CYCLIC POLYNOMIALS IN MIXED-WEIGHTED DIRICHLET-TYPE SPACES

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ABSTRACT. For $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, we investigate cyclic polynomials in the mixed-weighted Dirichlet-type space $\mathscr{D}_{\boldsymbol{\alpha}}$ of holomorphic functions $f(z_1, z_2) = \sum_{m,n\geq 0} a_{m,n} z_1^m z_2^n$ on the unit bidisk satisfying

$$\sum_{m,n\geq 0} |a_{m,n}|^2 (m+1)^{\alpha_1} (n+1)^{\alpha_2} (m+n+1)^{\alpha_3} < \infty.$$

We give a complete characterization of cyclic polynomials in terms of their zero sets and the parameters $\alpha_1, \alpha_2, \alpha_3$. Our results extend previously known characterizations for anisotropic and total-degree Dirichlet-type spaces on the unit bidisk.

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1. INTRODUCTION

The study of cyclic vectors in holomorphic function spaces lies at the intersection of operator theory, approximation theory, and complex function theory. In the classical Hardy space $H^2(\mathbb{D})$, by Beurling's theorem, a function f is cyclic if and only if it is outer. In the classical Dirichlet space, the corresponding problem is substantially more challenging. The celebrated Brown–Shields conjecture asserts that a function in the Dirichlet space is cyclic if and only if it is outer and its boundary zero set has logarithmic capacity zero [6]. This conjecture highlights the central role played by boundary zero sets in cyclicity questions. The situation becomes even more involved in several

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variables. Although cyclic polynomials in the Hardy space $H^2(\mathbb{D}^d)$ are precisely those having no zeros in $\mathbb{D}^d$ [15], their cyclicity in Dirichlet-type spaces may depend crucially on both the geometry of their zero sets on the distinguished boundary and the parameters defining the space. For a detailed discussion in one and several variables, we refer the reader to [8, 9, 11, 17].

On the unit bidisk, an important family of examples is given by the anisotropic Dirichlet-type spaces

$$\mathfrak{D}_{(\beta_1,\beta_2)} = \Big\{ \sum_{m,n\geq 0} a_{m,n} z_1^m z_2^n : \sum_{m,n\geq 0} (m+1)^{\beta_1}(n+1)^{\beta_2}|a_{m,n}|^2 < \infty \Big\}.$$

where $\beta_1, \beta_2 \in \mathbb{R}$. Bénéteau et al. [1] obtained a complete characterization of cyclic polynomials in the isotropic product-weight spaces $\mathfrak{D}_{(\beta,\beta)}$, $\beta \in \mathbb{R}$. This characterization was subsequently extended to the anisotropic setting by Knese et al. in [12].

Theorem 1.1 ([12])**.** *Let $\beta_1, \beta_2 \in \mathbb{R}$, and let $p \in \mathbb{C}[z_1, z_2]$ be irreducible, depend on both variables, and have no zeros in $\mathbb{D}^2$. The following characterization holds in $\mathfrak{D}_{(\beta_1,\beta_2)}$:*

(i) If $\beta_1 + \beta_2 \leq 1$, then p is cyclic.
(ii) If $\beta_1 + \beta_2 > 1$ and $\min\{\beta_1, \beta_2\} \leq 1$, then p is cyclic if and only if $\mathcal{Z}(p) \cap \mathbb{T}^2$ is finite.
(iii) If $\min\{\beta_1, \beta_2\} > 1$, then p is cyclic if and only if $\mathcal{Z}(p) \cap \mathbb{T}^2 = \varnothing$.

A different scale of Dirichlet-type spaces on the unit bidisk is obtained by replacing the product weight with a total-degree weight:

$$\mathcal{D}_\gamma = \Big\{ \sum_{m,n\geq 0} a_{m,n} z_1^m z_2^n : \sum_{m,n\geq 0} (m+n+1)^{\gamma}|a_{m,n}|^2 < \infty \Big\},$$

These spaces have been considered in [2, 4, 3]. A complete characterization of cyclic polynomials in these spaces was obtained in [14, 18].

Theorem 1.2. *Let $\gamma \in \mathbb{R}$, and let $p \in \mathbb{C}[z_1, z_2]$ be irreducible and have no zeros in $\mathbb{D}^2$. The following characterization holds in $\mathcal{D}_\gamma$:*

(i) If $\gamma \leq 1$, then p is cyclic.
(ii) If $1 < \gamma \leq 2$, then p is cyclic if and only if $\mathcal{Z}(p) \cap \mathbb{T}^2$ is finite.
(iii) If $\gamma > 2$, then p is cyclic if and only if $\mathcal{Z}(p) \cap \mathbb{T}^2 = \varnothing$.

Related results on cyclicity and invariant subspaces in one-variable Dirichlet spaces can be found in [6, 7]. In several variables, relevant work includes polynomial cyclicity in Hardy spaces on polydisks [15], outer functions in function algebras on the bidisk [10], and cyclic polynomials in Dirichlet-type spaces on the unit ball of $\mathbb{C}^2$ [13].

The characterizations in Theorems 1.1 and 1.2 both relate cyclicity to the size of the zero set on $\mathbb{T}^2$, but the parameter thresholds differ. Consider an irreducible polynomial depending on both variables and having no zeros in $\mathbb{D}^2$. In the anisotropic product-weight setting, a polynomial with infinitely many zeros on $\mathbb{T}^2$ is cyclic precisely when $\beta_1 + \beta_2 \leq 1$, whereas a polynomial with a nonempty finite zero set on $\mathbb{T}^2$ is cyclic precisely when $\min\{\beta_1, \beta_2\} \leq 1$. In the total-degree setting, the corresponding conditions are $\gamma \leq 1$ and $\gamma \leq 2$, respectively. These results lead to the question of whether the two characterizations can be obtained as special cases of a more general theorem.

This motivated us to study mixed-weighted Dirichlet-type spaces on the unit bidisk.

1.1. **Mixed-weighted Dirichlet-type spaces.** Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$. For $m, n \geq 0$, define

$$w_{\boldsymbol{\alpha}}(m,n) := (m+1)^{\alpha_1}(n+1)^{\alpha_2}(m+n+1)^{\alpha_3}.$$

The mixed-weighted Dirichlet-type space $\mathscr{D}_{\boldsymbol{\alpha}}$ consists of all holomorphic functions

$$f(z_1, z_2) = \sum_{m,n\geq 0} a_{m,n} z_1^m z_2^n$$

on $\mathbb{D}^2$ satisfying

$$\|f\|^2_{\boldsymbol{\alpha}} := \sum_{m,n\geq 0} w_{\boldsymbol{\alpha}}(m,n)|a_{m,n}|^2 < \infty.$$

This is a reproducing kernel Hilbert space, with inner product

$$\langle f, g\rangle_{\boldsymbol{\alpha}} = \sum_{m,n\geq 0} w_{\boldsymbol{\alpha}}(m,n) a_{m,n}\overline{b_{m,n}}, \qquad g(z_1,z_2) = \sum_{m,n\geq 0} b_{m,n} z_1^m z_2^n.$$

For $\zeta = (\zeta_1, \zeta_2) \in \mathbb{D}^2$, the reproducing kernel at ζ is

$$K_\zeta(z_1,z_2) = \sum_{m,n\geq 0} \frac{\overline{\zeta_1}^m \overline{\zeta_2}^n}{w_{\boldsymbol{\alpha}}(m,n)} z_1^m z_2^n. \tag{1.1}$$

It belongs to $\mathscr{D}_{\boldsymbol{\alpha}}$ and satisfies

$$f(\zeta) = \langle f, K_\zeta\rangle_{\boldsymbol{\alpha}}, \qquad f \in \mathscr{D}_{\boldsymbol{\alpha}}.$$

The normalized monomials

$$\left\{ \frac{z_1^m z_2^n}{\sqrt{w_{\boldsymbol{\alpha}}(m,n)}} : m, n \geq 0 \right\}$$

form an orthonormal basis for $\mathscr{D}_{\boldsymbol{\alpha}}$. In particular, polynomials are dense in $\mathscr{D}_{\boldsymbol{\alpha}}$.

The mixed-weighted scale contains both previously studied families:

$$\mathscr{D}_{(\alpha_1,\alpha_2,0)} = \mathfrak{D}_{(\alpha_1,\alpha_2)} \qquad \text{and} \qquad \mathscr{D}_{(0,0,\alpha_3)} = \mathcal{D}_{\alpha_3}.$$

Also for fixed $\boldsymbol{\alpha} \in \mathbb{R}^3$, consider the family of weights

$$\mathfrak{w}_\gamma(m,n) = (m+1)^{\gamma\alpha_1}(n+1)^{\gamma\alpha_2}(m+n+1)^{(1-\gamma)\alpha_3}, \qquad 0 \leq \gamma \leq 1.$$

The corresponding spaces are $\mathscr{D}_{(\gamma\alpha_1,\gamma\alpha_2,(1-\gamma)\alpha_3)}$. This family continuously interpolates between the total-degree and product weights:

$$\begin{aligned} \mathfrak{w}_0(m,n) &= (m+n+1)^{\alpha_3}, \\ \mathfrak{w}_1(m,n) &= (m+1)^{\alpha_1}(n+1)^{\alpha_2}, \end{aligned} \qquad m,n \geq 0.$$

Our results describe how cyclicity changes as the weight varies continuously from a total-degree weight to a product weight. Moreover, we give a complete characterization of cyclic polynomials in the mixed-weighted Dirichlet-type spaces $\mathscr{D}_{\boldsymbol{\alpha}}$, recovering the known anisotropic product-weight and total-degree characterizations as special cases.

Since

$$\sup_{m,n\geq 0}\frac{w_{\boldsymbol{\alpha}}(m+1,n)}{w_{\boldsymbol{\alpha}}(m,n)}<\infty \qquad \text{and} \qquad \sup_{m,n\geq 0}\frac{w_{\boldsymbol{\alpha}}(m,n+1)}{w_{\boldsymbol{\alpha}}(m,n)}<\infty,$$

the coordinate multiplication operators

$$M_{z_1}f(z_1,z_2)=z_1f(z_1,z_2) \qquad \text{and} \qquad M_{z_2}f(z_1,z_2)=z_2f(z_1,z_2)$$

are bounded on $\mathscr{D}_{\boldsymbol{\alpha}}$. Consequently, every polynomial is a multiplier of $\mathscr{D}_{\boldsymbol{\alpha}}$.

Definition 1.3. For $f\in\mathscr{D}_{\boldsymbol{\alpha}}$, let

$$[f]_{\boldsymbol{\alpha}}=\overline{\operatorname{span}\{z_1^mz_2^nf:m,n\geq 0\}}^{\mathscr{D}_{\boldsymbol{\alpha}}}.$$

The function f is said to be *cyclic* in $\mathscr{D}_{\boldsymbol{\alpha}}$ if

$$[f]_{\boldsymbol{\alpha}}=\mathscr{D}_{\boldsymbol{\alpha}}.$$

We will use the following elementary observations about cyclicity.

(i) Since polynomials are dense and multiplication by polynomials is bounded, a function $f\in\mathscr{D}_{\boldsymbol{\alpha}}$ is cyclic if and only if there exists a sequence $\{q_j\}_{j\geq 1}\subseteq\mathbb{C}[z_1,z_2]$ such that

$$\|q_jf-1\|_{\boldsymbol{\alpha}}\longrightarrow 0.$$

(ii) Since polynomials are bounded multipliers of $\mathscr{D}_{\boldsymbol{\alpha}}$, a factorization $p=p_1\cdots p_k$ into nonconstant polynomials satisfies

$$p \text{ is cyclic} \iff p_j \text{ is cyclic for every } j=1,\dots,k.$$

Thus, to characterize cyclic polynomials, it suffices to consider irreducible polynomials.

(iii) Since point evaluations on $\mathbb{D}^2$ are bounded, a polynomial vanishing at a point of $\mathbb{D}^2$ cannot be cyclic. Indeed, every polynomial multiple vanishes at that point and therefore cannot approximate the constant function 1 in norm. Hence a necessary condition for cyclicity of p is

$$\mathcal{Z}(p)\cap\mathbb{D}^2=\varnothing.$$

Throughout the paper, $\mathbb{Z}_+=\mathbb{N}_0=\{0,1,2,\dots\}$. For two nonnegative sequences $(u_{m,n})$ and $(v_{m,n})$, we write $u_{m,n}\asymp v_{m,n}$ if there exist constants $c,C>0$ such that

$$c\,v_{m,n}\leq u_{m,n}\leq C\,v_{m,n}$$

for all indices in the stated range. Similarly, $u_{m,n}\lesssim v_{m,n}$ means that $u_{m,n}\leq C\,v_{m,n}$, whereas $u_{m,n}\gtrsim v_{m,n}$ means that $u_{m,n}\geq c\,v_{m,n}$. These constants may depend on fixed parameters, such as $\boldsymbol{\alpha}$, but not on m,n. We use the same notation for comparisons of norms, with constants independent of the function being considered.

A finite zero set is allowed to be empty unless explicitly stated otherwise.

1.2. **Three effective parameters.** The mixed weight naturally gives rise to three effective exponents:

$$A = \alpha_1 + \alpha_3, \qquad B = \alpha_2 + \alpha_3, \qquad S = \alpha_1 + \alpha_2 + \alpha_3. \tag{1.2}$$

Their geometric significance becomes clear from the restrictions of the weight to different regions of the lattice $\mathbb{Z}_+^2$. Along the first coordinate axis,

$$w_{\boldsymbol{\alpha}}(m, 0) = (m+1)^A.$$

Along the second coordinate axis,

$$w_{\boldsymbol{\alpha}}(0, n) = (n+1)^B.$$

Along the diagonal,

$$w_{\boldsymbol{\alpha}}(k, k) = (k+1)^{\alpha_1+\alpha_2}(2k+1)^{\alpha_3} \asymp (k+1)^S.$$

Thus, A and B describe the growth or decay of the weight along the coordinate axes, whereas S describes its behavior when m and n grow at comparable rates.

1.3. **Main Result.** The following is the main result of this paper which gives a characterization of cyclic irreducible polynomials in mix-weighted Dirichlet-type spaces.

Theorem 1.4. *Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, set*

$$w_{\boldsymbol{\alpha}}(m, n) = (m+1)^{\alpha_1}(n+1)^{\alpha_2}(m+n+1)^{\alpha_3}, \qquad m, n \in \mathbb{Z}_+,$$

and let $\mathscr{D}_{\boldsymbol{\alpha}}$ denote the corresponding Dirichlet-type space on the bidisk. Set

$$A = \alpha_1 + \alpha_3, \qquad B = \alpha_2 + \alpha_3, \qquad S = \alpha_1 + \alpha_2 + \alpha_3.$$

Suppose that $p \in \mathbb{C}[z_1, z_2]$ is irreducible, depends on both variables, and has no zeros in $\mathbb{D}^2$.

- *(i) If $S \le 1$, then p is cyclic.*
- *(ii) If $S > 1$ and $\min\left\{A, B, \frac{S}{2}\right\} \le 1$, then p is cyclic if and only if it has finitely many zeros on the two-torus $\mathbb{T}^2$.*
- *(iii) If $\min\left\{A, B, \frac{S}{2}\right\} > 1$, then p is cyclic if and only if it has no zeros on $\mathbb{T}^2$.*

Our main contribution is a complete characterization of cyclic polynomials in $\mathscr{D}_{\boldsymbol{\alpha}}$ for every $\boldsymbol{\alpha} \in \mathbb{R}^3$, including parameter ranges not covered by direct norm comparisons with previously studied spaces. Two steps in the proof of Theorem 1.4 are central to this extension:

- For nonempty finite toral zero sets, we establish a sharp cyclicity criterion for the model polynomial $2 - z_1 - z_2$. We adapt the generating-function argument in [12, Appendix A] and combine it with weighted summability estimates for polynomial values on $\mathbb{N}_0^2$. We then use rotations of this model polynomial in the smooth-quotient construction to obtain the finite-zero classification.
- For irreducible polynomials p depending on both variables, having no zeros in $\mathbb{D}^2$, and having infinitely many zeros on $\mathbb{T}^2$, we prove noncyclicity whenever $S > 1$ by constructing a nonzero bounded linear functional supported on a regular arc of $\mathcal{Z}(p) \cap \mathbb{T}^2$. This functional approach differs from the proof of [12, Theorem 5] in the anisotropic setting, which reduces

noncyclicity to the model polynomial $1 - z_1 z_2$ through algebraic branch substitutions and the symmetrization lemma [12, Lemma 8].

The proofs of the remaining parts of the characterization in Theorem 1.4 adapt methods from [12] and our earlier work [14].

1.4. **Organization of the paper.** In Section 2, we characterize convergence of $\sum_{m,n\in\mathbb{N}_0} \frac{1}{w_{\boldsymbol{\alpha}}(m,n)}$. This determines when evaluation at points of $\mathbb{T}^2$ is bounded and provides a noncyclicity criterion for polynomials vanishing there. We also establish norm comparisons between $\mathscr{D}_{\boldsymbol{\alpha}}$ and anisotropic Dirichlet-type spaces and discuss their consequences for cyclicity.

In Section 3, we determine the cyclicity of the model polynomials $1-z_1$, $1-z_1z_2$, $2-z_1-z_2$, together with their rotations and monomial generalizations. These results will be used in the proof of Theorem 1.4.

Section 4 proves Theorem 1.4(i), which establishes cyclicity when $S \leq 1$. In Section 5, we characterize cyclicity for irreducible polynomials depending on both variables, having no zeros in $\mathbb{D}^2$, and having a nonempty finite zero set on $\mathbb{T}^2$.

In Section 6, we consider polynomials satisfying the same irreducibility and zero-free assumptions but having infinitely many zeros on $\mathbb{T}^2$. We first construct a real-analytic arc of zeros whose slope in angular coordinates is strictly negative and bounded away from zero. Integration by parts and Parseval's identity then give a bounded linear functional supported on this arc, proving noncyclicity when $S > 1$.

In Section 7, we combine the preceding results to complete the proof of Theorem 1.4.

Finally, Section 8 gives shorter proofs of noncyclicity in two parameter ranges by comparison with anisotropic Dirichlet-type spaces. It also establishes a Hilbert-sum representation of $\mathscr{D}_{\boldsymbol{\alpha}}$ when $\alpha_3 \leq 0$.

2. Preliminaries

In this section, we present some necessary conditions for cyclicity in $\mathscr{D}_{\boldsymbol{\alpha}}$. We also establish connections between $\mathscr{D}_{\boldsymbol{\alpha}}$ and anisotropic Dirichlet-type spaces.

We call a polynomial *stable* if it has no zeros in $\mathbb{D}^2$. We first recall a geometric property of stable irreducible polynomials.

Lemma 2.1 ([12, Lemma 3])**.** *Let $p \in \mathbb{C}[z_1, z_2]$ be irreducible, depend on both variables, and have no zeros in $\mathbb{D}^2$. Then $\mathcal{Z}(p) \cap \big((\mathbb{D}\times\mathbb{T}) \cup (\mathbb{T}\times\mathbb{D})\big) = \varnothing$. Consequently, for every $r \in (0,1)$, the polynomial $p_r(z_1,z_2) = p(rz_1, z_2)$ has no zeros on $\overline{\mathbb{D}}^2$.*

2.1. **Bounded evaluation on the distinguished boundary.** For $\tau = (\tau_1, \tau_2) \in \mathbb{T}^2$, evaluation at τ extends continuously to $\mathscr{D}_{\boldsymbol{\alpha}}$ if and only if

$$\sum_{m,n\geq 0} \frac{1}{w_{\boldsymbol{\alpha}}(m,n)} < \infty,$$

since

$$\|K_\tau(z_1,z_2)\|^2_{\boldsymbol{\alpha}} = \sum_{m,n\geq 0} \frac{1}{w_{\boldsymbol{\alpha}}(m,n)}, \tag{2.1}$$

where $K_\tau(z_1, z_2)$ is as in (1.1).

Next, we characterize convergence of the series in (2.1).

Proposition 2.2. *Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, and let A, B, S be as in (1.2). Then*

$$\sum_{m,n\geq 0} \frac{1}{(m+1)^{\alpha_1}(n+1)^{\alpha_2}(m+n+1)^{\alpha_3}} < \infty$$

if and only if $A > 1$, $B > 1$, and $S > 2$.

Proof. We first split the sum into the regions $\Omega_1 = \left\{(m,n) \in \mathbb{Z}_+^2 : m > n\right\}$ and $\Omega_2 = \left\{(m,n) \in \mathbb{Z}_+^2 : n \geq m\right\}$. On Ω_1,

$$m + n + 1 \asymp m + 1.$$

Therefore,

$$\sum_{(m,n)\in\Omega_1} \frac{1}{w_{\boldsymbol{\alpha}}(m,n)} \asymp \sum_{m\geq 0} (m+1)^{-A} \sum_{n=0}^{m} (n+1)^{-\alpha_2}. \tag{2.2}$$

The inner sum satisfies

$$\sum_{n=0}^{m} (n+1)^{-\alpha_2} \asymp \begin{cases} 1, & \alpha_2 > 1, \\ \log(m+2), & \alpha_2 = 1, \\ (m+1)^{1-\alpha_2}, & \alpha_2 < 1. \end{cases} \tag{2.3}$$

Indeed, comparison with an integral gives

$$\sum_{n=0}^{m} (n+1)^{-\alpha_2} \asymp 1 + \int_1^{m+1} x^{-\alpha_2}\, dx.$$

If $\alpha_2 > 1$, the integral is bounded independently of m, so the right-hand side is comparable to 1. If $\alpha_2 = 1$, it equals $1 + \log(m+1)$, which is comparable to $\log(m+2)$. Finally, if $\alpha_2 < 1$, then

$$1 + \int_1^{m+1} x^{-\alpha_2}\, dx = 1 + \frac{(m+1)^{1-\alpha_2} - 1}{1-\alpha_2} \asymp (m+1)^{1-\alpha_2}.$$

In each case, the comparison constants are independent of m.

By (2.3), the series (2.2) converges if and only if $A > 1$ when $\alpha_2 \geq 1$. When $\alpha_2 < 1$, it is comparable to

$$\sum_{m\geq 0} (m+1)^{1-A-\alpha_2} = \sum_{m\geq 0} (m+1)^{1-S},$$

since $A + \alpha_2 = S$. Thus, in this case, (2.2) converges if and only if $S > 2$.

By symmetry, the subseries over Ω_2 converges if and only if

$$\begin{cases} B > 1, & \text{if } \alpha_1 \geq 1, \\ S > 2, & \text{if } \alpha_1 < 1. \end{cases}$$

Since $S = A + \alpha_2$, the condition for convergence on Ω_1 is equivalent to requiring both $A > 1$ and $S > 2$. Indeed, if $\alpha_2 \geq 1$, then $A > 1$ implies $S > 2$; if $\alpha_2 < 1$, then $S > 2$ implies $A = S - \alpha_2 > 1$. Similarly, convergence on Ω_2 is equivalent to requiring both $B > 1$ and $S > 2$.

Since $\Omega_1 \cup \Omega_2 = \mathbb{Z}_+^2$, the full series converges if and only if both subseries converge. Therefore,

$$\sum_{m,n\geq 0} \frac{1}{w_{\boldsymbol{\alpha}}(m,n)} < \infty \iff A > 1,\ B > 1,\ S > 2.$$

This completes the proof. □

The following consequence proves the necessity in Theorem 1.4(iii).

Corollary 2.3. *Let $\boldsymbol{\alpha} \in \mathbb{R}^3$, and let A, B, S be as in (1.2). Suppose that $A > 1$, $B > 1$, and $S > 2$. If $p \in \mathbb{C}[z_1, z_2]$ satisfies $\mathcal{Z}(p) \cap \mathbb{T}^2 \neq \varnothing$, then p is noncyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$.*

Proof. Choose $\tau \in \mathcal{Z}(p) \cap \mathbb{T}^2$. By Proposition 2.2, evaluation at τ is bounded. Since $(qp)(\tau) = 0$ for every polynomial q, the functional $f \longmapsto f(\tau)$ annihilates $[p]_{\boldsymbol{\alpha}}$. However, $1(\tau) = 1$. Therefore, $1 \notin [p]_{\boldsymbol{\alpha}}$, and hence p is not cyclic. □

Remark 2.4. By Proposition 2.2, evaluation on $\mathbb{T}^2$ fails to be bounded precisely when $A \leq 1$ or $B \leq 1$ or $S \leq 2$. Thus bounded point evaluations on $\mathbb{T}^2$ provide no obstruction to cyclicity in this parameter region. Theorem 5.2 shows that this condition is sufficient for cyclicity when the polynomial is irreducible, depends on both variables, is zero-free in $\mathbb{D}^2$, and has a nonempty finite toral zero set.

2.2. **Relations with anisotropic Dirichlet-type spaces.** We compare the norm of $\mathscr{D}_{\boldsymbol{\alpha}}$ with those of anisotropic Dirichlet-type spaces to obtain cyclicity and noncyclicity criteria for polynomials.

Lemma 2.5. *Let $\boldsymbol{\alpha} \in \mathbb{R}^3$, and let A, B be as in (1.2). Then the following statements hold.*

(i) If $\alpha_3 \leq 0$, then

$$\|f\|_{\boldsymbol{\alpha}} \leq \|f\|_{\mathfrak{D}_{(A,\alpha_2)}}, \qquad f \in \mathfrak{D}_{(A,\alpha_2)},$$

and

$$\|f\|_{\boldsymbol{\alpha}} \leq \|f\|_{\mathfrak{D}_{(\alpha_1,B)}}, \qquad f \in \mathfrak{D}_{(\alpha_1,B)}.$$

In particular, $\mathfrak{D}_{(A,\alpha_2)}$ and $\mathfrak{D}_{(\alpha_1,B)}$ embed continuously into $\mathscr{D}_{\boldsymbol{\alpha}}$.

(ii) If $\alpha_3 > 0$, then

$$\mathscr{D}_{\boldsymbol{\alpha}} = \mathfrak{D}_{(A,\alpha_2)} \cap \mathfrak{D}_{(\alpha_1,B)}$$

with equivalent norms:

$$\|f\|_{\boldsymbol{\alpha}}^2 \asymp \|f\|_{\mathfrak{D}_{(A,\alpha_2)}}^2 + \|f\|_{\mathfrak{D}_{(\alpha_1,B)}}^2.$$

Proof. Suppose first that $\alpha_3 \leq 0$. Since

$$(m+n+1)^{\alpha_3} \leq (m+1)^{\alpha_3}, \qquad (m+n+1)^{\alpha_3} \leq (n+1)^{\alpha_3},$$

we have

$$w_{\boldsymbol{\alpha}}(m,n) \leq (m+1)^A (n+1)^{\alpha_2}, \qquad w_{\boldsymbol{\alpha}}(m,n) \leq (m+1)^{\alpha_1} (n+1)^B.$$

Summing these inequalities against the squared Taylor coefficients proves (i).

Suppose now that $\alpha_3 > 0$. Put $M = \max\{m+1, n+1\}$. The inequalities

$$M \leq m+n+1 \leq 2M, \qquad M^{\alpha_3} \leq (m+1)^{\alpha_3} + (n+1)^{\alpha_3} \leq 2M^{\alpha_3}$$

give

$$(m+n+1)^{\alpha_3} \asymp (m+1)^{\alpha_3} + (n+1)^{\alpha_3}.$$

Multiplying by $(m+1)^{\alpha_1}(n+1)^{\alpha_2}$, we obtain

$$w_{\boldsymbol{\alpha}}(m,n) \asymp (m+1)^{A}(n+1)^{\alpha_2} + (m+1)^{\alpha_1}(n+1)^{B}.$$

Summing against the squared Taylor coefficients proves the norm equivalence in (ii). The mixed-weight norm is therefore finite if and only if both anisotropic norms are finite, which also proves the equality of spaces. □

Remark 2.6. Let $\boldsymbol{\alpha} \in \mathbb{R}^3$, and let A, B, S be as in (1.2). Let $p \in \mathbb{C}[z_1, z_2]$ be irreducible, depend on both variables, and have no zeros in $\mathbb{D}^2$. The anisotropic spaces $\mathfrak{D}_{(A,\alpha_2)}$ and $\mathfrak{D}_{(\alpha_1,B)}$ have the same parameter sum, $A + \alpha_2 = \alpha_1 + B = S$.

(i) Suppose that $\alpha_3 > 0$, $S > 1$, and p has infinitely many zeros on $\mathbb{T}^2$. By Theorem 1.1(ii)–(iii), p is noncyclic in $\mathfrak{D}_{(A,\alpha_2)}$ and $\mathfrak{D}_{(\alpha_1,B)}$. If p were cyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$, the norm comparison in Lemma 2.5(ii) would imply that p is cyclic in $\mathfrak{D}_{(A,\alpha_2)}$, a contradiction. Thus p is noncyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$.

(ii) Suppose that $\alpha_3 > 0$ and $S \leq 1$. By Theorem 1.1(i), p is cyclic in $\mathfrak{D}_{(A,\alpha_2)}$ and $\mathfrak{D}_{(\alpha_1,B)}$. This alone does not establish cyclicity in their intersection, since the approximating sequences may differ. By Lemma 2.5(ii), cyclicity in $\mathscr{D}_{\boldsymbol{\alpha}}$ is equivalent to the existence of a single sequence of polynomials (q_j) such that

$$\|pq_j - 1\|_{\mathfrak{D}_{(A,\alpha_2)}} \longrightarrow 0, \qquad \|pq_j - 1\|_{\mathfrak{D}_{(\alpha_1,B)}} \longrightarrow 0.$$

(iii) Suppose that $\alpha_3 \leq 0$ and $S \leq 1$. Since $A + \alpha_2 = S$, Theorem 1.1(i) implies that p is cyclic in $\mathfrak{D}_{(A,\alpha_2)}$. By Lemma 2.5(i), any sequence of polynomial multiples of p converging to 1 in $\mathfrak{D}_{(A,\alpha_2)}$ also converges to 1 in $\mathscr{D}_{\boldsymbol{\alpha}}$. Hence p is cyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$.

(iv) For $\alpha_3 \leq 0$, the inclusions in Lemma 2.5(i) can be strengthened to the identity

$$\mathscr{D}_{\boldsymbol{\alpha}} = \mathfrak{D}_{(A,\alpha_2)} + \mathfrak{D}_{(\alpha_1,B)},$$

with equivalence of Hilbert-sum norms. This identification is established in Proposition 8.2.

3. Cyclicity of model polynomials

In this section, we study the cyclicity of model polynomials. The results will be used in subsequent sections to prove parts of Theorem 1.4. The main result of this section is the following theorem.

Theorem 3.1. *Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, $\zeta, \eta \in \mathbb{T}$, and let j, k be positive integers. Then the following statements hold.*

(i) *$\zeta - z_1$ is cyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$ if and only if $\alpha_1 + \alpha_3 \leq 1$.*
(ii) *$\eta - z_2$ is cyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$ if and only if $\alpha_2 + \alpha_3 \leq 1$.*
(iii) *$1 - z_1^j z_2^k$ is cyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$ if and only if $\alpha_1 + \alpha_2 + \alpha_3 \leq 1$.*
(iv) *$2 - \overline{\zeta} z_1 - \overline{\eta} z_2$ is cyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$ if and only if at least one of the inequalities*

$$\alpha_1 + \alpha_3 \leq 1, \qquad \alpha_2 + \alpha_3 \leq 1, \qquad \alpha_1 + \alpha_2 + \alpha_3 \leq 2 \tag{3.1}$$

holds.

Taking $j = k = 1$ in part (iii) gives the criterion for $1 - z_1 z_2$. We prove parts (i)–(iii) by reduction to one variable. For part (iv), we study the coefficients of functions orthogonal to polynomial multiples of $2 - z_1 - z_2$. The auxiliary results below show that the corresponding arrays have the form $b_{m,n} = P(m,n)$ for a polynomial $P \in \mathbb{C}[x,y]$, and that the weighted summability condition forces $P = 0$ in the stated parameter region.

3.1. **Reduction to one variable.** For $\beta \in \mathbb{R}$, define the one-variable Dirichlet-type space

$$D_\beta = \Big\{ f = \sum_{m \geq 0} c_m z^m \colon \|f\|_{D_\beta}^2 := \sum_{m \geq 0} (m+1)^\beta |c_m|^2 < \infty \Big\}.$$

Proof of Theorem 3.1(i)–(iii). We first recall the one-variable criterion due to Brown and Shields [6]:

$$1 - z \text{ is cyclic in } D_\beta \quad \Longleftrightarrow \quad \beta \leq 1. \tag{3.2}$$

For part (i), rotation of the first variable reduces the proof to $\zeta = 1$. Given $q \in \mathbb{C}[z_1, z_2]$, write

$$q(z_1, z_2) = q_0(z_1) + z_2 q_1(z_1, z_2),$$

where $q_0 \in \mathbb{C}[z_1]$ and $q_1 \in \mathbb{C}[z_1, z_2]$. Orthogonality of monomials with distinct powers of z_2 gives

$$\|(1 - z_1)q - 1\|_{\boldsymbol{\alpha}}^2 = \|(1 - z_1)q_0 - 1\|_{\boldsymbol{\alpha}}^2 + \|(1 - z_1) z_2 q_1\|_{\boldsymbol{\alpha}}^2.$$

Consequently,

$$\inf_{q \in \mathbb{C}[z_1, z_2]} \|(1 - z_1)q - 1\|_{\boldsymbol{\alpha}} = \inf_{q_0 \in \mathbb{C}[z_1]} \|(1 - z_1)q_0 - 1\|_{\boldsymbol{\alpha}}.$$

For $g(z_1) = \sum_{m \geq 0} c_m z_1^m$, we have

$$\|g\|_{\boldsymbol{\alpha}}^2 = \sum_{m \geq 0} (m+1)^{\alpha_1 + \alpha_3} |c_m|^2.$$

It therefore suffices to determine whether $1 - z$ is cyclic in the one-variable space $D_{\alpha_1 + \alpha_3}$. Part (i) now follows from (3.2) with $\beta = \alpha_1 + \alpha_3$. Interchanging the variables proves part (ii).

For part (iii), let $t = z_1^j z_2^k$. Every polynomial q has a unique decomposition

$$q = q_0(t) + q_\perp,$$

where $q_0 \in \mathbb{C}[z]$ and $q_\perp$ contains no monomial of the form $z_1^{j\ell} z_2^{k\ell}$, $\ell \geq 0$. Multiplication by t preserves this property of $q_\perp$. Therefore,

$$\|(1 - t)q - 1\|_{\boldsymbol{\alpha}}^2 = \|(1 - t)q_0(t) - 1\|_{\boldsymbol{\alpha}}^2 + \|(1 - t)q_\perp\|_{\boldsymbol{\alpha}}^2.$$

It follows that approximation of 1 by polynomial multiples of $1 - t$ reduces to polynomials in t. For $f(z) = \sum_{\ell \geq 0} c_\ell z^\ell$, we have

$$\|f(t)\|_{\boldsymbol{\alpha}}^2 = \sum_{\ell \geq 0} (j\ell + 1)^{\alpha_1} (k\ell + 1)^{\alpha_2} ((j+k)\ell + 1)^{\alpha_3} |c_\ell|^2.$$

Since $j, k \geq 1$ are fixed,

$$(j\ell + 1)^{\alpha_1} (k\ell + 1)^{\alpha_2} ((j+k)\ell + 1)^{\alpha_3} \asymp (\ell + 1)^{\alpha_1 + \alpha_2 + \alpha_3},$$

with constants independent of ℓ. Hence cyclicity of $1-t$ in $\mathscr{D}_{\boldsymbol{\alpha}}$ is equivalent to cyclicity of $1-z$ in $D_{\alpha_1+\alpha_2+\alpha_3}$. Part (iii) now follows from (3.2). □

Remark 3.2. The zero set of $1-z_1z_2$ on $\mathbb{T}^2$ is $\{(\zeta,\overline{\zeta}) : \zeta \in \mathbb{T}\}$, a curve on which both coordinates vary. Part (iii) shows that this polynomial is cyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$ precisely when $\alpha_1+\alpha_2+\alpha_3 \le 1$.

3.2. **Polynomial solutions of the recurrence.** We now prepare the proof of Theorem 3.1(iv). Define

$$p_0(z_1,z_2) = 2 - z_1 - z_2.$$

Its toral zero set consists of the single point $(1,1)$. Our argument adapts the generating-function method of [12, Appendix A], also used in [14, Section 3], to the mixed-weight setting. We first express orthogonality to $[p_0]_{\boldsymbol{\alpha}}$ as a recurrence relation.

Lemma 3.3. *Let*

$$f(z_1,z_2) = \sum_{m,n\ge 0} \frac{b_{m,n}}{w_{\boldsymbol{\alpha}}(m,n)} z_1^m z_2^n \in \mathscr{D}_{\boldsymbol{\alpha}}.$$

Then $f \in [p_0]_{\boldsymbol{\alpha}}^{\perp}$ if and only if

$$2b_{m,n} = b_{m+1,n} + b_{m,n+1}, \qquad m,n \ge 0. \tag{3.3}$$

Moreover,

$$\sum_{m,n\ge 0} \frac{|b_{m,n}|^2}{w_{\boldsymbol{\alpha}}(m,n)} < \infty. \tag{3.4}$$

Proof. The summability condition (3.4) follows from $f \in \mathscr{D}_{\boldsymbol{\alpha}}$:

$$\|f\|_{\boldsymbol{\alpha}}^2 = \sum_{m,n\ge 0} w_{\boldsymbol{\alpha}}(m,n) \left| \frac{b_{m,n}}{w_{\boldsymbol{\alpha}}(m,n)} \right|^2 = \sum_{m,n\ge 0} \frac{|b_{m,n}|^2}{w_{\boldsymbol{\alpha}}(m,n)}.$$

By

$$z_1^m z_2^n p_0 = 2z_1^m z_2^n - z_1^{m+1} z_2^n - z_1^m z_2^{n+1},$$

we obtain

$$\langle f, z_1^m z_2^n p_0 \rangle_{\boldsymbol{\alpha}} = 2b_{m,n} - b_{m+1,n} - b_{m,n+1}.$$

Thus $f \perp [p_0]_{\boldsymbol{\alpha}}$ if and only if (3.3) holds for all $m,n \ge 0$. □

The next lemma shows that every array satisfying the recurrence (3.3) and the weighted summability condition (3.4) consists of the values of a polynomial on the nonnegative integer lattice.

Lemma 3.4. *Suppose that a complex array $\{b_{m,n}\}_{m,n\ge 0}$ satisfies*

$$2b_{m,n} = b_{m+1,n} + b_{m,n+1}, \qquad m,n \ge 0, \tag{3.5}$$

and

$$\sum_{m,n\ge 0} \frac{|b_{m,n}|^2}{w_{\boldsymbol{\alpha}}(m,n)} < \infty. \tag{3.6}$$

Then there exists a polynomial $P \in \mathbb{C}[x,y]$ such that

$$b_{m,n} = P(m,n), \qquad m,n \ge 0.$$

Proof. Define the generating function

$$g(z_1, z_2) = \sum_{m,n\geq 0} b_{m,n} z_1^m z_2^n.$$

We divide the proof into three claims.

Claim 1. The series defining g represents a holomorphic function on $\mathbb{D}^2$, and there exists a polynomial h such that

$$g(z, 0) = \frac{1}{z-1} h\left(\frac{z}{z-1}\right), \qquad z \in \mathbb{D}.$$

Proof of Claim 1. We give the generating-function argument following [12, Appendix A, pp. 44–45].

Set

$$\gamma = \sum_{j=1}^{3} \max\{\alpha_j, 0\}, \qquad M = \max\{1, \gamma\}.$$

Since $m + n + 1 \leq (m + 1)(n + 1)$, we have

$$w_{\boldsymbol{\alpha}}(m, n) \leq (m + n + 1)^{\gamma} \leq (m + 1)^M (n + 1)^M.$$

Consequently, (3.6) implies that

$$K^2 := \sum_{m,n\geq 0} \frac{|b_{m,n}|^2}{(m+1)^M (n+1)^M} < \infty.$$

Consider the generating function

$$g(z_1, z_2) = \sum_{m,n\geq 0} b_{m,n} z_1^m z_2^n.$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned}\sum_{m,n\geq 0} |b_{m,n}|\, |z_1|^m |z_2|^n &\leq K \left(\sum_{m,n\geq 0} (m+1)^M (n+1)^M |z_1|^{2m} |z_2|^{2n} \right)^{1/2} \\ &= K \left(\sum_{m\geq 0} (m+1)^M |z_1|^{2m} \right)^{1/2} \left(\sum_{n\geq 0} (n+1)^M |z_2|^{2n} \right)^{1/2}.\end{aligned}$$

The two series on the right converge uniformly on compact subsets of $\mathbb{D}$. Hence the series defining g converges absolutely and locally uniformly on $\mathbb{D}^2$, and therefore $g \in \mathrm{Hol}(\mathbb{D}^2)$.

Multiplying (3.5) by $z_1^{m+1} z_2^{n+1}$ and summing over $m, n \geq 0$, we obtain

$$2 z_1 z_2 g(z_1, z_2) = z_2\big(g(z_1, z_2) - g(0, z_2)\big) + z_1\big(g(z_1, z_2) - g(z_1, 0)\big).$$

Thus

$$(z_1 + z_2 - 2 z_1 z_2) g(z_1, z_2) = z_1 g(z_1, 0) + z_2 g(0, z_2).$$

For $\zeta \neq \pm 1$, put

$$z_1 = \frac{\zeta}{\zeta - 1}, \qquad z_2 = \frac{\zeta}{\zeta + 1}.$$

A direct computation gives

$$\left|\frac{\zeta}{\zeta-1}\right|<1 \iff \operatorname{Re}\zeta<\frac{1}{2}, \qquad \left|\frac{\zeta}{\zeta+1}\right|<1 \iff \operatorname{Re}\zeta>-\frac{1}{2},$$

and

$$\frac{\zeta}{\zeta-1}+\frac{\zeta}{\zeta+1}-2\frac{\zeta}{\zeta-1}\frac{\zeta}{\zeta+1}=0.$$

Substituting into the generating-function identity yields

$$\frac{\zeta}{\zeta-1}g\left(\frac{\zeta}{\zeta-1},0\right)+\frac{\zeta}{\zeta+1}g\left(0,\frac{\zeta}{\zeta+1}\right)=0$$

on the strip $-\frac{1}{2}<\operatorname{Re}\zeta<\frac{1}{2}$. Dividing by ζ when $\zeta\neq 0$, and then using continuity at $\zeta=0$, gives

$$\frac{1}{\zeta-1}g\left(\frac{\zeta}{\zeta-1},0\right)=-\frac{1}{\zeta+1}g\left(0,\frac{\zeta}{\zeta+1}\right).$$

The two expressions are holomorphic on their respective half-planes and agree on their intersection. Consequently,

$$h(\zeta)=\begin{cases}\dfrac{1}{\zeta-1}g\left(\dfrac{\zeta}{\zeta-1},0\right), & \operatorname{Re}\zeta<\frac{1}{2},\\[2ex] -\dfrac{1}{\zeta+1}g\left(0,\dfrac{\zeta}{\zeta+1}\right), & \operatorname{Re}\zeta>-\frac{1}{2}\end{cases}$$

defines an entire function.

We next show that h has polynomial growth. Choose an integer $N\geq 1$ such that $2N-1\geq M$. For every $\ell\geq 0$,

$$(\ell+1)^M\leq(\ell+1)^{2N-1}\leq(2N-1)!\binom{\ell+2N-1}{2N-1}.$$

Hence, for $0\leq r<1$,

$$\sum_{\ell\geq 0}(\ell+1)^M r^{2\ell}\leq\frac{(2N-1)!}{(1-r^2)^{2N}}.$$

Restricting the definition of K to the coordinate axes gives

$$\sum_{m\geq 0}\frac{|b_{m,0}|^2}{(m+1)^M}\leq K^2, \qquad \sum_{n\geq 0}\frac{|b_{0,n}|^2}{(n+1)^M}\leq K^2.$$

Thus, by the Cauchy–Schwarz inequality and the preceding estimate,

$$\begin{aligned}|g(z,0)|&\leq\left(\sum_{m\geq 0}\frac{|b_{m,0}|^2}{(m+1)^M}\right)^{1/2}\left(\sum_{m\geq 0}(m+1)^M|z|^{2m}\right)^{1/2}\\&\leq\frac{K\sqrt{(2N-1)!}}{(1-|z|^2)^N}.\end{aligned}$$

The same estimate holds for $g(0,z)$. Consequently,

$$|g(z,0)|+|g(0,z)|\leq\frac{C}{(1-|z|^2)^N}, \qquad z\in\mathbb{D},$$

where C is independent of z.

If $\operatorname{Re}\zeta \le 0$, then

$$1-\left|\frac{\zeta}{\zeta-1}\right|^2 = \frac{1-2\operatorname{Re}\zeta}{|\zeta-1|^2} \ge \frac{1}{|\zeta-1|^2}.$$

Using the first representation of h, we obtain

$$\begin{aligned}|h(\zeta)| &\le \frac{C}{|\zeta-1|}\left(1-\left|\frac{\zeta}{\zeta-1}\right|^2\right)^{-N}\\ &\le C|\zeta-1|^{2N-1} \le C(1+|\zeta|)^{2N-1}.\end{aligned}$$

Similarly, if $\operatorname{Re}\zeta \ge 0$, then

$$1-\left|\frac{\zeta}{\zeta+1}\right|^2 = \frac{1+2\operatorname{Re}\zeta}{|\zeta+1|^2} \ge \frac{1}{|\zeta+1|^2}.$$

The second representation of h gives

$$|h(\zeta)| \le C|\zeta+1|^{2N-1} \le C(1+|\zeta|)^{2N-1}.$$

Thus

$$|h(\zeta)| \le C(1+|\zeta|)^{2N-1}, \qquad \zeta\in\mathbb{C}.$$

For every integer $k > 2N-1$, Cauchy's estimates imply

$$|h^{(k)}(0)| \le \frac{k!}{R^k}\max_{|\zeta|=R}|h(\zeta)| \le Ck!\frac{(1+R)^{2N-1}}{R^k} \longrightarrow 0$$

as $R\to\infty$. Therefore h is a polynomial of degree at most $2N-1$.

Finally, for $z\in\mathbb{D}$, the substitution $\zeta = z/(z-1)$ in the first representation of h gives

$$g(z,0) = \frac{1}{z-1}h\left(\frac{z}{z-1}\right).$$

Likewise, substituting $\zeta = z/(1-z)$ into the second representation gives

$$g(0,z) = \frac{1}{z-1}h\left(\frac{z}{1-z}\right).$$

This proves Claim 1. ■

Claim 2. There exists a polynomial $P_0\in\mathbb{C}[x]$ such that

$$b_{m,0} = P_0(m), \qquad m\ge 0.$$

Proof of Claim 2. Write $h(\xi) = \sum_{j=0}^{d} c_j\xi^j$. By Claim 1,

$$g(z,0) = \frac{1}{z-1}h\left(\frac{z}{z-1}\right) = \sum_{j=0}^{d} c_j\frac{z^j}{(z-1)^{j+1}}.$$

For each $j\ge 0$, the binomial-series identity gives

$$\frac{1}{(1-z)^{j+1}} = \sum_{\ell\ge 0}\binom{\ell+j}{j}z^\ell, \qquad |z|<1.$$

Multiplying by $(-1)^{j+1}z^j$ and setting $m=\ell+j$, we obtain

$$\frac{z^j}{(z-1)^{j+1}} = (-1)^{j+1}\sum_{m\ge j}\binom{m}{j}z^m.$$

Substituting these expansions into the expression for $g(z,0)$ and collecting coefficients, we find

$$g(z,0) = \sum_{m\geq 0}\left(\sum_{j=0}^{d}(-1)^{j+1}c_j\binom{m}{j}\right)z^m,$$

where $\binom{m}{j} = 0$ for $m < j$. Comparing this with $g(z,0) = \sum_{m\geq 0} b_{m,0}z^m$ gives

$$b_{m,0} = \sum_{j=0}^{d}(-1)^{j+1}c_j\binom{m}{j}.$$

Thus

$$P_0(x) = \sum_{j=0}^{d}(-1)^{j+1}c_j\binom{x}{j}$$

has the required values. Here $\binom{x}{0} = 1$, $\binom{x}{j} = \frac{x(x-1)\cdots(x-j+1)}{j!}$, $j \geq 1$, so P_0 is indeed a polynomial. This proves Claim 2. ∎

Claim 3. There exists a polynomial $P \in \mathbb{C}[x,y]$ such that

$$b_{m,n} = P(m,n), \qquad m,n \geq 0.$$

Proof of Claim 3. If $P_0 = 0$, (3.5) implies inductively that $b_{m,n} = 0$ for all $m,n \geq 0$, and we take $P = 0$. Otherwise, put $d_0 = \deg P_0$ and define the forward-difference operator $\Delta : \mathbb{C}[x] \to \mathbb{C}[x]$ by

$$(\Delta Q)(x) = Q(x+1) - Q(x), \qquad Q \in \mathbb{C}[x].$$

Rewriting (3.5) as

$$b_{m,n+1} = 2b_{m,n} - b_{m+1,n},$$

we first prove that increasing the second index corresponds to applying $I - \Delta$ to the polynomial in the first index. Let

$$((I-\Delta)Q)(x) = 2Q(x) - Q(x+1), \qquad Q \in \mathbb{C}[x]. \tag{3.7}$$

where I denotes the identity operator on $\mathbb{C}[x]$. We show by induction on n that

$$b_{m,n} = \big((I-\Delta)^n P_0\big)(m), \qquad m,n \geq 0. \tag{3.8}$$

For $n = 0$, this is the identity established in Claim 2. Suppose that the formula holds for some $n \geq 0$ and every $m \geq 0$. By (3.5),

$$\begin{aligned} b_{m,n+1} &= 2b_{m,n} - b_{m+1,n} \\ &= 2\big((I-\Delta)^n P_0\big)(m) - \big((I-\Delta)^n P_0\big)(m+1) \\ &= \big((I-\Delta)((I-\Delta)^n P_0)\big)(m) \\ &= \big((I-\Delta)^{n+1} P_0\big)(m). \end{aligned}$$

In the third equality, we applied (3.7) with $Q = (I-\Delta)^n P_0$ and $x = m$. This completes the induction.

Each application of Δ lowers the degree of a nonconstant polynomial by one and annihilates constants. Hence $\Delta^j P_0 = 0$ for $j > d_0$. Applying the binomial theorem to $(I - \Delta)^n$ in (3.8), we obtain

$$b_{m,n} = \sum_{j=0}^{d_0} (-1)^j \binom{n}{j} (\Delta^j P_0)(m)$$

with $\binom{n}{j} = 0$ when $j > n$. Therefore,

$$P(x, y) = \sum_{j=0}^{d_0} (-1)^j \binom{y}{j} (\Delta^j P_0)(x)$$

is a polynomial satisfying $P(m, n) = b_{m,n}$ for every $m, n \geq 0$. This proves Claim 3 and completes the proof of the lemma. □

3.3. **Divergence of weighted polynomial sums.** To prove the sufficiency direction of Theorem 3.1(iv), it suffices to show that $[2 - z_1 - z_2]_{\boldsymbol{\alpha}}^{\perp} = \{0\}$. By Lemmas 3.3 and 3.4, this follows if, under the stated parameter conditions, no nonzero polynomial P satisfies

$$\sum_{m,n \geq 0} \frac{|P(m, n)|^2}{w_{\boldsymbol{\alpha}}(m, n)} < \infty.$$

The next lemma establishes this fact.

Lemma 3.5. *Let $\boldsymbol{\alpha} \in \mathbb{R}^3$ and let $P \in \mathbb{C}[x, y]$ be nonzero. If at least one of the inequalities*

$$\alpha_1 + \alpha_3 \leq 1, \qquad \alpha_2 + \alpha_3 \leq 1, \qquad \alpha_1 + \alpha_2 + \alpha_3 \leq 2$$

holds, then

$$\sum_{m,n \geq 0} \frac{|P(m, n)|^2}{w_{\boldsymbol{\alpha}}(m, n)} = \infty.$$

Proof. Suppose first that $\alpha_1 + \alpha_3 \leq 1$. Choose a nonnegative integer n_0 such that $P(\cdot, n_0)$ is a nonzero polynomial, and let its degree be d_0. Such an n_0 exists because a nonzero polynomial cannot vanish identically on every horizontal line with nonnegative integer height. For all sufficiently large m,

$$|P(m, n_0)| \gtrsim m^{d_0}, \qquad w_{\boldsymbol{\alpha}}(m, n_0) \asymp m^{\alpha_1 + \alpha_3}.$$

Thus, for some sufficiently large integer M,

$$\sum_{m \geq M} \frac{|P(m, n_0)|^2}{w_{\boldsymbol{\alpha}}(m, n_0)} \gtrsim \sum_{m \geq M} m^{2d_0 - \alpha_1 - \alpha_3} = \infty,$$

since $2d_0 - \alpha_1 - \alpha_3 \geq -1$. The case $\alpha_2 + \alpha_3 \leq 1$ follows by symmetry.

Suppose next that $\alpha_1 + \alpha_2 + \alpha_3 \leq 2$. Let $d = \deg P$ and let P_d be its homogeneous part of degree d. The polynomial $P_d(1, t)$ is nonzero, so there exist $0 < c_1 < c_2$ and $\delta > 0$ such that

$$|P_d(1, t)| \geq \delta, \qquad t \in [c_1, c_2].$$

Since $P - P_d$ has degree at most $d-1$ and $n/m \in [c_1, c_2]$, there is a constant $C > 0$, independent of m, n, such that

$$\left| m^{-d} P(m,n) - P_d(1, n/m) \right| \le \frac{C}{m}, \qquad m \ge 1, \quad c_1 m \le n \le c_2 m.$$

Thus the error tends to zero as $m \to \infty$, with the same bound for every integer n in the stated range. Hence $|P(m,n)| \gtrsim m^d$ on this region for sufficiently large m. Moreover,

$$w_{\boldsymbol{\alpha}}(m,n) \asymp m^{\alpha_1+\alpha_2+\alpha_3} \quad \text{when } c_1 m \le n \le c_2 m.$$

For all sufficiently large m,

$$\#\{n \in \mathbb{Z} : c_1 m \le n \le c_2 m\} = \lfloor c_2 m \rfloor - \lceil c_1 m \rceil + 1 \ge \frac{c_2 - c_1}{2}\, m.$$

Consequently, choosing M sufficiently large, we obtain

$$\begin{aligned}
\sum_{m,n \ge 0} \frac{|P(m,n)|^2}{w_{\boldsymbol{\alpha}}(m,n)} &\ge \sum_{m \ge M} \sum_{\lceil c_1 m \rceil \le n \le \lfloor c_2 m \rfloor} \frac{|P(m,n)|^2}{w_{\boldsymbol{\alpha}}(m,n)} \\
&\gtrsim \sum_{m \ge M} \sum_{\lceil c_1 m \rceil \le n \le \lfloor c_2 m \rfloor} m^{2d-\alpha_1-\alpha_2-\alpha_3} \\
&= \sum_{m \ge M} \big(\lfloor c_2 m \rfloor - \lceil c_1 m \rceil + 1 \big) m^{2d-\alpha_1-\alpha_2-\alpha_3} \\
&\gtrsim \sum_{m \ge M} m^{2d+1-\alpha_1-\alpha_2-\alpha_3} = \infty.
\end{aligned}$$

This completes the proof of the lemma. □

3.4. **Completion of the proof of Theorem 3.1.**

Proof of Theorem 3.1(iv). Rotations of the variables preserve the norm of $\mathscr{D}_{\boldsymbol{\alpha}}$ and send polynomial multiples to polynomial multiples. It therefore suffices to consider $p_0 = 2 - z_1 - z_2$.

Suppose that none of the inequalities in (3.1) holds. By Proposition 2.2,

$$\sum_{m,n \ge 0} \frac{1}{w_{\boldsymbol{\alpha}}(m,n)} < \infty.$$

Hence by the bounded-evaluation criterion established before Proposition 2.2, evaluation at $(1,1)$ defines a bounded linear functional on $\mathscr{D}_{\boldsymbol{\alpha}}$. It vanishes on every polynomial multiple of p_0, but not on the constant function 1, so p_0 is noncyclic.

Conversely, suppose that at least one inequality in (3.1) holds. Let $f \in [p_0]_{\boldsymbol{\alpha}}^{\perp}$ and write

$$f(z_1, z_2) = \sum_{m,n \ge 0} \frac{b_{m,n}}{w_{\boldsymbol{\alpha}}(m,n)} z_1^m z_2^n.$$

By Lemma 3.3, the array $(b_{m,n})$ satisfies the recurrence (3.3) and the summability condition (3.4). Lemma 3.4 then gives $b_{m,n} = P(m,n)$ for a polynomial $P \in \mathbb{C}[x,y]$. By Lemma 3.5, the summability condition forces $P = 0$. Hence $f = 0$, so $[p_0]_{\boldsymbol{\alpha}}^{\perp} = \{0\}$. It follows that p_0 is cyclic, completing the proof. □

4. The case $S \leq 1$ of Theorem 1.4

In this section, we prove Theorem 1.4(i). We begin with a standard criterion for weak convergence in reproducing kernel Hilbert spaces. For a Hilbert space $\mathcal{H}$, we write $f_k \rightharpoonup f$ if

$$\langle f_k, g\rangle_{\mathcal{H}} \longrightarrow \langle f, g\rangle_{\mathcal{H}} \qquad \text{for every } g \in \mathcal{H}.$$

Lemma 4.1. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space of functions on a set X. Suppose that $\{f_k\}_{k\geq 1} \subseteq \mathcal{H}$ satisfies $\sup_{k\geq 1} \|f_k\|_{\mathcal{H}} < \infty$ and $f_k(x) \longrightarrow f(x)$, $x \in X$, for some $f \in \mathcal{H}$. Then $f_k \rightharpoonup f$ in $\mathcal{H}$.*

Proof. For $x \in X$, let K_x denote the reproducing kernel at x. Then

$$\langle f_k, K_x\rangle_{\mathcal{H}} = f_k(x) \longrightarrow f(x) = \langle f, K_x\rangle_{\mathcal{H}}.$$

Hence $\langle f_k, g\rangle_{\mathcal{H}} \longrightarrow \langle f, g\rangle_{\mathcal{H}}$ for every $g \in \operatorname{span}\{K_x : x \in X\}$. By [16, Proposition 2.1], the linear span of $\{K_x : x \in X\}$ is dense in $\mathcal{H}$. Since $\sup_{k\geq 1} \|f_k\|_{\mathcal{H}} < \infty$, the convergence extends to every $g \in \mathcal{H}$. □

We can now prove Theorem 1.4(i).

Theorem 4.2. *Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, and A, B, S be as in (1.2). Let $p \in \mathbb{C}[z_1, z_2]$ be irreducible, depend on both variables, and have no zeros in $\mathbb{D}^2$. If $S \leq 1$, then p is cyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$.*

Proof. Choose $r_k \uparrow 1$ and set

$$F_k(z_1, z_2) = \frac{p(z_1, z_2)}{p(r_k z_1, z_2)}.$$

Since $A + \alpha_2 = \alpha_1 + B = S \leq 1$, [12, Lemma 2] gives $F_k \rightharpoonup 1$ in both $\mathfrak{D}_{(A,\alpha_2)}$ and $\mathfrak{D}_{(\alpha_1,B)}$. Since weakly convergent sequences are bounded in norm by the uniform boundedness principle, we obtain

$$\sup_k \|F_k\|_{\mathfrak{D}_{(A,\alpha_2)}} < \infty, \qquad \sup_k \|F_k\|_{\mathfrak{D}_{(\alpha_1,B)}} < \infty.$$

For every $\alpha_3 \in \mathbb{R}$, Lemma 2.5(i)–(ii) gives

$$\|F_k\|^2_{\boldsymbol{\alpha}} \lesssim \|F_k\|^2_{\mathfrak{D}_{(A,\alpha_2)}} + \|F_k\|^2_{\mathfrak{D}_{(\alpha_1,B)}}.$$

Thus (F_k) is bounded in $\mathscr{D}_{\boldsymbol{\alpha}}$. Since $F_k \to 1$ pointwise on $\mathbb{D}^2$, Lemma 4.1 implies that $F_k \rightharpoonup 1$ in $\mathscr{D}_{\boldsymbol{\alpha}}$.

We conclude as in the proof of [12, Theorem 4]. Namely, by Lemma 2.1, $1/p(r_k z_1, z_2)$ is holomorphic on a neighborhood of $\overline{\mathbb{D}}^2$. Its Taylor coefficients decay exponentially by Cauchy's estimates, whereas $w_{\boldsymbol{\alpha}}(m, n)$ grows at most polynomially in $m + n$. Consequently, $1/p(r_k z_1, z_2)$ belongs to $\mathscr{D}_{\boldsymbol{\alpha}}$. Polynomial approximation of this reciprocal and boundedness of multiplication by p show that $F_k \in [p]_{\boldsymbol{\alpha}}$. Since $[p]_{\boldsymbol{\alpha}}$ is weakly closed, its weak limit 1 also belongs to $[p]_{\boldsymbol{\alpha}}$. Therefore, p is cyclic. □

5. Polynomials having finitely many zeros on $\mathbb{T}^2$

In this section, we characterize the cyclicity of polynomials in $\mathscr{D}_{\boldsymbol{\alpha}}$ having finitely many zeros on $\mathbb{T}^2$. We first construct a quotient belonging to $\mathscr{D}_{\boldsymbol{\alpha}}$.

Lemma 5.1. *Let $p \in \mathbb{C}[z_1, z_2]$ be irreducible, depend on both variables, and have no zeros in $\mathbb{D}^2$. Suppose that*

$$\mathcal{Z}(p) \cap \mathbb{T}^2 = \{(\zeta_j, \eta_j) : j = 1, \dots, N\}, \qquad N \geq 1,$$

and define

$$L_j(z_1, z_2) = 2 - \overline{\zeta_j} z_1 - \overline{\eta_j} z_2.$$

Then, for every $\boldsymbol{\alpha} \in \mathbb{R}^3$, there exists an integer $M \geq 1$ such that

$$Q_M = \frac{\prod_{j=1}^N L_j^M}{p} \in \mathscr{D}_{\boldsymbol{\alpha}}.$$

Proof. Choose an integer $s \geq 1$ such that

$$2s \geq \max\{\alpha_1^+ + \alpha_3^+, \alpha_2^+ + \alpha_3^+\}, \qquad t^+ = \max\{t, 0\}.$$

By [14, Lemma 4.2], for sufficiently large M, the boundary quotient Q_M extends to a function in $C^{2s}(\mathbb{T}^2)$. Then, the Fourier coefficients of Q_M satisfy

$$\sum_{m,n \geq 0} |\widehat{Q}_M(m,n)|^2 (m+1)^{2s} (n+1)^{2s} < \infty.$$

Since $m + n + 1 \leq (m+1)(n+1)$ for $m, n \geq 0$, our choice of s gives

$$\begin{aligned} w_{\boldsymbol{\alpha}}(m,n) &\leq (m+1)^{\alpha_1^+} (n+1)^{\alpha_2^+} (m+n+1)^{\alpha_3^+} \\ &\leq (m+1)^{\alpha_1^+ + \alpha_3^+} (n+1)^{\alpha_2^+ + \alpha_3^+} \\ &\leq (m+1)^{2s} (n+1)^{2s}. \end{aligned}$$

Consequently,

$$\|Q_M\|_{\boldsymbol{\alpha}}^2 = \sum_{m,n \geq 0} w_{\boldsymbol{\alpha}}(m,n) |\widehat{Q}_M(m,n)|^2 < \infty,$$

and $Q_M \in \mathscr{D}_{\boldsymbol{\alpha}}$. □

The following theorem characterizes cyclicity in $\mathscr{D}_{\boldsymbol{\alpha}}$ for polynomials with a nonempty finite zero set on $\mathbb{T}^2$.

Theorem 5.2. *Let $\boldsymbol{\alpha} \in \mathbb{R}^3$, and let A, B, S be as in (1.2). Let $p \in \mathbb{C}[z_1, z_2]$ be irreducible, depend on both variables, and have no zeros in $\mathbb{D}^2$. Suppose that*

$$0 < \#\big(\mathcal{Z}(p) \cap \mathbb{T}^2\big) < \infty.$$

Then p is cyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$ if and only if $A \leq 1$ or $B \leq 1$ or $S \leq 2$.

Proof. If $A > 1$, $B > 1$, and $S > 2$, then Corollary 2.3 implies that p is noncyclic, since it has a zero on $\mathbb{T}^2$. This proves necessity.

Conversely, suppose that $A \leq 1$ or $B \leq 1$ or $S \leq 2$. Write

$$\mathcal{Z}(p) \cap \mathbb{T}^2 = \{(\zeta_j, \eta_j) : j = 1, \dots, N\}$$

and define

$$L_j(z_1, z_2) = 2 - \overline{\zeta_j} z_1 - \overline{\eta_j} z_2, \qquad j = 1, \dots, N.$$

By Theorem 3.1(iv), each L_j is cyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$. Moreover, Lemma 5.1 gives a positive integer M such that

$$Q = \frac{\prod_{j=1}^N L_j^M}{p} \in \mathscr{D}_{\boldsymbol{\alpha}}.$$

Polynomial approximation of Q and boundedness of multiplication by p imply that

$$\prod_{j=1}^{N} L_j^M = pQ \in [p]_{\boldsymbol{\alpha}}.$$

A finite product of cyclic polynomial multipliers is cyclic, so $\prod_{j=1}^{N} L_j^M$ is cyclic. Thus the closed invariant subspace $[p]_{\boldsymbol{\alpha}}$ contains a cyclic polynomial, and hence p is cyclic. □

Remark 5.3. Note that if $\max\{A, B\} \leq 1$, the we can use the model polynomials $z_i - \zeta_i$, $\zeta_i \in \mathbb{T}$ to prove the cyclicity of p in Theorem 5.2.

6. Polynomials having infinitely many zeros on $\mathbb{T}^2$

We prove noncyclicity of a polynomial p in $\mathscr{D}_{\boldsymbol{\alpha}}$ when $\alpha_1 + \alpha_2 + \alpha_3 > 1$ by constructing a bounded linear functional supported on a smooth arc of zeros of p on $\mathbb{T}^2$. This functional annihilates every polynomial multiple of p but does not vanish on the constant function 1.

We first recall the geometric argument for the existence of a suitable arc from [1, Section 2.2 and proof of Theorem 5.2]. An essential additional step is to show that its angular parametrization $t \mapsto (e^{it}, e^{i\varphi(t)})$ can be chosen with $-C \leq \varphi'(t) \leq -c < 0$. In particular, the derivative is uniformly bounded away from zero. These bounds allow us to combine integration by parts with Parseval's identity to establish boundedness of the functional when $\alpha_1 + \alpha_2 + \alpha_3 > 1$.

6.1. **Local geometry of the toral zero set.**

The following proposition establishes a geometric property of polynomial zero sets that will be used in the proof of the main result of this section.

Proposition 6.1. *Let $p \in \mathbb{C}[z_1, z_2]$ be irreducible, depend on both variables, have no zeros in $\mathbb{D}^2$, and have infinitely many zeros on $\mathbb{T}^2$. Then there exist a nonempty open interval $I \subset \mathbb{R}$, a function $\varphi \in C^\infty(I; \mathbb{R})$, and constants $c, C > 0$ such that:*

(i) The smooth arc

$$\Gamma = \{(e^{it}, e^{i\varphi(t)}) : t \in I\}$$

is contained in $\mathcal{Z}(p) \cap \mathbb{T}^2$.

(ii) The derivative of φ satisfies

$$-C \leq \varphi'(t) \leq -c < 0, \qquad t \in I. \tag{6.1}$$

Proof. Part (i) follows from the geometric discussion in [1, Section 2.2 and proof of Theorem 5.2]. For the reader's convenience, we include the argument, also ensuring that the angular derivative is nonzero.

The set $\mathcal{Z}(p) \cap \mathbb{T}^2$ is semialgebraic. By [5, Proposition 2.9.10], it is a finite disjoint union of connected Nash submanifolds, which are real analytic. Since this set is infinite, at least one of these submanifolds has positive dimension and therefore contains a regular real-analytic arc. Moreover, since p is irreducible and depends on both variables, each partial derivative p_{z_j} is nonzero, has smaller degree than p, and is therefore relatively prime to p. By

Bézout's theorem, the sets $\mathcal{Z}(p) \cap \mathcal{Z}(p_{z_j})$, $j = 1, 2$, are finite. We may thus restrict the arc so that both partial derivatives are nonzero at every point of it.

Using local angular coordinates, write the arc as

$$\gamma(s) = \big(e^{i\theta_1(s)}, e^{i\theta_2(s)}\big), \qquad s \in J,$$

where $J \subset \mathbb{R}$ is a nonempty open interval, θ_1, θ_2 are real analytic, and $\gamma'(s) \neq 0$. Differentiating $p(\gamma(s)) = 0$ gives

$$p_{z_1}(\gamma(s))\, ie^{i\theta_1(s)}\theta_1'(s) + p_{z_2}(\gamma(s))\, ie^{i\theta_2(s)}\theta_2'(s) = 0.$$

Since both partial derivatives are nonzero, the vanishing of either $\theta_1'(s)$ or $\theta_2'(s)$ would force the other to vanish, contradicting $\gamma'(s) \neq 0$. Thus both angular derivatives are nonzero.

Since θ_1' is continuous and nonzero on J, θ_1 is strictly monotone and has a real-analytic inverse on $I_0 = \theta_1(J)$. We may therefore use $t = \theta_1(s)$ as a parameter. Setting $\varphi = \theta_2 \circ \theta_1^{-1}$, we obtain

$$\{(e^{it}, e^{i\varphi(t)}) : t \in I_0\} \subseteq \mathcal{Z}(p) \cap \mathbb{T}^2,$$

where φ is real analytic and

$$\varphi'(\theta_1(s)) = \frac{\theta_2'(s)}{\theta_1'(s)} \neq 0, \qquad s \in J.$$

This establishes (i).

To prove (ii), we first show that $\varphi' < 0$. Since $p_{z_2} \neq 0$ on the arc, the holomorphic implicit function theorem, after restricting I_0 if necessary, provides a nonvanishing holomorphic function h on a neighborhood of $\{e^{it} : t \in I_0\}$ such that

$$p(z, h(z)) = 0, \qquad h(e^{it}) = e^{i\varphi(t)}.$$

For each fixed $t \in I_0$ and $r < 1$ sufficiently close to 1, we have

$$p(re^{it}, h(re^{it})) = 0.$$

Since $|re^{it}| < 1$ and p has no zeros in $\mathbb{D}^2$, it follows that $|h(re^{it})| \geq 1$. On the other hand, $|h(e^{it})| = 1$. Consequently,

$$\frac{\partial}{\partial r} \log|h(re^{it})|\bigg|_{r=1} = \lim_{r\uparrow 1} \frac{\log|h(re^{it})| - \log|h(e^{it})|}{r-1} \leq 0.$$

Here the derivative exists because h is holomorphic and nonvanishing near e^{it}.

Differentiating the identity $h(e^{it}) = e^{i\varphi(t)}$ gives

$$ie^{it}h'(e^{it}) = i\varphi'(t)h(e^{it}).$$

Dividing by $ih(e^{it})$ and taking real parts, we obtain

$$\varphi'(t) = \operatorname{Re}\frac{e^{it}h'(e^{it})}{h(e^{it})} = \frac{\partial}{\partial r}\log|h(re^{it})|\bigg|_{r=1} \leq 0.$$

Since $\varphi'(t) \neq 0$, we conclude that $\varphi'(t) < 0$ throughout I_0.

Finally, choose a nonempty open interval I whose closure is a compact subset of I_0. Continuity and strict negativity of φ' give

$$c = \min_{t\in\overline{I}}\big(-\varphi'(t)\big) > 0, \qquad C = \max_{t\in\overline{I}}\big(-\varphi'(t)\big) < \infty.$$

Restricting φ to I, these constants satisfy (6.1), proving (ii). □

Remark 6.2. Let I, φ, c, C be as in Proposition 6.1. By part (ii), for $m, n \geq 0$, the derivative of $t \mapsto mt + n\varphi(t)$ can vanish at a point of I only if

$$cn \leq m \leq Cn.$$

In this region, $m + 1 \asymp n + 1 \asymp m + n + 1$, and consequently

$$w_{\boldsymbol{\alpha}}(m, n) \asymp (n + 1)^{\alpha_1+\alpha_2+\alpha_3}.$$

This comparison will be used in the proof of Lemma 6.4 below to establish boundedness of the functional supported on the arc when $\alpha_1 + \alpha_2 + \alpha_3 > 1$.

6.2. **A bounded functional supported on an arc.**

Lemma 6.3. *Let $I \subset \mathbb{R}$ be an open interval, and let $\varphi \in C^\infty(I; \mathbb{R})$ satisfy $-C \leq \varphi' \leq -c < 0$ for some $c, C > 0$. For $\chi \in C_c^\infty(I)$, set*

$$c_{m,n} = \int_I e^{i(mt+n\varphi(t))}\chi(t)\, dt, \qquad m, n \geq 0.$$

Choose $0 < c_0 < c$ and $C_0 > C$, and define

$$\mathcal{C} = \{(m, n) \in \mathbb{N}_0^2 : c_0 n \leq m \leq C_0 n\}.$$

For every integer $N \geq 1$, there is a constant $K_N > 0$ such that

$$|c_{m,n}| \leq K_N(1 + m + n)^{-N}, \qquad (m, n) \notin \mathcal{C}.$$

Proof. We divide the argument into three claims. The first gives a lower bound for the derivative outside the cone $\mathcal{C}$.

Claim 1. For every $(m, n) \in \mathbb{N}_0^2 \setminus \mathcal{C}$ and $t \in I$,

$$|m + n\varphi'(t)| \geq \delta(m + n),$$

where

$$\delta = \min\left\{\frac{c - c_0}{1 + c_0}, \frac{C_0 - C}{1 + C_0}\right\} > 0$$

is independent of m, n, and t.

Proof of Claim 1. If $(m, n) \notin \mathcal{C}$, then either $m < c_0 n$ or $m > C_0 n$.

If $m < c_0 n$, then $m + n\varphi'(t) < 0$. Since $m + n \leq (1 + c_0)n$, we obtain

$$\begin{aligned} |m + n\varphi'(t)| &= -m - n\varphi'(t) \\ &\geq cn - m \\ &\geq (c - c_0)n \\ &\geq \frac{c - c_0}{1 + c_0}(m + n). \end{aligned}$$

If $m > C_0 n$, then $m + n\varphi'(t) > 0$. Since $n \leq (m + n)/(1 + C_0)$, we obtain

$$\begin{aligned} |m + n\varphi'(t)| &= m + n\varphi'(t) \\ &\geq m - Cn \\ &= (m + n) - (C + 1)n \\ &\geq \frac{C_0 - C}{1 + C_0}(m + n). \end{aligned}$$

The claim follows from the definition of δ. ■

For $(m,n) \in \mathbb{N}_0^2 \setminus \mathcal{C}$, set

$$\rho = m + n, \qquad \Psi_{m,n}(t) = \frac{mt + n\varphi(t)}{\rho}.$$

Since $(0,0) \in \mathcal{C}$, we have $\rho \geq 1$. We next establish derivative bounds for $\Psi_{m,n}$ that are uniform in m, n.

Claim 2. For every $(m,n) \in \mathbb{N}_0^2 \setminus \mathcal{C}$, we have

$$|\Psi'_{m,n}(t)| \geq \delta, \qquad t \in I.$$

Moreover, there are constants $B_k, D_k > 0$, independent of m, n, such that

$$\sup_{t \in \operatorname{supp}\chi} |\Psi^{(k)}_{m,n}(t)| \leq B_k, \quad k \geq 1,$$

$$\sup_{t \in \operatorname{supp}\chi} \left| \frac{d^k}{dt^k} \left(\frac{1}{\Psi'_{m,n}(t)} \right) \right| \leq D_k, \quad k \geq 0.$$

Proof of Claim 2. By Claim 1,

$$|\Psi'_{m,n}(t)| = \frac{|m + n\varphi'(t)|}{m+n} \geq \delta, \qquad t \in I.$$

Differentiating $\Psi_{m,n}$ gives

$$\Psi'_{m,n}(t) = \frac{m}{m+n} + \frac{n}{m+n}\varphi'(t),$$
$$\Psi^{(k)}_{m,n}(t) = \frac{n}{m+n}\varphi^{(k)}(t), \qquad k \geq 2.$$

Since $0 \leq m/(m+n), n/(m+n) \leq 1$, we have $|\Psi'_{m,n}(t)| \leq 1 + C$ and $|\Psi^{(k)}_{m,n}(t)| \leq |\varphi^{(k)}(t)|$ for $k \geq 2$. Each derivative of φ is continuous and therefore bounded on the compact set $\operatorname{supp}\chi$. Thus, for every integer $k \geq 1$, there is a constant $B_k > 0$, independent of m, n, such that

$$\sup_{t \in \operatorname{supp}\chi} |\Psi^{(k)}_{m,n}(t)| \leq B_k.$$

The bound $|\Psi'_{m,n}| \geq \delta$ gives

$$\sup_{t \in \operatorname{supp}\chi} \left| \frac{1}{\Psi'_{m,n}(t)} \right| \leq \frac{1}{\delta} =: D_0.$$

For every integer $k \geq 1$, repeated differentiation gives

$$\frac{d^k}{dt^k} \left(\frac{1}{\Psi'_{m,n}} \right) = \sum_{\ell=1}^{k} \sum_{\substack{j_1 + \cdots + j_\ell = k \\ j_1, \ldots, j_\ell \geq 1}} \frac{(-1)^\ell k!}{j_1! \cdots j_\ell!} \frac{\prod_{r=1}^{\ell} \Psi^{(j_r+1)}_{m,n}}{(\Psi'_{m,n})^{\ell+1}}.$$

On $\operatorname{supp}\chi$, each numerator is bounded by $\prod_{r=1}^{\ell} B_{j_r+1}$, while the absolute value of the denominator is at least $\delta^{\ell+1}$. Since the sum is finite for each fixed k, there is a constant $D_k > 0$, independent of m, n, such that

$$\sup_{t \in \operatorname{supp}\chi} \left| \frac{d^k}{dt^k} \left(\frac{1}{\Psi'_{m,n}(t)} \right) \right| \leq D_k.$$

Together with the estimate for D_0, this proves Claim 2. ∎

The final claim gives an integral representation obtained by repeated integration by parts.

Claim 3. For every integer $N \geq 1$ and every $(m,n) \in \mathbb{N}_0^2 \setminus \mathcal{C}$, there is a function $b_N \in C_c^\infty(I)$, supported in $\operatorname{supp}\chi$, such that

$$c_{m,n} = \rho^{-N} \int_I e^{i\rho\Psi_{m,n}(t)} b_N(t)\,dt, \qquad \int_I |b_N(t)|\,dt \leq C_N.$$

Here b_N may depend on m,n, but the constant C_N does not.

Proof of Claim 3. Fix $(m,n) \in \mathbb{N}_0^2 \setminus \mathcal{C}$. Using

$$e^{i\rho\Psi_{m,n}(t)} = \frac{1}{i\rho\Psi'_{m,n}(t)} \frac{d}{dt} e^{i\rho\Psi_{m,n}(t)},$$

integration by parts gives

$$\begin{aligned} c_{m,n} &= \frac{1}{\rho} \int_I \frac{\chi(t)}{i\Psi'_{m,n}(t)} \frac{d}{dt} e^{i\rho\Psi_{m,n}(t)}\,dt \\ &= -\frac{1}{\rho} \int_I e^{i\rho\Psi_{m,n}(t)} \frac{d}{dt}\left(\frac{\chi(t)}{i\Psi'_{m,n}(t)}\right) dt. \end{aligned}$$

There is no boundary term because χ is compactly supported in I.

Define recursively

$$b_0 = \chi, \qquad b_{j+1} = -\frac{d}{dt}\left(\frac{b_j}{i\Psi'_{m,n}}\right), \qquad j \geq 0.$$

Each b_j is smooth and supported in $\operatorname{supp}\chi$, so no boundary terms arise in subsequent integrations. After N integrations by parts, we obtain

$$c_{m,n} = \rho^{-N} \int_I e^{i\rho\Psi_{m,n}(t)} b_N(t)\,dt.$$

It remains to bound the integral of $|b_N|$ uniformly in m,n. Set $K = \operatorname{supp}\chi$. By the Leibniz rule and the bounds D_ℓ from Claim 2,

$$\sup_{t\in K} |b_{j+1}^{(k)}(t)| \leq \sum_{\ell=0}^{k+1} \binom{k+1}{\ell} D_\ell \sup_{t\in K} |b_j^{(k+1-\ell)}(t)|, \qquad j,k \geq 0.$$

Since $b_0 = \chi$ is fixed and all its derivatives are bounded on K, induction on j shows that every derivative of b_j is uniformly bounded on K, independently of m,n. In particular, there is a constant $E_N > 0$ such that

$$\sup_{t\in K} |b_N(t)| \leq E_N.$$

Because b_N is supported in K, we conclude that

$$\int_I |b_N(t)|\,dt \leq |K| E_N =: C_N,$$

where $|K|$ denotes the Lebesgue measure of K. The constant C_N is independent of m,n. ∎

Finally, since $\rho = m+n \geq 1$ outside $\mathcal{C}$, we have $1+\rho \leq 2\rho$. Thus Claim 3 and $|e^{i\rho\Psi_{m,n}(t)}| = 1$ give

$$|c_{m,n}| \leq C_N \rho^{-N} \leq 2^N C_N (1+\rho)^{-N} = K_N (1+m+n)^{-N},$$

where $K_N = 2^N C_N$ is independent of m,n. This completes the proof. □

We write $I \Subset J$ when the closure of I is a compact subset of J.

The following lemma shows that integration against a smooth density with compact support on an arc satisfying (6.1) defines a bounded linear functional on $\mathscr{D}_{\boldsymbol{\alpha}}$ whenever $\alpha_1 + \alpha_2 + \alpha_3 > 1$.

Lemma 6.4. *Let $\boldsymbol{\alpha} \in \mathbb{R}^3$ and suppose that $S = \alpha_1 + \alpha_2 + \alpha_3 > 1$. Let $I \Subset J = (a, a + 2\pi)$, and let $\varphi \in C^\infty(I; \mathbb{R})$ satisfy $-C \le \varphi' \le -c < 0$ for some $c, C > 0$. For every $\chi \in C_c^\infty(I)$, the functional*

$$\Lambda_\chi(f) = \int_I f(e^{it}, e^{i\varphi(t)})\chi(t)\,dt,$$

initially defined on polynomials, extends continuously to $\mathscr{D}_{\boldsymbol{\alpha}}$.

Proof. For $m \in \mathbb{Z}$ and $n \ge 0$, put

$$c_{m,n} = \int_I e^{i(mt + n\varphi(t))}\chi(t)\,dt.$$

We prove that

$$\sum_{m,n\ge 0} \frac{|c_{m,n}|^2}{w_{\boldsymbol{\alpha}}(m,n)} < \infty. \tag{6.2}$$

Choose $\mathcal{C}$ as in Lemma 6.3. Outside $\mathcal{C}$, that lemma applies with any N. Since

$$w_{\boldsymbol{\alpha}}(m,n)^{-1} \le (1 + m + n)^R, \qquad R = \sum_{j=1}^{3} \max\{-\alpha_j, 0\},$$

choosing $2N - R > 2$ gives

$$\begin{aligned}\sum_{(m,n)\in\mathbb{N}_0^2\setminus\mathcal{C}} \frac{|c_{m,n}|^2}{w_{\boldsymbol{\alpha}}(m,n)} &\lesssim \sum_{m,n\ge 0} (1 + m + n)^{R-2N} \\ &= \sum_{k\ge 0} (k+1)^{R-2N+1} < \infty.\end{aligned}$$

Inside $\mathcal{C}$, we have $m + 1 \asymp n + 1 \asymp m + n + 1$, so

$$w_{\boldsymbol{\alpha}}(m,n) \asymp (n+1)^{\alpha_1+\alpha_2+\alpha_3}.$$

For fixed n, extend $g_n(t) = \chi(t)e^{in\varphi(t)}$ by zero from I to J. With the convention $\widehat{g}_n(k) = (2\pi)^{-1}\int_J g_n(t)e^{-ikt}\,dt$, we have $c_{m,n} = 2\pi\widehat{g}_n(-m)$. Parseval's identity therefore gives

$$\sum_{m\in\mathbb{Z}} |c_{m,n}|^2 = 2\pi \int_J |g_n(t)|^2\,dt = 2\pi\|\chi\|_{L^2(I)}^2.$$

Consequently,

$$\begin{aligned}\sum_{(m,n)\in\mathcal{C}} \frac{|c_{m,n}|^2}{w_{\boldsymbol{\alpha}}(m,n)} &\lesssim \sum_{n\ge 0} (n+1)^{-\alpha_1-\alpha_2-\alpha_3} \sum_{m\in\mathbb{Z}} |c_{m,n}|^2 \\ &= 2\pi\|\chi\|_{L^2(I)}^2 \sum_{n\ge 0} (n+1)^{-\alpha_1-\alpha_2-\alpha_3} < \infty,\end{aligned}$$

because $\alpha_1 + \alpha_2 + \alpha_3 > 1$. This proves (6.2).

For a polynomial $f = \sum a_{m,n} z_1^m z_2^n$, Cauchy–Schwarz now gives

$$|\Lambda_\chi(f)| = \left| \sum_{m,n\geq 0} a_{m,n} c_{m,n} \right| \leq \|f\|_{\boldsymbol{\alpha}} \left(\sum_{m,n\geq 0} \frac{|c_{m,n}|^2}{w_{\boldsymbol{\alpha}}(m,n)} \right)^{1/2}.$$

By density of polynomials, Λ_χ extends uniquely to a bounded linear functional on $\mathscr{D}_{\boldsymbol{\alpha}}$. □

6.3. **Noncyclicity.**

Theorem 6.5. *Let $\boldsymbol{\alpha} \in \mathbb{R}^3$, and let A, B, S be as in (1.2). Let $p \in \mathbb{C}[z_1, z_2]$ be irreducible, depend on both variables, and have no zeros in $\mathbb{D}^2$. Suppose that $\mathcal{Z}(p) \cap \mathbb{T}^2$ is infinite. If $S > 1$, then p is noncyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$.*

Proof. By Proposition 6.1, the set of zeros of p on $\mathbb{T}^2$ contains a smooth arc

$$\Gamma = \{(e^{it}, e^{i\varphi(t)}) : t \in I\}$$

satisfying (6.1). After restricting I if necessary, we may assume that $I \Subset (a, a + 2\pi)$ for some $a \in \mathbb{R}$.

Choose a nonnegative function $\chi \in C_c^\infty(I)$ such that $\int_I \chi(t)\, dt = 1$. Since $S > 1$, Lemma 6.4 shows that the functional

$$\Lambda_\chi(f) = \int_I f(e^{it}, e^{i\varphi(t)}) \chi(t)\, dt,$$

initially defined on polynomials, extends uniquely to a bounded linear functional on $\mathscr{D}_{\boldsymbol{\alpha}}$.

Since p vanishes on Γ, we have

$$\Lambda_\chi(pq) = 0 \quad \text{for every } q \in \mathbb{C}[z_1, z_2], \qquad \Lambda_\chi(1) = 1.$$

By continuity, Λ_χ vanishes on $[p]_{\boldsymbol{\alpha}}$. Hence $1 \notin [p]_{\boldsymbol{\alpha}}$, so p is noncyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$. □

7. Completion of the proof of Theorem 1.4

We combine the results of the preceding sections to complete the proof of Theorem 1.4.

Proof. If $\mathcal{Z}(p) \cap \mathbb{T}^2 = \varnothing$, then Lemma 2.1 shows that p has no zeros on $\overline{\mathbb{D}}^2$. Hence $1/p$ is holomorphic on a neighborhood of the closed bidisk and belongs to $\mathscr{D}_{\boldsymbol{\alpha}}$. It follows that $1 = p(1/p) \in [p]_{\boldsymbol{\alpha}}$, so p is cyclic for every $\boldsymbol{\alpha} \in \mathbb{R}^3$.

Part (i) is Theorem 4.2. For part (ii), Theorem 5.2 gives cyclicity for a nonempty finite toral zero set, and Theorem 6.5 excludes an infinite toral zero set. The empty case was settled above. Finally, under the hypotheses of part (iii), we have $A > 1$, $B > 1$, and $S > 2$, so Corollary 2.3 excludes every nonempty toral zero set. The converse again follows from the empty case. □

8. Final Remarks

We conclude with two further consequences of norm comparisons with anisotropic Dirichlet-type spaces. First, these comparisons give a shorter proof of Theorem 6.5 in two parameter ranges. We then establish a Hilbert-sum representation of $\mathscr{D}_{\boldsymbol{\alpha}}$ when $\alpha_3 \leq 0$.

Proposition 8.1. *Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, and set $S = \alpha_1 + \alpha_2 + \alpha_3$. Let $p \in \mathbb{C}[z_1, z_2]$ be irreducible, depend on both variables, have no zeros in $\mathbb{D}^2$, and have infinitely many zeros on $\mathbb{T}^2$. Then the following statements hold.*

(i) If $\alpha_3 \geq 0$ and $S > 1$, then p is noncyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$.
(ii) If $\alpha_3 < 0$ and $S + \alpha_3 > 1$, then p is noncyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$.

Proof. Suppose first that $\alpha_3 \geq 0$. Fix $t \in [0, \alpha_3]$ and set

$$\beta_1 = \alpha_1 + t, \qquad \beta_2 = \alpha_2 + \alpha_3 - t.$$

Since

$$(m+n+1)^{\alpha_3} \geq (m+1)^t (n+1)^{\alpha_3 - t},$$

we obtain

$$w_{\boldsymbol{\alpha}}(m,n) \geq (m+1)^{\beta_1}(n+1)^{\beta_2}.$$

Thus

$$\mathscr{D}_{\boldsymbol{\alpha}} \hookrightarrow \mathfrak{D}_{(\beta_1,\beta_2)}.$$

Since $\beta_1 + \beta_2 = S > 1$ and p has infinitely many zeros on $\mathbb{T}^2$, Theorem 1.1(ii)–(iii) implies that p is noncyclic in $\mathfrak{D}_{(\beta_1,\beta_2)}$. If p were cyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$, the continuous inclusion would make it cyclic in $\mathfrak{D}_{(\beta_1,\beta_2)}$, a contradiction.

Suppose next that $\alpha_3 < 0$. Since

$$m + n + 1 \leq (m+1)(n+1),$$

we have

$$(m+n+1)^{\alpha_3} \geq (m+1)^{\alpha_3}(n+1)^{\alpha_3}.$$

Consequently,

$$\mathscr{D}_{\boldsymbol{\alpha}} \hookrightarrow \mathfrak{D}_{(\alpha_1+\alpha_3,\alpha_2+\alpha_3)}.$$

The sum of the two anisotropic parameters is

$$\alpha_1 + \alpha_2 + 2\alpha_3 = S + \alpha_3.$$

If $S + \alpha_3 > 1$, then Theorem 1.1(ii)–(iii) implies that p is noncyclic in $\mathfrak{D}_{(\alpha_1+\alpha_3,\alpha_2+\alpha_3)}$. The continuous inclusion again implies that p is noncyclic in $\mathscr{D}_{\boldsymbol{\alpha}}$. □

For $\alpha_3 \leq 0$, the continuous inclusions in Lemma 2.5(i) can be strengthened to the following Hilbert-sum representation.

Proposition 8.2. *Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, and set $A = \alpha_1 + \alpha_3$ and $B = \alpha_2 + \alpha_3$. If $\alpha_3 \leq 0$, then*

$$\mathscr{D}_{\boldsymbol{\alpha}} = \mathfrak{D}_{(A,\alpha_2)} + \mathfrak{D}_{(\alpha_1,B)}$$

with equivalent norms:

$$\|f\|^2_{\boldsymbol{\alpha}} \asymp \inf_{\substack{f=g+h \\ g\in\mathfrak{D}_{(A,\alpha_2)} \\ h\in\mathfrak{D}_{(\alpha_1,B)}}} \left(\|g\|^2_{\mathfrak{D}_{(A,\alpha_2)}} + \|h\|^2_{\mathfrak{D}_{(\alpha_1,B)}} \right).$$

Proof. Put

$$u_{m,n} = (m+1)^A (n+1)^{\alpha_2}, \qquad v_{m,n} = (m+1)^{\alpha_1}(n+1)^B,$$

and let $r = -\alpha_3 \geq 0$. The inequalities

$$\max\{m+1, n+1\} \leq m+n+1 \leq 2\max\{m+1, n+1\}$$

imply that

$$(m+n+1)^r \asymp (m+1)^r + (n+1)^r,$$

with constants independent of m, n. This comparison also holds when $r = 0$. Multiplying by $(m+1)^{-\alpha_1}(n+1)^{-\alpha_2}$, we obtain

$$\begin{aligned} w_{\boldsymbol{\alpha}}(m,n)^{-1} &= (m+1)^{-\alpha_1}(n+1)^{-\alpha_2}(m+n+1)^r \\ &\asymp (m+1)^{-\alpha_1+r}(n+1)^{-\alpha_2} + (m+1)^{-\alpha_1}(n+1)^{-\alpha_2+r} \\ &= (m+1)^{-A}(n+1)^{-\alpha_2} + (m+1)^{-\alpha_1}(n+1)^{-B} \\ &= u_{m,n}^{-1} + v_{m,n}^{-1}. \end{aligned}$$

Here we used $A = \alpha_1 - r$ and $B = \alpha_2 - r$. Therefore

$$w_{\boldsymbol{\alpha}}(m,n) \asymp \frac{u_{m,n} v_{m,n}}{u_{m,n} + v_{m,n}}.$$

We next establish the scalar minimization identity underlying the Hilbert-sum norm. For $u, v > 0$ and $a, b, c \in \mathbb{C}$ with $a + b = c$, completing the square gives

$$u|a|^2 + v|b|^2 = (u+v)\left|a - \frac{v}{u+v}c\right|^2 + \frac{uv}{u+v}|c|^2.$$

The first term is nonnegative and vanishes precisely when $a = vc/(u+v)$. Consequently,

$$\min_{a+b=c}\left(u|a|^2 + v|b|^2\right) = \frac{uv}{u+v}|c|^2,$$

with the minimum attained at

$$a = \frac{v}{u+v}c, \qquad b = \frac{u}{u+v}c.$$

Suppose now that $f = g + h$, where

$$g = \sum_{m,n\geq 0} a_{m,n} z_1^m z_2^n \in \mathfrak{D}_{(A,\alpha_2)},$$

$$h = \sum_{m,n\geq 0} b_{m,n} z_1^m z_2^n \in \mathfrak{D}_{(\alpha_1,B)}.$$

The Taylor coefficients of f are $c_{m,n} = a_{m,n} + b_{m,n}$. Applying the scalar inequality to each coefficient and summing, we find

$$\begin{aligned} \sum_{m,n\geq 0} \frac{u_{m,n} v_{m,n}}{u_{m,n} + v_{m,n}} |c_{m,n}|^2 &\leq \sum_{m,n\geq 0} \left(u_{m,n}|a_{m,n}|^2 + v_{m,n}|b_{m,n}|^2\right) \\ &= \|g\|^2_{\mathfrak{D}_{(A,\alpha_2)}} + \|h\|^2_{\mathfrak{D}_{(\alpha_1,B)}} < \infty. \end{aligned}$$

By the weight comparison, $f \in \mathscr{D}_{\boldsymbol{\alpha}}$. This proves that the sum of the two anisotropic spaces is contained in $\mathscr{D}_{\boldsymbol{\alpha}}$.

Conversely, let

$$f = \sum_{m,n\geq 0} c_{m,n} z_1^m z_2^n \in \mathscr{D}_{\boldsymbol{\alpha}},$$

and choose the coefficientwise minimizers

$$a_{m,n} = \frac{v_{m,n}}{u_{m,n} + v_{m,n}} c_{m,n}, \qquad b_{m,n} = \frac{u_{m,n}}{u_{m,n} + v_{m,n}} c_{m,n}.$$

Since $u_{m,n}, v_{m,n} > 0$, we have $|a_{m,n}| \le |c_{m,n}|$ and $|b_{m,n}| \le |c_{m,n}|$. Thus the power series

$$g = \sum_{m,n\ge 0} a_{m,n} z_1^m z_2^n, \qquad h = \sum_{m,n\ge 0} b_{m,n} z_1^m z_2^n$$

converge absolutely and locally uniformly on $\mathbb{D}^2$, as does the Taylor series of f. They therefore define holomorphic functions, and $a_{m,n} + b_{m,n} = c_{m,n}$ gives $f = g + h$.

For this choice of coefficients, the scalar minimum is attained at every index, so

$$\begin{aligned}\sum_{m,n\ge 0} \left(u_{m,n}|a_{m,n}|^2 + v_{m,n}|b_{m,n}|^2\right) &= \sum_{m,n\ge 0} \frac{u_{m,n}v_{m,n}}{u_{m,n} + v_{m,n}} |c_{m,n}|^2 \\ &\asymp \|f\|_{\boldsymbol{\alpha}}^2 < \infty.\end{aligned}$$

Hence $g \in \mathfrak{D}_{(A,\alpha_2)}$ and $h \in \mathfrak{D}_{(\alpha_1,B)}$, proving the reverse inclusion.

Finally, the lower bound for arbitrary decompositions and the explicit minimizing decomposition give

$$\begin{aligned}&\inf_{\substack{f=g+h \\ g\in\mathfrak{D}_{(A,\alpha_2)} \\ h\in\mathfrak{D}_{(\alpha_1,B)}}} \left(\|g\|^2_{\mathfrak{D}_{(A,\alpha_2)}} + \|h\|^2_{\mathfrak{D}_{(\alpha_1,B)}}\right) \\ &\qquad = \sum_{m,n\ge 0} \frac{u_{m,n}v_{m,n}}{u_{m,n} + v_{m,n}} |c_{m,n}|^2 \asymp \|f\|_{\boldsymbol{\alpha}}^2.\end{aligned}$$

This proves the asserted equality of spaces and equivalence of norms. □

Data Availability. No data were used or generated in this study.

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