas via imprecise probabilities proposed by Schmelzer [24, 26]. Copulas are used there to describe the dependence between random sets. As a matter of fact, it turns out that sets of copulas are needed to describe such dependence instead of a single copula. Moreover, the same author also shows in [25] that in the special case of minitive belief functions Sklar's theorem actually holds, which means that a single copula is sufficient to express the dependence relation.

4. Imprecise copulas – a well justified definition

4.1. **ALGEN algorithm.** The key technique giving the justification to our definition of the imprecise copula is a method presented by the first author and Stopar in [17] called Algebraic Obstacles in the Geometry of Negative Volumes (ALGEN for short). This algorithm has been developed through a number of papers by the same authors; the most useful for us will be [16]. Let us start with some notation, continuing from Subsection 2.1:

Let A and B be a pair of real valued functions defined on $[0,1]^n$ such that $A \leqslant B$, i.e., $A(\mathbf{x}) \leqslant B(\mathbf{x})$ for all $\mathbf{x} \in [0,1]^n$. Given a rectangle $R := \prod_{i=1}^n [x_i, y_i]$, where $x_i \leq y_i$, we define for any point $\mathbf{x} \in [0,1]^n$ its *multiplicity* by

$$m_R(\mathbf{x}) = \begin{cases} \operatorname{sign}(\mathbf{x}), & \text{if } \mathbf{x} \in \operatorname{ver} R; \\ 0, & \text{otherwise.} \end{cases}, \text{ so that } \quad V_A(R) = \sum_{\mathbf{x} \in [0,1]^n} A(\mathbf{x}) m_R(\mathbf{x}).$$

Clearly this sum is finite. We extend this definition of m to any $\mathbf{R} \in \mathfrak{R}^n_{[0,1]}$, the set of all finite collections of rectangles, i.e., $\mathbf{R} = [R_i]_{i=1}^k$, where R_i for $i \in [k]$ is an arbitrary rectangle with vertices in $[0,1]^n$. Let us point out that we use the term collection here to emphasize that we allow it to contain repeated copies of any given member. Disjoint unions will be applied on collections in the usual way, so that $\mathbf{R}_1 \sqcup \mathbf{R}_2$ means the collection of all members of both collections repeated as many times as in both combined. For any $\mathbf{R} \in \mathfrak{R}^n_{[0,1]}$ we let $m_{\mathbf{R}}(\mathbf{x}) = \sum_{i=1}^k m_{R_i}(\mathbf{x})$ and $V_A(\mathbf{R}) := \sum_{i=1}^k V_A(R_i)$. This definition for the volume of an element $\mathbf{R} \in \mathfrak{R}^n_{[0,1]}$ can be

expressed easily with the so introduced function $m_{\mathbf{R}}$ in the same way as the volume of a rectangle

$$V_A(\mathbf{R}) = \sum_{i=1}^{k} V_A(R_i) = \sum_{i=1}^{k} \sum_{\mathbf{x}\in[0,1]^n} A(\mathbf{x}) m_{R_i}(\mathbf{x}) = \sum_{\mathbf{x}\in[0,1]^n} \sum_{i=1}^{k} A(\mathbf{x}) m_{R_i}(\mathbf{x})$$

$$= \sum_{\mathbf{x}\in[0,1]^n} A(\mathbf{x}) \sum_{i=1}^{k} m_{R_i}(\mathbf{x}) = \sum_{\mathbf{x}\in[0,1]^n} A(\mathbf{x}) m_{\mathbf{R}}(\mathbf{x}).$$

It is not hard to see from the definition and it is also proven in [16, Lemma 10] that the multiplicity of a given point and the volume corresponding to a given real valued function A with respect to members of $\mathfrak{R}^n_{[0,1]}$ are additive, i.e.,

$$m_{R_1 \sqcup R_2}(\mathbf{x}) = m_{R_1}(\mathbf{x}) + m_{R_2}(\mathbf{x}) \quad \text{and} \quad V_A(\mathbf{R}_1 \sqcup \mathbf{R}_2) = V_A(\mathbf{R}_1) + V_A(\mathbf{R}_2).$$

Test functions. Let us define a function L of $\mathbf{R} \in \mathfrak{R}^n_{[0,1]}$, and functions P_M and P_O of $\mathbf{x} \in [0,1]^n$, all depending also on the real valued functions A and B

$$L^{(A,B)}(\mathbf{R}) = \sum_{\substack{\mathbf{y}\in[0,1]^n \\ m_{\mathbf{R}}(\mathbf{y})>0}} B(\mathbf{y}) m_{\mathbf{R}}(\mathbf{y}) + \sum_{\substack{\mathbf{y}\in[0,1]^n \\ m_{\mathbf{R}}(\mathbf{y})<0}} A(\mathbf{y}) m_{\mathbf{R}}(\mathbf{y})$$

$$P_M^{(A,B)}(\mathbf{x}) = \inf_{\substack{\mathbf{R}\in\mathfrak{R}^n \\ m_{\mathbf{R}}(\mathbf{x})>0}} \frac{L^{(A,B)}(R)}{m_{\mathbf{R}}(\mathbf{x})} \quad \text{and} \quad P_O^{(A,B)}(\mathbf{x}) = \inf_{\substack{\mathbf{R}\in\mathfrak{R}^n \\ m_{\mathbf{R}}(\mathbf{x})<0}} \frac{L^{(A,B)}(R)}{-m_{\mathbf{R}}(\mathbf{x})},$$

where infimum of an empty set is assumed equal to $+\infty$. For any quasi-copulas $A \leqslant B$ we write

$$\mathcal{C}(A,B) = \{C \mid C \text{ copula}, \ A \leqslant C \leqslant B\}. \tag{4.1}$$

The following results were proven in [16, Theorem 15] and [16, Theorem 16].

Coherence and avoidung sure loss for quasi-copulas:

(1) $\mathcal{C}(A,B) \neq \emptyset$ if and only if $L^{(A,B)}(\mathbf{R}) \geqslant 0$ for all $\mathbf{R} \in \mathfrak{R}^n_{[0,1]}$. And if $\mathcal{C}(A,B) \neq \emptyset$, then

(2) $B = \sup \mathcal{C}(A,B)$ if and only if $B(\mathbf{x}) - A(\mathbf{x}) \leqslant P_O^{(A,B)}(\mathbf{x})$ for all $\mathbf{x} \in [0,1]^n$.

(3) $A = \inf \mathcal{C}(A,B)$ if and only if $B(\mathbf{x}) - A(\mathbf{x}) \leqslant P_M^{(A,B)}(\mathbf{x})$ for all $\mathbf{x} \in [0,1]^n$.

The notions "avoiding sure loss" and "coherence" were introduced by Walley [25, 26] as desired properties of intervals representing imprecise probability, meaning that the interval contains the true probability, respectively that it is made of lower and upper envelope of true probabilities. The reader should look for the exact definitions of these terms in his work cited above and think of this only as a possible interpretation of these notions. Our object of interest is given by equation (4.1), an interval of copulas to become our imprecise copula. Using test function L point (1) above tells that this object is avoiding sure loss exactly when function L is positive on $\mathfrak{R}^n_{[0,1]}$. Using test function P_O point (2) tells us exactly when the upper bound B is upper coherent. And using test function P_M point (3) tells us exactly when the lower bound A is lower coherent. So, we call the interval $\mathcal{C}(A,B)$ of (4.1) an **imprecise copula** iff nonempty. In this case we will denote it by $\mathcal{C}_{A,B}$. Here is our theorem.

Theorem 7. *An interval $\mathcal{C}(A,B)$ of (4.1) is an imprecise copula if and only if $L^{(A,B)}(\mathbf{R}) \geqslant 0$ for all $\mathbf{R} \in \mathfrak{R}^n_{[0,1]}$. In this case we have:*

(1) Its upper bound B is upper coherent if and only if $B(\mathbf{x}) - A(\mathbf{x}) \leqslant P_O^{(A,B)}(\mathbf{x})$ for all $\mathbf{x} \in [0,1]^n$.

(2) Its lower bound A is lower coherent if and only if $B(\mathbf{x}) - A(\mathbf{x}) \leqslant P_M^{(A,B)}(\mathbf{x})$ for all $\mathbf{x} \in [0,1]^n$.

This is the first of our justifications for this definition. We are presenting the second one now.

4.2. **The imprecise Sklar's theorem.** There has been different tries to give the Sklar type theorem in the imprecise setting, mostly bivariate: Montes et al. [20], the first author with Škulj [19], and with Stopar [15]. We will present here a theorem based on [16, Theorem 21]. But let us start with the original. We assume a random vector $\boldsymbol{X} = (X_1, \ldots, X_d)$ and its joint distribution function $H(\boldsymbol{x}) := P(X_1 \leq x_1, \ldots, X_d \leq x_d))$ as well as its marginals $F_j(x_j) = P(X_j \leq x_j), j = 1, \ldots, d$, for $\boldsymbol{x} = (x_1, \ldots, x_d)$.

Theorem. [27] (1959, Sklar) *Let $\boldsymbol{X}, H$, and $F_j, j = 1, \ldots, d$, be as above. Then there exists a d-variate copula C such that*

$$H(\boldsymbol{x}) = C(F_1(x_1), \ldots, F_d(x_d)) \quad \text{for every} \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{4.2}$$

Moreover, if C is any d-variate copula and $F_j, j = 1, \ldots, d$, are arbitrary univariate distributions, then composition (4.2) *defines a joint distribution of a d-dimensional random vector.*

So, for any multivariate distribution function there exists a copula such that the function is a composition of this copula and its univariate marginals. Copula C is sometimes called the copula of function H; in the rare case that C is not uniquely determined by H we just choose one of them. In search for an imprecise version of this theorem, one first needs to choose an appropriate counterpart to joint distributions. We follow [16] starting with their Section 3. Their approach allows either the usual probability background or the finitely additive one preferred in the imprecise probability community and develop all the notions parallel for either case. Components of a formal random vector $\boldsymbol{X} = (X_1, \ldots, X_d)$ have univariate distributions we write in a vector $F_{\boldsymbol{X}}(\boldsymbol{x}) = (F_{X_1}(x_1), \ldots, F_{X_d}(x_d))$. We fix this vector throughout this subsection. When extending the notion of p-boxes [7, 30] as in [16, Section 3] one can start with a set $\mathcal{F}$ of d-variate distributions. It turns out that

$$\underline{\mathcal{F}} = \inf\{F; F \in \mathcal{F}\} \quad \text{and} \quad \overline{\mathcal{F}} = \sup\{F; F \in \mathcal{F}\}$$

are quasi-distributions in the sense of [16] and we can extend the starting set of distributions up to a p-box

$$\mathcal{F} = \{F; F \text{ } d\text{-variate distribution}, \underline{\mathcal{F}} \leq F \leq \overline{\mathcal{F}}\} := (\underline{\mathcal{F}}, \overline{\mathcal{F}}).$$

Here is an example of a p-box of d-variate distributions. Let $\mathcal{C}_{A,B}$ be an imprecise copula induced by a pair of d-variate quasi-copulas $A \leq B$ and let $F_{\boldsymbol{X}}$ be as above. Their composition

$$\mathcal{C}_{A,B}\,(F_{\boldsymbol{X}}) := \{C(F_{\boldsymbol{X}}); C \text{ a copula}, C \in \mathcal{C}_{A,B}\}$$

is then a p-box and its two bounds are

$$\underline{\mathcal{C}_{A,B}} = A(F_{\boldsymbol{X}}), \quad \overline{\mathcal{C}_{A,B}} = B(F_{\boldsymbol{X}})$$

respectively as it is not hard to see. We denote by $\mathcal{D}^{F_{\boldsymbol{X}}}$ the family of all joint distributions with marginals $F_{\boldsymbol{X}}$. Similarly, we denote by $\widetilde{\mathcal{D}}^{F_{\boldsymbol{X}}}$ the family of all quasi-distributions with marginals $F_{\boldsymbol{X}}$. A version of Sklar's theorem [16, Theorem 8] says:

For any d-variate function F in $\mathcal{D}^{F_{\boldsymbol{X}}}$, respectively $\widetilde{\mathcal{D}}^{F_{\boldsymbol{X}}}$, there exists a copula, respectively a quasi-copula, C such that

$$F(\boldsymbol{x}) = C(F_{\boldsymbol{X}}(\boldsymbol{x})).$$

Here is the main result we want to present in this section and is an easy cosequence of [16, Theorem 21]. We fix a vector of marginals $F_{\boldsymbol{X}}$ and a p-box of distributions $\mathcal{F} \subseteq \mathcal{D}^{F_{\boldsymbol{X}}}$.

Theorem 8 (Imprecise Sklar theorem)**.** *Given $\mathcal{F}$ and $F_{\boldsymbol{X}}$ as above there exist d-variate quasi-copulas $A \leq B$ satisfying*

$$\underline{\mathcal{F}} = A(F_{\boldsymbol{X}}) \quad \text{and} \quad \overline{\mathcal{F}} = B(F_{\boldsymbol{X}})$$

such that $\mathcal{C}_{A,B}$ is an imprecise copula and its composition with $F_{\boldsymbol{X}}$ equals $\mathcal{F}$:

$$\mathcal{C}_{A,B}\,(F_{\boldsymbol{X}}) = \mathcal{F}. \tag{4.3}$$

Moreover, if $\mathcal{C}_{A,B}$ is an imprecise copula and $F_{\boldsymbol{X}}$ is a vector of marginals, then composition (4.3) *defines a p-box.*

5. From three-dimensional arrays to strong quasi-copula type arrays

In this section we address an *extension problem* in dimension three. Given an arbitrary three-dimensional array $\mathcal{P} \in \mathbb{R}^{n\times n\times n}$, we ask whether $\mathcal{P}$ can be embedded—up to an affine rescaling—into a larger array whose induced box-volumes satisfy the strong positivity requirements of Section 2 and therefore determine a quasi-copula. Our first main result answers this question in the affirmative and, moreover, provides an explicit construction. Specifically, Theorem 9 constructs a *strong* quasi-copula type array $\mathcal{R} \in \mathbb{R}^{(2n+1)\times(2n+1)\times(2n+1)}$ that contains a prescribed multiple of $\mathcal{P}$ as a designated subarray on $[n] \times [n+1; 2n] \times [n+1; 2n]$. The construction proceeds in three steps: we first shift $\mathcal{P}$ to control potential negativity; we then introduce auxiliary quasi-copula type slices to enforce the required two-axis partial-sum bounds; and finally we normalize to achieve the desired marginal sums. Altogether, this yields a concrete embedding mechanism that transforms arbitrary data into a strong quasi-copula type structure.

Theorem 9. *Let $n \in \mathbb{N}$ and let*

$$\mathcal{P} \equiv \big[\mathcal{P}_{i,j,k}\big]_{i,j,k} \in \mathbb{R}^{n\times n\times n}$$

be a given 3-dimensional array. Then there exists a 3-dimensional strong *quasi-copula type array*

$$\mathcal{R} \in \mathbb{R}^{(2n+1)\times(2n+1)\times(2n+1)}$$

that contains a multiple of $\mathcal{P}$ as a subarray at the positions

$$[n] \times [n+1; 2n] \times [n+1; 2n].$$

Construction of $\mathcal{R}$. *Set*

$$\nu := \max\Big(\max_{i,j,k\in[n]} \mathcal{P}_{i,j,k},\, 0\Big), \qquad \widetilde{\mathcal{P}} := \mathcal{P} - \nu\,\mathbb{L}^{(n,n,n)}.$$

Let $M := 2n + 1$ and define

$$\mu := \max_{i\in[n]}\Big(-\sum \widetilde{\mathcal{P}}^{(i)}_{(1)}\Big). \tag{5.1}$$

For $i \in [M]$ define the mode-1 sections of $\widetilde{\mathcal{R}}$ by

$$\widetilde{\mathcal{R}}^{(i)}_{(1)} := \begin{cases} \begin{bmatrix} \mathbf{0} & \widetilde{c}_i & \widetilde{\alpha}_i \\ \mu\widetilde{A}_i & \widetilde{\mathcal{P}}^{(i)}_{(1)} & \widetilde{d}_i \\ \mathbf{0} & \mu\widetilde{B}_i & \mathbf{0} \end{bmatrix}, & \text{for } i \in [n], \\ \mu\,\widetilde{\mathcal{F}}_i, & \text{for } i \in [n+1; 2n+1], \end{cases}$$

where $\frac{1}{n}\widetilde{A}_i, \frac{1}{n}\widetilde{B}_i \in \mathbb{R}^{n\times n}$ and $\frac{1}{M}\widetilde{\mathcal{F}}_i \in \mathbb{R}^{M\times M}$ are arbitrary two-dimensional quasi-copula type arrays, and

$$\begin{aligned} \widetilde{c}_i &:= \begin{bmatrix} -\sum \widetilde{\mathcal{P}}^{(i,1)}_{(1,2)} & -\sum \widetilde{\mathcal{P}}^{(i,2)}_{(1,2)} & \cdots & -\sum \widetilde{\mathcal{P}}^{(i,n)}_{(1,2)} \end{bmatrix} \in \mathbb{R}^{1\times n}, \\ \widetilde{d}_i &:= \begin{bmatrix} -\sum \widetilde{\mathcal{P}}^{(i,1)}_{(1,3)} & -\sum \widetilde{\mathcal{P}}^{(i,2)}_{(1,3)} & \cdots & -\sum \widetilde{\mathcal{P}}^{(i,n)}_{(1,3)} \end{bmatrix}^T \in \mathbb{R}^{n\times 1}, \\ \widetilde{\alpha}_i &:= \mu + \sum \widetilde{\mathcal{P}}^{(i)}_{(1)} \in \mathbb{R}. \end{aligned} \tag{5.2}$$

Finally, set

$$\mathcal{R} := \frac{1}{M^2\,(\mu + \nu M)}\Big(\widetilde{\mathcal{R}} + \nu\mathbb{L}^{(M,M,M)}\Big).$$

Proof. We prove that for each $i \in [M]$ the two-dimensional array $M\,\mathcal{R}^{(i)}_{(1)} \in \mathbb{R}^{M\times M}$ is a quasi-copula type array. Then Theorem 6 implies that $\mathcal{R}$ is a 3-dimensional quasi-copula type array.

For $i \in [n+1; M]$ this is immediate from the definition $M\,\mathcal{R}^{(i)}_{(1)} \propto \widetilde{\mathcal{F}}_i + \nu\mathbb{L}^{(M)}$. Hence, throughout the remainder of the proof we fix $i \in [n]$.

Step 1: Verification of (3.2)**.** It suffices to check that every one-dimensional section of $M\,\mathcal{R}^{(i)}_{(1)}$ has total sum $1/M$, i.e., for all $j \in [M]$,

$$\sum \Big(M\mathcal{R}^{(i)}_{(1)}\Big)^{(j)}_{(1)} = \frac{1}{M} \qquad \text{and} \qquad \sum \Big(M\mathcal{R}^{(i)}_{(1)}\Big)^{(j)}_{(2)} = \frac{1}{M}. \tag{5.3}$$

We verify the first identity; the second one follows analogously from the same block structure. Let $\xi := M(\mu + \nu M)$. Then, by the definition of $\mathcal{R}$,

$$\sum \big(M\mathcal{R}^{(i)}_{(1)}\big)^{(j)}_{(1)} = \frac{1}{\xi}\Big(\sum (\widetilde{\mathcal{R}}^{(i)}_{(1)})^{(j)}_{(1)} + \nu \sum (\mathbb{L}^{(M)})^{(j)}_{(1)}\Big) = \frac{1}{\xi}\,(\mu + \nu M) = \frac{1}{M},$$

as required.

Step 2: Verification of (3.3)**.** We need to show that for all $j_1, k_1 \in [M]$,

$$0 \le \sum_{k=1}^{k_1} \Big(\big(M\mathcal{R}^{(i)}_{(1)}\big)^{(j_1)}_{(1)}\Big)_k \le \frac{1}{M}, \qquad 0 \le \sum_{j=1}^{j_1} \Big(\big(M\mathcal{R}^{(i)}_{(1)}\big)^{(k_1)}_{(2)}\Big)_j \le \frac{1}{M}. \tag{5.4}$$

Again, we verify the first inequality; the second follows by the same argument.

Set $\xi := M(\mu + \nu M)$ and distinguish cases according to j_1.

Case 1: $j_1 \in [n]$**.**

- **Case 1.1:** $k_1 \in [n]$**.** Only the $\nu\mathbb{L}^{(M)}$-part contributes, hence

$$\sum_{k=1}^{k_1} \Big(\big(M\mathcal{R}^{(i)}_{(1)}\big)^{(j_1)}_{(1)}\Big)_k = \frac{1}{\xi}\sum_{k=1}^{k_1} \nu = \frac{\nu k_1}{\xi} \in \Big[0, \frac{\nu n}{\xi}\Big] \subseteq \Big[0, \frac{1}{M}\Big].$$

- **Case 1.2:** $k_1 \in [n+1; 2n+1]$**.** Here additionally the $\mu\widetilde{A}_i$-block contributes:

$$\sum_{k=1}^{k_1} \Big(\big(M\mathcal{R}^{(i)}_{(1)}\big)^{(j_1)}_{(1)}\Big)_k = \frac{1}{\xi}\sum_{k=1}^{k_1-n} \mu\Big(\widetilde{A}_i\Big)^{(j_1)}_{(1)}{}_k + \frac{\nu k_1}{\xi}.$$

  Since $\widetilde{A}_i$ is quasi-copula type, its partial sums lie in $[0,1]$, hence the expression lies in $[0, (\mu + \nu M)/\xi] = [0, 1/M]$.

Case 2: $j_1 \in [n+1; 2n]$**.**

- **Case 2.1:** $k_1 \in [n]$**.** The relevant blocks are $\mu\widetilde{B}_i$ and $\nu\mathbb{L}^{(M)}$, hence

$$\sum_{k=1}^{k_1} \Big(\big(M\mathcal{R}^{(i)}_{(1)}\big)^{(j_1)}_{(1)}\Big)_k = \frac{1}{\xi}\sum_{k=1}^{k_1} \mu\Big(\widetilde{B}_i\Big)^{(j_1)}_{(1)}{}_k + \frac{\nu k_1}{\xi} \in \Big[0, \frac{\mu + \nu n}{\xi}\Big] \subseteq \Big[0, \frac{1}{M}\Big].$$

- **Case 2.2:** $k_1 \in [n+1; 2n+1]$**.** Now the $\widetilde{\mathcal{P}}^{(i)}_{(1)}$-block also contributes. Splitting the sum at n gives

$$\sum_{k=1}^{k_1} \Big(\big(M\mathcal{R}^{(i)}_{(1)}\big)^{(j_1)}_{(1)}\Big)_k = \frac{1}{\xi}\left(\mu \sum_{k=1}^{n} \Big(\widetilde{B}_i\Big)^{(j_1)}_{(1)}{}_k + \sum_{k=1}^{k_1-n} \Big(\widetilde{\mathcal{P}}^{(i)}_{(1)}\Big)^{(j_1)}_{(1)}{}_k + \nu k_1\right).$$

  By the definition of μ in (5.1), the middle term is bounded below by $-\mu$, hence the expression lies in $[0, (\mu + \nu M)/\xi] = [0, 1/M]$.
- **Case 2.3:** $k_1 = 2n+1$**.** Using the definition of $\widetilde{c}_i$ in (5.2), the $\widetilde{\mathcal{P}}$-contribution cancels with the c-entry, and we obtain

$$\sum_{k=1}^{2n+1} \Big(\big(M\mathcal{R}^{(i)}_{(1)}\big)^{(j_1)}_{(1)}\Big)_k = \frac{\mu + \nu M}{\xi} = \frac{1}{M}.$$

Case 3: $j_1 = 2n+1$**.** All entries in the corresponding section are nonnegative, so it suffices to take $k_1 = 2n+1$, and then

$$\sum_{k=1}^{2n+1} \Big(\big(M\mathcal{R}^{(i)}_{(1)}\big)^{(2n+1)}_{(1)} \Big)_k = \frac{\mu + \nu M}{\xi} = \frac{1}{M}.$$

This completes the verification of (5.4), and hence $M\mathcal{R}^{(i)}_{(1)}$ is quasi-copula type. □

Remark 10. Theorem 9 extends directly to rectangular arrays $\mathcal{P} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ with n_1, n_2, n_3 not necessarily equal. Indeed, letting $n := \max\{n_1, n_2, n_3\}$, one may embed $\mathcal{P}$ into an array $\widehat{\mathcal{P}} \in \mathbb{R}^{n \times n \times n}$ by padding with zeros outside the original index set, and then apply Theorem 9 to $\widehat{\mathcal{P}}$.

Example 11. Let $\mathcal{P} \in \mathbb{R}^{2\times 2\times 2}$ be given via its mode-1 sections:

$$\mathcal{P}^{(1)}_{(1)} = \begin{bmatrix} -1 & 0 \\ -2 & -3 \end{bmatrix}, \qquad \mathcal{P}^{(2)}_{(1)} = \begin{bmatrix} -2 & -1 \\ -3 & -1 \end{bmatrix}.$$

Consequently, the mode-2 and mode-3 sections are

$$\mathcal{P}^{(1)}_{(2)} = \begin{bmatrix} -1 & -2 \\ -2 & -3 \end{bmatrix}, \qquad \mathcal{P}^{(2)}_{(2)} = \begin{bmatrix} 0 & -1 \\ -3 & -1 \end{bmatrix},$$

$$\mathcal{P}^{(1)}_{(3)} = \begin{bmatrix} -3 & -1 \\ -2 & -3 \end{bmatrix}, \qquad \mathcal{P}^{(2)}_{(3)} = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}.$$

We compute

$$\nu := \max\Big(\max_{i,j,k} \mathcal{P}_{i,j,k},\, 0 \Big) = 0,$$

and hence $\widetilde{\mathcal{P}} = \mathcal{P}$. Since $\nu = 0$, the construction yields $\mathcal{R} = \widetilde{\mathcal{R}} \in \mathbb{R}^{5\times 5\times 5}$. Moreover,

$$\sum \mathcal{P}^{(1)}_{(1)} = -6, \qquad \sum \mathcal{P}^{(2)}_{(1)} = -7,$$

so that $\mu = \max(6, 7) = 7$ (cf. (5.1)).

We choose $\widetilde{A}_i = \widetilde{B}_i = I_2$ for $i = 1, 2$ and $\widetilde{\mathcal{F}}_i = I_5$ for $i = 3, 4, 5$, where I_m denotes the $m \times m$ identity matrix. Finally,

$$(2n+1)^2\big(\mu + \nu(2n+1)\big) = 25 \cdot 7 = 175.$$

The mode-1 sections of $\mathcal{R}$ are therefore

$$\mathcal{R}^{(1)}_{(1)} = \frac{1}{175} \begin{bmatrix} 0 & 0 & 3 & 3 & 1 \\ 0 & 7 & -1 & 0 & 1 \\ 7 & 0 & -2 & -3 & 5 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 7 & 0 & 0 \end{bmatrix}, \qquad \mathcal{R}^{(2)}_{(1)} = \frac{1}{175} \begin{bmatrix} 0 & 0 & 5 & 2 & 0 \\ 0 & 7 & -2 & -1 & 3 \\ 7 & 0 & -3 & -1 & 4 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 7 & 0 & 0 \end{bmatrix},$$

$$\mathcal{R}^{(i)}_{(1)} = \frac{7}{175}\, I_5 \qquad \text{for } i \in \{3, 4, 5\}.$$

6. From three-dimensional arrays to quasi-copula type arrays

In the previous section, we constructed *strong* quasi-copula type arrays, meaning arrays whose induced volumes satisfy the two-axis positivity constraints of Theorem 4. In particular, this class includes the arrays produced by the stacking paradigm of Theorem 6, where higher-dimensional examples are assembled by stacking rescaled lower-dimensional quasi-copula type slices along a distinguished mode. Although this yields a convenient and readily verifiable subclass, it is inherently restrictive: the resulting dependence structure is imposed slice by slice and is therefore largely inherited from the lower-dimensional building blocks.

The purpose of this section is to move beyond the stacked setting and address the three-dimensional *extension problem* under the *minimal* quasi-copula type requirements specified in Theorem 3. Our second main result, Theorem 12, demonstrates that any array $\mathcal{P} \in \mathbb{R}^{n\times n\times n}$

can be embedded (up to scaling) into a larger tensor $\mathcal{T} \in \mathbb{R}^{(3n+2)^3}$ that induces a quasi-copula, without requiring the stronger positivity conditions. As a result, the construction developed here produces a strictly broader class of quasi-copula type arrays than those obtainable by stacking lower-dimensional quasi-copulas: it accommodates prescribed subtensors without assuming any *a priori* slice-wise quasi-copula structure.

Theorem 12. *Let $n \in \mathbb{N}$ and let*

$$\mathcal{P} \equiv \big[\mathcal{P}_{i,j,k}\big]_{i,j,k} \in \mathbb{R}^{n\times n\times n}$$

be a given 3-dimensional array. Then there exists a 3-dimensional quasi-copula type array

$$\mathcal{T} \in \mathbb{R}^{(3n+2)\times(3n+2)\times(3n+2)}$$

that contains a multiple of $\mathcal{P}$ as a subarray at the positions

$$[n+2; 2n+1] \times [2; n+1] \times [n+2; 2n+1].$$

Construction of $\mathcal{T}$. *Set*

$$\nu := \max\Big(\max_{i,j,k\in[n]} \mathcal{P}_{i,j,k},\, 0\Big), \qquad \widetilde{\mathcal{P}} := \mathcal{P} - \nu\,\mathbb{L}^{(n,n,n)}.$$

Let $N := 3n+2$ and define

$$\mu := \max_{i,k\in[n]}\Big(-n\sum \widetilde{\mathcal{P}}^{(i,k)}_{(1,3)}\Big).$$

For $i \in [N]$ define the mode-1 sections of $\widetilde{\mathcal{T}}$ by

$$\widetilde{\mathcal{T}}^{(i)}_{(1)} := \begin{cases} \mu\displaystyle\sum_{j=1}^{N} \mathbb{E}^{(N,N)}_{j,j}, & \text{if } i = 1,\\ \mu\Big(\mathbb{E}^{(N,N)}_{i,1} + \mathbb{E}^{(N,N)}_{1,i} + 3n\,\mathbb{E}^{(N,N)}_{i,i}\Big), & \text{if } i \in [2; n+1]\cup\{N\},\\ \begin{array}{c} \\ (N-i-n)R\\ 1R\\ (i-n-2)R\\ nR\\ nR\\ 1R \end{array} \begin{array}{c} \begin{array}{ccccc} 1C & nC & (i-n-2)C & 1C & (N-i)C \end{array}\\ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \widetilde{c}_i & \mathbf{0} & \widetilde{\beta}_i & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \widetilde{v} & \widetilde{\mathcal{P}}^{(i)}_{(1)} & \mathbf{0} & \widetilde{d}_i & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \widetilde{u}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \end{array}, & \text{if } i \in [n+2; 2n+1],\\ \mu\Big(\mathbb{E}^{(N,N)}_{1,i} + \mathbb{E}^{(N,N)}_{i,1} + 3n\,\mathbb{E}^{(N,N)}_{i,i-n}\Big), & \text{if } i \in [2n+2; 3n+1], \end{cases}$$

where C and R denote columns and rows, respectively, and

$$\begin{aligned} \widetilde{c}_i &:= \begin{bmatrix} -\sum \widetilde{\mathcal{P}}^{(i,1)}_{(1,2)} & -\sum \widetilde{\mathcal{P}}^{(i,2)}_{(1,2)} & \cdots & -\sum \widetilde{\mathcal{P}}^{(i,n)}_{(1,2)} \end{bmatrix} \in \mathbb{R}^{1\times n},\\ \widetilde{d}_i &:= \begin{bmatrix} -\sum \widetilde{\mathcal{P}}^{(i,1)}_{(1,3)} & -\sum \widetilde{\mathcal{P}}^{(i,2)}_{(1,3)} & \cdots & -\sum \widetilde{\mathcal{P}}^{(i,n)}_{(1,3)} \end{bmatrix}^T \in \mathbb{R}^{n\times 1},\\ \widetilde{v} &:= \begin{bmatrix} \frac{\mu}{n} & \cdots & \frac{\mu}{n} \end{bmatrix}^T \in \mathbb{R}^{n\times 1},\\ \widetilde{u}_i &:= [\underbrace{0,\ldots,0}_{i-1},\, \mu,\, \underbrace{0,\ldots,0}_{n-i}]^T \in \mathbb{R}^{n\times 1},\\ \widetilde{\beta}_i &:= 3n\mu + \sum \widetilde{\mathcal{P}}^{(i)}_{(1)} \in \mathbb{R}. \end{aligned}$$

Finally, define

$$\mathcal{T} := \frac{1}{N^2(\mu+\nu N)}\Big(\widetilde{\mathcal{T}} + \nu\mathbb{L}^{(N,N,N)}\Big).$$

Proof. We prove that $\mathcal{T}$ satisfies (3.2) and (3.3). By Theorem 3, it then follows that $\mathcal{T}$ is a quasi-copula type array.

Step 1: Mode-1 sections $\mathcal{T}_{(1)}^{(i)}$ satisfy (3.2) and (3.3). We show that for every $i \in [N]$,

$$\sum \mathcal{T}_{(1)}^{(i)} = \frac{1}{N} \qquad \text{and} \qquad 0 \le \sum_{j=1}^{j_1}\sum_{k=1}^{k_1} \big(\mathcal{T}_{(1)}^{(i)}\big)_{j,k} \le \frac{1}{N} \quad \text{for all } j_1, k_1 \in [N]. \tag{6.1}$$

We distinguish four cases according to i:

$$i = 1, \qquad i \in [2; n+1] \cup \{N\}, \qquad i \in [n+2; 2n+1], \qquad i \in [2n+2; 3n+1].$$

Case 1: $i = 1$. Since $\mathcal{T}_{(1)}^{(1)}$ has only nonnegative entries, it suffices to verify the equality in (6.1). Indeed,

$$\begin{aligned}\sum \mathcal{T}_{(1)}^{(1)} &= \frac{1}{N^2(\mu+\nu N)}\Big(\mu \sum \Big(\sum_{j=1}^{3n+2} \mathbb{E}_{j,j}^{(N,N)}\Big) + \nu \sum \big(\mathbb{L}^{(N)}\big)_{(1)}^{(1)}\Big) \\ &= \frac{1}{N^2(\mu+\nu N)}(\mu N + \nu N^2) = \frac{1}{N}.\end{aligned}$$

Case 2: $i \in [2; n+1] \cup \{N\}$. As in Case 1, $\mathcal{T}_{(1)}^{(i)}$ is entrywise nonnegative, so it suffices to verify the equality in (6.1). We compute

$$\begin{aligned}\sum \mathcal{T}_{(1)}^{(i)} &= \frac{1}{N^2(\mu+\nu N)}\Big(\mu \sum \big(\mathbb{E}_{i,1}^{(N,N)} + \mathbb{E}_{1,i}^{(N,N)} + 3n\,\mathbb{E}_{i,i}^{(N,N)}\big) + \nu \sum \big(\mathbb{L}^{(N)}\big)_{(1)}^{(i)}\Big) \\ &= \frac{1}{N^2(\mu+\nu N)}(\mu N + \nu N^2) = \frac{1}{N}.\end{aligned}$$

Case 3: $i \in [n+2; 2n+1]$. Set

$$\zeta := N^2(\mu + \nu N).$$

We first verify the equality in (6.1). Using the block structure, we obtain

$$\begin{aligned}\sum \mathcal{T}_{(1)}^{(i)} &= \frac{1}{\zeta}\Big(\sum \widetilde{v} + \sum \widetilde{c}_i + \sum \widetilde{\mathcal{P}}_{(1)}^{(i)} + \sum \widetilde{u}_i + \widetilde{\beta}_i + \sum \widetilde{d}_i + \nu \sum \big(\mathbb{L}^{(N)}\big)_{(1)}^{(i)}\Big) \\ &= \frac{1}{\zeta}\big(\mu N + \nu N^2\big) = \frac{1}{N}.\end{aligned}$$

It remains to establish the partial-sum bounds in (6.1). Fix $j_1, k_1 \in [N]$ and distinguish five cases according to j_1:

$$j_1 = 1, \qquad j_1 \in [2; n+1], \qquad j_1 \in [n+2; i-1], \qquad j_1 = i, \qquad j_1 \in [i+1; N].$$

Case 3.1: $j_1 = 1$. Since the entries of the fiber $\big(\mathcal{T}_{(1)}^{(i)}\big)_{(2)}^{(1)}$ are nonnegative, it suffices to take $k_1 = N$ and verify

$$\sum_{k=1}^{N} \big(\mathcal{T}_{(1)}^{(i)}\big)_{1,k} = \frac{1}{\zeta}\Big(\sum \widetilde{v} + \nu \sum \big((\mathbb{L}^{(N)})_{(1)}^{(i)}\big)_{1,k}\Big) = \frac{1}{\zeta}(\mu + \nu N) = \frac{1}{N^2} < \frac{1}{N}.$$

Case 3.2: $j_1 \in [2; n+1]$. We further distinguish five cases according to k_1:

$$k_1 \in [n+1], \quad k_1 \in [n+2; 2n+1], \quad k_1 \in [2n+2; i+n-1], \quad k_1 = i+n, \quad k_1 \in [i+n+1; N].$$

Case 3.2.1: $k_1 \in [n+1]$. All entries in the rectangle $[j_1] \times [k_1]$ are nonnegative, so it suffices to take $k_1 = n+1$ and estimate

$$\begin{aligned}\sum_{j=1}^{j_1}\sum_{k=1}^{n+1} \big(\mathcal{T}_{(1)}^{(i)}\big)_{j,k} &\le \frac{1}{\zeta}\Big(\sum \widetilde{u}_i + \nu \sum_{j=1}^{j_1}\sum_{k=1}^{n+1} \big((\mathbb{L}^{(N)})_{(1)}^{(i)}\big)_{j,k}\Big) \\ &= \frac{1}{\zeta}\big(\mu + \nu j_1(n+1)\big) < \frac{1}{\zeta}(\mu + \nu N^2) < \frac{1}{N}.\end{aligned}$$

Case 3.2.2: $k_1 \in [n+2; 2n+1]$**.** We have

$$\sum_{j=1}^{j_1}\sum_{k=1}^{k_1}\big(\mathcal{T}^{(i)}_{(1)}\big)_{j,k} = \frac{1}{\zeta}\Big((k_1-n-1)\frac{\mu}{n} + \sum_{k=1}^{k_1-n-1}\sum_{j=1}^{j_1-1}\big(\widetilde{\mathcal{P}}^{(i)}_{(1)}\big)_{j,k} + \sum_{j=1}^{j_1-1}(\widetilde{u}_i)_j + \nu j_1 k_1\Big).$$

By the definition of μ, we have $\sum_{j=1}^{j_1-1}(\widetilde{\mathcal{P}}^{(i)}_{(1)})_{j,k} \geq -\mu/n$, hence

$$\sum_{j=1}^{j_1}\sum_{k=1}^{k_1}\big(\mathcal{T}^{(i)}_{(1)}\big)_{j,k} \geq \frac{1}{\zeta}\Big((k_1-n-1)\frac{\mu}{n} - (k_1-n-1)\frac{\mu}{n}\Big) = 0,$$

which proves the left inequality in (6.1). Moreover,

$$\sum_{j=1}^{j_1}\sum_{k=1}^{k_1}\big(\mathcal{T}^{(i)}_{(1)}\big)_{j,k} < \frac{1}{\zeta}(\mu+\mu+\nu N^2) < \frac{1}{\zeta}(N\mu + N^2\nu) = \frac{1}{N},$$

which gives the right inequality.

Case 3.2.3: $k_1 \in [2n+2; i+n-1]$**.** Since $\widetilde{\mathcal{T}}^{(i,k)}_{(1,3)}$ is the zero array for each $k \in [2n+2; i+n-1]$, this case reduces to Case 3.2.2.

Case 3.2.4: $k_1 = i+n$**.** Using the definition of $\widetilde{c}_i$ to cancel the $\widetilde{\mathcal{P}}$-contribution, we obtain

$$\sum_{j=1}^{j_1}\sum_{k=1}^{i+n}\big(\mathcal{T}^{(i)}_{(1)}\big)_{j,k} = \frac{1}{\zeta}\Big(\sum\widetilde{v} + \sum_{j=1}^{j_1-1}(\widetilde{u}_i)_j + \nu j_1(i+n)\Big),$$

and hence

$$0 \leq \sum_{j=1}^{j_1}\sum_{k=1}^{i+n}\big(\mathcal{T}^{(i)}_{(1)}\big)_{j,k} < \frac{1}{\zeta}(\mu+\mu+\nu N^2) < \frac{1}{N}.$$

Case 3.2.5: $k_1 \in [i+n+1; N]$**.** Since $\widetilde{\mathcal{T}}^{(i,k)}_{(1,3)}$ is the zero array for each $k \in [i+n+1; N]$, this case reduces to Case 3.2.4.

Case 3.3: $j_1 \in [n+2; i-1]$**.** Since $\widetilde{\mathcal{T}}^{(i,j)}_{(1,2)}$ is the zero array for each $j \in [n+2; i-1]$, this case follows from Case 3.2.

Case 3.4: $j_1 = i$**.** The fiber $\widetilde{\mathcal{T}}^{(i,i)}_{(1,2)}$ is entrywise nonnegative, so the lower bound follows from Case 3.3. For the upper bound it suffices to take $k_1 = N$, and then

$$\sum_{j=1}^{i}\sum_{k=1}^{N}\big(\mathcal{T}^{(i)}_{(1)}\big)_{j,k} = \sum_{j=1}^{N}\sum_{k=1}^{N}\big(\mathcal{T}^{(i)}_{(1)}\big)_{j,k} = \sum\mathcal{T}^{(i)}_{(1)} = \frac{1}{N},$$

using that $(\mathcal{T}^{(i)}_{(1)})_{j,k} = 0$ for $j > i$.

Case 3.5: $j_1 \in [i+1; N]$**.** Since $\widetilde{\mathcal{T}}^{(i,j_1)}_{(1,2)}$ is the zero array for $j_1 > i$, this case reduces to Case 3.4.

Case 4: $i \in [2n+2; 3n+1]$**.** This case is analogous to Case 2, with the term $3n\,\mathbb{E}^{(N,N)}_{i,i}$ replaced by $3n\,\mathbb{E}^{(N,N)}_{i,i-n}$ in the computation of $\sum\mathcal{T}^{(i)}_{(1)}$.

Step 2: Mode-2 sections $\mathcal{T}^{(j)}_{(2)}$ satisfy (3.2) **and** (3.3)**.** We first recover the explicit form of the slices $\mathcal{T}^{(j)}_{(2)}$ for $j \in [M]$. Since

$$\mathcal{T}^{(j)}_{(2)} = \frac{1}{\zeta}\Big(\widetilde{\mathcal{T}}^{(j)}_{(2)} + \nu\big(\mathbb{L}^{(N)}\big)^{(j)}_{(2)}\Big),$$

it is sufficient to determine the slices $\widetilde{\mathcal{T}}^{(j)}_{(2)}$ of $\widetilde{\mathcal{T}}$.

Claim 1. For each $j \in [M]$, the slice $\widetilde{\mathcal{T}}^{(j)}_{(2)}$ equals

$$\begin{cases} \mu \sum_{i=1}^{n+1} \mathbb{E}^{(N,N)}_{i,i} + \frac{\mu}{n} \sum_{i=n+2}^{2n+1} \sum_{k=1}^{n} \mathbb{E}^{(N,N)}_{i,n+1+k} + \mu \sum_{i=2n+2}^{N} \mathbb{E}^{(N,N)}_{i,i}, & \text{if } j = 1, \\ \begin{array}{c} \\ 1\mathrm{R} \\ n\mathrm{R} \\ n\mathrm{R} \\ (n+1-j)\mathrm{R} \\ 1\mathrm{R} \\ (j-1)\mathrm{R} \\ 1\mathrm{R} \end{array} \begin{array}{c} \begin{array}{cccccc} 1\mathrm{C} & (j-2)\mathrm{C} & 1\mathrm{C} & (n+1-j)\mathrm{C} & n\mathrm{C} & (N-2n-1)\mathrm{C} \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{C}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathcal{P}}^{(j-1)}_{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mu & \mathbf{0} & 3n\mu & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mu & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}, \\ & \qquad \text{if } j \in [2, n+1], \\ \mu\big(\mathbb{E}^{(N,N)}_{1,j} + \mathbb{E}^{(N,N)}_{j,1}\big) + \Big(3n\mu + \sum \widetilde{\mathcal{P}}^{(j-n-1)}_{(1)}\Big)\mathbb{E}^{(N,N)}_{j,j+n} + \sum_{k=1}^{n} \Big(\sum \widetilde{\mathcal{P}}^{(j-n-1,k)}_{(1,3)}\Big)\mathbb{E}^{(N,N)}_{j,n+1+k}, \\ & \qquad \text{if } j \in [n+2, 2n+1], \\ \mu\big(\mathbb{E}^{(N,N)}_{1,j} + \mathbb{E}^{(N,N)}_{j,1} + 3n\,\mathbb{E}^{(N,N)}_{j,j-n}\big), \qquad \text{if } j \in [2n+2, 3n+1]. \end{cases}$$

Here the symbols C and R indicate columns and rows, respectively, and

$$\widetilde{C}_j := -\sum_{i=1}^{n} \Big(\sum \widetilde{\mathcal{P}}^{(i,j-1)}_{(1,2)}\Big)\mathbb{E}^{(n,n)}_{i,i}.$$

Proof of Claim 1. Recall that $\widetilde{\mathcal{T}}$ admits the expansion

$$\begin{aligned} \widetilde{\mathcal{T}} = \mu \sum_{j=1}^{3n+2} \mathbb{E}^{(N)}_{1,j,j} + \sum_{i=2}^{n+1} \mu\big(\mathbb{E}^{(N)}_{i,i,1} + \mathbb{E}^{(N)}_{i,1,i} + 3n\mathbb{E}^{(N)}_{i,i,i}\big) \\ + \sum_{i=n+2}^{2n+1} \Big(\mu\mathbb{E}^{(N)}_{i,i,1} + \frac{\mu}{n}\sum_{j=1}^{n} \mathbb{E}^{(N)}_{i,1,n+1+j} + \sum_{j,k=1}^{n} (\widetilde{\mathcal{P}}^{(i-n-1)}_{(1)})_{j,k}\mathbb{E}^{(N)}_{i,1+j,n+1+k} \\ - \sum_{k=1}^{n} \Big(\sum \widetilde{\mathcal{P}}^{(i-n-1,k)}_{(1,3)}\Big)\mathbb{E}^{(N)}_{i,i,n+1+k} - \sum_{j=1}^{n} \Big(\sum \widetilde{\mathcal{P}}^{(i-n-1,j)}_{(1,2)}\Big)\mathbb{E}^{(N)}_{i,1+j,n+i} \\ + \Big(3n\mu + \sum \widetilde{\mathcal{P}}^{(i-n-1)}_{(1)}\Big)\mathbb{E}^{(N)}_{i,i,n+i}\Big) \\ + \sum_{i=2n+2}^{3n+1} \mu\big(\mathbb{E}^{(N)}_{i,i,1} + \mathbb{E}^{(N)}_{i,1,i} + 3n\mathbb{E}^{(N)}_{i,i,i-n}\big) \\ + \mu\big(\mathbb{E}^{(N)}_{3n+2,3n+2,1} + \mathbb{E}^{(N)}_{3n+2,1,3n+2} + 3n\mathbb{E}^{(N)}_{3n+2,3n+2,3n+2}\big). \end{aligned}$$

Reordering the tensor $\widetilde{\mathcal{T}}$ according to increasing values of the second index yields the alternative representation

$$\begin{aligned} \widetilde{\mathcal{T}} = \mu\Big(\sum_{i=1}^{n+1} \mathbb{E}^{(N)}_{i,1,i} + \frac{1}{n}\sum_{i=n+2}^{2n+1}\sum_{k=1}^{n} \mathbb{E}^{(N)}_{i,1,n+1+k} + \sum_{i=2n+2}^{3n+2} \mathbb{E}^{(N)}_{i,1,i}\Big) \\ + \sum_{j=2}^{n+1} \Big(\mu(\mathbb{E}^{(N)}_{1,j,j} + \mathbb{E}^{(N)}_{j,j,1} + 3n\mathbb{E}^{(N)}_{j,j,j}) \\ + \sum_{i=n+2}^{2n+1}\sum_{k=1}^{n} (\widetilde{\mathcal{P}}^{(i-n-1)}_{(1)})_{j-1,k}\mathbb{E}^{(N)}_{i,j,n+1+k} - \sum_{i=n+2}^{2n+1} \Big(\sum \widetilde{\mathcal{P}}^{(i-n-1,j-1)}_{(1,2)}\Big)\mathbb{E}^{(N)}_{i,j,n+i}\Big) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=n+2}^{2n+1}\Big(\mu(\mathbb{E}^{(N)}_{j,j,1}+\mathbb{E}^{(N)}_{1,j,j})-\sum_{k=1}^{n}\Big(\sum\widetilde{\mathcal{P}}^{(j-n-1,k)}_{(1,3)}\Big)\mathbb{E}^{(N)}_{j,j,n+1+k}\\
&\quad+\Big(3n\mu+\sum\widetilde{\mathcal{P}}^{(j-n-1)}_{(1)}\Big)\mathbb{E}^{(N)}_{j,j,j+n}\Big)\\
&+\sum_{j=2n+2}^{3n+1}\mu\big(\mathbb{E}^{(N)}_{1,j,j}+\mathbb{E}^{(N)}_{j,j,1}+3n\mathbb{E}^{(N)}_{j,j,j-n}\big)\\
&+\mu\big(\mathbb{E}^{(N)}_{1,3n+2,3n+2}+\mathbb{E}^{(N)}_{3n+2,3n+2,1}+3n\mathbb{E}^{(N)}_{3n+2,3n+2,3n+2}\big).
\end{aligned}
$$

Extracting the slices $\widetilde{\mathcal{T}}^{(j)}_{(2)}$ from this expression proves the claim. ■

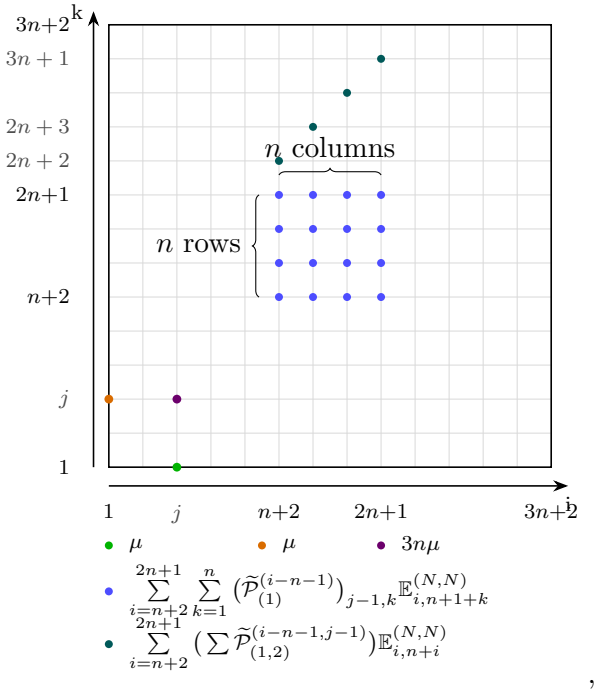
An argument completely analogous to that used in Step 1 shows that, for every $j,i_1,k_1\in[N]$,

$$\sum\mathcal{T}^{(j)}_{(2)}=\frac{1}{N},\qquad 0\le\sum_{i=1}^{i_1}\sum_{k=1}^{k_1}(\mathcal{T}^{(j)}_{(2)})_{i,k}\le\frac{1}{N}.$$

For the convenience of the reader, we conclude by illustrating the structure of the slices graphically. Let 1_m denote the $m\times m$ matrix with all entries equal to 1, and let J_m denote the $m\times m$ matrix with ones on the main antidiagonal. Then

$$\widetilde{\mathcal{T}}^{(1)}_{(2)}=\begin{bmatrix}&&\mu J_{n+1}\\&\frac{\mu}{n}1_n&\\ \mu J_{n+1}&&\end{bmatrix}.$$

Slices $\widetilde{\mathcal{T}}^{(j)}_{(2)}$ for $j\in[2;n+1]$ are equal to



,

Slices $\widetilde{\mathcal{T}}^{(j)}_{(2)}$ for $j\in[n+2;2n+1]$ are equal to

$$\begin{array}{c} \\ 1 \\ n+j \\ 2n+1 \\ \vdots \\ j \\ \vdots \\ n+2 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & j & 3n+2 \end{array} \\ \begin{pmatrix} & & \\ & 3n\mu+\sum\widetilde{\mathcal{P}}_{(1)}^{(j-n-1)} & \\ & -\sum\widetilde{\mathcal{P}}_{(1,2)}^{(n,j)} & \\ & \vdots & \\ \mu & -\sum\widetilde{\mathcal{P}}_{(1,2)}^{(j-n-1,j)} & \\ & \vdots & \\ & -\sum\widetilde{\mathcal{P}}_{(1,2)}^{(1,j)} & \\ & \mu & \end{pmatrix} \end{array}$$

Finally,

$$\widetilde{\mathcal{T}}_{(2)}^{(j)} = \begin{array}{c} \\ 3n+2 \\ j \\ j-n \\ j \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & j & 3n+2 \end{array} \\ \begin{pmatrix} & & \\ \mu & & \\ & 3n\mu & \\ & \mu & \end{pmatrix} \end{array} \text{ for } j\in[2n+2;3n+1], \qquad \widetilde{\mathcal{T}}_{(2)}^{(N)} = \begin{array}{c} \\ 3n+2 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 3n+2 \end{array} \\ \begin{pmatrix} \mu & 3n\mu \\ & \mu \end{pmatrix} \end{array}$$

Step 3: Mode-3 sections $\mathcal{T}_{(3)}^{(k)}$ satisfy (3.2) and (3.3). We first recover the explicit form of the slices $\mathcal{T}_{(3)}^{(k)}$ for $k\in[M]$. Since

$$\mathcal{T}_{(3)}^{(k)} = \frac{1}{\zeta}\Big(\widetilde{\mathcal{T}}_{(3)}^{(k)} + \nu\big(\mathbb{L}^{(N)}\big)_{(3)}^{(k)}\Big),$$

it suffices to determine the slices $\widetilde{\mathcal{T}}_{(3)}^{(k)}$ of $\widetilde{\mathcal{T}}$.

Claim 2. For each $k\in[M]$, the slice $\widetilde{\mathcal{T}}_{(3)}^{(k)}$ is given by

$$\begin{cases} \mu\displaystyle\sum_{i=1}^{n+1}\mathbb{E}_{i,i}^{(N,N)}, & \text{if } k=1, \\ \mu\Big(\mathbb{E}_{1,k}^{(N,N)}+\mathbb{E}_{k,1}^{(N,N)}+3n\,\mathbb{E}_{k,k}^{(N,N)}\Big), & \text{if } k\in[2,n+1], \\ \begin{array}{c} \\ (N-k)\mathrm{R} \\ 1\mathrm{R} \\ (k-n-2)\mathrm{R} \\ n\mathrm{R} \\ (n+1)\mathrm{R} \\ 1\mathrm{R} \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 1\mathrm{C} & n\mathrm{C} & n\mathrm{C} & (k-n-2)\mathrm{C} & 1\mathrm{C} & (N-n-k)\mathrm{C} \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 3n\mu & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \widetilde{v} & \widetilde{\mathcal{P}}_{(3)}^{(k-n-1)} & \widetilde{D}_k & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \widetilde{u}_{k-n-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}, & \\ & \text{if } k\in[n+2,2n+1], \\ \mu\big(\mathbb{E}_{1,k}^{(N,N)}+\mathbb{E}_{k,1}^{(N,N)}\big) - \displaystyle\sum_{j=1}^{n}\Big(\sum\widetilde{\mathcal{P}}_{(1,2)}^{(k-2n-1,j)}\Big)\mathbb{E}_{k-n,1+j}^{(N,N)} + \Big(3n\mu+\sum\widetilde{\mathcal{P}}_{(1)}^{(k-2n-1)}\Big)\mathbb{E}_{k-n,k-n}^{(N,N)}, & \\ & \text{if } k\in[2n+2,3n+1], \\ \mu\big(\mathbb{E}_{N,1}^{(N,N)}+\mathbb{E}_{1,N}^{(N,N)}+3n\,\mathbb{E}_{N,N}^{(N,N)}\big), & \text{if } k=N. \end{cases}$$

Here C and R indicate columns and rows, respectively, and

$$\widetilde{v} = \begin{bmatrix} \frac{\mu}{n} \\ \vdots \\ \frac{\mu}{n} \end{bmatrix} \in \mathbb{R}^n, \qquad \widetilde{u}_i = [\underbrace{0,\dots,0}_{i-1},\,\mu,\,\underbrace{0,\dots,0}_{n-i}], \qquad \widetilde{D}_k := -\sum_{i=1}^{n}\Big(\sum\widetilde{\mathcal{P}}_{(1,3)}^{(i,k-n-1)}\Big)\mathbb{E}_{i,i}^{(n,n)}.$$

Proof of Claim 2. Recall the expression for $\widetilde{\mathcal{T}}$ from the proof of Claim 1. Reordering $\widetilde{\mathcal{T}}$ according to increasing values of the third index yields the following representation:

$$\begin{aligned}
\widetilde{\mathcal{T}} = \mu \sum_{i=1}^{N} \mathbb{E}^{(N)}_{i,i,1} + \sum_{k=2}^{n+1} \mu\Big(\mathbb{E}^{(N)}_{1,k,k} + \mathbb{E}^{(N)}_{k,1,k} + 3n\mathbb{E}^{(N)}_{k,k,k}\Big) \\
+ \sum_{k=n+2}^{2n+1} \Big(\mu\mathbb{E}^{(N)}_{1,k,k} + \frac{\mu}{n} \sum_{i=n+2}^{2n+1} \mathbb{E}^{(N)}_{i,1,k} + \sum_{i=n+2}^{2n+1} \sum_{j=1}^{n} \big(\sum \widetilde{\mathcal{P}}^{(i-n-1)}_{(1)}\big)_{j,k-n-1} \mathbb{E}^{(N)}_{i,1+j,k} \\
- \sum_{i=n+2}^{2n+1} \big(\sum \widetilde{\mathcal{P}}^{(i-n-1,k-n-1)}_{(1,3)}\big)\mathbb{E}^{(N)}_{i,i,k} + 3n\mu\mathbb{E}^{(N)}_{k+n,k+n,k}\Big) \\
+ \sum_{k=2n+2}^{3n+1} \Big(\mu\big(\mathbb{E}^{(N)}_{1,k,k} + \mathbb{E}^{(N)}_{k,1,k}\big) - \sum_{j=1}^{n} \big(\sum \widetilde{\mathcal{P}}^{(k-2n-1,j)}_{(1,2)}\big)\mathbb{E}^{(N)}_{k-n,1+j,k} \\
+ \Big(3n\mu + \sum \widetilde{\mathcal{P}}^{(k-2n-1)}_{(1)}\Big)\mathbb{E}^{(N)}_{k-n,k-n,k}\Big) \\
+ \mu\Big(\mathbb{E}^{(N)}_{3n+2,1,3n+2} + \mathbb{E}^{(N)}_{1,3n+2,3n+2} + 3n\mathbb{E}^{(N)}_{3n+2,3n+2,3n+2}\Big).
\end{aligned}$$

Extracting the slices $\widetilde{\mathcal{T}}^{(k)}_{(3)}$ from this expression proves the claim. ■

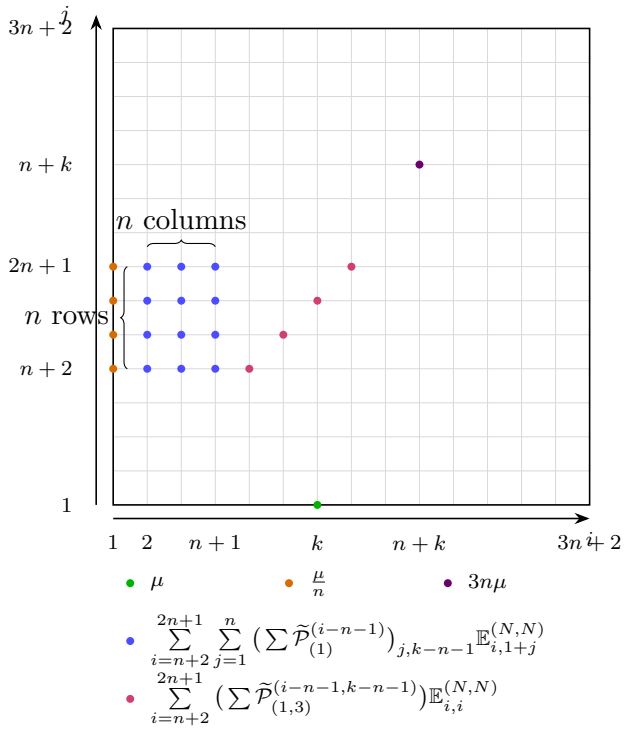
An argument analogous to that in Step 1 shows that for each $i_1, j_1, k \in [N]$,

$$\sum \mathcal{T}^{(k)}_{(3)} = \frac{1}{N}, \qquad 0 \le \sum_{i=1}^{i_1} \sum_{j=1}^{j_1} \big(\mathcal{T}^{(k)}_{(3)}\big)_{i,j} \le \frac{1}{N}.$$

For the reader's convenience, we illustrate the slices graphically. Let J_m denote the $m \times m$ matrix with ones on the main antidiagonal. Then

$$\widetilde{\mathcal{T}}^{(1)}_{(3)} = \mu J_{3n+2}, \qquad \widetilde{\mathcal{T}}^{(k)}_{(3)} = \begin{array}{c} \\ 3n+2 \\ k \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & k & 3n+2 \end{array} \\ \begin{pmatrix} & & \\ \mu & 3n\mu & \\ & \mu & \end{pmatrix} \end{array} \quad \text{for } k \in [2, n+1].$$

Slices $\widetilde{\mathcal{T}}^{(k)}_{(3)}$ for $k \in [n+2, 2n+1]$ are equal to



Slices $\widetilde{\mathcal{T}}_{(3)}^{(k)}$ for $k \in [2n+2, 3n+1]$ are equal to

$$\begin{array}{c} \\ 3n+2 \\ k \\ k-n \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 1 & 2 & & n+1 & k-n & k \quad 3n+2 \end{array} \\ \begin{pmatrix} & & & & & \\ \mu & & & & & \\ & -\sum\widetilde{\mathcal{P}}_{(1,2)}^{(k-2n-1,1)} & \cdots & -\sum\widetilde{\mathcal{P}}_{(1,2)}^{(k-2n-1,n)} & 3n\mu+\sum\widetilde{\mathcal{P}}_{(1)}^{(k-2n-1)} & \\ & & & & & \mu \end{pmatrix} \end{array}.$$

Finally,

$$\widetilde{\mathcal{T}}_{(3)}^{(N)} = \begin{array}{c} \\ 3n+2 \\ \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 3n+2 \end{array} \\ \begin{pmatrix} \mu & 3n\mu \\ & \\ & \mu \end{pmatrix} \end{array}.$$

□

Example 13. Let $\mathcal{P} \in \mathbb{R}^{2\times 2\times 2}$ be as in Example 11. In particular, its mode-1 sections are

$$\mathcal{P}_{(1)}^{(1)} = \begin{bmatrix} -1 & 0 \\ -2 & -3 \end{bmatrix}, \qquad \mathcal{P}_{(1)}^{(2)} = \begin{bmatrix} -2 & -1 \\ -3 & -1 \end{bmatrix}.$$

Since $\nu = 0$, we obtain $\mathcal{T} = \widetilde{\mathcal{T}} \in \mathbb{R}^{8\times 8\times 8}$. Moreover,

$$\sum \mathcal{P}_{(1,3)}^{(1,1)} = (-2)+(-3) = -5, \qquad \sum \mathcal{P}_{(1,3)}^{(1,2)} = (-1)+0 = -1,$$
$$\sum \mathcal{P}_{(1,3)}^{(2,1)} = (-3)+(-1) = -4, \qquad \sum \mathcal{P}_{(1,3)}^{(2,2)} = (-2)+(-1) = -3,$$

and hence

$$\mu = -2\min\{-5,-1,-4,-3\} = 10.$$

Furthermore,

$$(3n+2)^2\big(\mu+\nu(3n+2)\big) = 8^2\cdot 10 = 640.$$

Let J_m denote the $m \times m$ matrix with ones on the main antidiagonal and zeros elsewhere. The mode-1 sections of $\mathcal{T}$ are as follows:

$$\mathcal{T}^{(1)}_{(1)} = \frac{1}{64} J_8, \qquad \mathcal{T}^{(2)}_{(1)} = \frac{1}{640} \cdot \begin{array}{c} \\ 8 \\ \\ 2 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 8 \end{array} \\ \begin{pmatrix} & & \\ & & \\ 10 & 60 & \\ & 10 & \end{pmatrix} \end{array}, \qquad \mathcal{T}^{(3)}_{(1)} = \frac{1}{640} \cdot \begin{array}{c} \\ 8 \\ \\ 3 \\ \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 3 & 8 \end{array} \\ \begin{pmatrix} & & \\ 10 & 60 & \\ & & \\ & 10 & \end{pmatrix} \end{array},$$

$$\mathcal{T}^{(4)}_{(1)} = \frac{1}{640} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 54 & 0 & 0 & 0 & 0 \\ 5 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 5 & -2 & -3 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{T}^{(5)}_{(1)} = \frac{1}{640} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 & 53 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & -2 & -1 & 0 & 3 & 0 & 0 & 0 \\ 5 & -3 & -1 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{T}^{(6)}_{(1)} = \frac{1}{640} \cdot \begin{array}{c} \\ 8 \\ 6 \\ 4 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 6 & 8 \end{array} \\ \begin{pmatrix} & & \\ 10 & & \\ & & 60 \\ & 10 & \end{pmatrix} \end{array}, \qquad \mathcal{T}^{(7)}_{(1)} = \frac{1}{640} \cdot \begin{array}{c} \\ 8 \\ 7 \\ 5 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 7 & 8 \end{array} \\ \begin{pmatrix} & & \\ 10 & & \\ & 60 & \\ & 10 & \end{pmatrix} \end{array}, \qquad \mathcal{T}^{(8)}_{(1)} = \frac{1}{640} \cdot \begin{array}{c} \\ 8 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 8 \end{array} \\ \begin{pmatrix} 10 & 60 \\ & 10 \end{pmatrix} \end{array},$$

The mode-2 sections of $\mathcal{T}$ are:

$$\mathcal{T}^{(1)}_{(2)} = \frac{1}{640} \begin{bmatrix} & & 10J_3 \\ & 51_2 & \\ 10J_3 & & \end{bmatrix},$$

$$\mathcal{T}^{(2)}_{(2)} = \frac{1}{640} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 60 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{T}^{(3)}_{(2)} = \frac{1}{640} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & -1 & 0 & 0 & 0 \\ 10 & 0 & 60 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{T}^{(4)}_{(2)} = \frac{1}{640} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 54 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{T}^{(5)}_{(2)} = \frac{1}{640} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 53 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{T}^{(6)}_{(2)} = \frac{1}{640} \cdot \begin{array}{c} \\ 8 \\ 6 \\ 4 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 6 & 8 \end{array} \\ \begin{pmatrix} & & \\ 10 & & \\ & & 60 \\ & 10 & \end{pmatrix} \end{array}, \qquad \mathcal{T}^{(7)}_{(2)} = \frac{1}{640} \cdot \begin{array}{c} \\ 8 \\ 7 \\ 5 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 7 & 8 \end{array} \\ \begin{pmatrix} & & \\ 10 & & \\ & 60 & \\ & 10 & \end{pmatrix} \end{array}, \qquad \mathcal{T}^{(8)}_{(2)} = \frac{1}{640} \cdot \begin{array}{c} \\ 8 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 8 \end{array} \\ \begin{pmatrix} 10 & 60 \\ & 10 \end{pmatrix} \end{array},$$

Finally, the mode-3 sections of $\mathcal{T}$ are:

$$\mathcal{T}^{(1)}_{(3)} = \frac{1}{64} J_8, \quad \mathcal{T}^{(2)}_{(3)} = \frac{1}{640} \cdot \begin{array}{c} \\ 8 \\ \\ 2 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} & \\ 10 & 60 \\ & 10 \end{pmatrix} \end{array} \begin{array}{c} 8 \end{array}, \quad \mathcal{T}^{(3)}_{(3)} = \frac{1}{640} \cdot \begin{array}{c} \\ 8 \\ \\ 3 \\ \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 3 & 8 \end{array} \\ \begin{pmatrix} & & \\ 10 & 60 & \\ & & 10 \end{pmatrix} \end{array}$$

$$\mathcal{T}^{(4)}_{(3)} = \frac{1}{640} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 60 & 0 & 0 \\ 5 & -3 & -1 & 0 & 4 & 0 & 0 & 0 \\ 5 & -2 & -3 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{T}^{(5)}_{(3)} = \frac{1}{640} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 60 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & -2 & -1 & 0 & 3 & 0 & 0 & 0 \\ 5 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{T}^{(6)}_{(3)} = \frac{1}{640} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 54 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 \end{bmatrix}, \quad \mathcal{T}^{(7)}_{(3)} = \frac{1}{640} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 & 53 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 \end{bmatrix},$$

$$\mathcal{T}^{(8)}_{(3)} = \frac{1}{640} \cdot \begin{array}{c} \\ 8 \\ \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 8 \end{array} \\ \begin{pmatrix} 10 & 60 \\ & \\ & 10 \end{pmatrix} \end{array}.$$

7. Extending multidimensional arrays to quasi-copula type arrays

This section generalizes the two extension constructions from the previous sections from dimension 3 to arbitrary order $k \geq 4$. More precisely, Theorem 9 provides a *strictly positive* quasi-copula type extension in three dimensions, whereas Theorem 12 yields an extension that is quasi-copula type without imposing the stronger positivity requirements. Theorem 14 shows that both constructions persist in every dimension.

Starting from an arbitrary k-array $\mathcal{P} \in \mathbb{R}^{n \times \cdots \times n}$, we construct larger arrays $\mathcal{R}$ and $\mathcal{T}$ of side lengths $2n+1$ and $3n+2$, respectively, which embed a prescribed scalar multiple of $\mathcal{P}$ as a designated subarray while enforcing the quasi-copula type conditions. The proof highlights the modular nature of the approach: viewing k-dimensional arrays as stacks of $(k-1)$-dimensional sections and applying Theorem 6 inductively, we iterate the three-dimensional building blocks to obtain valid extensions in higher dimensions.

Theorem 14. *Let $k, n \in \mathbb{N}$, and let*

$$\mathcal{P} \equiv \left[\mathcal{P}_{i_1,\ldots,i_k}\right]_{i_1,\ldots,i_k} \in \mathbb{R}^{n \times \cdots \times n}$$

be a given k-dimensional array. Then the following statements hold:

(1) There exists a k-dimensional strictly positive *quasi-copula type array*

$$\mathcal{R} \in \mathbb{R}^{(2n+1) \times \cdots \times (2n+1)}$$

which contains a scalar multiple of $\mathcal{P}$ as a subarray at the positions

$$[n] \times \prod_{i=2}^{3} [n+1; 2n] \times \prod_{i=4}^{k} [n].$$

(2) There exists a k-dimensional quasi-copula type array

$$\mathcal{T} \in \mathbb{R}^{(3n+2)\times\cdots\times(3n+2)}$$

which contains a scalar multiple of $\mathcal{P}$ as a subarray at the positions

$$[n+2;2n+1]\times[2;n+1]\times[n+2;2n+1]\times\prod_{i=4}^{k}[n].$$

Concrete realizations of $\mathcal{R}$ and $\mathcal{T}$ are given below. Set

$$\nu := \max\Big(\max_{i_1,\ldots,i_k\in[n]}\mathcal{P}_{i_1,\ldots,i_k},\,0\Big), \qquad \widetilde{\mathcal{P}} := \mathcal{P} - \nu\,\mathbb{L}^{(n)}.$$

(i) Let $M := 2n+1$, define

$$\mu := \max_{i_1,i_4,\ldots,i_k\in[n]}\Big(-\sum\widetilde{\mathcal{P}}^{(i_1,i_4,\ldots,i_k)}_{(1,4,\ldots,k)}\Big),$$

and let

$$\widehat{\mathcal{P}} \equiv \big[\widehat{\mathcal{P}}_{i_1,\ldots,i_k}\big]_{i_1,\ldots,i_k} \in \mathbb{R}^{n\times n\times\prod_{i=3}^{k}M}$$

be a k-dimensional array that contains $\widetilde{\mathcal{P}}$ as a subarray at positions $[n]\times\cdots\times[n]$, with all other entries equal to 0. Define the mode-1 slices of $\widetilde{\mathcal{R}}$ by

$$\widetilde{\mathcal{R}}^{(i)}_{(1)} := \begin{cases} \begin{bmatrix} \mathbf{0} & \widetilde{c}_i & \widetilde{\alpha}_i \\ \mu\widetilde{A}_i & \widehat{\mathcal{P}}^{(i)}_{(1)} & \widetilde{d}_i \\ \mathbf{0} & \mu\widetilde{B}_i & \mathbf{0} \end{bmatrix}, & \text{if } i\in[n], \\ \mu\,\widetilde{\mathcal{F}}_i, & \text{if } i\in[n+1;2n+1]. \end{cases}$$

Here:

- *$\frac{1}{n}\widetilde{A}_i, \frac{1}{n}\widetilde{B}_i \in \mathbb{R}^{n\times n\times\prod_{r=3}^{k-1}M}$ are $(k-1)$-dimensional arrays such that for each $(i_3,\ldots,i_{k-1})\in[M]\times\cdots\times[M]$, the slices*

$$\Big(\tfrac{1}{n}\widetilde{A}_i\Big)^{(i_3,\ldots,i_{k-1})}_{(3,\ldots,k-1)}, \qquad \Big(\tfrac{1}{n}\widetilde{B}_i\Big)^{(i_3,\ldots,i_{k-1})}_{(3,\ldots,k-1)}$$

  are arbitrary 2-dimensional quasi-copula type arrays.
- *$\frac{1}{M}\widetilde{\mathcal{F}}_i \in \mathbb{R}^{M\times\cdots\times M}$ are arbitrary $(k-1)$-dimensional quasi-copula type arrays.*
- *$\widetilde{c}_i \in \mathbb{R}^{n\times1\times\prod_{r=3}^{k-1}M}$, $\widetilde{d}_i \in \mathbb{R}^{1\times n\times\prod_{r=3}^{k-1}M}$, and $\widetilde{\alpha}_i \in \mathbb{R}^{1\times1\times\prod_{r=3}^{k-1}M}$ are arrays such that for each $(i_3,\ldots,i_{k-1})\in[M]\times\cdots\times[M]$, the corresponding slices satisfy*

$$\big(\widetilde{c}_i\big)^{(i_3,\ldots,i_{k-1})}_{(3,\ldots,k-1)} = \Big[-\sum\widehat{\mathcal{P}}^{(i,1,i_3,\ldots,i_{k-1})}_{(1,2,4,\ldots,k)} \quad\cdots\quad -\sum\widehat{\mathcal{P}}^{(i,n,i_3,\ldots,i_{k-1})}_{(1,2,4,\ldots,k)}\Big] \in \mathbb{R}^{1\times n},$$

$$\big(\widetilde{d}_i\big)^{(i_3,\ldots,i_{k-1})}_{(3,\ldots,k-1)} = \Big[-\sum\widehat{\mathcal{P}}^{(i,1,i_3,\ldots,i_{k-1})}_{(1,3,4,\ldots,k)} \quad\cdots\quad -\sum\widehat{\mathcal{P}}^{(i,n,i_3,\ldots,i_{k-1})}_{(1,3,4,\ldots,k)}\Big]^T \in \mathbb{R}^{n\times1},$$

$$\big(\widetilde{\alpha}_i\big)^{(i_3,\ldots,i_{k-1})}_{(3,\ldots,k-1)} = \mu + \sum\widehat{\mathcal{P}}^{(i,i_3,\ldots,i_{k-1})}_{(1,4,\ldots,k)} \in \mathbb{R}.$$

Finally, set

$$\mathcal{R} := \frac{1}{M^{k-1}(\mu+\nu M)}\Big(\widetilde{\mathcal{R}} + \nu\mathbb{L}^{(M)}\Big).$$

(ii) Let $N := 3n+2$, define

$$\mu := \max_{i_1,i_3,\ldots,i_k\in[n]}\Big(-n\sum\widetilde{\mathcal{P}}^{(i_1,i_3,\ldots,i_k)}_{(1,3,\ldots,k)}\Big),$$

and let

$$\widehat{\mathcal{P}} \equiv \big[\widehat{\mathcal{P}}_{i_1,\ldots,i_k}\big]_{i_1,\ldots,i_k} \in \mathbb{R}^{n\times n\times\prod_{r=3}^{k}N}$$

be a k-dimensional array containing $\widetilde{\mathcal{P}}$ as a subarray at positions $[n]\times\cdots\times[n]$, with all other entries equal to 0. Define the mode-1 slices of $\widetilde{\mathcal{T}}$ by

$$\widetilde{\mathcal{T}}^{(i)}_{(1)} := \begin{cases} \mu \displaystyle\sum_{\substack{i_r=1,\\ r\in[3;k]}}^{N} \sum_{j=1}^{N} \mathbb{E}^{(N)}_{j,j,i_3,\dots,i_k}, & \text{if } i=1,\\ \mu \displaystyle\sum_{\substack{i_r=1,\\ r\in[3;k]}}^{N} \Big(\mathbb{E}^{(N)}_{i,1,i_3,\dots,i_k} + \mathbb{E}^{(N)}_{1,i,i_3,\dots,i_k} + 3n\,\mathbb{E}^{(N)}_{i,i,i_3,\dots,i_k}\Big), & \text{if } i\in[2;n+1]\cup\{3n+2\},\\ \begin{array}{c} \\ (N-i-n)R\\ 1R\\ (i-n-2)R\\ nR\\ nR\\ R \end{array} \begin{array}{c} \begin{array}{ccccc} 1C & nC & (i-n-2)C & 1C & (N-i)C \end{array}\\ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \widetilde{c}_i & \mathbf{0} & \widetilde{\beta}_i & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \widetilde{v} & \widehat{\mathcal{P}}^{(i)}_{(1)} & \mathbf{0} & \widetilde{d}_i & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \widetilde{u}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \end{array}, & \text{if } i\in[n+2;2n+1],\\ \mu \displaystyle\sum_{\substack{i_r=1,\\ r\in[3;k]}}^{N} \Big(\mathbb{E}^{(N)}_{1,i,i_3,\dots,i_k} + \mathbb{E}^{(N)}_{i,1,i_3,\dots,i_k} + 3n\,\mathbb{E}^{(N)}_{i,i-n,i_3,\dots,i_k}\Big), & \text{for } i\in[2n+2;3n+1], \end{cases}$$

Here C and R denote columns and rows, respectively. Moreover, $\widetilde{v},\widetilde{d}_i\in\mathbb{R}^{1\times n\times\prod_{r=3}^{k-1}N}$, $\widetilde{c}_i,\widetilde{u}_i\in\mathbb{R}^{n\times1\times\prod_{r=3}^{k-1}N}$, and $\widetilde{\beta}_i\in\mathbb{R}^{1\times1\times\prod_{r=3}^{k-1}N}$ are defined so that for each $(i_3,\dots,i_{k-1})\in[N]\times\cdots\times[N]$, the corresponding slices satisfy

$$\begin{aligned} \big(\widetilde{v}\big)^{(i_3,\dots,i_{k-1})}_{(3,\dots,k-1)} &:= \begin{bmatrix}\frac{\mu}{n} & \cdots & \frac{\mu}{n}\end{bmatrix}^T \in\mathbb{R}^{n\times1},\\ \big(\widetilde{c}_i\big)^{(i_3,\dots,i_{k-1})}_{(3,\dots,k-1)} &:= \Big[-\sum \widehat{\mathcal{P}}^{(i,1,i_3,\dots,i_{k-1})}_{(1,2,4,\dots,k)} \quad\cdots\quad -\sum \widehat{\mathcal{P}}^{(i,n,i_3,\dots,i_{k-1})}_{(1,2,4,\dots,k)}\Big]\in\mathbb{R}^{1\times n},\\ \big(\widetilde{u}_i\big)^{(i_3,\dots,i_{k-1})}_{(3,\dots,k-1)} &:= [\underbrace{0,\dots,0}_{i-1},\,\mu,\,\underbrace{0,\dots,0}_{n-i}]\in\mathbb{R}^{n\times1},\\ \big(\widetilde{d}_i\big)^{(i_3,\dots,i_{k-1})}_{(3,\dots,k-1)} &:= \Big[-\sum \widehat{\mathcal{P}}^{(i,1,i_3,\dots,i_{k-1})}_{(1,3,4,\dots,k)} \quad\cdots\quad -\sum \widehat{\mathcal{P}}^{(i,n,i_3,\dots,i_{k-1})}_{(1,3,4,\dots,k)}\Big]^T\in\mathbb{R}^{n\times1},\\ \big(\widetilde{\beta}_i\big)^{(i_3,\dots,i_{k-1})}_{(3,\dots,k-1)} &:= 3n\mu + \sum \widehat{\mathcal{P}}^{(i,i_3,\dots,i_{k-1})}_{(1,4,\dots,k)}\in\mathbb{R}. \end{aligned}$$

Finally, set

$$\mathcal{T} := \frac{1}{N^{k-1}(\mu+\nu N)}\Big(\widetilde{\mathcal{T}} + \nu\mathbb{L}^{(N)}\Big).$$

Proof. We prove the statement for $\mathcal{R}$; the proof for $\mathcal{T}$ is analogous. The argument proceeds by induction on k.

Base case $k=4$. For each $i\in[n]$, the array $M\,\mathcal{R}^{(i)}_{(1)}\in\mathbb{R}^{M\times M\times M}$ is quasi-copula type by Theorem 9. For $i\in[n+1;2n+1]$, the same conclusion holds by construction of $\mathcal{R}$. Hence every 3-dimensional slice $M\,\mathcal{R}^{(i)}_{(1)}$ is quasi-copula type, and Theorem 6 implies that $\mathcal{R}$ itself is quasi-copula type. This establishes the base case.

Induction step. Assume the statement holds for some $k_0\geq4$. We show that it holds for k_0+1.

Let $i,j\in[M]$ be arbitrary. Then

$$\mathcal{R}^{(i,j)}_{(1,k_0+1)} = \big(\mathcal{R}^{(j)}_{(k_0+1)}\big)^{(i)}_{(1)} = \big(\mathcal{R}^{(i)}_{(1)}\big)^{(j)}_{(k_0)}.$$

Fix $j \in [n]$. The second identity shows that $M\,\mathcal{R}^{(j)}_{(k_0+1)}$ is obtained from $\mathcal{P}^{(j)}_{(k_0+1)}$ exactly as $\mathcal{R}$ is obtained from $\mathcal{P}$. Thus, by the induction hypothesis, $M\,\mathcal{R}^{(j)}_{(k_0+1)}$ is quasi-copula type. If instead $j \in [n+1; M]$, then $M\,\mathcal{R}^{(j)}_{(k_0+1)}$ is quasi-copula type by construction.

Therefore all k_0-dimensional sections of $\mathcal{R}$ satisfy the assumptions of Theorem 6, and hence $\mathcal{R}$ is quasi-copula type. □

Remark 15. Theorem 14 extends verbatim to arrays $\mathcal{P} \in \mathbb{R}^{n_1 \times \cdots \times n_k}$ with not necessarily equal side lengths. Indeed, set $n := \max_i n_i$ and embed $\mathcal{P}$ into an array $\widehat{\mathcal{P}} \in \mathbb{R}^{n \times \cdots \times n}$ by padding entries outside the original index set with zeros.

8. Concluding remarks and future research

In this paper, we developed a discrete tensor approach for constructing multidimensional quasi-copulas from prescribed signed mass patterns. Using the boundary-box volume characterization, we translated quasi-copula constraints into finite conditions on arrays—mode-wise marginal equalities and bounded multi-index partial sums—yielding a characterization of quasi-copula type arrays. We also introduced a stronger two-axis positivity criterion and a stacking principle for producing higher-dimensional examples from lower-dimensional building blocks.

Our main results provide explicit and terminating embedding constructions. In dimension three, any $\mathcal{P} \in \mathbb{R}^{n \times n \times n}$ embeds (up to affine rescaling) into a *strong* quasi-copula type tensor of side length $2n+1$, and—without imposing the stronger positivity constraints—into a quasi-copula type tensor of side length $3n+2$, yielding a strictly richer class of admissible extensions. Both constructions extend to arbitrary order $k \geq 4$ with the same prescribed extension sizes.

We conclude with a few directions for future research. A first open problem is *optimality*: can one minimize the total added nonnegative mass, or reduce the guaranteed extension size below $2n+1$ and $3n+2$? A second direction concerns the *geometry and freedom* of extensions: for a fixed embedded subtensor, how large is the family of valid completions, and can one identify extremal or canonical choices? Finally, it would be very interesting to find an embedding construction that avoids the initial sign-control step (i.e., subtracting a constant multiple of the all-ones array to enforce a convenient shift of the prescribed patch) and instead extends the original tensor directly, without this preliminary modification.

References

[1] J.J. Arias-García, R. Mesiar, B. De Baets, A hitchhiker's guide to quasi-copulas, *Fuzzy Sets and Syst.* **393** (2020), 1–28.

[2] T. Augustin, F.P.A. Coolen, G. de Cooman, M.C.M. Troffaes, 2014. *Introduction to imprecise probabilities.* John Wiley & Sons.

[3] M. Beer, S. Ferson, V. Kreinovich, *Imprecise probabilities in engineering analyses*, Mechanical Systems and Signal Processing, **37** (2013) 4–29.

[4] F.P.A. Coolen, On the Use of Imprecise Probabilities in Reliability, *Quality and Reliability Engineering International*, **20** (2004), 193–202.

[5] G. de Cooman, F. Hermans, E. Quaeghebeur, Imprecise Markov chains and their limit behavior. *Probability in the Engineering and Informational Sciences*, **23.4**(2009), 597–635.

[6] F. Durante and C. Sempi, *Principles of Copula Theory*, CRC Press, 2015.

[7] ž S. Ferson, V. Kreinovich, L. Ginzburg, D.S. Myers, K. Sentz, *Constructing probability boxes and Dempster-Shafer structures.* Technical report SAND2002-4015, 2003.

[8] C. Jansen, G. Schollmeyer, T. Augustin, Concepts for decision making under severe uncertainty with partial ordinal and partial cardinal preferences, *International Journal of Approximate Reasoning*, **98** (2018), 112–131.

[9] P. Kumar, *Copula Functions and Applications in Engineering.* In: Deep, K., Jain, M., Salhi, S. (eds) Logistics, Supply Chain and Financial Predictive Analytics. Asset Analytics. Springer, Singapore, (2019). https://doi.org/10.1007/978-981-13-0872-7-15.

[10] T. Košir, E. Perrone, Discrete imprecise copulas, *Fuzzy Sets and Syst.* **504** (2025), 109251.

[11] E. Miranda, I. Montes, Shapley and Banzhaf values as probability transformations, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, **26** (2018), 917–947.

[12] I. Montes, E. Miranda, S. Montes, Decision making with imprecise probabilities and utilities by means of statistical preference and stochastic dominance, *European Journal of Operational Research*, **234** (2014), 209–220.

[13] R. Nau, *Imprecise probabilities in Non-cooperative games.* In Proceedings of ISIPTA 2011.

[14] M. Oberguggenberger, J. King, B. Schmelzer, Classical and imprecise probability methods for sensitivity analysis in engineering: a case study, *International Journal of Approximate Reasoning*, **50** (2009), 680–693.
[15] M. Omladič, N. Stopar, Final solution to the problem of relating a true copula to an imprecise copula, *Fuzzy Sets and Syst.* **393** (2020), 96–112.
[16] M. Omladič, N. Stopar, Multivariate imprecise Sklar type theorems, *Fuzzy Sets and Syst.* **428** (2022), 80–101.
[17] M. Omladič, N. Stopar, Dedekind-MacNeille completion of multivariate copulas via ALGEN method, *Fuzzy Sets and Syst.* **441** (2022), 321–334.
[18] M. Omladič, N. Stopar, Freedom in constructing quasi-copulas vs. copulas, *Fuzzy Sets and Syst.* 514 (2025), 109399,
[19] M. Omladič, D. Škulj, Constructing copulas from shock models with imprecise distributions, *International J. Approx. Reasoning*, **118** (2020), 27–46.
[20] I. Montes, E. Miranda, R. Pelessoni, P. Vicig, Sklar's theorem in an imprecise setting, *Fuzzy Sets and Syst.* 278 (2015), 48–66.
[21] R. Pelessoni, P. Vicig, Convex Imprecise Previsions, *Reliable Computing*, **9** (2003), 465–485.
[22] R. Pelessoni, P. Vicig, I. Montes, E. Miranda, Bivariate p-boxes. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, **24.02** (2016), 229–263.
[23] J. A. Rodríguez-Lallena, M. Úbeda-Flores, Some new characterizations and properties of quasi-copulas, *Fuzzy Sets and Syst.* **160** (2009), 717–725.
[24] B. Schmelzer, Joint distributions of random sets and their relation to copulas, *International Journal of Approximate Reasoning*, **65** (2015), 59–69.
[25] B. Schmelzer, Sklar's theorem for minitive belief functions, *International Journal of Approximate Reasoning*, **63** (2015), 48–61.
[26] B. Schmelzer, Multivariate capacity functional vs. capacity functionals on product spaces, *Fuzzy Sets and Syst.* **364** (2019), 1–35.
[27] A. Sklar, Fonctions de répartition à n dimensions et leurs marges, *Publ. Inst. Stat. Univ. Paris*, **8** (1959), 229–231.
[28] D. Škulj, Discrete time Markov chains with interval probabilities. *International Journal of Approximate Reasoning*, **50.9** (2009), 1314–1329.
[29] M. C. M. Troffaes, *Decision making under uncertainty using imprecise probabilities*, International Journal of Approximate Reasoning, **45** (2007), 17–29.
[30] M. C. M. Troffaes, S. Destercke *Probability boxes on totally preordered spaces for multivariate modelling*, *International Journal of Approximate Reasoning*, **52** (2011), 767–791.
[31] L. V. Utkin, F. P. A. Coolen, *Imprecise Reliability: An Introductory Overview.* In: Levitin G. (eds) Computational Intelligence in Reliability Engineering. Studies in Computational Intelligence, **40** (2007). Springer, Berlin, Heidelberg.
[32] P. Vicig, Financial risk measurement with imprecise probabilities, *International Journal of Approximate Reasoning*, **49** (2008), 159–174.
[33] P. Walley. Coherent lower (and upper) probabilities. Technical Report 22, University of Warwick, Coventry, 1981.
[34] P. Walley, Statistical Reasoning with Imprecise Probabilities. Chapman and Hall, London, 1991.
[35] L. Yu, S. Destercke, M. Sallak, W. Schon, *Comparing system reliability with ill-known probabilities*, Proceedings of IPMU 2016, 619–629.
[36] L. Zhang, VP Singh. Copulas and Their Applications in Water Resources Engineering. Cambridge University Press; 2019.

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