

MOMENT THEORY APPROACH TO GAUSSIAN QUADRATURES WITH PRESCRIBED NODES

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ABSTRACT. Let μ be a positive Borel measure on the real line and let L be the linear functional on univariate polynomials of bounded degree, defined as integration with respect to μ . In [BKRSV20], the characterization of all minimal quadrature rules of μ in terms of the roots of a bivariate polynomial is given and two determinantal representations of this polynomial are established. In particular, the authors solved the question of the existence of a minimal quadrature rule with one prescribed node, leaving open the extension to more prescribed nodes [BKRSV20, Problem 1]. In this paper, we solve this problem using moment theory as the main tool.

1. INTRODUCTION

Given a real sequence

$$(1.1) \quad \gamma \equiv \gamma^{(D)} = (\gamma_0, \gamma_1, \dots, \gamma_D) \in \mathbb{R}^{D+1}$$

of degree D , $D \in \mathbb{N} \cup \{0\}$, the \mathbb{R} -truncated moment problem (\mathbb{R} -TMP) for γ asks to characterize the existence of a positive Borel measure μ on \mathbb{R} , such that

$$(1.2) \quad \gamma_i = \int_{\mathbb{R}} x^i d\mu \quad \text{for } i \in \mathbb{N} \cup \{0\}, 0 \leq i \leq D.$$

If such a measure exists, we say that γ has a \mathbb{R} -representing measure (\mathbb{R} -rm).

The solution to the \mathbb{R} -TMP in terms of the properties of the corresponding Hankel matrix, called **moment matrix**, was established in [CF91] relying on certificates of nonnegativity for polynomials, while special cases were already solved in [AK62, KN77, Ioh82]. See also [Sch17, Chapter 9] for a nice overview. In [CF91] the question of the existence and uniqueness of the solution is settled. In the nonsingular case, i.e., when the moment matrix is positive definite, there is a one-parametric family of finitely atomic representing measures, supported on the smallest possible number of atoms equal to the rank of the moment matrix. Such a solution is called **Gaussian quadrature rule (GQR)** in numerical analysis and is important because it allows efficient computation of definite integrals of univariate functions. In [BKRSV20], the authors investigated the question of characterizing the GQRs in terms of symmetric determinantal representations involving moment matrices. Their main result [BKRSV20, Theorem 1.4], obtained using convex analysis and algebraic geometry, characterizes when a given atom, also called a **node**, is a part of a GQR and how to determine other nodes in terms of the determinant of a certain univariate matrix polynomial. Here we emphasize that they allow node at infinity, called **evaluation at ∞** , and the corresponding quadrature rule is called **generalized GQR (gGQR)**. A characterization for the case that all nodes are

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real, in terms of the invertibility of a certain matrix, is also given. A characterization of (g)GQRs with finitely many prescribed nodes remains an open problem:

Problem 1 ([BKRSV20, Problem 1]). *Given $d_1, d_2 \geq 1$, a degree $D := d_1 + 2d_2 - 1$ sequence $\gamma^{(D)}$ as in (1.1) such that the moment matrix $M_{\lfloor \frac{D}{2} \rfloor}$ is positive definite and real numbers x_1, \dots, x_{d_1} , when does a (g)GQR with $d_1 + d_2$ nodes, containing x_1, \dots, x_{d_1} , exist?*

In the formulation of Problem 1 from [BKRSV20], $d_1 = n - 1$, $d_2 = \ell + 1$ and the sequence $\gamma^{(D)}$ is assumed to come from some positive measure. Since $M_{\lfloor \frac{D}{2} \rfloor}$ is assumed to be positive definite, the same automatically holds in the formulation above (see e.g., Theorem 2.1 below). The choice of degree D in Problem 1 is natural due to the fact that for each prescribed node only its density is unknown, while for each non-prescribed node both the value of the node and its density are unknown. In total, there are exactly $D + 1$ variables, which is equal to the length of $\gamma^{(D)}$.

A necessary condition for the solution to Problem 1 is [BKRSV20, Proposition 4.1], but it is not sufficient for $d_1 > 1$ as demonstrated by [BKRSV20, Example 4.2]. Finding other necessary conditions to settle the problem was the main motivation for this paper.

The main result of this paper is the solution to Problem 1, obtained using moment theory. Section 2 introduces the notation and tools needed to establish the main results. The solution to the GQR case of Problem 1 is Theorem 3.1, while the gGQR case is covered by Theorem 3.2. The main technique to establish these results is to observe the appropriate univariate sequence of γ , associated with the monic polynomial that has zeros exactly in the prescribed nodes. This sequence uniquely determines both the remaining nodes in the potential representing measure and the extension of the original sequence. Then the solution follows by applying the solution to the \mathbb{R} -TMP from [CF91] to the extended sequence. In Examples 3.6 and 3.7 we demonstrate the application of Theorems 3.1 and 3.2 on numerical examples.

2. PRELIMINARIES

We write $\mathbb{R}^{n \times m}$ for the set of $n \times m$ real matrices. For a polynomial $f \in \mathbb{R}[x]$ we denote by $\mathcal{Z}(f) := \{x \in \mathbb{R} : f(x) = 0\}$ its set of zeros.

2.1. Moment matrix. Let $D = 2d$, $d \in \mathbb{N}$. For $\ell \in \mathbb{N}$, $\ell \leq d$ the Hankel matrix of size $(\ell + 1) \times (\ell + 1)$ corresponding to the sequence γ as in (1.1), with columns and rows indexed by the monomials $1, X, \dots, X^\ell$, is defined by

$$(2.1) \quad M_\ell := (\gamma_{i+j-2})_{i,j=1}^{\ell+1} = \begin{matrix} & I & X & X^2 & \dots & X^\ell \\ I & \left(\begin{array}{cccccc} \gamma_0 & \gamma_1 & \gamma_2 & \dots & \gamma_\ell \\ \gamma_1 & \gamma_2 & \dots & \dots & \gamma_{\ell+1} \\ \gamma_2 & \dots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \gamma_{2\ell-1} \\ X^\ell & \gamma_\ell & \gamma_{\ell+1} & \dots & \gamma_{2\ell-1} & \gamma_{2\ell} \end{array} \right) \end{matrix}$$

and is called the ℓ -th truncated moment matrix of γ . For $i, j \in \mathbb{N} \cup \{0\}$, $i \leq d$, $j \leq d$, let

$$(2.2) \quad \mathbf{v}_i^{(j)} := (\gamma_{i+r-1})_{1 \leq r \leq j+1} \in \mathbb{R}^{j+1}.$$

Using this notation, we have that

$$M_d = \begin{pmatrix} \mathbf{v}_0^{(d)} & \dots & \mathbf{v}_d^{(d)} \end{pmatrix}.$$

For $p(x) = \sum_{i=0}^{\ell} a_i x^i \in \mathbb{R}[x]$ we define the **evaluation** $p(X)$ on the columns of the matrix M_ℓ by $p(X) = a_0 I + \sum_{i=1}^{\ell} a_i X^i$, where I and X^i represent the columns of M_ℓ indexed by these monomials. Then $p(X)$ is a vector from the linear span of the columns of M_ℓ . If this vector is the zero one, then we say p is a **column relation** of M_ℓ .

As in [CF91], the **rank** of γ , denoted by $\text{rank } \gamma$, is defined by

$$\text{rank } \gamma = \begin{cases} d+1, & \text{if } M_d \text{ is nonsingular,} \\ \min \left\{ i : \mathbf{v}_i^{(d)} \in \text{span} \{ \mathbf{v}_0^{(d)}, \dots, \mathbf{v}_{i-1}^{(d)} \} \right\}, & \text{if } M_d \text{ is singular.} \end{cases}$$

If $\text{rank } \gamma < d+1$, we say that γ is **singular**. Else γ is **nonsingular**. We call γ **positively recursively generated (prg)** if for $r = \text{rank } m$ the following two conditions hold:

(1) M_{r-1} is positive definite.

(2) If $r < d+1$, denoting $(\varphi_0, \dots, \varphi_{r-1}) := M_{r-1}^{-1} \mathbf{v}_r^{(r-1)}$, the equality

$$(2.3) \quad \gamma_j = \varphi_0 \gamma_{j-r} + \dots + \varphi_{r-1} \gamma_{j-1}$$

holds for $j = r, \dots, 2d$.

If γ is singular and prg, we call

$$(2.4) \quad p_{(\gamma)}(\mathbf{x}) := \mathbf{x}^r - \sum_{i=0}^{r-1} \varphi_i \mathbf{x}^i \in \mathbb{R}[\mathbf{x}],$$

with φ_i as in (2), the **generating polynomial** of γ .

2.2. Localizing moment matrices. Let γ be as in (1.1). The **Riesz functional** $L : \mathbb{R}[x]_{<D} \rightarrow \mathbb{R}$ of γ is defined by $L(x^i) := \gamma_i$ for each i . For $f \in \mathbb{R}[x]_{\leq D}$ an **f -localizing moment matrix \mathcal{H}_f of γ** is a real square matrix of size $s(D, f) \times s(D, f)$, where $s(D, f) = \lfloor \frac{D - \deg f}{2} \rfloor + 1$, with the (i, j) -th entry equal to $L(fx^{i+j-2})$. We write

$$(2.5) \quad f \cdot \gamma := (\gamma_0^{(f)}, \gamma_1^{(f)}, \dots, \gamma_{D-\deg f}^{(f)}), \quad \gamma_i^{(f)} := L(fx^i).$$

Note that \mathcal{H}_f corresponds to the moment matrix of the sequence $f \cdot \gamma$. We denote the Riesz functional of $f \cdot \gamma$ by L_f and call it an **f -localizing Riesz functional of γ** .

We write $\mathcal{H}_f(\ell)$ for the ℓ -th truncated moment matrix of $f \cdot \gamma$, i.e.,

$$\mathcal{H}_f(\ell) := \left(\gamma_{i+j-2}^{(f)} \right)_{i,j=1}^{\ell+1} = \begin{matrix} & I & X & X^2 & \dots & X^\ell \\ I & \left(\begin{array}{cccccc} \gamma_0^{(f)} & \gamma_1^{(f)} & \gamma_2^{(f)} & \dots & \gamma_\ell^{(f)} \\ \gamma_1^{(f)} & \gamma_2^{(f)} & \dots & \dots & \gamma_{\ell+1}^{(f)} \\ \gamma_2^{(f)} & \dots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \gamma_{2\ell-1}^{(f)} \\ \gamma_\ell^{(f)} & \gamma_{\ell+1}^{(f)} & \dots & \gamma_{2\ell-1}^{(f)} & \gamma_{2\ell}^{(f)} \end{array} \right) \\ X & & & & & \\ X^2 & & & & & \\ \vdots & & & & & \\ X^\ell & & & & & \end{matrix}$$

2.3. Atomic measures. For $x \in \mathbb{R}$, δ_x stands for the Dirac measure supported on x . By a **finitely atomic positive measure** on \mathbb{R} we mean a measure of the form $\mu = \sum_{j=1}^{\ell} \rho_j \delta_{x_j}$, where $\ell \in \mathbb{N} \cup \{0\}$, each $\rho_j > 0$ and each $x_j \in \mathbb{R}$. The points x_j are called **atoms** of the measure μ and the constants ρ_j the corresponding **densities**.

We now recall the definition of the evaluation at ∞ from [BKRSV20, Definition 1.1]. The **evaluation at ∞** is the linear functional, defined by

$$(2.6) \quad \mathbf{ev}_\infty : \mathbb{R}[\mathbf{x}]_{\leq D} \rightarrow \mathbb{R}, \quad \sum_{k=0}^D f_k \mathbf{x}^k \mapsto f_D.$$

Allowing \mathbf{ev}_∞ to be a part of a finitely atomic positive measure μ , we call μ a **generalized measure**. When such a measure represents a sequence γ as in (1.1) by (1.2), then μ is called a $(\mathbb{R} \cup \{\infty\})$ -rm for γ .

2.4. Solution to the \mathbb{R} -TMP. Let $x_1, \dots, x_m \in \mathbb{R}$. We denote by $V_{(x_1, \dots, x_m)} \in \mathbb{R}^{m \times m}$ the Vandermondo matrix

$$V_{(x_1, \dots, x_m)} := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_m \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{m-1} & x_2^{m-1} & \cdots & x_m^{m-1} \end{pmatrix}.$$

The following is a solution to the \mathbb{R} -TMP of degree $2d$.

Theorem 2.1 ([CF91, Theorems 3.9 and 3.10]). *Let $d \in \mathbb{N}$ and $\gamma = (\gamma_0, \dots, \gamma_{2d}) \in \mathbb{R}^{2d+1}$ with $\gamma_0 > 0$. The following statements are equivalent:*

- (1) *There exists a \mathbb{R} -representing measure for γ .*
- (2) *There exists a $(\text{rank } \gamma)$ -atomic \mathbb{R} -representing measure for γ .*
- (3) *γ is positively recursively generated.*
- (4) *M_d is positive semidefinite and $\text{rank } M_d = \text{rank } \gamma$.*
- (5) *One of the following statements holds:*
 - (a) *M_d is positive definite.*
 - (b) *M_d is positive semidefinite and $\text{rank } M_d = \text{rank } M_{d-1}$.*

Moreover, if a \mathbb{R} -representing measure for γ exists, then:

- (i) *If $r \leq d$, then the \mathbb{R} -representing measure is unique and of the form $\mu = \sum_{i=1}^r \rho_i \delta_{x_i}$, where x_1, \dots, x_r are the roots of $p(\gamma)$ as in (2.4) and $(\rho_i)_{i=0}^r = V_{(x_1, \dots, x_m)}^{-1} \mathbf{v}_0^{(r-1)}$.*
- (ii) *If $r = d + 1$, then there are infinitely many \mathbb{R} -representing measures for γ . All $(d + 1)$ -atomic ones are obtained by choosing $\gamma_{2k+1} \in \mathbb{R}$ arbitrarily, defining $\gamma_{2k+2} := (\mathbf{v}_{d+1}^{(d)})^T (M_d)^{-1} \mathbf{v}_{d+1}^{(d)}$, and using (i) for $\tilde{\gamma} := (\gamma_0, \dots, \gamma_{2d+1}, \gamma_{2d+2}) \in \mathbb{R}^{2k+3}$.*

2.5. Companion matrix. For a univariate polynomial $p(\mathbf{x}) = \mathbf{x}^k - \sum_{i=0}^{k-1} \lambda_i \mathbf{x}^i \in \mathbb{R}[\mathbf{x}]$ let

$$(2.7) \quad C_{p(\mathbf{x})} = \begin{pmatrix} 0 & \cdots & & 0 & \lambda_0 \\ 1 & 0 & & 0 & \lambda_1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & 1 & 0 & \lambda_{k-2} \\ 0 & \cdots & \cdots & 0 & 1 & \lambda_{k-1} \end{pmatrix} \in \mathbb{R}^{k \times k}$$

be its **companion matrix**, i.e., $p(\mathbf{x}) = \det(\mathbf{x}I_k - C_{p(\mathbf{x})})$, where I_k stands for the identity matrix of size k .

2.6. Elementary symmetric polynomials. Let $d_1 \in \mathbb{N}$ and

$$e_i(\mathbf{x}_1, \dots, \mathbf{x}_{d_1}) := \sum_{1 \leq j_1 < j_2 < \dots < j_i \leq d_1} \mathbf{x}_{j_1} \mathbf{x}_{j_2} \cdots \mathbf{x}_{j_i}$$

stand for the i -th elementary symmetric polynomial in variables $\mathbf{x}_1, \dots, \mathbf{x}_{d_1}$. Given distinct real numbers x_1, \dots, x_{d_1} , let e_i stand for $e_i(x_1, \dots, x_{d_1})$. In particular,

$$(2.8) \quad e_0 = 1, \quad e_1 = \sum_{i=1}^{d_1} x_i, \quad e_2 = \sum_{1 \leq i < j \leq d_1} x_i x_j, \quad \dots, \quad e_{d_1} = x_1 x_2 \cdots x_{d_1}.$$

3. MINIMAL MEASURES WITH FINITELY MANY PRESCRIBED ATOMS

The main results of this section are solutions to Problem 1 from Introduction, i.e., Theorem 3.1 covers the case where only real atoms are allowed in the representing measure, while in Theorem 3.2 the functional \mathbf{ev}_∞ (see (2.6)) is also allowed in the support of the measure.

Theorem 3.1. *Let $d_1, d_2 \in \mathbb{N}$ and $x_1, \dots, x_{d_1} \in \mathbb{R}$ be distinct real numbers. Let $D := d_1 + 2d_2 - 1$. Assume that $\gamma \equiv \gamma^{(D)} = (\gamma_0, \dots, \gamma_D) \in \mathbb{R}^{D+1}$ is a sequence such that the moment matrix $M_{\lfloor \frac{D}{2} \rfloor}$ is positive definite. Let $e_i := e_i^{(d_1)}$ be as in (2.8) and*

$$f(\mathbf{x}) := \prod_{i=1}^{d_1} (\mathbf{x} - x_i) = \sum_{i=0}^{d_1} (-1)^i e_i \mathbf{x}^{d_1-i}.$$

The following statements are equivalent:

- (1) There exists a $(d_1 + d_2)$ -atomic \mathbb{R} -representing measure for γ with d_1 atoms equal to x_1, \dots, x_{d_1} .
- (2) The following conditions hold:
 - (a) The localizing matrix $\mathcal{H}_f(d_2 - 1)$ is invertible.
 - (b) Denote

$$(3.1) \quad (\lambda_0 \quad \lambda_1 \quad \cdots \quad \lambda_{d_2-1})^T = (\mathcal{H}_f(d_2 - 1))^{-1} \left(\sum_{i=0}^{d_1} (-1)^i e_i \mathbf{v}_{d_1+d_2-i}^{(d_2-1)} \right)$$

where $\mathbf{v}_{d_1+d_2-i}^{(d_2-1)}$ are as in (2.2), let

$$(3.2) \quad h(\mathbf{x}) := f(\mathbf{x}) \underbrace{\left(\mathbf{x}^{d_2} - \sum_{i=0}^{d_2-1} \lambda_i \mathbf{x}^i \right)}_{g(\mathbf{x})} = \mathbf{x}^{d_1+d_2} - \sum_{i=0}^{d_1+d_2-1} \varphi_i \mathbf{x}^i,$$

and let

$$\tilde{\gamma} := \{\tilde{\gamma}_u\}_{u=0}^{D+d_1-1}$$

be the extension of γ , defined by $\tilde{\gamma}_u = \gamma_u$ for $0 \leq u \leq D$ and

$$(3.3) \quad \tilde{\gamma}_{D+\ell} = \sum_{i=1}^{d_1+d_2} \varphi_{d_1+d_2-i} \tilde{\gamma}_{D+\ell-i} \quad \text{for } \ell = 1, \dots, d_1 - 1.$$

The moment matrix $M_{d_1+d_2-1}$ of $\tilde{\gamma}$ is positive definite.

Moreover, if the equivalent statements (1), (2) hold, then the other atoms in the measure are the zeros of the polynomial $g(\mathbf{x})$.

Theorem 3.2. *Let $d_1, d_2 \in \mathbb{N}$ and $x_1, \dots, x_{d_1} \in \mathbb{R}$ be distinct real numbers. Let $D := d_1 + 2d_2 - 1$. Assume that $\gamma \equiv \gamma^{(D)} = (\gamma_0, \dots, \gamma_D) \in \mathbb{R}^{D+1}$ is a sequence such that the moment matrix $M_{\lfloor \frac{D}{2} \rfloor}$ is positive definite. The following statements are equivalent:*

- (1) *There exists a $(d_1 + d_2)$ -atomic $(\mathbb{R} \cup \{\infty\})$ -representing measure μ for γ with d_1 atoms equal to x_1, \dots, x_{d_1} .*
- (2) *One of the following statements hold:*
 - (a) *There exists a $(d_1 + d_2)$ -atomic \mathbb{R} -representing measure μ for γ with d_1 atoms equal to x_1, \dots, x_{d_1} , obtained by Theorem 3.1.*
 - (b) *There exists a $(d_1 + d_2 - 1)$ -atomic \mathbb{R} -representing measure μ for the sequence $\hat{\gamma} \equiv (\gamma_0, \dots, \gamma_{D-2}) \in \mathbb{R}^{D-1}$ with d_1 atoms equal to x_1, \dots, x_{d_1} , which also represents γ_{D-1} and $\gamma_D - \alpha$ for some $\alpha > 0$. Namely, denoting $e_i := e_i^{(d_1)}$ as in (2.8) and $f(\mathbf{x}) := \prod_{i=1}^{d_1} (\mathbf{x} - x_i)$, the following conditions hold:*
 - (i) *The localizing matrix $\mathcal{H}_f(d_2 - 2)$ is invertible.*
 - (ii) *Denote*

$$(\lambda_0 \quad \lambda_1 \quad \cdots \quad \lambda_{d_2-2})^T = (\mathcal{H}_f(d_2 - 2))^{-1} \left(\sum_{i=0}^{d_1} (-1)^i e_i \mathbf{v}_{d_1+d_2-1-i}^{(d_2-2)} \right)$$

where $\mathbf{v}_{d_1+d_2-1-i}^{(d_2-2)}$ are as in (2.2), let

$$h(\mathbf{x}) := f(\mathbf{x}) \left(\mathbf{x}^{d_2-1} - \sum_{i=0}^{d_2-2} \lambda_i \mathbf{x}^i \right) = \mathbf{x}^{d_1+d_2-1} - \sum_{i=0}^{d_1+d_2-2} \varphi_i \mathbf{x}^i,$$

and let

$$\tilde{\gamma} := \{\tilde{\gamma}_u\}_{u=0}^{D+d_1-3}$$

be defined by $\tilde{\gamma}_u = \gamma_u$ for $0 \leq u \leq D - 2$ and

$$\tilde{\gamma}_{D-2+\ell} = \sum_{i=1}^{d_1+d_2-1} \varphi_{d_1+d_2-1-i} \tilde{\gamma}_{D-2+\ell-i} \quad \text{for } \ell = 1, \dots, d_1 - 1.$$

The moment matrix $M_{d_1+d_2-2}$ of $\tilde{\gamma}$ is positive definite,

$$\tilde{\gamma}_{D-1} = \gamma_{D-1} \quad \text{and} \quad \tilde{\gamma}_D < \gamma_D.$$

Remark 3.3. (1) In the case $d_1 = 1$, there is nothing to check in (2b) of Theorem 3.1 and the invertibility of $\mathcal{H}_f(d_2 - 1)$ is equivalent to the existence of a \mathbb{R} -rm. This agrees with Theorem [BKRSV20, Theorem 1.4]. The remaining nodes in [BKRSV20, Theorem 1(a)] are solutions of $F(x_1, y) = 0$ where $F(\mathbf{t}, \mathbf{y}) := \det(\mathcal{H}_{(\mathbf{x}-\mathbf{t})(\mathbf{x}-\mathbf{y})}(d_2 - 1))$. Note that

$$\mathcal{H}_{(\mathbf{x}-\mathbf{y})f(\mathbf{x})}(d_2 - 1) = \mathcal{H}_{\mathbf{x}f(\mathbf{x})}(d_2 - 1) - \mathbf{y}\mathcal{H}_{f(\mathbf{x})}(d_2 - 1) = \mathcal{H}_{f(\mathbf{x})}(d_2 - 1)(C_{g(\mathbf{x})} - \mathbf{y}I),$$

where $C_{g(\mathbf{x})}$ is the companion matrix of $g(\mathbf{x})$ from (3.2). So $F(x_1, y) = 0$ indeed corresponds to $g(y) = 0$, which should be true by the moreover part of Theorem 3.1.

- (2) Assume that $d_1 = 1$ and $\mathcal{H}_f(d_2 - 1)$ is singular. Let $(\varphi_0 \quad \cdots \quad \varphi_{d_2-1})^T = M_{d_2-1}^{-1} \mathbf{v}_{d_2}^{(d_2-1)}$ and $p(\mathbf{x}) = \mathbf{x}^{d_2} - \sum_{i=0}^{d_2-1} \varphi_i \mathbf{x}^i$. Since $\mathcal{H}_{\mathbf{x}}(d_2 - 1) = M_{d_2-1} C_{p(\mathbf{x})}$, where $C_{p(\mathbf{x})}$ is the companion matrix of $p(\mathbf{x})$, it follows that $\mathcal{H}_{f(\mathbf{x})}(d_2 - 1) = M_{d_2-1} (C_{p(\mathbf{x})} - x_1 I)$. Hence, x_1 is a zero of $p(\mathbf{x})$ and by Theorem 2.1 zeroes of $p(\mathbf{x})$ represent the support of the d_2 -atomic \mathbb{R} -rm for $(\gamma_0, \dots, \gamma_{D-1}, \gamma_D - M_{d_2}/M_{d_2-1})$, where M_{d_2}/M_{d_2-1} stands for the Schur complement of

M_{d_2-1} in M_{d_2} . Since $\alpha := \gamma_D - M_{d_2}/M_{d_2-1}$ is positive, the remaining atom is ev_∞ with the density α .

- (3) Note that the representing measures obtained by Theorems 3.1 and 3.2 are unique.
- (4) Theorem 3.1 and 3.2 assume $d_1 + d_2$ atoms with x_1, \dots, x_{d_1} prescribed are necessary to represent γ . It might happen that there already exists a (d_1+i) -atomic representing measure for some $1 \leq i < d_2$. One can also handle these cases by the following simple adaptation. Namely, for a given i Theorem 3.1 is applied to the truncation $\gamma^{(D_i)} = \{\gamma_j\}_{j=0}^{D_i}$ where $D_i = d_1 + 2i - 1$ of γ . Since the measure obtained by Theorem 3.1 is unique, one only checks whether this measure also represents $\gamma_{D_i+1}, \dots, \gamma_{D-1}$ and $\gamma_D - \alpha$ for some $\alpha \geq 0$. If $\alpha > 0$, then ev_∞ must also be added as an atom with the density α .

The following two lemmas are important technical tools in the proof of Theorem 3.1.

Lemma 3.4. *Let $k, d \in \mathbb{N}$ with $k < d$. Given distinct real numbers x_1, \dots, x_k , write*

$$A_i := \begin{pmatrix} -x_i & 1 & 0 & \cdots & 0 \\ 0 & -x_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -x_i & 1 \end{pmatrix} \in \mathbb{R}^{(d-i+1) \times (d-i+2)} \quad \text{for } i = 1, \dots, k.$$

and

$$(3.4) \quad B_k := A_k A_{k-1} \cdots A_1 \in \mathbb{R}^{(d-k+1) \times (d+1)}.$$

Let $e_i := e_i^{(k)}$ be as in (2.8) (with $d_1 = k$). Then

$$(3.5) \quad B_k = \begin{pmatrix} (-1)^k e_k & (-1)^{k-1} e_{k-1} & \cdots & (-1)^{k-i} e_{k-i} & \cdots & e_0 & 0 & \cdots & \cdots & 0 \\ 0 & (-1)^k e_k & (-1)^{k-1} e_{k-1} & \cdots & (-1)^{k-i} e_{k-i} & \cdots & e_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & (-1)^k e_k & (-1)^{k-1} e_{k-1} & \cdots & (-1)^{k-i} e_{k-i} & \cdots & e_0 \end{pmatrix}.$$

Equivalently, writing $B_k = (b_{ij})_{i,j}$, we have that

$$(3.6) \quad b_{ij} = \begin{cases} (-1)^{k+i-j} e_{k+i-j}, & 1 \leq i \leq j \leq i+k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We prove the result inductively on k . The basis case $k = 1$ is clear. Now, assume that (3.5) holds for $k-1 < d-1$ and prove it for $k < d$. Namely, assuming that

$$B_{k-1} = (b_{ij}^{(k-1)})_{i,j} \in \mathbb{R}^{(d-k+2) \times (d+1)}$$

with

$$b_{ij}^{(k-1)} = \begin{cases} (-1)^{k-1+i-j} e_{k-1+i-j}^{(k-1)}, & 1 \leq i \leq j \leq i+k-1, \\ 0, & \text{otherwise,} \end{cases}$$

we have to prove that $B_k = A_k B_{k-1}$ is of the form (3.5) or equivalently (3.6) holds. We have that

$$\begin{aligned}
b_{ij} &= \underbrace{\begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}_{i-1}}_{i-1} \begin{pmatrix} -x_k & 1 & 0 & \cdots & 0 \end{pmatrix}_{d-k-i} \begin{pmatrix} (-1)^{k-j} e_{k-j}^{(k-1)} \\ (-1)^{k-j+1} e_{k-j+1}^{(k-1)} \\ \vdots \\ (-1)^{k-j+d-k} e_{d-j-1}^{(k-1)} \end{pmatrix} \\
&= (-1)^{k+i-j-1} (-x_k) e_{k-j+i-1}^{(k-1)} + (-1)^{k+i-j} e_{k-j+i}^{(k-1)} \\
&= (-1)^{k+i-j} (x_k e_{k-j+i-1}^{(k-1)} + e_{k-j+i}^{(k-1)}) \\
&= (-1)^{k+i-j} e_{k+i-j}^{(k)}.
\end{aligned}$$

This completes the proof. \square

Lemma 3.5. *Let $k, d \in \mathbb{N}$ with $k < d$. Let $\gamma \equiv (\gamma_0, \gamma_1, \dots, \gamma_{2d}) \in \mathbb{R}^{2d+1}$ with a corresponding moment matrix M_d . Given distinct real numbers x_1, \dots, x_k , let $e_i := e_i^{(k)}$ be as in (2.8) (with $d_1 = k$) and $f(\mathbf{x}) := \prod_{i=1}^k (\mathbf{x} - x_i) = \sum_{i=0}^k (-1)^i e_i \mathbf{x}^{k-i}$. Let B_k be as in (3.4) and $\gamma_i^{(f)}$ as in (2.5). Then*

$$(B_k M_d)_{ij} = \gamma_{i+j-2}^{(f)} \quad \text{for } 1 \leq i \leq d - k + 1, 1 \leq j \leq d + 1,$$

and

$$B_k M_d B_k^T = \sum_{i=0}^k (-1)^i e_i \mathcal{H}_{\mathbf{x}^{k-i} f(\mathbf{x})}(d - k).$$

Moreover, if M_d is positive definite, then $B_k M_d B_k^T$ is invertible.

Proof. By definition of $\gamma_i^{(f)}$, we have to prove that

$$(3.7) \quad (B_k M_d)_{ij} = L(f \mathbf{x}^{i+j-2}) \quad \text{for each } 1 \leq i \leq d - k + 1, 1 \leq j \leq d + 1,$$

where L is a Riesz functional of γ . Let $\mathbf{v}_i^{(j)} := (\gamma_{i+r-1})_{1 \leq r \leq j+1}$ and let L be the Riesz functional of γ . We have that:

$$\begin{aligned}
(B_k M_d)_{ij} &= \left(B_k \begin{pmatrix} \mathbf{v}_0^{(d)} & \cdots & \mathbf{v}_{j-1}^{(d)} & \cdots & \mathbf{v}_d^{(d)} \end{pmatrix} \right)_{ij} \\
&= \sum_{\ell=1}^{d+1} b_{i\ell} \gamma_{j+\ell-2} \\
&\stackrel{(3.6)}{=} \sum_{\ell=i}^{i+k} (-1)^{k+i-\ell} e_{k+i-\ell} \gamma_{j+\ell-2} \\
&= L \left(\sum_{\ell=i}^{i+k} (-1)^{k+i-\ell} e_{k+i-\ell} \mathbf{x}^{j+\ell-2} \right) \\
&= L \left(\left(\sum_{\ell=i}^{i+k} (-1)^{k+i-\ell} e_{k+i-\ell} \mathbf{x}^{\ell-i} \right) \mathbf{x}^{i+j-2} \right) \\
&= {}^6 L \left(\left(\sum_{\hat{\ell}=0}^k (-1)^{k-\hat{\ell}} e_{k-\hat{\ell}} \widehat{\mathbf{x}}^{\hat{\ell}} \right) \mathbf{x}^{i+j-2} \right)
\end{aligned}$$

$$\begin{aligned}
&= {}^7 L \left(\left(\sum_{\tilde{\ell}=0}^k (-1)^{\tilde{\ell}} e_{\tilde{\ell}} \mathbf{x}^{k-\tilde{\ell}} \right) \mathbf{x}^{i+j-2} \right) \\
&= L(f \mathbf{x}^{i+j-2}),
\end{aligned}$$

where in the sixth (resp. the seventh) equality we introduced a new variable $\widehat{\ell} = \ell - i$ (resp. $\widetilde{\ell} = k - \widehat{\ell}$).

Further on,

$$\begin{aligned}
B_k M_d B_k^T &= (\gamma_{i+j-2}^{(f)})_{\substack{1 \leq i \leq d-k+1, \\ 1 \leq j \leq d+1}} B_k^T \\
&= \sum_{\ell=0}^k (-1)^{k-\ell} e_{k-\ell} (\gamma_{i+j-2+\ell}^{(f)})_{i,j=1}^{d-k+1} \\
&= \sum_{\ell=0}^k (-1)^{k-\ell} e_{k-\ell} \mathcal{H}_{\mathbf{x}^\ell f(\mathbf{x})}(d-k) \\
&= \sum_{i=0}^k (-1)^i e_i \mathcal{H}_{\mathbf{x}^{k-i} f(\mathbf{x})}(d-k),
\end{aligned}$$

where we introduced a new variable $i = k - \ell$ in the last equality.

The moreover part follows by noticing that $\text{rank } B_k = d - k + 1$. Indeed, if all x_i are nonzero, then $e_k \neq 0$. So the first diagonal of B_k is nonzero and $\text{rank } B_k = d - k + 1$. If one of x_i is 0, then we may assume that $x_1 = 0$ and hence $e_{k-1} = x_2 \cdots x_k \neq 0$. Thus the second diagonal of B_k is nonzero and $\text{rank } B_k = d - k + 1$. Now for $v \in \mathbb{R}^{d-k+1}$ we have

$$\begin{aligned}
B_k M_d B_k^T v = 0 &\Rightarrow v^T B_k M_d B_k^T v = (B_k^T v)^T M_d (B_k^T v) = 0 \\
&\Rightarrow B_k^T v = 0 \\
&\Rightarrow v = 0,
\end{aligned}$$

where we used that M_d is positive definite in the second and the fact that B_k^T has a trivial kernel (being of full column rank) in the last line. Therefore $B_k M_d B_k^T$ is invertible. \square

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. First, we prove the implication (1) \Rightarrow (2). Let $\mu = \sum_{i=1}^{d_1+d_2} \rho_i \delta_{y_i}$ be a $(d_1 + d_2)$ -atomic \mathbb{R} -rm for γ with $\rho_i > 0$ and $y_i = x_i$ for $i = 1, \dots, d_1$. We extend $M_{\lfloor \frac{D}{2} \rfloor}$ to $M_{d_1+d_2}$ by computing $\gamma_{D+1}, \gamma_{D+2}, \dots, \gamma_{D+d_1+1}$ with respect to the measure μ :

$$M_{d_1+d_2} = (\vec{X})^T \begin{pmatrix} I & X & \cdots & X^{\lfloor \frac{D}{2} \rfloor} & \cdots & X^{d_1+d_2} \\ \mathbf{v}_0^{(d_2-1)} & \mathbf{v}_1^{(d_2-1)} & \cdots & \mathbf{v}_{\lfloor \frac{D}{2} \rfloor}^{(d_2-1)} & \cdots & \mathbf{v}_{d_1+d_2}^{(d_2-1)} \\ X^{d_2} & \gamma_{d_2} & \gamma_{d_2+1} & \cdots & \gamma_{\lfloor \frac{D}{2} \rfloor + d_2} & \cdots & \gamma_{D+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ X^{d_1+d_2} & \gamma_{d_1+d_2} & \gamma_{d_1+d_2+1} & \cdots & \gamma_{\lfloor \frac{D}{2} \rfloor + d_1+d_2} & \cdots & \gamma_{D+d_1+1} \end{pmatrix},$$

where $\vec{X} = (I \ X \ \dots \ X^{d_2-1})$. Let

$$\begin{aligned}
\tilde{g}(\mathbf{x}) &:= \prod_{i=d_1+1}^{d_1+d_2} (\mathbf{x} - y_i) = \mathbf{x}^{d_2} - \sum_{i=0}^{d_2-1} \tilde{\lambda}_i \mathbf{x}^i, \\
\tilde{h}(\mathbf{x}) &:= \prod_{i=1}^{d_1+d_2} (\mathbf{x} - y_i) = \left(\prod_{i=1}^{d_1} (\mathbf{x} - x_i) \right) \tilde{g}(\mathbf{x}) \\
(3.8) \quad &= \left(\sum_{i=0}^{d_1} (-1)^i e_i \mathbf{x}^{d_1-i} \right) \left(\mathbf{x}^{d_2} - \sum_{i=0}^{d_2-1} \tilde{\lambda}_i \mathbf{x}^i \right) \\
&= \left(\sum_{i=0}^{d_1} (-1)^i e_i \mathbf{x}^{d_1-i} \right) \mathbf{x}^{d_2} - \left(\sum_{i=0}^{d_1} (-1)^i e_i \mathbf{x}^{d_1-i} \right) \left(\sum_{i=0}^{d_2-1} \tilde{\lambda}_i \mathbf{x}^i \right).
\end{aligned}$$

We will prove that $\tilde{g}(\mathbf{x}) = g(\mathbf{x})$ and $\tilde{h}(\mathbf{x}) = h(\mathbf{x})$, where $g(\mathbf{x})$ and $h(\mathbf{x})$ are as in (3.2). By Theorem 2.1, $M_{d_1+d_2-1}$ is invertible and $\tilde{h}(X) = \mathbf{0}$ in $M_{d_1+d_2}$. By Lemma 3.5 used for $k = d_1$ and $d := d_1 + d_2 - 1$, we get that

$$(3.9) \quad \sum_{i=0}^{d_1} (-1)^i e_i \mathcal{H}_{\mathbf{x}^{d_1-i} f(\mathbf{x})}(d_2 - 1) \quad \text{is invertible.}$$

Let $C_{\tilde{g}(\mathbf{x})}$ is the companion matrix of $\tilde{g}(\mathbf{x})$ (see (2.7)). From the third equality for $\tilde{h}(\mathbf{x})$ in (3.8) above it follows that

$$(3.10) \quad \mathcal{H}_{\mathbf{x}^{d_1-i} f(\mathbf{x})}(d_2 - 1) = \mathcal{H}_{\mathbf{x}^{d_1-i-1} f(\mathbf{x})}(d_2 - 1) C_{\tilde{g}(\mathbf{x})} \quad \text{for } i = 0, \dots, d_1 - 1.$$

Using (3.10) in (3.9), it follows that

$$\mathcal{H}_f(d_2 - 1) \left(\sum_{i=0}^{d_1} (-1)^i e_i C_{\tilde{g}(\mathbf{x})}^{d_1-i} \right) \quad \text{is invertible.}$$

In particular, $\mathcal{H}_f(d_2 - 1)$ is invertible. (3.10) used for $i = d_1 - 1$, implies that $\tilde{\lambda}_i = \lambda_i$ for each i , where λ_i are as in (3.1). This proves that $g(\mathbf{x}) = \tilde{g}(\mathbf{x})$ and $h(\mathbf{x}) = \tilde{h}(\mathbf{x})$. Finally, observing the $d_1 - 1$ entries in rows $X^{d_2}, \dots, X^{d_1+d_2-2}$ of the column $X^{d_1+d_2}$ of $M_{d_1+d_2}$ gives equalities (3.3). This concludes the proof of the implication (1) \Rightarrow (2).

It remains to prove the implication (2) \Rightarrow (1). We define $\gamma_{D+d_1}, \gamma_{D+d_1+1}$ by (3.3) for $\ell = d_1, d_1 + 1$. By assumptions of the theorem, the moment matrix $M_{d_1+d_2}$ satisfies the column relation

$$X^{d_1+d_2} - \sum_{i=0}^{d_1+d_2-1} \varphi_i X^i = \mathbf{0}.$$

Since $M_{d_1+d_2-1}$ is positive definite, Theorem 2.1 implies that there is a $(d_1 + d_2)$ -atomic \mathbb{R} -rm for γ supported on the zeroes of this column relation. These are x_1, \dots, x_{d_1} and the zeroes of $g(\mathbf{x})$. \square

Finally, we prove Theorem 3.2.

Proof of Theorem 3.2. The nontrivial implication is (1) \Rightarrow (2). We only need to prove that if (2a) does not hold, then (2b) holds. Assume that (2a) does not hold. Then one of the atoms in a

$(d_1 + d_2)$ -atomic $(\mathbb{R} \cup \{\infty\})$ -rm μ for γ must be ∞ . Hence, $\mu = \sum_{j=1}^{d_1+d_2-1} \rho_j \delta_{y_j} + \rho_{d_1+d_2} \mathbf{e}\mathbf{v}_\infty$ for some $\rho_j > 0$ and $y_j \in \mathbb{R}$ with $y_1 = x_1, \dots, y_{d_1} = x_{d_1}$. Thus, we have

$$\gamma_i = \sum_{j=1}^{d_1+d_2-1} \rho_j y_j^i + \rho_{d_1+d_2} \mathbf{e}\mathbf{v}_\infty(x^i) = \begin{cases} \sum_{j=1}^{d_1+d_2-1} \rho_j y_j^i, & \text{if } i < D, \\ \sum_{j=1}^{d_1+d_2-1} \rho_j y_j^i + \rho_{d_1+d_2}, & \text{if } i = D. \end{cases}$$

Write

$$\gamma = \underbrace{(\gamma_0, \gamma_1, \dots, \gamma_{D-2}, 0, 0)}_{(\hat{\gamma}, 0, 0)} + \underbrace{(0, \dots, 0, \gamma_{D-1}, \rho_{d_1+d_2})}_{D-1}.$$

Since $\hat{\gamma}$ has a unique $(d_1 + d_2 - 1)$ -atomic \mathbb{R} -rm $\sum_{j=1}^{d_1+d_2-1} \rho_j \delta_{y_j}$ such that $y_j = x_j$ for $j = 1, \dots, d_1$, the statement (2b) follows by Theorem 3.1. \square

The next numerical example (obtained using [Wol]) shows the violation of the conditions in Theorems 3.1 and 3.2, which prevents the existence of the measure for a given sequence of degree 9 with two prescribed atoms. This is [BKRSV20, Example 2], where the authors show that the necessary condition $\det(\mathcal{H}_{(x-y)f(x)}(d_2 - 1)) = 0$ for the existence of a measure containing x_1, \dots, x_{d_1} and y in the support is not also sufficient (see [BKRSV20, Proposition 4.1]).

Example 3.6. Let $\gamma \equiv \{\gamma_i\}_{i=0}^9 \in \mathbb{R}^{10}$ be a sequence defined by $\gamma_i = i!$. Namely,

$$\gamma = (1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880).$$

Let $x_1 = \frac{1}{3}$, $x_2 = 11$ and $d_1 = 2, d_2 = 4$. The question is whether there is a $(d_1 + d_2)$ -atomic $(\mathbb{R} \cup \{\infty\})$ -rm for γ with two atoms equal to x_1, x_2 .

First we will demonstrate which condition in Theorem 3.1 is violated, preventing the existence of a 6-atomic \mathbb{R} -rm for γ having atoms x_1, x_2 in the support. Using the notation of Theorem 3.1, let $f(\mathbf{x}) = (\mathbf{x} - \frac{1}{3})(\mathbf{x} - 11)$ and

$$\mathcal{H}_f(3) = \begin{pmatrix} -\frac{17}{3} & -13 & -\frac{110}{3} & -130 \\ -13 & -\frac{110}{3} & -130 & -552 \\ -\frac{110}{3} & -130 & -552 & -2680 \\ -130 & -552 & -2680 & -14160 \end{pmatrix}.$$

The eigenvalues of $\mathcal{H}_f(3)$ are $-14691.9, -61.04, -1.54, 0.18$ and hence $\mathcal{H}_f(3)$ is invertible. The last column of $\mathcal{H}_f(4)$, restricted to the first 4 rows, is

$$v = (-552 \quad -2680 \quad -14160 \quad -75600)^T.$$

Hence, $(\mathcal{H}_f(3))^{-1}v = (-\frac{46998216}{137503}, \frac{41197920}{137503}, -\frac{11282760}{137503}, \frac{1695024}{137503})^T$. So

$$g(\mathbf{x}) = \mathbf{x}^4 - \frac{1695024}{137503} \mathbf{x}^3 + \frac{11282760}{137503} \mathbf{x}^2 - \frac{41197920}{137503} \mathbf{x} + \frac{46998216}{137503}$$

and $h(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x}) = \mathbf{x}^6 - \sum_{i=0}^5 \varphi_i \mathbf{x}^i$ is equal to

$$h(\mathbf{x}) = \mathbf{x}^6 - \frac{9760174}{412509} \mathbf{x}^5 + \frac{92991629}{412509} \mathbf{x}^4 - \frac{175284288}{137503} \mathbf{x}^3 + \frac{555278096}{137503} \mathbf{x}^2 - \frac{683705488}{137503} \mathbf{x} + \frac{172326792}{137503}.$$

The candidate for γ_{10} , coming from the potential measure for γ , is

$$\gamma_{10} = \sum_{i=1}^6 \varphi_{6-i} \gamma_{10-i} = \frac{492324551232}{137503}.$$

The extended moment matrix M_5 is then equal to

$$M_5 = \begin{pmatrix} 1 & 1 & 2 & 6 & 24 & 120 \\ 1 & 2 & 6 & 24 & 120 & 720 \\ 2 & 6 & 24 & 120 & 720 & 5040 \\ 6 & 24 & 120 & 720 & 5040 & 40320 \\ 24 & 120 & 720 & 5040 & 40320 & 362880 \\ 120 & 720 & 5040 & 40320 & 362880 & \frac{492324551232}{137503} \end{pmatrix}.$$

Its eigenvalues are $3.62 \cdot 10^6, 3774.42, 14.69, 0.808, 0.0398, -0.436$. Since the smallest one is negative, M_5 is not positive definite, which is needed for the existence of the measure.

Finally we will show that also if we allow the atom at ∞ there is still no 6-atomic $(\mathbb{R} \cup \{\infty\})$ -rm for γ with atoms x_1, x_2 in the support. It remains to consider the case when ∞ is necessarily one of the atoms. So we have to find a candidate for a 5-atomic \mathbb{R} -rm with x_1, x_2 as atoms for the truncated sequence $\hat{\gamma} = \{\gamma_i\}_{i=0}^7$ and then check whether the measure obtained also represents γ_8 and if its moment of degree 9 is strictly smaller than γ_9 . We have that

$$\mathcal{H}_f(2) = \begin{pmatrix} -\frac{17}{3} & -13 & -\frac{110}{3} \\ -13 & -\frac{110}{3} & -130 \\ -\frac{110}{3} & -130 & -552 \end{pmatrix}.$$

The eigenvalues of $\mathcal{H}_f(2)$ are $-585.52, -8.76, -0.054$ and hence $\mathcal{H}_f(2)$ is invertible. The last column of $\mathcal{H}_f(3)$, restricted to the first 3 rows, is

$$v_2 = (-130 \quad -552 \quad -2680)^T.$$

Hence, $(\mathcal{H}_f(2))^{-1} v_2 = \left(\frac{204774}{1861} \quad -\frac{172062}{1861} \quad \frac{35955}{1861} \right)^T$. So

$$g_2(\mathbf{x}) = \mathbf{x}^3 - \frac{35955}{1861} \mathbf{x}^2 + \frac{172062}{1861} \mathbf{x} - \frac{204774}{1861}$$

and $h_2(\mathbf{x}) = f(\mathbf{x})g_2(\mathbf{x}) = \mathbf{x}^5 - \sum_{i=0}^4 \tilde{\varphi}_i \mathbf{x}^i$ is equal to

$$h_2(\mathbf{x}) = \mathbf{x}^5 - \frac{171139}{5583} \mathbf{x}^4 + \frac{1759127}{5583} \mathbf{x}^3 - \frac{2286645}{1861} \mathbf{x}^2 + \frac{2951666}{1861} \mathbf{x} - \frac{750838}{1861}.$$

The candidate for $\hat{\gamma}_8$, coming from the potential measure for $\hat{\gamma}$, is

$$\hat{\gamma}_8 = \sum_{i=1}^5 \tilde{\varphi}_{5-i} \gamma_{8-i} = \frac{73385484}{1861}.$$

Already at this point we see that $\hat{\gamma}_8 \neq \gamma_8$, so the condition of Theorem 3.2 is violated, but let us also compute M_4 to see that even the measure for $\hat{\gamma}$ does not exist. The extended moment matrix

M_4 is then equal to

$$M_4 = \begin{pmatrix} 1 & 1 & 2 & 6 & 24 \\ 1 & 2 & 6 & 24 & 120 \\ 2 & 6 & 24 & 120 & 720 \\ 6 & 24 & 120 & 720 & 5040 \\ 24 & 120 & 720 & 5040 & \frac{73385484}{1861} \end{pmatrix}.$$

Its eigenvalues are 40092.4, 86.03, 1.69, 0.25, -0.03 . Since the smallest one is negative, M_4 is not positive definite, which is needed for the existence of the measure.

Replacing $x_1 = \frac{1}{3}$ with $x_1 = 1$ in the previous example, there exist a 6-atomic \mathbb{R} -rm for γ with two atoms being $x_1 = 1, x_2 = 11$, by the following example.

Example 3.7. Let $\gamma \equiv \{\gamma_i\}_{i=0}^9 \in \mathbb{R}^{10}$ be the same sequence as in Example 3.6. Let $x_1 = 1, x_2 = 11$ and $d_1 = 2, d_2 = 4$. For $f(x) = (x-1)(x-11)$ we have

$$\mathcal{H}_f(3) = \begin{pmatrix} 1 & -7 & -26 & -102 \\ -7 & -26 & -102 & -456 \\ -26 & -102 & -456 & -2280 \\ -102 & -456 & -2280 & -12240 \end{pmatrix}.$$

The eigenvalues of $\mathcal{H}_f(3)$ are $-12683.9, -40.56, 3.28, 0.14$ and hence $\mathcal{H}_f(3)$ is invertible. The last column of $\mathcal{H}_f(4)$, restricted to the first 4 rows, is

$$v = (-456 \quad -2280 \quad -12240 \quad 65520)^T.$$

Hence, $(\mathcal{H}_f(3))^{-1}v = \left(\frac{95824}{1601} \quad -\frac{753912}{1601} \quad \frac{1476768}{1601} \quad -\frac{220344}{1601}\right)^T$. So

$$g(x) = x^4 - \frac{95824}{1601}x^3 + \frac{753912}{1601}x^2 - \frac{1476768}{1601}x + \frac{220344}{1601}$$

and $h(x) = f(x)g(x) = x^6 - \sum_{i=0}^5 \varphi_i x^5$ is equal to

$$h(x) = x^6 - \frac{115036}{1601}x^5 + \frac{1921411}{1601}x^4 - \frac{11577776}{1601}x^3 + \frac{26234592}{1601}x^2 - \frac{18888576}{1601}x + \frac{2423784}{1601}.$$

The candidate for γ_{10} , coming from the potential measure for γ , is

$$\gamma_{10} = \sum_{i=1}^6 \varphi_{6-i} \gamma_{10-i} = \frac{5944515264}{1601}.$$

The extended moment matrix M_5 is then equal to

$$M_5 = \begin{pmatrix} 1 & 1 & 2 & 6 & 24 & 120 \\ 1 & 2 & 6 & 24 & 120 & 720 \\ 2 & 6 & 24 & 120 & 720 & 5040 \\ 6 & 24 & 120 & 720 & 5040 & 40320 \\ 24 & 120 & 720 & 5040 & 40320 & 362880 \\ 120 & 720 & 5040 & 40320 & 362880 & \frac{5944515264}{1601} \end{pmatrix}.$$

Its eigenvalues are $3.75 \cdot 10^6, 5068.64, 38.02, 1.66, 0.35, 0.019$. So M_5 is positive definite and the measure for γ exists. The atoms are x_1, x_2 and the zeroes of $g(x)$, i.e., $x_3 \approx 0.16, x_4 \approx 2.81, x_5 \approx 5.91, x_6 \approx 50.97$.

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