TRUNCATED MOMENT PROBLEMS AND THE EXTENSION PROPERTY ON MONOMIAL CURVES

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ABSTRACT. In several papers, Stochel and Szafraniec studied moment problems on algebraic sets from an operator-theoretic perspective, investigating when positive definite sequences satisfying polynomial relations admit representing measures. Within this framework, Stochel introduced type A sets, and Bisgaard classified the plane curves defined by relations between two monomials that have this property. Curto and Fialkow introduced a stronger, truncated version of the type A property, requiring that the existence of a positive semidefinite extension of prescribed degree guarantees the existence of a representing measure. Motivated by Bisgaard's classification, we determine which plane curves defined by relations between two monomials satisfy this extension property. In the affirmative cases, we obtain explicit bounds on the required extension degree. In the negative cases, we construct truncated sequences that admit positive semidefinite extensions of arbitrarily high order but have no representing measure supported on the curve. These constructions yield explicit polynomials that are nonnegative on the corresponding curves but are not sums of squares in their coordinate rings. In the affirmative cases, we also derive explicit degree bounds for sums-of-squares certificates of strictly positive polynomials.

1. INTRODUCTION

In [SS94], Stochel and Szafraniec studied moment problems on algebraic sets from an operator-theoretic perspective, relating the existence of representing measures for positive definite sequences satisfying polynomial relations to normal extensions of algebraic operators. The operator-theoretic approach to moment problems was further developed in [JS05], where vector and operator multidimensional moment problems with jointly subnormal solutions were characterized in terms of their initial data. Related connections between polynomial relations modulo an ideal, multivariate three-term recurrences, and orthogonality with respect to measures supported on algebraic sets were developed in [CSS05]. Within this framework, Stochel introduced in [Sto92] the notions of algebraic sets of type A and type B. More precisely, let $\mathcal{Z}_p := \{x \in \mathbb{R}^d : p(x) = 0\}$ be the real zero set of a polynomial p. The set $\mathcal{Z}_p$ is said to be of *type A* if every linear functional on $\mathbb{R}[x_1, \ldots, x_d]$ that is nonnegative on squares and vanishes on the principal ideal (p) admits a positive representing measure supported on $\mathcal{Z}_p$. It is said to be of *type B* if the same conclusion holds for every such functional vanishing on the vanishing ideal

$$I(\mathcal{Z}_p) := \{P \in \mathbb{R}[x_1, \ldots, x_d] : P(x) = 0 \text{ for every } x \in \mathcal{Z}_p\}.$$

Using the Real Nullstellensatz, Bisgaard and Stochel proved in [BiS16, Section 3] that the type A and type B properties coincide. Consequently, the type A property depends only on the zero set $\mathcal{Z}_p$, not on the choice of its defining polynomial p.

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Stochel also proved in [Sto92] that several classes of plane algebraic curves are of type A, including curves of the forms $y = q(x)$ and $yq(x) = 1$, and every plane algebraic curve of degree at most 2. Using semigroup theory, Bisgaard characterized in [Bis98] which plane curves defined by relations between two monomials, $x^{i_1}y^{j_1} = x^{i_2}y^{j_2}$, are of type A. Moreover, Schmüdgen's result [Sch91] implies that every bounded algebraic set $\mathcal{Z}_p$ is of type A.

More generally, let $A_p := \mathbb{R}[x_1, \ldots, x_d]/(p)$. By [Sto92, Theorem 4.3], the set $\mathcal{Z}_p$ is of type A if and only if the image in A_p of the cone of sums of squares in $\mathbb{R}[x_1, \ldots, x_d]$ is dense in the cone of positive elements of A_p, with respect to any locally convex topology on A_p for which every linear functional is continuous.

Equivalently, $\mathcal{Z}_p$ is of type A if and only if the quadratic module generated by $Q := \{1, p, -p\}$ (see (2.4)) has the *strong moment property* in the sense of [Sch17, Definition 13.6]. The classification of real algebraic curves with the strong moment property follows from results of Scheiderer and Plaumann [Sch03, Pla10]; see, for example, [Pla10, Theorem 3.13 and Proposition 5.2].

The present paper is motivated by the study of plane curves defined by relations between two monomials from the perspective of truncated moment theory. A theorem of Stochel [Sto01] shows that solvability of all truncations of a full moment sequence implies solvability of the corresponding full moment problem. Thus, if every truncation of a given sequence has a representing measure supported on a fixed algebraic set, then the full sequence also has a representing measure supported on that set. More general extension criteria for positive definite mappings on symmetric subsets of $*$-semigroups, including applications to truncated moment problems, were established in [CSS11]. For another application of moment methods in operator theory, namely a two-moment characterization of spectral measures on the real line, see [PS23].

In [CF08], Curto and Fialkow introduced a strengthening of the type A property for truncated moment sequences, namely the extension property $(S_{n,k})$; see (2.1) below. In this framework, the generating families associated with a line, a circle, or an ellipse satisfy $(S_{n,1})$, whereas those associated with a parabola or a hyperbola satisfy $(S_{n,2})$. Moreover, Fialkow proved in [Fia11] that suitable generating families for curves of the forms $y = q(x)$ and $yq(x) = 1$ satisfy $(S_{n,k})$ for sufficiently large k. Improved bounds on k were subsequently obtained in [Zal22b]. Property $(S_{n,k})$ also has important consequences in real algebraic geometry: it yields degree-bounded weighted sums-of-squares representations for polynomials that are strictly positive on the underlying semialgebraic set.

In this paper, we investigate property $(S_{n,k})$ for the same families of monomial plane curves that appear in Bisgaard's classification of type A curves. More precisely, we consider polynomials p that are monomial multiples of binomials and determine when the associated generator family $\{1, p, -p\}$ satisfies property $(S_{n,k})$ for some finite k. Our results, together with the previously known cases, yield the classification summarized in Table 1. The table records whether a finite extension parameter exists, gives an admissible value of k in each affirmative case, and indicates the result in which the corresponding assertion is established.

In the negative cases, our constructions also produce explicit polynomials that are nonnegative on the underlying curves but are not sums of squares in the corresponding coordinate rings. In the affirmative cases, the extension bounds yield degree-bounded positivity certificates. More precisely, if $\{1, p, -p\}$ satisfies property $(S_{n,k})$, then every polynomial $f \in \mathcal{P}_{2n}$ that is strictly positive on $\mathcal{Z}_p$ admits a representation $f = \sum_{j=1}^{r} g_j^2 + ph$, where $g_1, \ldots, g_r, h \in \mathbb{R}[x, y]$ and $\deg g_j^2 \leq 2(n + k)$, $\deg(ph) \leq 2(n + k)$. This follows from [CF08, Theorem 1.5(i)]; explicit versions are given in

Corollaries 4.2, 4.5, and 4.7. Thus, our results both extend the theory of truncated moment problems on monomial curves and provide explicit information about sums-of-squares representations, including obstructions in the negative cases and degree bounds in the affirmative cases.

TABLE 1. Classification of the generator families $\{1, p, -p\}$ with respect to property $(S_{n,k})$, where $n \geq \deg p$. Unless stated otherwise, $\ell \in \mathbb{N}$. In the rows involving ℓ_1, ℓ_2, we assume $\ell_1, \ell_2 \in \mathbb{N} \setminus \{1\}$ and $\gcd(\ell_1, \ell_2) = 1$.

Defining polynomial $p(x, y)$	**Finite** k	**Admissible values of** k	**Result**
$x - y^\ell$ or $y - x^\ell$	Yes	$k \geq \max\{\ell - 1, 1\}$	[Zal22b, Th. 3.1]
$y(x - y^\ell)$ or $x(y - x^\ell)$	Yes	$k \geq \ell + \max\{\ell - 1, 2\}$	Th. 4.6
$y(y - x^\ell)$ or $x(x - y^\ell)$, $\ell \geq 2$	No	—	Th. 3.8
$xy(y - x^\ell)$ or $xy(x - y^\ell)$, $\ell \geq 2$	No	—	Th. 3.10
$q(x, y)(x^{\ell_1} - y^{\ell_2})$, $q \in \{1, x, y, xy\}$	No	—	Ths. 3.1, 3.5, and 3.6
$xy^\ell - 1$ or $x^\ell y - 1$	Yes	$k \geq \ell + 1$	[Zal22b, Th. 4.1]
$x^{\ell_1} y^{\ell_2} - 1$	Yes	$k \geq \ell_1 + \ell_2 - 1$	Th. 4.1
$q(x, y)(x^{\ell_1} y^{\ell_2} - 1)$, $q \in \{x, y, xy\}$	Yes	$k \geq 2(\ell_1 + \ell_2)$	Ths. 4.3, 4.4

Remark 1.1. We record several observations concerning the classification in Table 1.

(1) For a fixed $n \in \mathbb{N}$, the existence of a finite $k \in \mathbb{N}_0$ such that $\{1, p, -p\}$ satisfies property $(S_{n,k})$ depends only on the zero set $\mathcal{Z}_p$, rather than on the particular polynomial used to define it; see Remark 2.4. This explains why we restrict our attention to defining polynomials in which every irreducible factor occurs with multiplicity one.

(2) The assumption $n \geq \deg p$ ensures that the column $p(X, Y)$ occurs in $M(n)$. If a positive extension satisfies the localizing conditions associated with p and $-p$, then $p(X, Y) = 0$, together with all column relations obtained from this identity by recursive generation whenever the relevant products are defined. If $n < \deg p$, then the column $p(X, Y)$ does not occur in $M(n)$, and the corresponding cases require a separate analysis. We therefore restrict our attention to the principal setting $n \geq \deg p$.

1.1. **Reader's guide.** The paper is organized as follows. In Section 2, we introduce the necessary notation and recall the basic facts about truncated moment problems, moment and localizing matrices, and property $(S_{n,k})$.

In Section 3, we treat the negative cases summarized in Table 1. For each of these families, we construct truncated sequences having extension properties of every finite order but admitting no representing measure on the corresponding algebraic set. These constructions also yield explicit polynomials that are nonnegative on the curves but are not sums of squares in their coordinate rings.

In Section 4, we treat the affirmative cases summarized in Table 1. We establish property $(S_{n,k})$ for the displayed admissible extension parameters by reducing the corresponding truncated moment problems to one-dimensional moment problems. As a consequence, we obtain explicit degree bounds for sums-of-squares representations of polynomials that are strictly positive on the curves.

2. Preliminaries

In this section, we recall the basic concepts and results from the theory of truncated moment problems that will be used throughout the paper, including moment and localizing matrices, recursively generated moment matrices, and property $(S_{n,k})$.

Let $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $\alpha := (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, we write $|\alpha| := \alpha_1 + \cdots + \alpha_d$. Let

$$\beta \equiv \beta^{(2n)} = \{\beta_\alpha\}_{\alpha \in \mathbb{N}_0^d,\ |\alpha| \le 2n}$$

be a truncated multivariate sequence of degree $2n$ in d real variables, and let

$$\mathcal{P}_n := \{p \in \mathbb{R}[x_1, \ldots, x_d] : \deg p \le n\}.$$

The associated Riesz functional $L_\beta : \mathcal{P}_{2n} \to \mathbb{R}$ is defined by

$$L_\beta\Big(\sum_{|\alpha| \le 2n} a_\alpha x^\alpha\Big) := \sum_{|\alpha| \le 2n} a_\alpha \beta_\alpha.$$

Let

$$\mathcal{B}_n := \{x^\alpha : \alpha \in \mathbb{N}_0^d,\ |\alpha| \le n\}$$

denote the monomial basis of $\mathcal{P}_n$, ordered first by total degree and then lexicographically within each degree.

The moment matrix $M(n) \equiv M(n; \beta)$ is the symmetric matrix indexed by $\mathcal{B}_n \times \mathcal{B}_n$ with entries

$$M(n)_{x^\alpha, x^{\alpha'}} := L_\beta(x^\alpha x^{\alpha'}), \qquad |\alpha|, |\alpha'| \le n.$$

We write

$$\operatorname{Col} M(n) := \{M(n)v : v \in \mathbb{R}^{\dim \mathcal{P}_n}\} \subseteq \mathbb{R}^{\dim \mathcal{P}_n}$$

and refer to it as the **column space** of $M(n)$.

For every $x^\alpha \in \mathcal{B}_n$, let X^α denote the column of $M(n)$ indexed by x^α. If $p(x) = \sum_{|\alpha| \le n} a_\alpha x^\alpha \in \mathcal{P}_n$, we define

$$p(X) := \sum_{|\alpha| \le n} a_\alpha X^\alpha \in \operatorname{Col} M(n).$$

The moment matrix $M(n)$ is said to be **recursively generated** if whenever $p, q, pq \in \mathcal{P}_n$ and $p(X) = 0$ in $\operatorname{Col} M(n)$, then $(pq)(X) = 0$ in $\operatorname{Col} M(n)$.

Let $q \in \mathbb{R}[x_1, \ldots, x_d]$ be a polynomial of degree $\delta \le 2n$. The **localizing matrix** associated with q, denoted by $M_q(n) \equiv M_q(n; \beta)$, is the symmetric matrix indexed by $\mathcal{B}_{n - \lceil \frac{\delta}{2} \rceil} \times \mathcal{B}_{n - \lceil \frac{\delta}{2} \rceil}$ with entries

$$M_q(n)_{x^\alpha, x^{\alpha'}} := L_\beta\Big(q x^{\alpha + \alpha'}\Big), \qquad |\alpha|, |\alpha'| \le n - \left\lceil \frac{\delta}{2} \right\rceil.$$

Let

$$Q := \{q_0 := 1, q_1, \ldots, q_m\} \subset \mathbb{R}[x_1, \ldots, x_d]$$

and define the semialgebraic set

$$K_Q := \{x \in \mathbb{R}^d : q_i(x) \ge 0,\ i = 0, 1, \ldots, m\}.$$

Let $k \in \mathbb{N}_0$. We say that a moment matrix $M(n+k)$ is a **positive extension** of $M(n)$ if $M(n+k) \succeq 0$ and its principal submatrix indexed by $\mathcal{B}_n$ coincides with $M(n)$. Following Curto and Fialkow [CF08], we say that Q satisfies **property** $(S_{n,k})$ if, for every truncated sequence $\beta^{(2n)}$, the following equivalence holds:

$$\beta^{(2n)} \text{ has a } K_Q\text{–representing measure} \iff \begin{array}{l} M(n) \text{ admits a positive extension } M(n+k), \\ \text{such that } M_{q_i}(n+k) \succeq 0, \quad i = 1, \ldots, m. \end{array} \tag{2.1}$$

If the right-hand side of (2.1) holds, we say that $\beta^{(2n)}$ has the k**–extension property relative to** Q.

Remark 2.1. The validity of property $(S_{n,k})$ may depend on the particular choice of generators $Q = \{1, q_1, \ldots, q_m\}$ defining K_Q, and not only on the semialgebraic set K_Q itself. We therefore attribute property $(S_{n,k})$ to Q throughout the paper.

For full positive functionals, the property that vanishing on the principal ideal generated by a defining polynomial depends only on its real zero set is an immediate consequence of the fact that the annihilator ideal of a positive functional is real; see [BiS16, Lemmas 2.4 and 2.5]. More precisely, if $p, q \in \mathbb{R}[x_1, \ldots, x_d]$ satisfy $\mathcal{Z}_p = \mathcal{Z}_q$, then a full positive functional vanishes on (p) if and only if it vanishes on (q). The analogous statement in the truncated setting requires additional care. The truncated annihilator is defined only up to a fixed degree and need not be an ideal. We therefore first establish a finite-degree real-radical transfer principle. Its proof uses Real Nullstellensatz certificates and enlarges the truncation order so that all the polynomials occurring in these certificates lie in the domain of the extended Riesz functional.

Lemma 2.2. *Let $a, b \in \mathbb{R}[x_1, \ldots, x_d]$ satisfy $a \in I(\mathcal{Z}_b)$, and let $r \in \mathbb{N}_0$. Then there exists $N_0 \in \mathbb{N}$ such that, for every $N \geq N_0$ and every linear functional $L : \mathcal{P}_{2N} \to \mathbb{R}$ that is nonnegative on squares and satisfies*

$$L(bfg) = 0 \qquad \text{for all } f, g \in \mathcal{P}_{N - \lceil \frac{\deg b}{2} \rceil},$$

one has

$$L(auv) = 0 \qquad \text{for all } u, v \in \mathcal{P}_r.$$

Proof. Let $\mathcal{B}_r = \{u_1, \ldots, u_m\}$ be the monomial basis of $\mathcal{P}_r$, and set

$$\mathcal{U} := \{u_i : 1 \leq i \leq m\} \cup \{u_i + u_j : 1 \leq i < j \leq m\}.$$

For every $v \in \mathcal{U}$, the inclusion $a \in I(\mathcal{Z}_b)$ implies $av^2 \in I(\mathcal{Z}_b)$. By the Real Nullstellensatz [Mar08, p. 26], there exist $M_v \in \mathbb{N}$ and polynomials $s_{v,1}, \ldots, s_{v,m_v}$ such that

$$(av^2)^{2M_v} + \sum_{j=1}^{m_v} s_{v,j}^2 \in (b).$$

Choose $\ell_v \in \mathbb{N}$ such that $2^{\ell_v} \geq 2M_v$, and set $q_v := 2^{\ell_v - 1} - M_v \in \mathbb{N}_0$. Multiplying the preceding certificate by $(av^2)^{2q_v}$, and then renaming the resulting polynomials, gives

$$(av^2)^{2^{\ell_v}} + \sum_{j=1}^{m_v} s_{v,j}^2 = bh_v. \tag{2.2}$$

Since $\mathcal{U}$ is finite, we may choose N_0 sufficiently large that, for every $v \in \mathcal{U}$ and $1 \leq j \leq m_v$,

$$2^{\ell_v - 1} \deg(av^2) \leq N_0, \qquad \deg s_{v,j} \leq N_0, \qquad \deg h_v \leq 2\left(N_0 - \left\lceil \frac{\deg b}{2} \right\rceil\right). \tag{2.3}$$

We also require $\deg(auv) \le 2N_0$ for all $u, v \in \mathcal{P}_r$.

Fix $N \ge N_0$, let L satisfy the assumptions, and set $r_b := N - \lceil \deg b/2 \rceil$. Since every monomial of degree at most $2r_b$ is a product of two monomials of degree at most r_b, linearity gives $L(bh) = 0$ for every $h \in \mathcal{P}_{2r_b}$. In particular, (2.3) implies $L(bh_v) = 0$ for every $v \in \mathcal{U}$.

Applying L to (2.2), we obtain

$$L\big((av^2)^{2^{\ell_v}}\big) + \sum_{j=1}^{m_v} L(s_{v,j}^2) = 0.$$

Every term on the left is nonnegative, and hence $L((av^2)^{2^{\ell_v}}) = 0$.

Positivity of L gives the Cauchy–Schwarz inequality

$$|L(g)|^2 \le L(g^2)L(1) \qquad (g \in \mathcal{P}_N).$$

Applying it successively to

$$(av^2)^{2^{\ell_v - 1}},\ (av^2)^{2^{\ell_v - 2}}, \ldots, av^2$$

gives $L(av^2) = 0$ for every $v \in \mathcal{U}$. In particular,

$$L(au_i^2) = 0, \qquad L\big(a(u_i + u_j)^2\big) = 0 \quad (1 \le i < j \le m).$$

Polarization therefore gives

$$2L(au_iu_j) = L\big(a(u_i + u_j)^2\big) - L(au_i^2) - L(au_j^2) = 0.$$

Together with the diagonal identities, this shows that $L(au_iu_j) = 0$ for all $1 \le i, j \le m$. The conclusion $L(auv) = 0$ for arbitrary $u, v \in \mathcal{P}_r$ now follows by bilinearity. $\square$

The lemma shows that, after passing to a sufficiently high positive extension, the truncated annihilation of one defining polynomial implies the truncated annihilation of every polynomial vanishing on the same algebraic set. We now apply it to property $(S_{n,k})$.

Proposition 2.3. *Let $p, q \in \mathbb{R}[x_1, \ldots, x_d]$ satisfy*

$$\mathcal{Z}_p = \mathcal{Z}_q,$$

and let $n \ge \max\{\deg p, \deg q\}$. Then the following statements are equivalent:

(1) The generator family $\{1, p, -p\}$ satisfies property $(S_{n,k})$ for some $k \in \mathbb{N}_0$.

(2) The generator family $\{1, q, -q\}$ satisfies property $(S_{n,\widetilde{k}})$ for some $\widetilde{k} \in \mathbb{N}_0$.

Proof. By symmetry, it suffices to prove (1) $\Rightarrow$ (2). Suppose that $\{1, p, -p\}$ satisfies property $(S_{n,k})$ for some $k \in \mathbb{N}_0$, and set

$$r_p := n + k - \left\lceil \frac{\deg p}{2} \right\rceil.$$

Since $\mathcal{Z}_p = \mathcal{Z}_q$, we have $p \in I(\mathcal{Z}_q)$. Apply Lemma 2.2 with $a = p$, $b = q$, and $r = r_p$, and let N_0 be the integer provided by the lemma.

Choose $\widetilde{k} \in \mathbb{N}_0$ sufficiently large that, with $\widetilde{n} := n + \widetilde{k}$,

$$\widetilde{n} \ge \max\{n + k, N_0\}.$$

We claim that $\{1, q, -q\}$ satisfies property $(S_{n,\widetilde{k}})$.

Let $\beta^{(2n)}$ be a truncated sequence having the $\widetilde{k}$-extension property relative to $\{1, q, -q\}$. Thus, $M(n)$ admits a positive extension $M(\widetilde{n})$ satisfying

$$M_q(\widetilde{n}) \succeq 0, \qquad M_{-q}(\widetilde{n}) \succeq 0.$$

Since $M_{-q}(\widetilde{n}) = -M_q(\widetilde{n})$, these inequalities are equivalent to $M_q(\widetilde{n}) = 0$.

Let $L := L_{\beta^{(2\widetilde{n})}}$ be the Riesz functional associated with this extension. The identity $M_q(\widetilde{n}) = 0$ means that

$$L(qfg) = 0 \qquad \text{for all } f, g \in \mathcal{P}_{\widetilde{n}-\lceil \frac{\deg q}{2} \rceil}.$$

Since $\widetilde{n} \geq N_0$, Lemma 2.2 gives

$$L(puv) = 0 \qquad \text{for all } u, v \in \mathcal{P}_{r_p}.$$

Equivalently, $M_p(n+k) = 0$, and hence

$$M_p(n+k) \succeq 0, \qquad M_{-p}(n+k) \succeq 0.$$

Since $\widetilde{n} \geq n+k$, the truncation of $M(\widetilde{n})$ to degree $2(n+k)$ is a positive extension $M(n+k)$ of $M(n)$. Therefore, $\beta^{(2n)}$ has the k-extension property relative to $\{1, p, -p\}$. Since $\{1, p, -p\}$ satisfies property $(S_{n,k})$, the sequence $\beta^{(2n)}$ admits a $\mathcal{Z}_p$-representing measure. Because $\mathcal{Z}_p = \mathcal{Z}_q$, the same measure is also a $\mathcal{Z}_q$-representing measure.

We have thus proved that the $\widetilde{k}$-extension property relative to $\{1, q, -q\}$ implies the existence of a $\mathcal{Z}_q$-representing measure.

Conversely, suppose that $\beta^{(2n)}$ has a $\mathcal{Z}_q$-representing measure. By the Richter–Tchakaloff theorem [Ric57] (see also [Sch17, Theorem 1.24]), $\beta^{(2n)}$ has a finitely atomic representing measure supported on $\mathcal{Z}_q$. Such a measure has moments of every order and therefore defines positive moment extensions of every order. Since its support is contained in $\mathcal{Z}_q$, the corresponding localizing matrices satisfy $M_q(\widetilde{n}) = M_{-q}(\widetilde{n}) = 0$. Thus, $\beta^{(2n)}$ has the $\widetilde{k}$-extension property relative to $\{1, q, -q\}$.

It follows that $\{1, q, -q\}$ satisfies property $(S_{n,\widetilde{k}})$. This proves (1) ⇒ (2). The reverse implication follows by interchanging p and q. □

Remark 2.4. As an immediate consequence of Proposition 2.3, for every $r \in \mathbb{N}$ and every $n \geq r \deg p$, the existence of some finite k such that $\{1, p, -p\}$ satisfies property $(S_{n,k})$ is equivalent to the existence of some finite $\widetilde{k}$ such that $\{1, p^r, -p^r\}$ satisfies property $(S_{n,\widetilde{k}})$. More generally, if

$$p = c\, p_1^{m_1} \cdots p_s^{m_s}, \qquad c \in \mathbb{R} \setminus \{0\},$$

where $p_1, \ldots, p_s$ are irreducible polynomials and $m_1, \ldots, m_s \in \mathbb{N}$, then $\mathcal{Z}_p = \mathcal{Z}_{p_1 \cdots p_s}$. Therefore, when studying whether there exists some finite k such that $\{1, p, -p\}$ satisfies $(S_{n,k})$, it is enough to assume that all irreducible factors of p appear with multiplicity one. Proposition 2.3 preserves only the existence of some finite extension parameter; it does not imply that the same parameter works for p and its square-free part. Thus, the failure of property $(S_{n,k})$ for every $k \in \mathbb{N}_0$ is preserved when passing between p and its square-free part, whereas a positive result may involve different finite extension parameters.

The n**-th truncated quadratic module** generated by Q, denoted by $\Sigma_{Q,n} \subset \mathcal{P}_{2n}$, is defined as

$$\Sigma_{Q,n} := \left\{ \sum_{i=0}^{m} \sum_{j=1}^{N_i} q_i f_{ij}^2 : N_i \in \mathbb{N}_0,\ \deg(q_i f_{ij}^2) \leq 2n \right\}. \tag{2.4}$$

The following theorem relates the truncated quadratic modules $\Sigma_{Q,*}$ to property $(S_{n,k})$.

Theorem 2.5 ([CF08, Theorems 1.5, 1.6])**.** *With the notation above:*

(i) If Q satisfies $(S_{n,k})$, then every polynomial $p \in \mathcal{P}_{2n}$ that is strictly positive on K_Q belongs to $\Sigma_{Q,n+k}$.

(ii) If $k \geq 1$ and every polynomial $p \in \mathcal{P}_{2n+2}$ that is strictly positive on K_Q belongs to $\Sigma_{Q,n+k}$, then Q satisfies $(S_{n,k})$.

(iii) If K_Q is compact and every polynomial $p \in \mathcal{P}_{2n}$ that is strictly positive on K_Q belongs to $\Sigma_{Q,n}$, then Q satisfies $(S_{n,0})$.

Several generator families for which property $(S_{n,k})$ is known to hold are summarized in Table 2. In addition, for $d = 1$, the generator family $Q = \{1\}$, for which $K_Q = \mathbb{R}$, satisfies property $(S_{n,1})$ for every $n \in \mathbb{N}$ [CF91, Th. 3.9].

TABLE 2. Known cases in which $Q = \{1, p, -p\}$ satisfies property $(S_{n,k})$, where $n \geq \deg p$.

Polynomial p	Extension parameter k	Reference
$p(x) = (x-a)(b-x)$, $a < b$	0	[KN77, Th. II.2.3]
$p(x,y) = 1 - x^2 - y^2$	1	[CF08, Prop. 3.10]
$p(x,y) = ax + by + c$, $(a,b) \neq (0,0)$	1	[CF08, Prop. 3.11]
$p(x,y)$, $\deg p = 2$, defining an ellipse	1	[CF08, Prop. 3.13]
$p(x,y)$, $\deg p = 2$, defining a parabola or a hyperbola	2	[CF08, Prop. 3.13]
$p(x,y) = yq(x) - 1$, $\deg q \geq 1$	k_1	[Fia11, Th. 4.1]
$p(x,y) = yx^\ell - 1$, $\ell \geq 1$	$\ell + 1$	[Zal22b, Th. 4.1]
$p(x,y) = y - q(x)$, $\deg q \geq 3$	$\deg q - 1$	[Zal22b, Th. 3.1]

$k_1 := (2n+2)(2+\deg q) - (n+1+\deg q)$.

3. GENERATOR FAMILIES $\{1, p, -p\}$ FAILING PROPERTY $(S_{n,k})$

In this section, we study plane curves for which property $(S_{n,k})$ fails for every finite k. We begin with curves defined by $x^{\ell_1} - y^{\ell_2}$ and their monomial multiples (Theorems 3.1, 3.5, and 3.6), for which we construct explicit sequences exhibiting this failure. As a consequence, we obtain polynomials that are nonnegative on the corresponding curves but are not sums of squares in their coordinate rings; see Corollaries 3.3 and 3.7. We then consider the asymmetric families $p(x,y) = y(y - x^\ell)$ and $p(x,y) = xy(y - x^\ell)$ (Theorems 3.8 and 3.10) and show that the same obstruction persists. The corresponding nonnegative polynomials that are not sums of squares are given in Corollary 3.11.

Theorem 3.1. *Let $\ell_1, \ell_2 \in \mathbb{N} \setminus \{1\}$ be relatively prime, and let*

$$p(x,y) := x^{\ell_1} - y^{\ell_2}.$$

Then, for every $n \geq \max\{\ell_1, \ell_2\}$, there exists a truncated sequence $\beta^{(2n)}$ having the k-extension property relative to $\{1, p, -p\}$ for every $k \in \mathbb{N}$, but admitting no $\mathcal{Z}_p$-representing measure. Consequently, $\{1, p, -p\}$ does not satisfy property $(S_{n,k})$ for any $k \in \mathbb{N}_0$.

We first establish an auxiliary semigroup-Hankel construction that will be used in the proof of Theorem 3.1. It allows us to prescribe a negative moment while preserving the positive definiteness of every finite Hankel restriction indexed by the relevant additive semigroup.

Lemma 3.2. *Let $S \subseteq \mathbb{N}_0$ be a nonempty additive subsemigroup, and let $d \in \mathbb{N}_0$ satisfy $d/2 \notin S$. Then there exists a sequence*

$$\gamma = (\gamma_s)_{s \in (S+S) \cup \{d\}}$$

such that $\gamma_d < 0$ and, for every finite subset $F \subseteq S$, the matrix

$$H_\gamma|_F := (\gamma_{r+s})_{r,s \in F}$$

is positive definite.

Proof. Since $S \subseteq \mathbb{N}_0$, its elements can be enumerated in increasing order:

$$s_0 < s_1 < s_2 < \cdots .$$

If S is finite, the same argument applies and terminates after finitely many steps.

We prescribe an arbitrary negative value for γ_d and construct the remaining moments inductively so that

$$H_j := (\gamma_{s_a + s_b})_{a,b=0}^{j} \succ 0$$

at every stage.

Since $d/2 \notin S$, we have $2s_0 \neq d$. We may therefore choose $\gamma_{2s_0} > 0$, which gives $H_0 = (\gamma_{2s_0}) \succ 0$.

Suppose that $H_{j-1} \succ 0$ has already been constructed. After adjoining the index s_j, the enlarged matrix has the block form

$$H_j = \begin{pmatrix} H_{j-1} & v_j \\ v_j^{\mathsf{T}} & \gamma_{2s_j} \end{pmatrix},$$

where

$$v_j := \left(\gamma_{s_0+s_j}, \ldots, \gamma_{s_{j-1}+s_j}\right)^{\mathsf{T}}.$$

Some entries of v_j may already have been prescribed at earlier stages, and one of them may equal the distinguished moment γ_d. We retain all previously prescribed values and assign arbitrary values, say zero, to the remaining semigroup sums, using the same value whenever two sums coincide.

The diagonal moment γ_{2s_j} has not been prescribed at an earlier stage. Indeed, every moment occurring in H_{j-1} has an index of the form $s_a + s_b$ with $a, b < j$, and hence

$$s_a + s_b < 2s_j.$$

Moreover, $2s_j \neq d$, because $d/2 \notin S$. We may therefore choose γ_{2s_j} sufficiently large that

$$\gamma_{2s_j} > v_j^{\mathsf{T}} H_{j-1}^{-1} v_j.$$

By the Schur complement criterion, this implies $H_j \succ 0$.

Proceeding inductively, we obtain a consistently defined sequence γ on $(S+S) \cup \{d\}$ such that $H_j \succ 0$ at every stage. Now let $F \subseteq S$ be finite. Then $F \subseteq \{s_0, \ldots, s_j\}$ for some j, so $H_\gamma|_F$ is a principal submatrix of H_j. Therefore,

$$H_\gamma|_F \succ 0.$$

The prescribed inequality $\gamma_d < 0$ is preserved throughout the construction. $\square$

A concrete obstruction on the cusp $x^2 = y^3$. Before proving Theorem 3.1, we illustrate the construction for $p(x, y) = x^2 - y^3$ and $n = 4$. Thus,

$$\mathcal{Z}_p = \{(x, y) \in \mathbb{R}^2 : x^2 = y^3\}.$$

We shall construct a truncated bivariate sequence

$$\beta^{(8)} = \{\beta_{i,j} : i, j \in \mathbb{N}_0,\ i + j \leq 8\}$$

that has the k-extension property relative to $\{1, p, -p\}$ for every $k \in \mathbb{N}$, but admits no $\mathcal{Z}_p$-representing measure.

The curve $\mathcal{Z}_p$ has the parametrization $x = t^3$, $y = t^2$, where $t \in \mathbb{R}$. Hence $x^i y^j = t^{3i+2j}$. The associated additive submonoid is

$$S := \{3i + 2j : i, j \in \mathbb{N}_0\} = \{0, 2, 3, 4, \ldots\}.$$

Since $2 \in S$, whereas $2/2 = 1 \notin S$, Lemma 3.2, applied with $d = 2$, yields a sequence $\gamma = (\gamma_s)_{s \in S}$ such that

$$\gamma_2 < 0 \qquad \text{and} \qquad H_\gamma|_F := (\gamma_{r+s})_{r,s \in F} \succ 0 \tag{3.1}$$

for every finite subset $F \subseteq S$.

Define an infinite bivariate sequence by

$$\beta_{i,j} := \gamma_{3i+2j}, \qquad i, j \in \mathbb{N}_0.$$

Its truncation to total degree eight is the sequence $\beta^{(8)}$ considered above.

For $r \in \mathbb{N}_0$, set

$$S_r := \{3i + 2j : i, j \in \mathbb{N}_0,\ i + j \leq r\}.$$

In particular,

$$S_4 = \{0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

The relation $X^2 = Y^3$ generates, among others, the relations $X^3 = XY^3$ and $X^2Y = Y^4$. We may therefore choose

$$\mathcal{B}_4 = \{1, Y, X, Y^2, XY, Y^3, XY^2, Y^4, XY^3, X^2Y^2, X^3Y, X^4\}$$

as a set containing one representative monomial for each exponent in S_4.

Under the parametrization $X = t^3$, $Y = t^2$, the elements of $\mathcal{B}_4$ correspond, in the displayed order, to

$$1, t^2, t^3, t^4, t^5, t^6, t^7, t^8, t^9, t^{10}, t^{11}, t^{12}.$$

Consequently, the restriction of $M(4; \beta)$ to the rows and columns indexed by $\mathcal{B}_4$ is equal, up to a simultaneous permutation of rows and columns, to

$$H_\gamma|_{S_4} = (\gamma_{r+s})_{r,s \in S_4}.$$

Indeed, if $X^aY^b, X^cY^d \in \mathcal{B}_4$, then

$$\beta_{a+c,b+d} = \gamma_{3(a+c)+2(b+d)} = \gamma_{(3a+2b)+(3c+2d)}.$$

By (3.1), $H_\gamma|_{S_4} \succ 0$.

More generally, for every $r \geq 4$, choose one monomial X^iY^j of degree at most r for each exponent in S_r. The corresponding principal restriction of $M(r; \beta)$ is, up to a simultaneous permutation of rows and columns, equal to $H_\gamma|_{S_r}$, which is positive definite by (3.1). The remaining moments are then assigned consistently so that every monomial column not indexed by $\mathcal{B}_r$ coincides with the representative column having the same parametrized exponent. It follows that

$$M(r; \beta) \succeq 0 \qquad \text{for every } r \geq 4.$$

Moreover, for all $i, j \in \mathbb{N}_0$,

$$\beta_{i+2,j} = \gamma_{3(i+2)+2j} = \gamma_{3i+2j+6} = \gamma_{3i+2(j+3)} = \beta_{i,j+3}.$$

Thus $X^2 = Y^3$ is a recursively generated column relation in every moment matrix $M(r;\beta)$. Consequently,

$$M_p(r) = M_{-p}(r) = 0$$

whenever these localizing matrices are defined.

Therefore, for every $k \in \mathbb{N}$, the matrix $M(4+k;\beta)$ is a positive extension of $M(4;\beta)$ and satisfies

$$M_p(4+k) \succeq 0, \qquad M_{-p}(4+k) \succeq 0.$$

Hence $\beta^{(8)}$ has the k-extension property relative to $\{1, p, -p\}$ for every $k \in \mathbb{N}$.

It remains to show that $\beta^{(8)}$ admits no $\mathcal{Z}_p$-representing measure. Suppose, to the contrary, that μ is such a measure. Every point of $\mathcal{Z}_p$ has the form (t^3, t^2), and therefore $y = t^2 \geq 0$ on $\mathcal{Z}_p$. It follows that

$$\beta_{0,1} = \int_{\mathcal{Z}_p} y \, d\mu \geq 0.$$

On the other hand, (3.1) gives

$$\beta_{0,1} = \gamma_2 < 0,$$

which is a contradiction. Thus $\beta^{(8)}$ admits no $\mathcal{Z}_p$-representing measure.

This example is the even-odd case of Theorem 3.1, with $\ell_1 = 2$, $\ell_2 = 3$, and obstruction index $d = \ell_1 = 2$.

We now apply the preceding semigroup-Hankel construction to prove Theorem 3.1 for arbitrary relatively prime integers $\ell_1, \ell_2 > 1$.

Proof of Theorem 3.1. Parametrize $\mathcal{Z}_p$ by $x = t^{\ell_2}$, $y = t^{\ell_1}$, where $t \in \mathbb{R}$, and set

$$S := \{i\ell_2 + j\ell_1 : i, j \in \mathbb{N}_0\}.$$

We construct an infinite bivariate sequence $\beta = (\beta_{i,j})_{i,j \in \mathbb{N}_0}$ such that every moment matrix $M(r;\beta)$ is positive semidefinite and $M_p(r) = M_{-p}(r) = 0$, but whose truncation $\beta^{(2n)}$ admits no $\mathcal{Z}_p$-representing measure.

We distinguish three cases.

Case 1. Suppose that ℓ_1 is odd and ℓ_2 is even, and set $d := \ell_2$. Then $d \in S$, whereas $d/2 \notin S$. Indeed, if $\ell_2 < \ell_1$, then $0 < \ell_2/2 < \ell_2 < \ell_1$, so $\ell_2/2$ cannot be a nonzero element of S. If $\ell_1 < \ell_2$ and

$$\frac{\ell_2}{2} = a\ell_2 + b\ell_1 \qquad (a, b \in \mathbb{N}_0),$$

then $a = 0$, and hence $\ell_2 = 2b\ell_1$. This contradicts $\gcd(\ell_1, \ell_2) = 1$ and $\ell_1 > 1$.

Case 2. Suppose that ℓ_1 is even and ℓ_2 is odd, and set $d := \ell_1$. By the same argument, with ℓ_1 and ℓ_2 interchanged, $d \in S$ and $d/2 \notin S$.

Case 3. Suppose that both ℓ_1 and ℓ_2 are odd, and set $d := \ell_1 + \ell_2$. Clearly, $d \in S$. We claim that $d/2 \notin S$. Otherwise, there would exist $a, b \in \mathbb{N}_0$ such that $\frac{\ell_1+\ell_2}{2} = a\ell_2 + b\ell_1$, or equivalently, $(2a-1)\ell_2 = (1-2b)\ell_1$. If $a, b \geq 1$, the two sides have opposite signs. If $a = 0$, then $\ell_2 = (2b-1)\ell_1$, contradicting coprimality and $\ell_1 > 1$. Similarly, if $b = 0$, then $\ell_1 = (2a-1)\ell_2$, which yields the same contradiction. Therefore, $d/2 \notin S$.

In all three cases, $d \in S$ and $d/2 \notin S$. Hence Lemma 3.2 provides a sequence

$$\gamma = (\gamma_s)_{s \in (S+S) \cup \{d\}}$$

such that

$$\gamma_d < 0 \qquad \text{and} \qquad (\gamma_{u+v})_{u,v \in F} \succ 0 \quad \text{for every finite subset } F \subseteq S. \tag{3.2}$$

Since $0 \in S$, we have $S + S = S$. Moreover, $d \in S$, and therefore

$$(S+S) \cup \{d\} = S.$$

Thus we may regard γ simply as a sequence $\gamma = (\gamma_s)_{s \in S}$.

Define

$$\beta_{i,j} := \gamma_{i\ell_2 + j\ell_1}, \qquad i, j \in \mathbb{N}_0.$$

For $r \in \mathbb{N}_0$, let

$$S_r := \{i\ell_2 + j\ell_1 : i, j \in \mathbb{N}_0,\ i + j \le r\}.$$

Choose one monomial X^iY^j of degree at most r for each element of S_r, and denote the resulting set by $\mathcal{B}_r$. Up to a simultaneous permutation of rows and columns, the restriction of $M(r;\beta)$ to $\mathcal{B}_r$ is

$$(\gamma_{u+v})_{u,v \in S_r},$$

which is positive definite by (3.2).

Every monomial of degree at most r has the same image under the parametrization as exactly one representative in $\mathcal{B}_r$. Consequently, every column of $M(r;\beta)$ coincides with one of the columns indexed by $\mathcal{B}_r$. It follows that

$$M(r;\beta) \succeq 0 \qquad \text{for every } r \in \mathbb{N}_0.$$

For all $i, j \in \mathbb{N}_0$, we have

$$\beta_{i+\ell_1,j} = \gamma_{i\ell_2 + j\ell_1 + \ell_1\ell_2} = \beta_{i,j+\ell_2}.$$

Thus $X^{\ell_1} = Y^{\ell_2}$ is a recursively generated column relation in every moment matrix $M(r;\beta)$. Consequently, $M_p(r) = M_{-p}(r) = 0$ whenever these localizing matrices are defined.

For every $k \in \mathbb{N}$, the matrix $M(n+k;\beta)$ is therefore a positive extension of $M(n;\beta)$ satisfying

$$M_p(n+k) \succeq 0, \qquad M_{-p}(n+k) \succeq 0.$$

Hence $\beta^{(2n)}$ has the k-extension property relative to $\{1, p, -p\}$ for every $k \in \mathbb{N}$.

Define

$$f := \begin{cases} x, & \text{in Case 1,} \\ y, & \text{in Case 2,} \\ xy, & \text{in Case 3.} \end{cases}$$

By construction, $L_\beta(f) = \gamma_d < 0$. Under the parametrization $x = t^{\ell_2}$, $y = t^{\ell_1}$, we have

$$f(t^{\ell_2}, t^{\ell_1}) = \begin{cases} \left(t^{\ell_2/2}\right)^2, & \text{if } \ell_1 \text{ is odd and } \ell_2 \text{ is even,} \\ \left(t^{\ell_1/2}\right)^2, & \text{if } \ell_1 \text{ is even and } \ell_2 \text{ is odd,} \\ \left(t^{(\ell_1+\ell_2)/2}\right)^2, & \text{if } \ell_1 \text{ and } \ell_2 \text{ are both odd.} \end{cases}$$

Hence $f \ge 0$ on $\mathcal{Z}_p$. If $\beta^{(2n)}$ admitted a $\mathcal{Z}_p$-representing measure μ, then

$$0 > L_\beta(f) = \int_{\mathcal{Z}_p} f \, d\mu \ge 0,$$

a contradiction. Therefore, $\beta^{(2n)}$ admits no $\mathcal{Z}_p$-representing measure, although it has the k-extension property relative to $\{1, p, -p\}$ for every $k \in \mathbb{N}_0$. $\square$

Corollary 3.3. *Let* $\ell_1, \ell_2 \in \mathbb{N} \setminus \{1\}$ *satisfy* $\gcd(\ell_1, \ell_2) = 1$*, and let* $p(x,y) := x^{\ell_1} - y^{\ell_2}$*. Define*

$$f(x,y) := \begin{cases} x, & \textit{if } \ell_1 \textit{ is odd and } \ell_2 \textit{ is even}, \\ y, & \textit{if } \ell_1 \textit{ is even and } \ell_2 \textit{ is odd}, \\ xy, & \textit{if } \ell_1 \textit{ and } \ell_2 \textit{ are both odd}. \end{cases}$$

Then f *is nonnegative on* $\mathcal{Z}_p$*, but*

$$f \notin \Sigma_{\{1,p,-p\},n} \qquad \textit{for every } n \in \mathbb{N}.$$

Equivalently, the class of f *is not a sum of squares in the coordinate ring* $\mathbb{R}[x,y]/(p)$*.*

Proof. The computation at the end of the proof of Theorem 3.1 shows that, under the parametrization $x = t^{\ell_2}$, $y = t^{\ell_1}$, the polynomial $f(t^{\ell_2}, t^{\ell_1})$ is a square in each of the three parity cases. Hence $f \geq 0$ on $\mathcal{Z}_p$.

Let L_β be the Riesz functional constructed in that proof. By construction, $L_\beta(g^2) \geq 0$ and $L_\beta(ph) = 0$ for all $g, h \in \mathbb{R}[x,y]$, while $L_\beta(f) < 0$.

Suppose that the class of f were a sum of squares in $\mathbb{R}[x,y]/(p)$. Then $f = \sum_{i=1}^r g_i^2 + ph$ for some $g_1, \ldots, g_r, h \in \mathbb{R}[x,y]$. Applying L_β gives

$$0 > L_\beta(f) = \sum_{i=1}^r L_\beta(g_i^2) \geq 0,$$

a contradiction. Thus the class of f is not a sum of squares in $\mathbb{R}[x,y]/(p)$. In particular, $f \notin \Sigma_{\{1,p,-p\},n}$ for every $n \in \mathbb{N}$, since any such representation would give a sum-of-squares representation of the class of f modulo (p). $\square$

Remark 3.4. The polynomials in Corollary 3.3 arise from the simplest obstruction used in the proof of Theorem 3.1: a moment $\gamma_d < 0$ whose half-index $d/2$ does not belong to the additive semigroup

$$S = \{i\ell_2 + j\ell_1 : i, j \in \mathbb{N}_0\}.$$

Although the semigroup-Hankel matrices indexed by finite subsets of S are positive definite, γ_d would be a negative diagonal entry in the full Hankel matrix obtained by adjoining the monomial $T^{d/2}$. The corresponding polynomial x, y, or xy is nonnegative on $\mathcal{Z}_p$, but is evaluated negatively by L_β.

The same separating-functional argument produces further nonnegative polynomials that are not sums of squares in the coordinate ring. Let $a_1, \ldots, a_r \in \mathbb{N}_0$ be such that $a_i + a_j \in S$ for all i, j, and consider

$$G := (\gamma_{a_i + a_j})_{i,j=1}^r.$$

Suppose that G has a negative eigenvalue, and let $v = (v_1, \ldots, v_r)^\mathsf{T}$ be a corresponding eigenvector. If, under the parametrization $x = t^{\ell_2}$, $y = t^{\ell_1}$, the square

$$(v_1 t^{a_1} + \cdots + v_r t^{a_r})^2$$

is represented by a polynomial $f(x,y)$, then f is nonnegative on $\mathcal{Z}_p$. On the other hand,

$$L_\beta(f) = v^\mathsf{T} G v < 0.$$

The argument from Corollary 3.3 therefore shows that the class of f is not a sum of squares in $\mathbb{R}[x,y]/(p)$.

For a concrete example, suppose that ℓ_1 is odd and ℓ_2 is even, and consider

$$G := \begin{pmatrix} \gamma_{\ell_2} & \gamma_{\ell_1\ell_2/2} \\ \gamma_{\ell_1\ell_2/2} & \gamma_{(\ell_1-1)\ell_2} \end{pmatrix}.$$

Since its first diagonal entry satisfies $\gamma_{\ell_2} < 0$, the matrix G is not positive semidefinite. Let $v = (v_1, v_2)^{\mathsf{T}}$ be an eigenvector corresponding to a negative eigenvalue of G, and define

$$f_v(x,y) := v_1^2 x + 2v_1v_2 y^{\ell_2/2} + v_2^2 x^{\ell_1-1}.$$

Under the parametrization of $\mathcal{Z}_p$, this polynomial satisfies

$$\begin{aligned} f_v(t^{\ell_2}, t^{\ell_1}) &= v_1^2 t^{\ell_2} + 2v_1v_2 t^{\ell_1\ell_2/2} + v_2^2 t^{(\ell_1-1)\ell_2} \\ &= \left(v_1 t^{\ell_2/2} + v_2 t^{(\ell_1-1)\ell_2/2}\right)^2. \end{aligned}$$

Hence $f_v \geq 0$ on $\mathcal{Z}_p$. Nevertheless, $L_\beta(f_v) = v^{\mathsf{T}} G v < 0$. Therefore, the class of f_v is not a sum of squares in $\mathbb{R}[x,y]/(p)$.

We next show that the obstruction established in Theorem 3.1 persists after multiplying the defining polynomial by either coordinate variable.

Theorem 3.5. *Let $\ell_1, \ell_2 \in \mathbb{N} \setminus \{1\}$ satisfy $\gcd(\ell_1, \ell_2) = 1$, and let*

$$p(x,y) = q(x,y)\big(x^{\ell_1} - y^{\ell_2}\big), \qquad q(x,y) \in \{x, y\}.$$

Then, for every $n \geq \max\{\ell_1+1, \ell_2+1\}$, there exists a truncated sequence $\beta^{(2n)}$ having the k-extension property relative to $\{1, p, -p\}$ for every $k \in \mathbb{N}_0$, but admitting no $\mathcal{Z}_p$-representing measure.

Proof. By interchanging x and y, it suffices to consider $q(x,y) = y$, so that

$$p(x,y) = y\big(x^{\ell_1} - y^{\ell_2}\big).$$

Its zero set is

$$\mathcal{Z}_p = \mathcal{Z}_y \cup \mathcal{Z}_{x^{\ell_1}-y^{\ell_2}}.$$

The second component is parametrized by $x = t^{\ell_2}$, $y = t^{\ell_1}$, $t \in \mathbb{R}$. Consider the additive subsemigroup

$$S_+ := \{a\ell_2 + b\ell_1 : a \in \mathbb{N}_0,\ b \in \mathbb{N}\}.$$

We first record an auxiliary construction.

Auxiliary lemma. Suppose that $d \in S_+$ and $d/2 \notin S_+$. Then there exists an infinite bivariate sequence $\beta = (\beta_{i,j})_{i,j\in\mathbb{N}_0}$ such that

$$M(r;\beta) \succeq 0 \quad (r \in \mathbb{N}_0) \qquad \text{and} \qquad \beta_{i+\ell_1,j+1} = \beta_{i,j+\ell_2+1} \quad (i,j \in \mathbb{N}_0),$$

and $\beta_{a,b} < 0$ whenever $a \in \mathbb{N}_0$, $b \in \mathbb{N}$, and $a\ell_2 + b\ell_1 = d$.

Proof of the auxiliary lemma. Introduce

$$\mathcal{T} := \{\mathsf{P}_a : a \in \mathbb{N}_0\} \,\dot\cup\, \{\mathsf{M}_s : s \in S_+\},$$

with the commutative semigroup operation defined by $\mathsf{P}_a + \mathsf{P}_b := \mathsf{P}_{a+b}$, $\mathsf{P}_a + \mathsf{M}_s := \mathsf{M}_{a\ell_2+s}$, and $\mathsf{M}_s + \mathsf{M}_t := \mathsf{M}_{s+t}$. Thus P_a represents the pure monomial X^a, whereas M_s represents the class of monomials X^aY^b with $b \geq 1$ and $a\ell_2 + b\ell_1 = s$. In particular, pure and mixed classes are kept distinct.

Define $w(\mathsf{P}_a) := a\ell_2$ and $w(\mathsf{M}_s) := s$. Enumerate the elements of $\mathcal{T}$ as $\tau_0, \tau_1, \tau_2, \ldots$ in nondecreasing order of weight, with the mixed element placed before the pure element whenever the two have the same weight.

We construct $\varphi : \mathcal{T} \to \mathbb{R}$ such that $\varphi(\mathsf{M}_d) < 0$ and

$$\big(\varphi(\sigma+\tau)\big)_{\sigma,\tau\in F} \succ 0$$

for every finite $F \subseteq \mathcal{T}$.

Prescribe an arbitrary negative value for $\varphi(\mathsf{M}_d)$. Suppose inductively that

$$H_{j-1} := \big(\varphi(\tau_a+\tau_b)\big)_{a,b=0}^{j-1} \succ 0$$

has already been constructed. After adjoining τ_j, write

$$H_j = \begin{pmatrix} H_{j-1} & v_j \\ v_j^T & \varphi(2\tau_j) \end{pmatrix}.$$

Retain the values in v_j that have already been prescribed, and assign arbitrary values, say zero, to the remaining entries.

The diagonal value $\varphi(2\tau_j)$ has not been prescribed at an earlier stage. Indeed, if $2\tau_j = \tau_a + \tau_b$ for some $a, b < j$, then equality of weights forces $w(\tau_a) = w(\tau_b) = w(\tau_j)$. If τ_j is mixed, this is impossible because the mixed element is placed first at each weight. If τ_j is pure, the only possible earlier element of the same weight is mixed, but the sum of two mixed elements is mixed whereas $2\tau_j$ is pure. Moreover, $2\tau_j \neq \mathsf{M}_d$: this is clear when τ_j is pure, while in the mixed case equality would imply $d/2 \in S_+$.

Hence $\varphi(2\tau_j)$ is free, and we choose it so that

$$\varphi(2\tau_j) > v_j^T H_{j-1}^{-1} v_j.$$

The Schur complement criterion gives $H_j \succ 0$. Proceeding inductively yields the desired function φ.

Now define

$$\beta_{i,0} := \varphi(\mathsf{P}_i), \qquad \beta_{i,j} := \varphi\big(\mathsf{M}_{i\ell_2+j\ell_1}\big) \quad (j \geq 1).$$

Associate to each monomial X^iY^j the label

$$\lambda(i,j) := \begin{cases} \mathsf{P}_i, & j = 0, \\ \mathsf{M}_{i\ell_2+j\ell_1}, & j \geq 1. \end{cases}$$

Then $\lambda(i,j) + \lambda(a,b) = \lambda(i+a, j+b)$, and therefore

$$\beta_{i+a,j+b} = \varphi\big(\lambda(i,j) + \lambda(a,b)\big).$$

Thus every moment matrix $M(r;\beta)$ is the pullback of a finite positive definite matrix

$$\big(\varphi(\sigma+\tau)\big)_{\sigma,\tau\in F},$$

and hence

$$M(r;\beta) \succeq 0 \qquad (r \in \mathbb{N}_0).$$

Furthermore, for all $i, j \in \mathbb{N}_0$,

$$\beta_{i+\ell_1,j+1} = \beta_{i,j+\ell_2+1},$$

because the two moments correspond to the same mixed label. Finally, if $d = a\ell_2 + b\ell_1$ with $b \geq 1$, then $\beta_{a,b} = \varphi(\mathsf{M}_d) < 0$. This proves the auxiliary lemma. ■

We now choose the obstruction index d. Suppose first that ℓ_1 is odd and ℓ_2 is even, and set

$$d := \ell_2 + 2\ell_1.$$

Then $d \in S_+$. If $d/2 \in S_+$, then $\ell_2/2 + \ell_1 = a\ell_2 + b\ell_1$ for some $a \in \mathbb{N}_0$, $b \in \mathbb{N}$, and hence $(2a-1)\ell_2 = 2(1-b)\ell_1$. If $a \geq 1$, the two sides have opposite signs; hence $a = 0$, which gives $\ell_2 = 2(b-1)\ell_1$, contradicting coprimality and $\ell_1 > 1$. Thus $d/2 \notin S_+$.

Suppose next that ℓ_1 is even and ℓ_2 is odd, and set $d := \ell_1$. Then $d \in S_+$ and $d/2 \notin S_+$, since every element of S_+ is at least ℓ_1.

Finally, suppose that both ℓ_1 and ℓ_2 are odd, and set $d := \ell_1 + \ell_2$. If $d/2 \in S_+$, then $(\ell_1+\ell_2)/2 = a\ell_2 + b\ell_1$ for some $a \in \mathbb{N}_0$, $b \in \mathbb{N}$, and hence $(2a-1)\ell_2 = (1-2b)\ell_1$. Since $b \geq 1$, the right-hand side is negative, so $a = 0$, which gives $\ell_2 = (2b-1)\ell_1$, again contradicting coprimality and $\ell_1 > 1$. Thus $d/2 \notin S_+$.

In each case, the auxiliary lemma gives an infinite sequence $\beta = (\beta_{i,j})_{i,j\in\mathbb{N}_0}$ satisfying

$$M(r;\beta) \succeq 0 \qquad (r \in \mathbb{N}_0)$$

and

$$\beta_{i+\ell_1,j+1} = \beta_{i,j+\ell_2+1} \qquad (i,j \in \mathbb{N}_0).$$

Hence

$$Y(X^{\ell_1} - Y^{\ell_2}) = 0$$

is a recursively generated column relation at every order, and therefore

$$M_p(r) = M_{-p}(r) = 0$$

whenever the localizing matrices are defined.

Define

$$f := \begin{cases} xy^2, & \ell_1 \text{ odd},\ \ell_2 \text{ even}, \\ y, & \ell_1 \text{ even},\ \ell_2 \text{ odd}, \\ xy, & \ell_1, \ell_2 \text{ odd}. \end{cases}$$

By construction, $L_\beta(f) < 0$. Under the parametrization $x = t^{\ell_2}$, $y = t^{\ell_1}$,

$$f(t^{\ell_2}, t^{\ell_1}) = \begin{cases} \left(t^{\ell_2/2+\ell_1}\right)^2, & \ell_1 \text{ odd},\ \ell_2 \text{ even}, \\ \left(t^{\ell_1/2}\right)^2, & \ell_1 \text{ even},\ \ell_2 \text{ odd}, \\ \left(t^{(\ell_1+\ell_2)/2}\right)^2, & \ell_1, \ell_2 \text{ odd}. \end{cases}$$

Moreover, f vanishes on $\mathcal{Z}_y$. Hence $f \geq 0$ on $\mathcal{Z}_p$, whereas $L_\beta(f) < 0$.

Let $\beta^{(2n)}$ be the degree-$2n$ truncation of β. Since all moment matrices are positive semidefinite and $M_p(r) = M_{-p}(r) = 0$ at every order, $M(n+k;\beta)$ is a positive extension of $M(n;\beta)$ satisfying the required localizing conditions for every $k \in \mathbb{N}_0$. Thus $\beta^{(2n)}$ has the k-extension property relative to $\{1, p, -p\}$ for every $k \in \mathbb{N}_0$.

If $\beta^{(2n)}$ admitted a $\mathcal{Z}_p$-representing measure μ, then

$$0 > L_\beta(f) = \int_{\mathcal{Z}_p} f\, d\mu \geq 0,$$

a contradiction. Therefore, $\beta^{(2n)}$ admits no $\mathcal{Z}_p$-representing measure. □

We next show that the same obstruction persists when both coordinate axes are adjoined to the monomial curve.

Theorem 3.6. *Let $\ell_1, \ell_2 \in \mathbb{N} \setminus \{1\}$ satisfy $\gcd(\ell_1, \ell_2) = 1$, and let*

$$p(x, y) := xy\big(x^{\ell_1} - y^{\ell_2}\big).$$

Then, for every $n \geq \max\{\ell_1 + 2, \ell_2 + 2\}$, there exists a truncated sequence $\beta^{(2n)}$ having the k-extension property relative to $\{1, p, -p\}$ for every $k \in \mathbb{N}_0$, but admitting no $\mathcal{Z}_p$-representing measure.

Proof. The zero set of p is

$$\mathcal{Z}_p = \mathcal{Z}_x \cup \mathcal{Z}_y \cup \mathcal{Z}_{x^{\ell_1} - y^{\ell_2}}.$$

The last component is parametrized by $x = t^{\ell_2}$, $y = t^{\ell_1}$, $t \in \mathbb{R}$. Consider the additive subsemigroup

$$S_{++} := \{a\ell_2 + b\ell_1 : a, b \in \mathbb{N}\}.$$

Its elements are precisely the exponents associated with monomials having positive powers of both variables.

We first record an auxiliary construction.

Auxiliary lemma. Suppose that $d \in S_{++}$ and $d/2 \notin S_{++}$. Then there exists an infinite bivariate sequence $\beta = (\beta_{i,j})_{i,j \in \mathbb{N}_0}$ such that

$$M(r; \beta) \succeq 0 \quad (r \in \mathbb{N}_0) \qquad \text{and} \qquad \beta_{i+\ell_1+1, j+1} = \beta_{i+1, j+\ell_2+1} \quad (i, j \in \mathbb{N}_0),$$

and $\beta_{a,b} < 0$ whenever $a, b \in \mathbb{N}$ and $a\ell_2 + b\ell_1 = d$.

Proof of the auxiliary lemma. Introduce the set

$$\mathcal{T} := \{\mathsf{e}\} \,\dot\cup\, \{\mathsf{X}_a : a \in \mathbb{N}\} \,\dot\cup\, \{\mathsf{Y}_b : b \in \mathbb{N}\} \,\dot\cup\, \{\mathsf{M}_s : s \in S_{++}\}.$$

We make $\mathcal{T}$ into a commutative semigroup with identity e by setting

$$\begin{aligned} \mathsf{X}_a + \mathsf{X}_c &:= \mathsf{X}_{a+c}, & \mathsf{Y}_b + \mathsf{Y}_e &:= \mathsf{Y}_{b+e}, \\ \mathsf{X}_a + \mathsf{Y}_b &:= \mathsf{M}_{a\ell_2 + b\ell_1}, & \mathsf{X}_a + \mathsf{M}_s &:= \mathsf{M}_{a\ell_2 + s}, \\ \mathsf{Y}_b + \mathsf{M}_s &:= \mathsf{M}_{b\ell_1 + s}, & \mathsf{M}_s + \mathsf{M}_t &:= \mathsf{M}_{s+t}. \end{aligned}$$

Thus X_a and Y_b represent the pure monomials X^a and Y^b, whereas M_s represents the class of mixed monomials X^aY^b, $a, b \geq 1$, having parametrized exponent $a\ell_2 + b\ell_1 = s$. In particular, pure and mixed classes are kept distinct.

Define a weight on $\mathcal{T}$ by

$$w(\mathsf{e}) := 0, \qquad w(\mathsf{X}_a) := a\ell_2, \qquad w(\mathsf{Y}_b) := b\ell_1, \qquad w(\mathsf{M}_s) := s.$$

Enumerate $\mathcal{T}$ as $\tau_0, \tau_1, \tau_2, \ldots$ in nondecreasing order of weight, with $\tau_0 = \mathsf{e}$, and with the mixed element placed before the pure elements whenever several elements have the same weight.

We construct a function $\varphi : \mathcal{T} \to \mathbb{R}$ such that $\varphi(\mathsf{M}_d) < 0$ and

$$\big(\varphi(\sigma + \tau)\big)_{\sigma, \tau \in F} \succ 0$$

for every finite subset $F \subseteq \mathcal{T}$.

Prescribe an arbitrary negative value for $\varphi(\mathsf{M}_d)$. Suppose inductively that

$$H_{j-1} := \big(\varphi(\tau_a + \tau_b)\big)_{a,b=0}^{j-1} \succ 0$$

has already been constructed. After adjoining τ_j, write

$$H_j = \begin{pmatrix} H_{j-1} & v_j \\ v_j^T & \varphi(2\tau_j) \end{pmatrix}.$$

We retain all values in v_j that have already been prescribed and assign arbitrary values, say zero, to the remaining semigroup sums, using the same value whenever two sums coincide.

The diagonal value $\varphi(2\tau_j)$ is still free. Indeed, if $2\tau_j$ coincided with a sum of two elements already present at this stage, other than $\tau_j + \tau_j$, equality of weights would force both summands to have the same weight as τ_j. If $\tau_j = \mathsf{M}_s$ is mixed, this is impossible because the mixed element is placed first among the elements of weight s. If $\tau_j = \mathsf{X}_a$ is pure, only the sum of two pure X-elements can be pure of type X, and there is only one pure X-element of a given weight. The same argument applies to a pure Y-element. Finally, if $\tau_j = \mathsf{e}$, its diagonal is e itself and is chosen positive at the initial step.

Moreover, $2\tau_j \neq \mathsf{M}_d$. This is immediate when τ_j is pure or $\tau_j = \mathsf{e}$. If $\tau_j = \mathsf{M}_s$, equality would imply $2s = d$, contrary to $d/2 \notin S_{++}$.

Hence $\varphi(2\tau_j)$ can be chosen sufficiently large that

$$\varphi(2\tau_j) > v_j^T H_{j-1}^{-1} v_j.$$

The Schur complement criterion gives $H_j \succ 0$. Proceeding inductively yields φ such that every finite matrix $\big(\varphi(\sigma+\tau)\big)_{\sigma,\tau\in F}$ is positive definite.

Associate to each monomial X^iY^j the label

$$\lambda(i,j) := \begin{cases} \mathsf{e}, & i = j = 0, \\ \mathsf{X}_i, & i \geq 1,\ j = 0, \\ \mathsf{Y}_j, & i = 0,\ j \geq 1, \\ \mathsf{M}_{i\ell_2 + j\ell_1}, & i,j \geq 1. \end{cases}$$

Then

$$\lambda(i,j) + \lambda(a,b) = \lambda(i+a, j+b)$$

for all $i,j,a,b \in \mathbb{N}_0$. Define

$$\beta_{i,j} := \varphi\big(\lambda(i,j)\big).$$

It follows that

$$\beta_{i+a,j+b} = \varphi\big(\lambda(i,j) + \lambda(a,b)\big).$$

Thus every moment matrix $M(r;\beta)$ is the pullback of a finite positive definite matrix

$$\big(\varphi(\sigma+\tau)\big)_{\sigma,\tau\in F},$$

and hence

$$M(r;\beta) \succeq 0 \qquad (r \in \mathbb{N}_0).$$

Furthermore, for all $i,j \in \mathbb{N}_0$,

$$\beta_{i+\ell_1+1,j+1} = \beta_{i+1,j+\ell_2+1},$$

since both moments correspond to the mixed label with exponent

$$(i+1)\ell_2 + (j+1)\ell_1 + \ell_1\ell_2.$$

Finally, if $d = a\ell_2 + b\ell_1$ with $a,b \in \mathbb{N}$, then $\beta_{a,b} = \varphi(\mathsf{M}_d) < 0$. This proves the auxiliary lemma. ■

We now choose the obstruction index d, distinguishing three parity cases.

Suppose first that ℓ_1 is odd and ℓ_2 is even, and set

$$d := \ell_2 + 2\ell_1.$$

Then $d \in S_{++}$, corresponding to the moment $\beta_{1,2}$, while

$$\frac{d}{2} = \frac{\ell_2}{2} + \ell_1 < \ell_2 + \ell_1.$$

Since every element of S_{++} is at least $\ell_1 + \ell_2$, it follows that $d/2 \notin S_{++}$.

Suppose next that ℓ_1 is even and ℓ_2 is odd, and set

$$d := 2\ell_2 + \ell_1.$$

Then $d \in S_{++}$, corresponding to $\beta_{2,1}$, whereas $d/2 = \ell_2 + \ell_1/2 < \ell_1 + \ell_2$. Hence $d/2 \notin S_{++}$.

Finally, suppose that both ℓ_1 and ℓ_2 are odd, and set

$$d := \ell_1 + \ell_2.$$

Then $d \in S_{++}$, corresponding to $\beta_{1,1}$, and $d/2 < \ell_1 + \ell_2$, so again $d/2 \notin S_{++}$.

In each case, the auxiliary lemma gives a single infinite sequence $\beta = (\beta_{i,j})_{i,j \in \mathbb{N}_0}$ satisfying

$$M(r;\beta) \succeq 0 \quad (r \in \mathbb{N}_0) \qquad \text{and} \qquad \beta_{i+\ell_1+1,j+1} = \beta_{i+1,j+\ell_2+1} \quad (i,j \in \mathbb{N}_0).$$

Thus

$$XY(X^{\ell_1} - Y^{\ell_2}) = 0$$

is a recursively generated column relation at every order. Consequently,

$$M_p(r) = M_{-p}(r) = 0$$

whenever the localizing matrices are defined.

Define

$$f := \begin{cases} xy^2, & \ell_1 \text{ odd}, \ \ell_2 \text{ even}, \\ x^2y, & \ell_1 \text{ even}, \ \ell_2 \text{ odd}, \\ xy, & \ell_1, \ell_2 \text{ odd}. \end{cases}$$

By construction, $L_\beta(f) < 0$. On the component $\mathcal{Z}_{x^{\ell_1} - y^{\ell_2}}$, parametrized by $x = t^{\ell_2}$ and $y = t^{\ell_1}$, we have

$$f(t^{\ell_2}, t^{\ell_1}) = \begin{cases} \left(t^{\ell_2/2+\ell_1}\right)^2, & \ell_1 \text{ odd}, \ \ell_2 \text{ even}, \\ \left(t^{\ell_2+\ell_1/2}\right)^2, & \ell_1 \text{ even}, \ \ell_2 \text{ odd}, \\ \left(t^{(\ell_1+\ell_2)/2}\right)^2, & \ell_1, \ell_2 \text{ odd}. \end{cases}$$

Moreover, f vanishes on both coordinate axes. Hence $f \geq 0$ on $\mathcal{Z}_p$, whereas $L_\beta(f) < 0$.

Let $\beta^{(2n)}$ be the degree-$2n$ truncation of β. Since all moment matrices are positive semidefinite and $M_p(r) = M_{-p}(r) = 0$ at every order, $M(n+k;\beta)$ is a positive extension of $M(n;\beta)$ satisfying the required localizing conditions for every $k \in \mathbb{N}_0$. Thus $\beta^{(2n)}$ has the k-extension property relative to $\{1, p, -p\}$ for every $k \in \mathbb{N}_0$.

If $\beta^{(2n)}$ admitted a $\mathcal{Z}_p$-representing measure μ, then

$$0 > L_\beta(f) = \int_{\mathcal{Z}_p} f \, d\mu \geq 0,$$

a contradiction. Therefore, $\beta^{(2n)}$ admits no $\mathcal{Z}_p$-representing measure. $\square$

The preceding constructions also yield explicit nonnegative polynomials that fail to be sums of squares in the corresponding coordinate rings.

Corollary 3.7. *Let $\ell_1, \ell_2 \in \mathbb{N} \setminus \{1\}$ satisfy $\gcd(\ell_1, \ell_2) = 1$, and let*

$$p(x,y) = q(x,y)\big(x^{\ell_1} - y^{\ell_2}\big), \qquad q(x,y) \in \{x, y, xy\}.$$

Define

$$f(x,y) := \begin{cases} x, & \textit{if } q = x,\ \ell_1 \textit{ is odd, and } \ell_2 \textit{ is even}, \\ xy^2, & \textit{if } q \in \{y, xy\},\ \ell_1 \textit{ is odd, and } \ell_2 \textit{ is even}, \\ y, & \textit{if } q = y,\ \ell_1 \textit{ is even, and } \ell_2 \textit{ is odd}, \\ x^2y, & \textit{if } q \in \{x, xy\},\ \ell_1 \textit{ is even, and } \ell_2 \textit{ is odd}, \\ xy, & \textit{if } \ell_1 \textit{ and } \ell_2 \textit{ are both odd}. \end{cases}$$

Then f is nonnegative on $\mathcal{Z}_p$, but

$$f \notin \Sigma_{\{1,p,-p\},n} \qquad \textit{for every } n \in \mathbb{N}.$$

Equivalently, the class of f is not a sum of squares in the coordinate ring $\mathbb{R}[x,y]/(p)$.

The proof is identical to that of Corollary 3.3, using the separating Riesz functionals and the corresponding obstruction moments constructed in Theorems 3.5 and 3.6.

We next consider the reducible curve obtained by adjoining the line $\mathcal{Z}_y$ to $\mathcal{Z}_{y-x^\ell}$, and show that the obstruction to property $(S_{n,k})$ persists.

Theorem 3.8. *Let $p(x,y) := y(y - x^\ell)$, where $\ell \in \mathbb{N} \setminus \{1\}$. Then, for every $n \geq \ell + 1$, there exists a truncated sequence $\beta^{(2n)}$ having the k-extension property relative to $\{1, p, -p\}$ for every $k \in \mathbb{N}_0$, but admitting no $\mathcal{Z}_p$-representing measure.*

Proof. Since $y(y-x^\ell) = -y(x^\ell - y)$, the generator families associated with $y(y-x^\ell)$ and $y(x^\ell - y)$ coincide, i.e., $\{1, p, -p\} = \{1, y(x^\ell - y), -y(x^\ell - y)\}$. Although Theorem 3.5 is stated under the assumption that both exponents are greater than one, its proof remains valid in the boundary case $\ell_1 = \ell$, $\ell_2 = 1$. Indeed, the component $\mathcal{Z}_{x^\ell - y}$ is parametrized by $x = t$, $y = t^\ell$, where $t \in \mathbb{R}$, and the corresponding mixed exponents form the additive subsemigroup $S_+ = \{a + b\ell : a \in \mathbb{N}_0,\ b \in \mathbb{N}\}$.

Suppose first that ℓ is even. Set $d := \ell$. Then $d \in S_+$, whereas $d/2 \notin S_+$, since every element of S_+ is at least ℓ. The auxiliary construction in the proof of Theorem 3.5 therefore gives an infinite sequence $\beta = (\beta_{i,j})_{i,j \in \mathbb{N}_0}$ such that all moment matrices are positive semidefinite, $\beta_{i+\ell, j+1} = \beta_{i,j+2}$ for all $i, j \in \mathbb{N}_0$, and $\beta_{0,1} = \varphi(\mathsf{M}_\ell) < 0$. Under the parametrization, $y = t^\ell = (t^{\ell/2})^2$, while $y = 0$ on $\mathcal{Z}_y$. Hence $y \geq 0$ on $\mathcal{Z}_p$, whereas $L_\beta(y) = \beta_{0,1} < 0$.

Suppose now that ℓ is odd. Set $d := \ell + 1$. Then $d \in S_+$, while $d/2 \notin S_+$, since $(\ell+1)/2 < \ell$ and every element of S_+ is at least ℓ. The same auxiliary construction gives an infinite sequence $\beta = (\beta_{i,j})_{i,j \in \mathbb{N}_0}$ such that all moment matrices are positive semidefinite, $\beta_{i+\ell, j+1} = \beta_{i,j+2}$ for all $i, j \in \mathbb{N}_0$, and $\beta_{1,1} = \varphi(\mathsf{M}_{\ell+1}) < 0$. Under the parametrization, $xy = t^{\ell+1} = (t^{(\ell+1)/2})^2$, while $xy = 0$ on $\mathcal{Z}_y$. Thus $xy \geq 0$ on $\mathcal{Z}_p$, whereas $L_\beta(xy) = \beta_{1,1} < 0$.

In either parity case, $\beta_{i+\ell, j+1} = \beta_{i,j+2}$ for all $i, j \in \mathbb{N}_0$. Thus $Y(X^\ell - Y) = 0$ is a recursively generated column relation at every order. Since $Y(Y - X^\ell) = -Y(X^\ell - Y)$, we have

$$M_p(r) = M_{-p}(r) = 0$$

whenever the localizing matrices are defined. Consequently, $\beta^{(2n)}$ has the k-extension property relative to $\{1, p, -p\}$ for every $k \in \mathbb{N}_0$.

In either parity case, the constructed Riesz functional takes a negative value on a polynomial that is nonnegative on $\mathcal{Z}_p$. Such a functional cannot arise from a $\mathcal{Z}_p$-representing measure. Therefore, $\beta^{(2n)}$ admits no $\mathcal{Z}_p$-representing measure. $\square$

Remark 3.9. The preceding construction is asymmetric and does not apply directly to $y(x-y^\ell)$. Indeed, the component $\mathcal{Z}_{x-y^\ell}$ is parametrized by $x=t^\ell$, $y=t$, so the mixed exponents are of the form $a\ell+b$, where $a\in\mathbb{N}_0$ and $b\in\mathbb{N}$. Thus the corresponding exponent set contains every positive integer. Consequently, the half-indices used to create the negative diagonal obstructions in the proof of Theorem 3.8 already belong to the relevant exponent set, and the same construction cannot be applied.

This reflects a genuine difference between the two families: $y(x-y^\ell)$ satisfies property $(S_{n,k})$ for a finite extension parameter k, as shown in Theorem 4.6.

We conclude this family of examples by showing that the same obstruction persists when both coordinate axes are adjoined to the curve $\mathcal{Z}_{y-x^\ell}$.

Theorem 3.10. *Let $p(x,y):=xy(y-x^\ell)$, where $\ell\in\mathbb{N}\setminus\{1\}$. Then, for every $n\geq\ell+2$, there exists a truncated sequence $\beta^{(2n)}$ having the k-extension property relative to $\{1,p,-p\}$ for every $k\in\mathbb{N}_0$, but admitting no $\mathcal{Z}_p$-representing measure.*

Proof. Since $xy(y-x^\ell)=-xy(x^\ell-y)$, we have

$$\{1,p,-p\}=\{1,xy(x^\ell-y),-xy(x^\ell-y)\}.$$

Although Theorem 3.6 is stated under the assumption that both exponents are greater than one, its auxiliary construction remains valid in the boundary case $\ell_1=\ell$, $\ell_2=1$. Indeed, the component $\mathcal{Z}_{x^\ell-y}$ is parametrized by $x=t$, $y=t^\ell$, where $t\in\mathbb{R}$, and the relevant mixed exponent semigroup becomes $S_{++}=\{a+b\ell : a,b\in\mathbb{N}\}$.

Suppose first that ℓ is even. Set $d:=\ell+2$. Then $d\in S_{++}$, corresponding to the moment $\beta_{2,1}$, while $d/2=\ell/2+1<\ell+1$. Since every element of S_{++} is at least $\ell+1$, we have $d/2\notin S_{++}$. The auxiliary construction in the proof of Theorem 3.6 therefore yields an infinite sequence $\beta=(\beta_{i,j})_{i,j\in\mathbb{N}_0}$ such that all moment matrices are positive semidefinite, $\beta_{i+\ell+1,j+1}=\beta_{i+1,j+2}$ for all $i,j\in\mathbb{N}_0$, and $\beta_{2,1}=\varphi(\mathsf{M}_{\ell+2})<0$. Under the parametrization, $x^2y=t^{\ell+2}=(t^{\ell/2+1})^2$, while x^2y vanishes on both coordinate axes. Hence $x^2y\geq 0$ on $\mathcal{Z}_p$, whereas $L_\beta(x^2y)=\beta_{2,1}<0$.

Suppose now that ℓ is odd. Set $d:=\ell+1$. Then $d\in S_{++}$, corresponding to the moment $\beta_{1,1}$, while $d/2=(\ell+1)/2<\ell+1$. Hence $d/2\notin S_{++}$. The same auxiliary construction gives an infinite sequence $\beta=(\beta_{i,j})_{i,j\in\mathbb{N}_0}$ such that all moment matrices are positive semidefinite, $\beta_{i+\ell+1,j+1}=\beta_{i+1,j+2}$ for all $i,j\in\mathbb{N}_0$, and $\beta_{1,1}=\varphi(\mathsf{M}_{\ell+1})<0$. Under the parametrization, $xy=t^{\ell+1}=(t^{(\ell+1)/2})^2$, while xy vanishes on both coordinate axes. Thus $xy\geq 0$ on $\mathcal{Z}_p$, whereas $L_\beta(xy)=\beta_{1,1}<0$.

In either parity case, $\beta_{i+\ell+1,j+1}=\beta_{i+1,j+2}$ for all $i,j\in\mathbb{N}_0$. Thus $XY(X^\ell-Y)=0$ is a recursively generated column relation at every order. Since $XY(Y-X^\ell)=-XY(X^\ell-Y)$, we have

$$M_p(r)=M_{-p}(r)=0$$

whenever the localizing matrices are defined. Consequently, $\beta^{(2n)}$ has the k-extension property relative to $\{1,p,-p\}$ for every $k\in\mathbb{N}_0$.

According to the parity of ℓ, the constructed Riesz functional takes a negative value on x^2y or xy, although the corresponding polynomial is nonnegative on $\mathcal{Z}_p$. Hence it cannot arise from a $\mathcal{Z}_p$-representing measure. Therefore, $\beta^{(2n)}$ admits no $\mathcal{Z}_p$-representing measure. $\square$

Corollary 3.11. *Let $\ell \in \mathbb{N} \setminus \{1\}$, and let*

$$p(x,y) \in \{y(y-x^\ell),\ x(x-y^\ell),\ xy(y-x^\ell),\ xy(x-y^\ell)\}.$$

Define

$$f(x,y) := \begin{cases} y, & \text{if } p(x,y) = y(y-x^\ell) \text{ and } \ell \text{ is even},\\ x, & \text{if } p(x,y) = x(x-y^\ell) \text{ and } \ell \text{ is even},\\ x^2y, & \text{if } p(x,y) = xy(y-x^\ell) \text{ and } \ell \text{ is even},\\ xy^2, & \text{if } p(x,y) = xy(x-y^\ell) \text{ and } \ell \text{ is even},\\ xy, & \text{if } \ell \text{ is odd}. \end{cases}$$

Then f is nonnegative on $\mathcal{Z}_p$, but

$$f \notin \Sigma_{\{1,p,-p\},n} \qquad \text{for every } n \in \mathbb{N}.$$

Equivalently, f is not a sum of squares in the coordinate ring $\mathbb{R}[x,y]/(p)$.

The proof is identical to that of Corollary 3.3, using the separating Riesz functionals and obstruction moments constructed in Theorems 3.8 and 3.10, together with the symmetric cases obtained by interchanging x and y.

4. Generator families $\{1,p,-p\}$ satisfying property $(S_{n,k})$

In this section, we establish property $(S_{n,k})$ for several families of planar algebraic curves and provide admissible bounds for the extension parameter k. We first consider curves defined by $x^{\ell_1}y^{\ell_2} - 1$ and their monomial multiples; see Theorems 4.1, 4.3, and 4.4. We then treat the reducible curves defined by $y(x-y^\ell)$ and, by symmetry, $x(y-x^\ell)$; see Theorem 4.6. In each case, the proof reduces the truncated moment problem on the curve to a one-dimensional moment problem through a suitable parametrization, yielding explicit control of the required extension order. As a consequence, we obtain degree-bounded sums-of-squares certificates for polynomials that are strictly positive on the corresponding algebraic sets; see Corollaries 4.2, 4.5, and 4.7.

We begin with the irreducible curves defined by $x^{\ell_1}y^{\ell_2} = 1$. Their Laurent parametrization reduces the corresponding truncated moment problem to a strong truncated Hamburger moment problem.

Theorem 4.1. *Let $\ell_1, \ell_2 \in \mathbb{N} \setminus \{1\}$ satisfy $\gcd(\ell_1,\ell_2) = 1$, and let $p(x,y) := x^{\ell_1}y^{\ell_2} - 1$. Then, for every $n \geq \ell_1 + \ell_2$, the generator family $\{1,p,-p\}$ satisfies property $(S_{n,\ell_1+\ell_2-1})$.*

Proof. Set $k := \ell_1 + \ell_2 - 1$ and put $N := n + k$. Let $\beta \equiv \beta^{(2n)}$ be a truncated sequence having the k-extension property relative to $\{1,p,-p\}$. Thus, $M(n)$ admits a positive extension $M(N)$ satisfying

$$M_p(N) \succeq 0, \qquad M_{-p}(N) \succeq 0.$$

Since $M_{-p}(N) = -M_p(N)$, these inequalities imply $M_p(N) = 0$. Equivalently, the columns of $M(N)$ satisfy the recursively generated relation $X^{\ell_1}Y^{\ell_2} = 1$ whenever the corresponding products are defined. Let $L_\beta : \mathcal{P}_{2N} \to \mathbb{R}$ denote the Riesz functional associated with the chosen extension.

We first record a consequence that will be used repeatedly:

$$L_\beta(ph) = 0 \qquad \text{whenever} \qquad \deg(ph) \leq 2N. \tag{4.1}$$

Indeed, it suffices by linearity to consider a monomial h. Under the indicated degree bound, we may factor $h = qr$ so that $\deg q \le N - \deg p$ and $\deg r \le N$. Recursive generation gives $(pq)(X,Y) = 0$, and pairing this column relation with $r(X,Y)$ gives $L_\beta(ph) = L_\beta(pqr) = 0$.

The curve $\mathcal{Z}_p$ is parametrized by

$$x = t^{\ell_2}, \qquad y = t^{-\ell_1}, \qquad t \in \mathbb{R} \setminus \{0\}. \tag{4.2}$$

Recall that a Laurent polynomial is a finite linear combination of integer powers of t, that is, an element of $\mathbb{R}[t, t^{-1}]$. If ν is a positive Borel measure on $\mathbb{R} \setminus \{0\}$, its Laurent moments are

$$\gamma_u := \int_{\mathbb{R}\setminus\{0\}} t^u \, d\nu(t), \qquad u \in \mathbb{Z},$$

whenever these integrals are finite. Positive exponents are the usual moments, whereas negative exponents record moments of $1/t$. More generally, if the moments are specified only for u in a set $E \subseteq \mathbb{Z}$, we call $\{\gamma_u : u \in E\}$ a partial Laurent moment sequence.

Under the parametrization (4.2), the monomial $x^i y^j$ becomes the Laurent monomial $t^{i\ell_2 - j\ell_1}$. Consequently, the bivariate moment $\beta_{i,j}$ corresponds to the Laurent moment $\gamma_{i\ell_2 - j\ell_1}$. The Laurent indices determined by $\beta^{(2n)}$ form the set

$$D_{2n} := \{i\ell_2 - j\ell_1 : i, j \in \mathbb{N}_0, \ i + j \le 2n\} \subseteq \mathbb{Z}.$$

For $u \in D_{2n}$, choose $i, j \in \mathbb{N}_0$ such that $u = i\ell_2 - j\ell_1$ and $i + j \le 2n$, and define $\gamma_u := \beta_{i,j}$. This definition is independent of the chosen representation. Indeed, suppose that $i\ell_2 - j\ell_1 = i'\ell_2 - j'\ell_1$. Since $\gcd(\ell_1, \ell_2) = 1$, there exists $r \in \mathbb{Z}$ such that

$$i - i' = r\ell_1, \qquad j - j' = r\ell_2.$$

It follows that $x^i y^j - x^{i'} y^{j'} \in (p)$. Moreover, both monomials have degree at most $2n$, so their difference can be written as ph with $\deg(ph) \le 2n \le 2N$. Hence (4.1) gives

$$\beta_{i,j} - \beta_{i',j'} = L_\beta(x^i y^j - x^{i'} y^{j'}) = 0.$$

Thus, $\gamma = \{\gamma_u : u \in D_{2n}\}$ is a well-defined partial Laurent sequence.

A representing measure for γ is a positive Borel measure ν on $\mathbb{R} \setminus \{0\}$ satisfying

$$\gamma_u = \int_{\mathbb{R}\setminus\{0\}} t^u \, d\nu(t), \qquad u \in D_{2n}.$$

Such a measure pushes forward under $\varphi(t) := (t^{\ell_2}, t^{-\ell_1})$ to a $\mathcal{Z}_p$-representing measure for $\beta^{(2n)}$.

The strong truncated Hamburger moment theorem that we shall use is stated for Laurent moments indexed by a full interval of integers, rather than by the possibly nonconsecutive set D_{2n}. We therefore construct an extension of γ to the interval

$$I := [-2n\ell_1 - 2, \ 2n\ell_2 + 2] \cap \mathbb{Z}.$$

We first claim that every integer $u \in [-n\ell_1 - 1, \ n\ell_2 + 1]$ can be written as

$$u = i\ell_2 - j\ell_1, \qquad i, j \in \mathbb{N}_0, \qquad i + j \le N.$$

Suppose first that $1 \le u \le n\ell_2$, and set $m := \lceil u/\ell_2 \rceil$. Then $1 \le m \le n$ and $(m-1)\ell_2 < u \le m\ell_2$. Choose $r \in \{0, \dots, \ell_1 - 1\}$ such that $(m + r)\ell_2 \equiv u \pmod{\ell_1}$, and set

$$s := \frac{(m + r)\ell_2 - u}{\ell_1}.$$

Then $s \in \mathbb{N}_0$, and

$$0 \le (m + r)\ell_2 - u < (r + 1)\ell_2 \le \ell_1 \ell_2.$$

Consequently, $s \le \ell_2 - 1$, and $u = (m+r)\ell_2 - s\ell_1$, where

$$(m+r) + s \le n + (\ell_1 - 1) + (\ell_2 - 1) = n + k - 1 < N + 1.$$

For the remaining positive endpoint $u = n\ell_2 + 1$, choose $r \in \{1, \ldots, \ell_1 - 1\}$ such that $r\ell_2 \equiv 1 \pmod{\ell_1}$, and set $s := (r\ell_2 - 1)/\ell_1$. Then $0 \le s \le \ell_2 - 1$, and

$$n\ell_2 + 1 = (n+r)\ell_2 - s\ell_1, \qquad (n+r) + s \le n + k = N.$$

The negative indices are treated symmetrically. Suppose that $-n\ell_1 \le u \le -1$, and put $w := -u$ and $m := \lceil w/\ell_1 \rceil$. Choose $r \in \{0, \ldots, \ell_2 - 1\}$ such that $(m+r)\ell_1 \equiv w \pmod{\ell_2}$, and set

$$s := \frac{(m+r)\ell_1 - w}{\ell_2}.$$

Then $0 \le s \le \ell_1 - 1$, and $u = s\ell_2 - (m+r)\ell_1$, where

$$s + (m+r) \le n + (\ell_1 - 1) + (\ell_2 - 1) = n + k - 1.$$

Finally, for $u = -n\ell_1 - 1$, choose $r \in \{1, \ldots, \ell_2 - 1\}$ such that $r\ell_1 \equiv 1 \pmod{\ell_2}$, and set $s := (r\ell_1 - 1)/\ell_2$. Then $0 \le s \le \ell_1 - 1$, and

$$-n\ell_1 - 1 = s\ell_2 - (n+r)\ell_1, \qquad s + (n+r) \le n + k = N.$$

The index $u = 0$ is represented by the monomial 1, proving the claim.

For every

$$u \in J := [-n\ell_1 - 1,\, n\ell_2 + 1] \cap \mathbb{Z},$$

choose a monomial $m_u := X^{i_u} Y^{j_u}$ such that $\deg m_u \le N$ and $i_u \ell_2 - j_u \ell_1 = u$. Consider the principal submatrix

$$H := \big(L_\beta(m_u m_v)\big)_{u,v \in J}$$

of $M(N)$. Since $M(N) \succeq 0$, we have $H \succeq 0$.

We next show that the entries of H depend only on $u+v$. Suppose that $u, v, u', v' \in J$ satisfy $u+v = u'+v'$. Then $m_u m_v$ and $m_{u'} m_{v'}$ have the same Laurent index under (4.2). Their difference therefore belongs to (p). Since both products have degree at most $2N$, we may write $m_u m_v - m_{u'} m_{v'} = ph$ with $\deg(ph) \le 2N$. By (4.1),

$$L_\beta(m_u m_v) = L_\beta(m_{u'} m_{v'}).$$

Since

$$J + J = [-2n\ell_1 - 2,\, 2n\ell_2 + 2] \cap \mathbb{Z} = I,$$

we may define, for $w \in I$,

$$\widetilde{\gamma}_w := L_\beta(m_u m_v),$$

where $u, v \in J$ are any indices satisfying $u + v = w$. The preceding argument shows that $\widetilde{\gamma}_w$ is independent of the chosen decomposition $w = u + v$. Thus $\widetilde{\gamma} = \{\widetilde{\gamma}_w : w \in I\}$ is well defined, and

$$H = H_{\widetilde{\gamma}} := \big(\widetilde{\gamma}_{u+v}\big)_{u,v \in J} \succeq 0.$$

It remains to verify that $\widetilde{\gamma}$ extends the original partial sequence γ. Let $q = i\ell_2 - j\ell_1 \in D_{2n}$, where $i + j \le 2n$. Since

$$D_{2n} \subseteq [-2n\ell_1,\, 2n\ell_2] \cap \mathbb{Z} \subseteq I,$$

choose $u, v \in J$ such that $u + v = q$. The monomials $X^i Y^j$ and $m_u m_v$ have the same Laurent index, so their difference belongs to (p). Moreover,

$$\deg(X^i Y^j) \le 2n \le 2N, \qquad \deg(m_u m_v) \le 2N.$$

Hence (4.1) gives

$$\widetilde{\gamma}_q = L_\beta(m_u m_v) = L_\beta(X^i Y^j) = \beta_{i,j} = \gamma_q.$$

Therefore, $\widetilde{\gamma}$ is a Laurent extension of γ.

Let

$$\widehat{\gamma} := \{\widetilde{\gamma}_u : -2n\ell_1 \le u \le 2n\ell_2\},$$

and let

$$H_{\widehat{\gamma}} := \big(\widetilde{\gamma}_{u+v}\big)_{-n\ell_1 \le u,v \le n\ell_2}$$

be its Laurent Hankel matrix. Since $H_{\widehat{\gamma}}$ is a principal submatrix of $H_{\widetilde{\gamma}}$, we have $H_{\widehat{\gamma}} \succeq 0$. Suppose first that $\widetilde{\gamma}_{-2n\ell_1} = 0$. This is the diagonal entry of $H_{\widetilde{\gamma}}$ indexed by $-n\ell_1$. Since $H_{\widetilde{\gamma}} \succeq 0$, the corresponding row and column vanish. It is easy to show inductively that $H_{\widehat{\gamma}} = 0$, and hence $\widehat{\gamma}$ is the zero sequence. Since $\widehat{\gamma}$ extends the original partial sequence γ, it follows that $\beta^{(2n)}$ is the zero sequence, which is represented by the zero measure. We may therefore assume from now on that $\widetilde{\gamma}_{-2n\ell_1} > 0$.

If $H_{\widehat{\gamma}} \succ 0$, then [Zal22a, Theorem 3.1(4a)] implies that $\widehat{\gamma}$ admits a representing measure supported on $\mathbb{R} \setminus \{0\}$. Suppose now that $H_{\widehat{\gamma}}$ is singular. The principal submatrices of $H_{\widetilde{\gamma}}$ indexed by

$$[-n\ell_1,\, n\ell_2 + 1] \qquad \text{and} \qquad [-n\ell_1 - 1,\, n\ell_2]$$

are positive semidefinite one-step extensions of $H_{\widehat{\gamma}}$ on the positive and negative sides, respectively. By [Zal22a, Proposition 2.1(4),(5)],

$$\operatorname{rank} H_{\widehat{\gamma}} = \operatorname{rank}\big(\widetilde{\gamma}_{u+v}\big)_{-n\ell_1 \le u,v \le n\ell_2 - 1} = \operatorname{rank}\big(\widetilde{\gamma}_{u+v}\big)_{-n\ell_1 + 1 \le u,v \le n\ell_2}.$$

Hence condition [Zal22a, Theorem 3.1(4b)] is satisfied. Therefore, in either case, [Zal22a, Theorem 3.1] yields a representing measure ν for $\widehat{\gamma}$, supported on $\mathbb{R} \setminus \{0\}$.

Push ν forward under $t \mapsto (t^{\ell_2}, t^{-\ell_1})$. The resulting measure μ is supported on $\mathcal{Z}_p$, and for $i, j \in \mathbb{N}_0$ with $i + j \le 2n$,

$$\int_{\mathcal{Z}_p} x^i y^j \, d\mu = \int_{\mathbb{R}\setminus\{0\}} t^{i\ell_2 - j\ell_1} \, d\nu(t) = \widehat{\gamma}_{i\ell_2 - j\ell_1} = \gamma_{i\ell_2 - j\ell_1} = \beta_{i,j}.$$

Thus, μ is a $\mathcal{Z}_p$-representing measure for $\beta^{(2n)}$.

We have proved that the k-extension property relative to $\{1, p, -p\}$, where $k = \ell_1 + \ell_2 - 1$, implies the existence of a $\mathcal{Z}_p$-representing measure. The converse follows from the standard necessity of the positive extension and localizing conditions. Therefore, $\{1, p, -p\}$ satisfies property $(S_{n,\ell_1+\ell_2-1})$. □

Corollary 4.2. *Let $\ell_1, \ell_2 \in \mathbb{N} \setminus \{1\}$ satisfy $\gcd(\ell_1, \ell_2) = 1$, and let*

$$p(x, y) := x^{\ell_1} y^{\ell_2} - 1.$$

Suppose that $n \ge \ell_1 + \ell_2$. Then every polynomial $f \in \mathcal{P}_{2n}$ that is strictly positive on $\mathcal{Z}_p$ admits a representation

$$f = \sum_{j=1}^{r} g_j^2 + ph,$$

where $g_1, \ldots, g_r, h \in \mathbb{R}[x, y]$ and $g_j^2,\ ph \in \mathcal{P}_{2(n+\ell_1+\ell_2-1)}$, $j = 1, \ldots, r$.

Proof. Set $k := \ell_1 + \ell_2 - 1$. By Theorem 4.1, the generator family $\{1, p, -p\}$ satisfies property $(S_{n,k})$. Hence, Theorem 2.5(i) implies that $f \in \Sigma_{\{1,p,-p\},n+k}$. Thus $f = \sigma_0 + p\sigma_1 - p\sigma_2$ for some sums of squares $\sigma_0, \sigma_1, \sigma_2$ satisfying the corresponding degree bounds. Writing $\sigma_0 = \sum_{j=1}^{r} g_j^2$ and $h := \sigma_1 - \sigma_2$, we obtain $f = \sum_{j=1}^{r} g_j^2 + ph$, with $g_j^2, ph \in \mathcal{P}_{2(n+k)}$, as required. $\square$

We next extend Theorem 4.1 to reducible curves obtained by adjoining one of the coordinate axes to $\mathcal{Z}_{x^{\ell_1}y^{\ell_2}-1}$.

Theorem 4.3. *Let $\ell_1, \ell_2 \in \mathbb{N} \setminus \{1\}$ satisfy $\gcd(\ell_1, \ell_2) = 1$, and let*

$$p(x, y) := q(x, y)\big(x^{\ell_1}y^{\ell_2} - 1\big), \qquad q(x, y) \in \{x, y\}.$$

Then, for every $n \geq \ell_1 + \ell_2 + 1$, the generator family $\{1, p, -p\}$ satisfies property $\big(S_{n,\,2(\ell_1+\ell_2)}\big)$.

Proof. By symmetry, it suffices to consider $p(x, y) = y(x^{\ell_1}y^{\ell_2} - 1)$. Set $s := \ell_1 + \ell_2$, $k := 2s$, $N := n + k = n + 2s$, and $g(x, y) := x^{\ell_1}y^{\ell_2} - 1$.

Let $\beta \equiv \beta^{(2n)}$ have the k-extension property relative to $\{1, p, -p\}$. Thus, $M(n)$ admits a positive extension $M(N)$ satisfying $M_p(N) \succeq 0$ and $M_{-p}(N) \succeq 0$. Since $M_{-p}(N) = -M_p(N)$, it follows that $M_p(N) = 0$. By polarization and recursive generation, the columns of $M(N)$ satisfy $X^iY^{j+1} = X^{i+\ell_1}Y^{j+\ell_2+1}$ whenever the monomials involved have degree at most N.

We shall use the following consequence of this column relation: $L_\beta(pr) = 0$ whenever $\deg(pr) \leq 2N$. Indeed, it is enough to consider a monomial r. Under the indicated degree bound, we may factor $r = r_1r_2$ so that $\deg(pr_1) \leq N$ and $\deg r_2 \leq N$. The recursively generated relation gives $(pr_1)(X) = 0$, and pairing this column relation with $r_2(X)$ yields $L_\beta(pr) = L_\beta(pr_1r_2) = 0$. Linearity gives the assertion for arbitrary polynomials r in the same degree range.

The zero set of p is $\mathcal{Z}_p = \mathcal{Z}_y \cup \mathcal{Z}_g$. We shall decompose the degree-$2n$ truncation of $M(N)$ into positive moment matrices corresponding to these two components.

Order the monomials of degree at most N as $\vec{X}^{(0,N)}, \vec{\mathcal{T}}_N$, where $\vec{X}^{(0,N)} = (1, X, \ldots, X^N)$ and $\vec{\mathcal{T}}_N$ consists of the monomials of degree at most N containing a positive power of Y. With respect to this ordering, write

$$M(N) = \begin{pmatrix} A^{(N)} & B^{(N)} \\ (B^{(N)})^{\mathsf{T}} & C^{(N)} \end{pmatrix}.$$

Thus, the rows and columns of $A^{(N)}$ are indexed by the monomials X^a, $0 \leq a \leq N$, where $X^0 = 1$; the rows of $B^{(N)}$ have the same indexing, while its columns and the rows and columns of $C^{(N)}$ are indexed by the monomials in $\mathcal{T}_N$.

Since $M(N) \succeq 0$, the generalized Schur-complement criterion gives $\operatorname{Ran}((B^{(N)})^{\mathsf{T}}) \subseteq \operatorname{Ran} C^{(N)}$ and $A^{(N)} - B^{(N)}(C^{(N)})^{\dagger}(B^{(N)})^{\mathsf{T}} \succeq 0$. Define $A_1^{(N)} := B^{(N)}(C^{(N)})^{\dagger}(B^{(N)})^{\mathsf{T}}$. We use monomial subscripts for its entries; thus, $(A_1^{(N)})_{X^a,X^b}$ denotes the entry in the row indexed by X^a and the column indexed by X^b. In particular, $(A_1^{(N)})_{X^a,1}$ means $(A_1^{(N)})_{X^a,X^0}$. We have

$$M(N) = \begin{pmatrix} A^{(N)} - A_1^{(N)} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A_1^{(N)} & B^{(N)} \\ (B^{(N)})^{\mathsf{T}} & C^{(N)} \end{pmatrix}, \tag{4.3}$$

and both summands are positive semidefinite. We shall show that, after suitable truncation, the first summand corresponds to $\mathcal{Z}_y$, while the second corresponds to $\mathcal{Z}_g$.

Set $R := N - s = n + s$. For each $a \in \{0, \ldots, R\}$, define $z_a := X^{a+\ell_1}Y^{\ell_2}$. Since $\deg z_a = a + s \leq R + s = N$, the monomial z_a belongs to $\mathcal{T}_N$.

Let $r \in \mathcal{T}_N$. Since r contains a positive power of Y, the quotient r/Y is a monomial, and $z_ar - X^ar = X^ar(X^{\ell_1}Y^{\ell_2} - 1) = pX^a(r/Y)$. Moreover, $\deg(z_ar) \leq R + s + N = 2N$. It follows that $L_\beta(X^a r) = L_\beta(z_a r)$. In terms of the blocks $B^{(N)}$ and $C^{(N)}$, this means

$$B^{(N)}_{X^a,\bullet} = C^{(N)}_{z_a,\bullet}, \qquad 0 \le a \le R, \tag{4.4}$$

where the right-hand side denotes the row of $C^{(N)}$ indexed by z_a.

Using (4.4) and $C^{(N)}(C^{(N)})^\dagger C^{(N)} = C^{(N)}$, we obtain, for $0 \le a, b \le R$,

$$\begin{aligned}(A_1^{(N)})_{X^a,X^b} &= B^{(N)}_{X^a,\bullet}(C^{(N)})^\dagger (B^{(N)}_{X^b,\bullet})^\mathsf{T} \\ &= e_{z_a}^\mathsf{T} C^{(N)}(C^{(N)})^\dagger C^{(N)} e_{z_b} \\ &= C^{(N)}_{z_a,z_b}.\end{aligned} \tag{4.5}$$

In particular, $(A_1^{(N)})_{X^a,X^b} = \beta_{a+b+2\ell_1,\, 2\ell_2}$, so these entries depend only on $a+b$. Thus the principal submatrix of $A_1^{(N)}$ indexed by $1, X, \ldots, X^R$ is Hankel.

Let

$$M^{[g]}(R) := \begin{pmatrix} A_1^{(R)} & B^{(R)} \\ (B^{(R)})^\mathsf{T} & C^{(R)} \end{pmatrix}$$

be the principal submatrix of the second summand in (4.3) indexed by all monomials of degree at most R, where $A_1^{(R)}, B^{(R)}$, and $C^{(R)}$ denote the corresponding restrictions of the blocks in (4.3). Since the second summand in (4.3) is positive semidefinite, $M^{[g]}(R) \succeq 0$. The entries of $B^{(R)}$ and $C^{(R)}$ are entries of the original moment matrix $M(N)$, while (4.5) shows that $A_1^{(R)}$ is Hankel. Consequently, $M^{[g]}(R)$ is a well-defined bivariate moment matrix.

We next show that its columns satisfy $X^{\ell_1}Y^{\ell_2} = 1$. Write $z_0 = X^{\ell_1}Y^{\ell_2}$. If the row is indexed by a pure monomial X^a, where $0 \le a \le R$, then (4.4) and (4.5) give

$$(M^{[g]}(R))_{X^a,1} = (A_1^{(N)})_{X^a,1} = C^{(N)}_{z_a,z_0} = B^{(N)}_{X^a,z_0} = (M^{[g]}(R))_{X^a,z_0}.$$

If the row is indexed by a monomial $r \in \mathcal{T}_R$, then (4.4), with $a = 0$, gives $(M^{[g]}(R))_{r,1} = B^{(N)}_{1,r} = C^{(N)}_{z_0,r} = (M^{[g]}(R))_{r,z_0}$. Thus the columns indexed by 1 and $X^{\ell_1}Y^{\ell_2}$ coincide. Since $M^{[g]}(R)$ is positive semidefinite, the standard kernel property for moment matrices shows that this relation is recursively generated.

The degree-$2n$ truncation of $M^{[g]}(R)$ therefore has a positive extension to order $n+s-1 \le R$ satisfying $X^{\ell_1}Y^{\ell_2} = 1$. Equivalently, the localizing matrices associated with g and $-g$ vanish. Hence this truncation has the $(s-1)$-extension property relative to $\{1, g, -g\}$. By Theorem 4.1, it admits a representing measure μ_g supported on $\mathcal{Z}_g = \mathcal{Z}_{x^{\ell_1}y^{\ell_2}-1}$.

It remains to treat the first summand in (4.3). Set $D^{(N)} := A^{(N)} - A_1^{(N)}$. Then $D^{(N)} \succeq 0$. Moreover, $A^{(N)}$ is Hankel, and (4.5) shows that the entries of $A_1^{(N)}$ depend only on the sum of their exponents whenever the corresponding monomials have degree at most R. It follows that

$$\big(D^{(N)}_{X^a,X^b}\big)_{0\le a,b\le R}$$

is a positive semidefinite univariate Hankel moment matrix.

Since $n+1 \le R$, the principal submatrix

$$D^{(n+1)} := \big(D^{(N)}_{X^a,X^b}\big)_{0\le a,b\le n+1}$$

is a positive semidefinite univariate Hankel moment matrix extending $D^{(n)} := (D^{(N)}_{X^a,X^b})_{0\le a,b\le n}$. Since the generator family $\{1\}$ satisfies property $(S_{n,1})$, the degree-$2n$ truncation corresponding to $D^{(n)}$ admits a representing measure ν on $\mathbb{R}$. Pushing ν forward under $t \mapsto (t, 0)$ gives a measure μ_y supported on $\mathcal{Z}_y$.

Restricting (4.3) to moments of degree at most $2n$ gives $\beta^{(2n)} = \beta_y^{(2n)} + \beta_g^{(2n)}$, where $\beta_y^{(2n)}$ and $\beta_g^{(2n)}$ are represented by μ_y and μ_g, respectively. Therefore $\mu := \mu_y + \mu_g$ represents $\beta^{(2n)}$, and $\operatorname{supp}\mu \subseteq \mathcal{Z}_y \cup \mathcal{Z}_g = \mathcal{Z}_p$.

We have proved that the $2(\ell_1 + \ell_2)$-extension property relative to $\{1, p, -p\}$ implies the existence of a $\mathcal{Z}_p$-representing measure. The converse follows from the standard necessity of positive extensions and the corresponding localizing conditions. Hence $\{1, p, -p\}$ satisfies property $\big(S_{n,\,2(\ell_1+\ell_2)}\big)$. $\square$

We finally consider the case in which both coordinate axes are adjoined to the curve $\mathcal{Z}_{x^{\ell_1}y^{\ell_2}-1}$.

Theorem 4.4. *Let $\ell_1, \ell_2 \in \mathbb{N} \setminus \{1\}$ satisfy $\gcd(\ell_1, \ell_2) = 1$, and let*

$$p(x, y) := xy\big(x^{\ell_1}y^{\ell_2} - 1\big).$$

Then, for every $n \geq \ell_1 + \ell_2 + 2$, the generator family $\{1, p, -p\}$ satisfies property $\big(S_{n,\,2(\ell_1+\ell_2)}\big)$.

Proof. Set $s := \ell_1 + \ell_2$, $k := 2s$, $N := n + k = n + 2s$, and $g(x, y) := x^{\ell_1}y^{\ell_2} - 1$.

Let $\beta \equiv \beta^{(2n)}$ have the k-extension property relative to $\{1, p, -p\}$. Thus, $M(n)$ admits a positive extension $M(N)$ satisfying $M_p(N) \succeq 0$ and $M_{-p}(N) \succeq 0$. Since $M_{-p}(N) = -M_p(N)$, we have $M_p(N) = 0$. By polarization and recursive generation, the columns of $M(N)$ satisfy $X^{i+1}Y^{j+1} = X^{i+\ell_1+1}Y^{j+\ell_2+1}$ whenever the monomials involved have degree at most N.

As in the proof of Theorem 4.3, the corresponding factorization argument gives

$$(*) \qquad L_\beta(pr) = 0 \qquad \text{whenever} \qquad \deg(pr) \leq 2N.$$

Indeed, one factors each monomial $r = r_1r_2$ so that $\deg(pr_1) \leq N$ and $\deg r_2 \leq N$, and then pairs the column relation $(pr_1)(X) = 0$ with $r_2(X)$.

The zero set of p is $\mathcal{Z}_p = \mathcal{Z}_x \cup \mathcal{Z}_y \cup \mathcal{Z}_g$. We shall decompose the degree-$2n$ truncation of $M(N)$ into positive moment matrices corresponding to these three components.

Order the monomials of degree at most N as $1, \vec{X}^{(1,N)}, \vec{Y}^{(1,N)}, \vec{\mathcal{T}}_N$, where $\vec{X}^{(1,N)} = (X, \ldots, X^N)$, $\vec{Y}^{(1,N)} = (Y, \ldots, Y^N)$, and $\vec{\mathcal{T}}_N$ consists of the mixed monomials X^iY^j with $i, j \geq 1$ and $i+j \leq N$. With respect to this ordering, write

$$M(N) = \begin{pmatrix} \beta_{0,0} & (a^{(N)})^{\mathsf{T}} & (b^{(N)})^{\mathsf{T}} & (c^{(N)})^{\mathsf{T}} \\ a^{(N)} & A^{(N)} & B^{(N)} & C^{(N)} \\ b^{(N)} & (B^{(N)})^{\mathsf{T}} & D^{(N)} & E^{(N)} \\ c^{(N)} & (C^{(N)})^{\mathsf{T}} & (E^{(N)})^{\mathsf{T}} & F^{(N)} \end{pmatrix}.$$

Thus, $A^{(N)}$, $D^{(N)}$, and $F^{(N)}$ are indexed by the monomials X^a, Y^b, and the mixed monomials, respectively. We use monomial subscripts throughout; for example, $A^{(N)}_{X^a, X^{a'}}$ denotes the entry in the row indexed by X^a and the column indexed by $X^{a'}$.

Set

$$Q^{(N)} := \begin{pmatrix} \beta_{0,0} & (a^{(N)})^{\mathsf{T}} & (b^{(N)})^{\mathsf{T}} \\ a^{(N)} & A^{(N)} & B^{(N)} \\ b^{(N)} & (B^{(N)})^{\mathsf{T}} & D^{(N)} \end{pmatrix}, \qquad G^{(N)} := \begin{pmatrix} (c^{(N)})^{\mathsf{T}} \\ C^{(N)} \\ E^{(N)} \end{pmatrix}.$$

Then $M(N) = \begin{pmatrix} Q^{(N)} & G^{(N)} \\ (G^{(N)})^{\mathsf{T}} & F^{(N)} \end{pmatrix}$. Since $M(N) \succeq 0$, the generalized Schur-complement criterion gives

$$S^{(N)} := Q^{(N)} - G^{(N)}(F^{(N)})^{\dagger}(G^{(N)})^{\mathsf{T}} \succeq 0.$$

Consequently,

$$M(N) = \begin{pmatrix} S^{(N)} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} Q_1^{(N)} & G^{(N)} \\ (G^{(N)})^\mathsf{T} & F^{(N)} \end{pmatrix}, \tag{4.6}$$

where $Q_1^{(N)} := G^{(N)}(F^{(N)})^\dagger(G^{(N)})^\mathsf{T}$, and both summands are positive semidefinite.

Write

$$Q_1^{(N)} = \begin{pmatrix} \eta_3 & (a_1^{(N)})^\mathsf{T} & (b_1^{(N)})^\mathsf{T} \\ a_1^{(N)} & A_1^{(N)} & B_1^{(N)} \\ b_1^{(N)} & (B_1^{(N)})^\mathsf{T} & D_1^{(N)} \end{pmatrix}.$$

Thus,

$$\begin{aligned} \eta_3 &:= (c^{(N)})^\mathsf{T}(F^{(N)})^\dagger c^{(N)}, \\ a_1^{(N)} &:= C^{(N)}(F^{(N)})^\dagger c^{(N)}, & b_1^{(N)} &:= E^{(N)}(F^{(N)})^\dagger c^{(N)}, \\ A_1^{(N)} &:= C^{(N)}(F^{(N)})^\dagger (C^{(N)})^\mathsf{T}, & B_1^{(N)} &:= C^{(N)}(F^{(N)})^\dagger (E^{(N)})^\mathsf{T}, \\ D_1^{(N)} &:= E^{(N)}(F^{(N)})^\dagger (E^{(N)})^\mathsf{T}. \end{aligned}$$

Set $R := N - s = n + s$. For $0 \le a, b \le R$, define $z_a := X^{a+\ell_1}Y^{\ell_2}$ and $w_b := X^{\ell_1}Y^{b+\ell_2}$. Then $z_0 = w_0$, and all these monomials have degree at most N.

The argument used to obtain the block-row identity in the proof of Theorem 4.3 applies in both coordinate directions. Indeed, if $r \in \mathcal{T}_N$, then

$$z_a r - X^a r = pX^a \frac{r}{XY}, \qquad w_b r - Y^b r = pY^b \frac{r}{XY},$$

and the degrees are at most $2N$. Hence $(*)$ gives

$$(c^{(N)})^\mathsf{T} = F^{(N)}_{z_0,\bullet}, \qquad C^{(N)}_{X^a,\bullet} = F^{(N)}_{z_a,\bullet}, \qquad E^{(N)}_{Y^b,\bullet} = F^{(N)}_{w_b,\bullet}, \tag{4.7}$$

for $1 \le a, b \le R$.

Applying the same Schur-completion calculation as in the proof of Theorem 4.3, now using $F^{(N)}(F^{(N)})^\dagger F^{(N)} = F^{(N)}$, gives

$$\begin{aligned} \eta_3 &= F^{(N)}_{z_0,z_0}, \\ (a_1^{(N)})_{X^a} &= F^{(N)}_{z_a,z_0}, & (b_1^{(N)})_{Y^b} &= F^{(N)}_{w_b,z_0}, \\ (A_1^{(N)})_{X^a,X^{a'}} &= F^{(N)}_{z_a,z_{a'}}, & (D_1^{(N)})_{Y^b,Y^{b'}} &= F^{(N)}_{w_b,w_{b'}}, \\ (B_1^{(N)})_{X^a,Y^b} &= F^{(N)}_{z_a,w_b}, \end{aligned} \tag{4.8}$$

whenever the exponents lie in $\{1, \dots, R\}$.

It remains to identify the completed X-Y block. For $1 \le a, b \le R$,

$$z_a w_b - X^a Y^b = pX^{a-1}Y^{b-1}\big(X^{\ell_1}Y^{\ell_2} + 1\big),$$

and $\deg(z_a w_b) = a + b + 2s \le 2N$. Thus $(*)$ and (4.8) give

$$(B_1^{(N)})_{X^a,Y^b} = F^{(N)}_{z_a,w_b} = B^{(N)}_{X^a,Y^b}, \qquad 1 \le a, b \le R. \tag{4.9}$$

Let $M^{[g]}(R)$ be the principal submatrix of the second summand in (4.6) indexed by all monomials of degree at most R. Then $M^{[g]}(R) \succeq 0$. The identities in (4.8) show that its completed pure-X and pure-Y blocks have the required Hankel structure, while (4.9) identifies its X-Y block with the corresponding block of $M(N)$. Consequently, $M^{[g]}(R)$ is a well-defined bivariate moment matrix.

The verification that its columns satisfy $X^{\ell_1}Y^{\ell_2}=1$ is the same as in the proof of Theorem 4.3. Namely, (4.7) and (4.8) show that the columns indexed by 1 and $z_0=X^{\ell_1}Y^{\ell_2}$ agree on the constant, pure-X, pure-Y, and mixed rows. Positivity then implies that this relation is recursively generated.

The degree-$2n$ truncation of $M^{[g]}(R)$ therefore has a positive extension to order $n+s-1\le R$ satisfying $X^{\ell_1}Y^{\ell_2}=1$. Hence it has the $(s-1)$-extension property relative to $\{1,g,-g\}$. By Theorem 4.1, it admits a representing measure μ_g supported on $\mathcal{Z}_g=\mathcal{Z}_{x^{\ell_1}y^{\ell_2}-1}$.

It remains to split the first summand in (4.6) between the coordinate axes. Let $S^{(R)}$ be the principal submatrix of $S^{(N)}$ indexed by $1,X,\dots,X^R,Y,\dots,Y^R$. Set

$$\begin{aligned} \rho &:= \beta_{0,0}-\eta_3, \\ u^{(R)} &:= a^{(R)}-a_1^{(R)}, \qquad & v^{(R)} &:= b^{(R)}-b_1^{(R)}, \\ P_X^{(R)} &:= A^{(R)}-A_1^{(R)}, \qquad & P_Y^{(R)} &:= D^{(R)}-D_1^{(R)}. \end{aligned}$$

By (4.9),

$$S^{(R)}=\begin{pmatrix} \rho & (u^{(R)})^{\mathsf{T}} & (v^{(R)})^{\mathsf{T}} \\ u^{(R)} & P_X^{(R)} & 0 \\ v^{(R)} & 0 & P_Y^{(R)} \end{pmatrix}\succeq 0. \tag{4.10}$$

The generalized Schur-complement criterion applied to (4.10) gives $\rho\ge\alpha_X+\alpha_Y$, where

$$\alpha_X:=(u^{(R)})^{\mathsf{T}}(P_X^{(R)})^{\dagger}u^{(R)}, \qquad \alpha_Y:=(v^{(R)})^{\mathsf{T}}(P_Y^{(R)})^{\dagger}v^{(R)}.$$

Choose $\eta_1,\eta_2\ge 0$ such that $\eta_1\ge\alpha_Y$, $\eta_2\ge\alpha_X$, and $\eta_1+\eta_2=\rho$. Then

$$H_x(R):=\begin{pmatrix} \eta_1 & (v^{(R)})^{\mathsf{T}} \\ v^{(R)} & P_Y^{(R)} \end{pmatrix}\succeq 0, \qquad H_y(R):=\begin{pmatrix} \eta_2 & (u^{(R)})^{\mathsf{T}} \\ u^{(R)} & P_X^{(R)} \end{pmatrix}\succeq 0.$$

Here $H_x(R)$ is indexed by $1,Y,\dots,Y^R$, while $H_y(R)$ is indexed by $1,X,\dots,X^R$.

As in the treatment of the axis component in the proof of Theorem 4.3, (4.8) shows that these are univariate Hankel moment matrices. Since $n+1\le R$, their principal submatrices of order $n+1$ give positive one-step extensions of their order-n truncations. Property $(S_{n,1})$ therefore yields representing measures on $\mathbb{R}$. Pushing the measure corresponding to $H_x(R)$ forward under $t\mapsto(0,t)$ gives a measure μ_x supported on $\mathcal{Z}_x$, while pushing the measure corresponding to $H_y(R)$ forward under $t\mapsto(t,0)$ gives a measure μ_y supported on $\mathcal{Z}_y$.

Restricting (4.6) and (4.10) to moments of degree at most $2n$ gives $\beta^{(2n)}=\beta_x^{(2n)}+\beta_y^{(2n)}+\beta_g^{(2n)}$. Consequently, $\mu:=\mu_x+\mu_y+\mu_g$ represents $\beta^{(2n)}$, and $\operatorname{supp}\mu\subseteq\mathcal{Z}_x\cup\mathcal{Z}_y\cup\mathcal{Z}_g=\mathcal{Z}_p$.

We have shown that the $2(\ell_1+\ell_2)$-extension property relative to $\{1,p,-p\}$ implies the existence of a $\mathcal{Z}_p$-representing measure. The converse follows from the standard necessity of positive extensions and the corresponding localizing conditions. Hence $\{1,p,-p\}$ satisfies property $\big(S_{n,\,2(\ell_1+\ell_2)}\big)$. $\square$

Corollary 4.5. *Let $\ell_1,\ell_2\in\mathbb{N}\setminus\{1\}$ satisfy $\gcd(\ell_1,\ell_2)=1$, and let*

$$p(x,y):=q(x,y)\big(x^{\ell_1}y^{\ell_2}-1\big), \qquad q(x,y)\in\{x,y,xy\}.$$

Suppose that $n\ge\deg p$. Then every $f\in\mathcal{P}_{2n}$ that is strictly positive on $\mathcal{Z}_p$ admits a representation

$$f=\sum_{j=1}^{r}g_j^2+ph,$$

where $g_1,\dots,g_r,h\in\mathbb{R}[x,y]$ and $g_j^2,\ ph\in\mathcal{P}_{2(n+2(\ell_1+\ell_2))}$, $j=1,\dots,r$.

The proof is identical to that of Corollary 4.2, using Theorem 4.3 when $q \in \{x, y\}$ and Theorem 4.4 when $q = xy$.

We conclude the section with the reducible curve obtained by adjoining the x-axis to $\mathcal{Z}_{x-y^\ell}$.

Theorem 4.6. *Let $p(x,y) := y(x - y^\ell)$, where $\ell \in \mathbb{N}$. Then, for every $n \geq \ell + 1$, the generator family $\{1, p, -p\}$ satisfies property $\left(S_{n,\ell+\max\{\ell-1,2\}}\right)$.*

Proof. Set $h := \max\{\ell-1, 2\}$, $k := \ell+h$, and $N := n+k = n+\ell+h$. Also, put $g(x,y) := x-y^\ell$.

Let $\beta \equiv \beta^{(2n)}$ have the k-extension property relative to $\{1, p, -p\}$. Thus, $M(n)$ admits a positive extension $M(N)$ satisfying $M_p(N) \succeq 0$ and $M_{-p}(N) \succeq 0$. Since $M_{-p}(N) = -M_p(N)$, it follows that $M_p(N) = 0$. By polarization and recursive generation, the columns of $M(N)$ satisfy

$$X^{i+1}Y^{j+1} = X^iY^{j+\ell+1}$$

whenever the monomials involved have degree at most N.

As in the proof of Theorem 4.3, the corresponding factorization argument gives

$$L_\beta(pr) = 0 \qquad \text{whenever} \qquad \deg(pr) \leq 2N. \tag{4.11}$$

Indeed, one factors each monomial $r = r_1r_2$ so that $\deg(pr_1) \leq N$ and $\deg r_2 \leq N$, and then pairs the column relation $(pr_1)(X) = 0$ with $r_2(X)$.

The zero set of p is $\mathcal{Z}_p = \mathcal{Z}_y \cup \mathcal{Z}_g$. We shall decompose the degree-$2n$ truncation of $M(N)$ into positive moment matrices corresponding to these two components.

Order the monomials of degree at most N as $\vec{X}^{(0,N)}, \vec{\mathcal{T}}_N$, where $\vec{X}^{(0,N)} = (1, X, \ldots, X^N)$ and $\vec{\mathcal{T}}_N$ consists of the monomials of degree at most N containing a positive power of Y. With respect to this ordering, write

$$M(N) = \begin{pmatrix} A^{(N)} & B^{(N)} \\ (B^{(N)})^\mathsf{T} & C^{(N)} \end{pmatrix}.$$

Thus, the rows and columns of $A^{(N)}$ are indexed by the monomials X^a, $0 \leq a \leq N$, where $X^0 = 1$, while the rows and columns of $C^{(N)}$ are indexed by the monomials in $\mathcal{T}_N$. We use monomial subscripts throughout; for example, $(A^{(N)})_{X^a,X^b}$ denotes the entry in the row indexed by X^a and the column indexed by X^b.

Since $M(N) \succeq 0$, the generalized Schur-complement criterion gives $\operatorname{Ran}((B^{(N)})^\mathsf{T}) \subseteq \operatorname{Ran} C^{(N)}$. Define

$$A_1^{(N)} := B^{(N)}(C^{(N)})^\dagger(B^{(N)})^\mathsf{T}.$$

Then

$$M(N) = \begin{pmatrix} A^{(N)} - A_1^{(N)} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A_1^{(N)} & B^{(N)} \\ (B^{(N)})^\mathsf{T} & C^{(N)} \end{pmatrix}, \tag{4.12}$$

and both summands are positive semidefinite. We shall show that, after suitable truncation, the first summand corresponds to $\mathcal{Z}_y$, while the second corresponds to $\mathcal{Z}_g$.

Set $R := N - \ell = n + h$. For each $a \in \{1, \ldots, R\}$, define

$$z_a := X^{a-1}Y^\ell.$$

Since $\deg z_a = a + \ell - 1 \leq R + \ell - 1 = N - 1$, the monomial z_a belongs to $\mathcal{T}_N$.

Let $r \in \mathcal{T}_N$. Since r contains a positive power of Y, the quotient r/Y is a monomial, and

$$X^ar - z_ar = X^{a-1}r(X - Y^\ell) = pX^{a-1}\frac{r}{Y}.$$

The terms on the left have degree at most $2N$, so (4.11) gives $L_\beta(X^a r) = L_\beta(z_a r)$. In terms of the blocks $B^{(N)}$ and $C^{(N)}$, this means

$$B^{(N)}_{X^a,\bullet} = C^{(N)}_{z_a,\bullet}, \qquad 1 \le a \le R. \tag{4.13}$$

Applying the same Schur-completion calculation as in the proof of Theorem 4.3, and using $C^{(N)}(C^{(N)})^\dagger C^{(N)} = C^{(N)}$, gives

$$(A_1^{(N)})_{X^a,X^b} = C^{(N)}_{z_a,z_b}, \qquad 1 \le a,b \le R. \tag{4.14}$$

The entries involving the constant monomial require a separate observation. Since $\operatorname{Ran}((B^{(N)})^\mathsf{T}) \subseteq \operatorname{Ran} C^{(N)}$, we have

$$B^{(N)}_{1,\bullet}(C^{(N)})^\dagger C^{(N)} = B^{(N)}_{1,\bullet}.$$

Consequently, for $1 \le a \le R$,

$$\begin{aligned} (A_1^{(N)})_{1,X^a} &= B^{(N)}_{1,\bullet}(C^{(N)})^\dagger (B^{(N)}_{X^a,\bullet})^\mathsf{T} \\ &= B^{(N)}_{1,\bullet}(C^{(N)})^\dagger C^{(N)} e_{z_a} \\ &= B^{(N)}_{1,z_a}. \end{aligned} \tag{4.15}$$

Equations (4.14) and (4.15) show that the relevant portion of $A_1^{(N)}$ is Hankel. Indeed,

$$(A_1^{(N)})_{1,X^a} = \beta_{a-1,\ell}, \qquad (A_1^{(N)})_{X^a,X^b} = \beta_{a+b-2,\,2\ell}.$$

Moreover, (4.11), applied to $X^{a+b-2}Y^\ell(X - Y^\ell)$, gives

$$\beta_{a+b-2,\,2\ell} = \beta_{a+b-1,\ell}.$$

Thus $(A_1^{(N)})_{X^a,X^b}$ depends only on $a+b$ and is consistent with the entries $(A_1^{(N)})_{1,X^{a+b}}$ whenever the latter are within the available range. Hence the principal submatrix of $A_1^{(N)}$ indexed by $1, X, \dots, X^R$ is Hankel.

Let $M^{[g]}(R)$ be the principal submatrix of the second summand in (4.12) indexed by all monomials of degree at most R. Explicitly,

$$M^{[g]}(R) := \begin{pmatrix} A_1^{(R)} & B^{(R)} \\ (B^{(R)})^\mathsf{T} & C^{(R)} \end{pmatrix},$$

where the blocks denote the corresponding restrictions of those in (4.12). As a principal submatrix of a positive semidefinite matrix, $M^{[g]}(R) \succeq 0$. The entries of $B^{(R)}$ and $C^{(R)}$ are entries of the original moment matrix, while the preceding identities show that $A_1^{(R)}$ is Hankel. Consequently, $M^{[g]}(R)$ is a well-defined bivariate moment matrix.

We next show that its columns satisfy $X = Y^\ell$. Since $z_1 = Y^\ell$, (4.13), with $a = 1$, shows that the columns indexed by X and Y^ℓ agree on all rows indexed by monomials in $\mathcal{T}_R$. If the row is indexed by X^a, where $1 \le a \le R$, then (4.13) and (4.14) give

$$(M^{[g]}(R))_{X^a,X} = (A_1^{(N)})_{X^a,X} = C^{(N)}_{z_a,z_1} = B^{(N)}_{X^a,z_1} = (M^{[g]}(R))_{X^a,Y^\ell}.$$

For the constant row, (4.15), with $a = 1$, gives

$$(M^{[g]}(R))_{1,X} = (A_1^{(N)})_{1,X} = B^{(N)}_{1,z_1} = (M^{[g]}(R))_{1,Y^\ell}.$$

Thus the columns indexed by X and Y^ℓ coincide. Since $M^{[g]}(R) \succeq 0$, the standard kernel property for moment matrices shows that this relation is recursively generated.

Set $r_0 := \max\{\ell-1,1\}$. Since $r_0 \le h$, the restriction of $M^{[g]}(R)$ to order $n+r_0$ is a positive r_0-extension of its degree-$2n$ truncation satisfying $X = Y^\ell$. Equivalently, the localizing matrices associated with g and $-g$ vanish. By [Zal22b, Theorem 3.1], this truncation admits a representing measure μ_g supported on $\mathcal{Z}_g = \mathcal{Z}_{x-y^\ell}$.

It remains to treat the first summand in (4.12). Set

$$D^{(N)} := A^{(N)} - A_1^{(N)}.$$

Then $D^{(N)} \succeq 0$. Moreover, $A^{(N)}$ is Hankel, and the preceding identities show that the principal submatrix of $A_1^{(N)}$ indexed by $1, X, \dots, X^R$ is Hankel. Therefore, the corresponding principal submatrix of $D^{(N)}$ is a positive semidefinite univariate Hankel moment matrix.

Since $h \ge 2$, we have $n+1 \le R = n+h$. Thus the principal submatrix of $D^{(N)}$ indexed by $1, X, \dots, X^{n+1}$ is a positive semidefinite univariate Hankel moment matrix extending the submatrix indexed by $1, X, \dots, X^n$. As in the axis argument in the proof of Theorem 4.3, property $(S_{n,1})$ gives a representing measure ν on $\mathbb{R}$ for its degree-$2n$ truncation. Pushing ν forward under $t \mapsto (t,0)$ gives a measure μ_y supported on $\mathcal{Z}_y$.

Restricting (4.12) to moments of degree at most $2n$ gives

$$\beta^{(2n)} = \beta_y^{(2n)} + \beta_g^{(2n)}.$$

Consequently, $\mu := \mu_y + \mu_g$ represents $\beta^{(2n)}$, and

$$\operatorname{supp} \mu \subseteq \mathcal{Z}_y \cup \mathcal{Z}_g = \mathcal{Z}_p.$$

We have proved that the $\ell + \max\{\ell-1,2\}$-extension property relative to $\{1,p,-p\}$ implies the existence of a $\mathcal{Z}_p$-representing measure. The converse follows from the standard necessity of positive extensions and the corresponding localizing conditions. Hence $\{1,p,-p\}$ satisfies property $\big(S_{n,\ell+\max\{\ell-1,2\}}\big)$. ☐

Corollary 4.7. *Let*

$$p(x,y) \in \{y(x-y^\ell),\, x(y-x^\ell)\}, \qquad \ell \in \mathbb{N},$$

and suppose that $n \ge \ell+1$. *Then every* $f \in \mathcal{P}_{2n}$ *that is strictly positive on* $\mathcal{Z}_p$ *admits a representation*

$$f = \sum_{j=1}^r g_j^2 + ph,$$

where $g_1, \dots, g_r, h \in \mathbb{R}[x,y]$ *and* $g_j^2,\ ph \in \mathcal{P}_{2(n+\ell+\max\{\ell-1,2\})}$, $j = 1, \dots, r$.

The proof is identical to that of Corollary 4.2, using Theorem 4.6 and, for $p = x(y-x^\ell)$, symmetry under the interchange of x and y.

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